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ABSTRACT

SOME PROBLEMS ARISING FROM MATHEMATICAL MODEL OF DUCTAL CARCINOMA IN SITU

Heng Li

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Ductal carcinoma in situ (DCIS) is the earliest form of breast cancer. Three mathematical models in the one dimensional case arising from DCIS are proposed. The first two models are in the form of parabolic equation with initial and known moving boundaries. Direct and inverse problems are considered in model 1, existence and uniqueness are proved by using tool from heat potential theory and Volterra integral equations. Also, we discuss the direct problem and nonlocal problem of model 2, existence and uniqueness are proved. And approximation solution of these problems are implemented by Ritz-Galerkin method, which is a first attempt to deal with such problems. Based on the finding of the previous two models, the more general free boundary problem model – nonlinear parabolic partial differential equation with initial, boundary and free boundary condition is presented. Well-posedness theorems are proved by applying knowledge of semigroup solution operators. Illustrative examples are included to demonstrate the validity and applicability of the technique for all three models.

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CHAPTER I

INTRODUCTION

Ductal carcinoma in situ denotes the initial stage of breast cancer, and the tumor cell is only confined within the breast ductal. “Ductal” means that the cancer takes place inside the milk ducts, “carcinoma” refers to any cancer that starts in the skin or other tissues. For breast, it lies beneath the skin and leads to the nipple, and the phrase “ in situ ” means “ in its original place.” Although it is non-invasive and not life-threatening and usually considered as stage 0 of breast cancer, it can increase the risk of breast cancer in future and sometimes lead to invasive cancer.

Based on the appearance of the tumor cells proliferating within the duct, DCIS can be classified into two categories: comedo and non-comedo. Usually, the non-comedo type DCIS tends to be less aggressive than the other one. There are 3 common non-comedo types of DCIS: (1) Solid DCIS: cancer cells completely fill the affected ducts; (2) Cribiform DCIS: cancer cells do not completely fill the affected breast ducts; there are gaps between the cells; (3) Papillary DCIS: cancer cells form themselves in a fern-like pattern within the affected breast ducts.

Over the past 40 years, tumor growth described by mathematical models has been a significant focus. The model for the growth of a tumor consisting of live cells was first proposed by Byrne and Chaplain [8, 9]; Ward and King [36, 37] assumed that the tumor is radially symmetric, and consists of a continuum of live and dead cells. Also they developed a velocity field in response to local volume variations due to cell movement. For other publications of tumor growth, for examples, see Adam, Burton, Friedman and Greenspan. [1, 7, 26, 28].

While there is considerable interest in modeling all aspects of solid tumor growth, very little attention has been devoted to early stages cancer, not to say D-CIS. It is very necessary and useful to model tumor growth in the very early stage. For example, when noticing a possible breast tumor, a sequence of check-up is necessary; or tumor is resected, but need continuous therapy for patient to make sure the undetectable metastases are being treated. All these situations are reasonably modeled as early stage tumors.

Franks and Byrne [25] developed a nutrient limited growth model to study D-CIS, which involves cell movement, interactions between the expansive forces created by tumor cell proliferation and the stresses that develop in the compliant basement membrane. They also determined effect of growth by treatment, protease production and the inclusion of the surrounding stroma. Also, Xu [38, 39, 40, 41, 42, 43] modified an existing tumor growth model by Byrne and Chaplain [8, 9] and adapted a nutrient diffusion limited model with a radially symmetric, cylindrical geometry to study DCIS, and he noted that the spatial patterns exhibited by stationary model solutions were consistent with morphologies commonly observed in DCIS. Also, he proposed some inverse problems related to clinical diagnosis.

It is well known that tumor growth strongly depends upon the availability of nutrients, which can only be obtained from the surrounding tissue or matrix via the tumor surface. And the distribution of nutrient is controlled by two processes – diffusion and consumption, so it is reasonable to model tumor growth by using dimensionless nutrient concentration $u(x,t)$ which satisfies a reaction-diffusion equation.

Assume Fick's laws is applied to model the diffusion of the nutrient into the interior of the tumor, with the diffusion coefficient D to be constant. Also assume that consumption rate is governed by two terms, the first one comes from normal (that is, nonmitotic) processes, $k_q(u)n$, while the second one describes the additional amount used during mitosis and is given by $\alpha k_p(u)n$, where n means the living-cell concentrations.

Combining these assumptions, we have the following reaction diffusion equation for u :

$$\frac{\partial u}{\partial t} + \nabla \cdot (c\mathrm{v}) = D\triangle u - k_q(u)n - \alpha k_p(u)n$$

where v is the local velocity of cells.

For simplification, $k_q(u), k_p(u)$ usually are linear functions of u. $k_q(u) = k_1 u, k_p(u) = k_2 u$. Also, assume $\mathrm{v} = 0, D = 1$, then simplified equation for nutrient concentration u becomes:

$$\frac{\partial u}{\partial t} = \triangle u - \lambda u, \quad \lambda = (k_1 + \alpha k_2)n$$

If we consider the external resource for the cell, F(x,t), then the above equation becomes

$$\frac{\partial u}{\partial t} = \triangle u - \lambda u + F(x,t) \qquad (I)$$

Note λ could be a constant, a function w.r.t x, t or both.

Also, we assume $u(x,t)$ satisfies the following initial and boundary and free boundary conditions:

$$u(x,0) = f(x), \quad \varphi_1(0) < x < \varphi_2(0), \qquad (II)$$

$$u(\varphi_1(t),t) = g_1(t), \quad 0 < t < T, \qquad (III)$$

$$u(\varphi_2(t),t) = g_2(t), \quad 0 < t < T, \qquad (IV)$$

Here we assume the tumor to be within the interval $[\varphi_1(t), \varphi_2(t)]$ at time t, where $x = \varphi_1(t)$ and $x = \varphi_2(t)$ are two growing boundary of tumor.

Finally, the mass conservation consideration implies following relationship:

$$\mu \int_{\varphi_1(t)}^{\varphi_2(t)} (u(x,t) - u_0)dx = \frac{\partial \varphi_2(t)}{\partial t}, \quad \varphi_2(0) = s_0 > 0 \qquad (V)$$

Here μ, u_0, and s_0 are known constant.

In this dissertation, we consider the problems related to the following 3 models in one dimensional case:

(1) Consider $(I)-(IV)$ as our model 1 by assuming the boundaries are known with $\lambda(t)$ at (I).

(2) Consider $(I)-(IV)$ as our model 2 by assuming the boundaries are known with $\lambda(x)$ at (I).

(3) Based on the research on previous two models, we consider more complicated free boundary problem model $(I)-(V)$ with general λ at (I) as model 3.

The rest of dissertation is organized as follows:

In Chapter II, dealing with direct problem of model 1, we establish the integral form of solution and proved the existence and uniqueness of solution by employing several transformations and heat potential theory. An algorithm for solving a system of Volterra equations with weakly singular kernel is presented. Finally, numerical approximation of direct problems with different types of boundaries are discussed.

In Chapter III, we discuss the direct problem of model 2. The Ritz-Galerkin method in Bernstein polynomial basis is implemented to obtain an approximate solution of non-classical parabolic equation subject to given initial and known moving boundary conditions. Also, existence and uniqueness of solution are discussed. Illustrative examples are included to demonstrate the validity and applicability of Galerkin method.

In Chapter IV, we consider a inverse problem of model 1 with determining the source parameter of parabolic equation. Corresponding existence and uniqueness theorem and results of numerical experiments are included.

In Chapter V, we discuss the nonlocal problem of model 2, where one of the boundary conditions is in the form of an integral. We prove the existence and uniqueness for the solution. To overcome the difficulty of integral in boundary condition, we still use the Ritz-Galerkin method and obtain the approximate solution.

In Chapter VI, we present well-posedness theorems of free boundary problem model - nonlinear parabolic partial differential equation with initial, boundary and free boundary conditions. To validate our findings, results of numerical experiments and simulation are also presented.

Finally, future work and directions are discussed in Chapter VII.

CHAPTER II

DIRECT PROBLEM FOR MODEL 1

A Introduction

In this chapter, we consider the direct problem of model 1. So far, there are many publications about parabolic equations with fixed value boundaries, but little research done for the kind of direct problem presented here. The organization of this chapter is as follows:

In section B, by employing several transformations and heat potential theory, we established the integral form of solution and proved the existence and uniqueness of solution. An algorithm for solving a system of Volterra equations with weakly singular kernel is presented in Section C. In section D, numerical approximation of direct problems with different types of boundaries are discussed.

B The Mathematical Problem

Let $D \subset R$ be a bounded domain with boundary $\Gamma := \partial D$. Consider an initial boundary value problem of heat equation as follows:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda(t)u(x,t) + F(x,t), \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < T; \tag{1}$$

with initial condition

$$u(x,0) = f(x), \quad \varphi_1(0) < x < \varphi_2(0); \tag{2}$$

and boundary conditions

$$u(\varphi_i(t),t) = g_i(t), \quad 0 < t < T, \quad i = 1,2, \tag{3}$$

Here $\lambda(t)u(x,t)$ denotes the nutrient consumption rate. The direct problem is finding function $u(x,t)$ satisfying $(1)-(3)$ for given $\lambda(t), f(x), g_i(t)$ *and* $\varphi_i(t) (i = 1, 2)$.

Now let's first introduce some definitions and lemmas. Then we transform our problem to a system of integral equation by using heat potential theory.

The function

$$G(x,t;\xi,\tau) = \frac{1}{2\sqrt{\pi(t-\tau)}} e^{-\frac{(x-\xi)^2}{4(t-\tau)}}$$

is called fundamental solution of the heat equation $\dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2}$. Straightforward differentiation shows that G satisfies the heat equation with respect to the variables x and t. With the aid of this fundamental solution, heat potentials are constructed, then we use these potentials for solving our initial boundary value problem.

Definition II.1 *For a function $\mu \in C(\Gamma \times [0,T])$ the double-layer heat potential is given by*

$$v(x,t) = \int_0^t \mu(\tau) \frac{\partial G}{\partial \xi}(x,t;\xi,\tau) d\tau$$

and the volume heat potential by

$$U(\rho) = \int_0^t d\tau \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} \rho(\xi,\tau) G(x,t;\xi,\tau) d\xi$$

Lemma II.1 *[32] [Jump relation for double-layer potential]*

The double-layer heat potential with continuous density μ can be continuously extended from $D \times (0,T]$ into $\overline{D} \times (0,T]$ with limiting values

$$v_{\pm}(x_0,t_0) = \lim_{(x,t)\to(x_0,t_0)} \int_0^t \mu(\tau) \frac{\partial G}{\partial \xi}(x_0,t_0;\xi,\tau) d\tau \pm \frac{\mu(t_0)}{2}$$

where "+" and "−" represent (x,t) approaches to (x_0,t_0) on Γ from right side and left side respectively.

Lemma II.2 *[27] The volume potential $U[F]$ satisfies*

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - F(x,t), \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < T;$$

with initial condition

$$u(x,0) = 0, \quad \varphi_1(0) < x < \varphi_2(0);$$

Definition II.2 *I is the domain of definition of the given function g and the desired function f. f is to be determined from the following equation: Volterra integral equation of the second kind:*

$$f(x) = g(x) + \int_a^x k(x, y, f(y))dy \quad for \quad x \in I$$

The function $k(x, y, z)$ is called the kernel. It is defined on the domain

$$(x, y, z) : x \in I, y \in [a, x], z \in R \subset I \times I \times R.$$

Lemma II.3 *[29][Existence and uniqueness of system of Volterra equations]*

We assume $g \in \mathcal{C}(I), k \in (\mathcal{I} \times \mathcal{I} \times \mathcal{R})$. Furthermore, let the kernel k satisfy the following Lipschitz condition: there is a function $L \in \mathcal{C}(I \times I)$ such that

$$|k(x, y, z) - k(x, y, z')| \leq L(x, y)|z - z'| \quad for\ all \quad (x, y) \in \mathcal{C}(I \times I), \quad z, z' \in R$$

then, there is exactly one solution $f \in \mathcal{C}(I)$ of the Volterra integral equation.

Remark: Since this is a vector equation, it is equivalent to a system consisting of the m scalar equation

$$f_i(x) = g_i(x) + \int_a^x k_i(x, y, z)dy \quad i = 1, ..., m$$

Therefore, we get the existence and uniqueness theorem for system of Volterra equations of second kind.

Now let's construct the solution involving the heat potential:

Employ the first transformation

$$w(x, t) = e^{\int_0^t \lambda(s)ds} u(x, t), \tag{4}$$

then problem (1)-(3) becomes:

$$\frac{\partial w}{\partial t} = \frac{\partial^2 w}{\partial x^2} + e^{\int_0^t \lambda(s)ds} F(x,t), \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < T; \tag{5}$$

$$w(x,0) = f(x), \quad \varphi_1(0) < x < \varphi_2(0), \tag{6}$$

$$w(\varphi_i(t),t) = e^{\int_0^t \lambda(s)ds} g_i(t), \quad 0 < t < T, \quad i = 1,2. \tag{7}$$

Next employ the second transformation

$$h(x,t) = w(x,t) - f(x), \tag{8}$$

the problem (5)-(7) becomes:

$$\frac{\partial h}{\partial t} = \frac{\partial^2 h}{\partial x^2} + e^{\int_0^t \lambda(s)ds} F(x,t) + f''(x), \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < T; \tag{9}$$

$$h(x,0) = 0, \quad \varphi_1(0) < x < \varphi_2(0), \tag{10}$$

$$h(\varphi_i(t),t) = e^{\int_0^t \lambda(s)ds} g_i(t) - f(\varphi_i(t)) := \hat{h}_i(t), \quad 0 < t < T, \quad i = 1,2. \tag{11}$$

By Lemma II.2, we know $U[-\rho(x,t)]$ satisfies (9) and (10).

where

$$\rho(x,t) = e^{\int_0^t \lambda(s)ds} F(x,t) + f''(x), \tag{12}$$

then employ the third transformation

$$v(x,t) = h(x,t) - \int_0^t d\tau \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} \rho(\xi,\tau) G(x,t;\xi,\tau) d\xi, \tag{13}$$

the problem (9)-(11) becomes:

$$\frac{\partial v}{\partial t} = \frac{\partial^2 v}{\partial x^2}, \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < T; \tag{14}$$

$$v(x, 0) = 0, \quad \varphi_1(0) < x < \varphi_2(0), \tag{15}$$

$$v(\varphi_i(t), t) = \hat{h}_i(t) - \int_0^t d\tau \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} \rho(\xi, \tau) G(\varphi_i(t), t; \xi, \tau) d\xi := \hat{v}_i(t), \quad 0 < t < T, i = 1, 2. \tag{16}$$

By using double-layer potential, $v(x, t)$ has the following sum form

$$v(x, t) = \int_0^t \mu_1(\tau) \frac{\partial G}{\partial \xi}(x, t; \varphi_1(\tau), \tau) d\tau + \int_0^t \mu_2(\tau) \frac{\partial G}{\partial \xi}(x, t; \varphi_2(\tau), \tau) d\tau. \tag{17}$$

In order to make the equation (17) be the solution of the problem (14)-(16), apply Lemma II.1 and boundary condition (16), and derive the following system of Volterra integral equations of the second kind

$$\begin{cases} \dfrac{\mu_1(t)}{2} + \displaystyle\int_0^t \mu_1(\tau) \frac{\partial G}{\partial \xi}(\varphi_1(t), t; \varphi_1(\tau), \tau) + \mu_2(\tau) \frac{\partial G}{\partial \xi}(\varphi_1(t), t; \varphi_2(\tau), \tau) d\tau = \hat{v}_1(t), \\ -\dfrac{\mu_2(t)}{2} + \displaystyle\int_0^t \mu_2(\tau) \frac{\partial G}{\partial \xi}(\varphi_2(t), t; \varphi_2(\tau), \tau) + \mu_1(\tau) \frac{\partial G}{\partial \xi}(\varphi_2(t), t; \varphi_1(\tau), \tau) d\tau = \hat{v}_2(t). \end{cases} \tag{18}$$

According to Lemma II.3, (18) has an unique solution provide $\varphi_1(t)$ and $\varphi_2(t)$ are Hölder continuous with exponent $\alpha > \frac{1}{2}$ in the interval $[0, T]$, that is

$$\varphi_1(t), \varphi_2(t) \in C^{0,\alpha}([0, T]) = \{f(x) |\, |f(x) - f(y)| \leq M|x - y|, \forall x, y \in [0, T], M > 0\},$$

and $\hat{v_1}(t)$ and $\hat{v_2}(t)$ are continuous in the interval $[0, T]$. Thus, we have the following theorem.

Theorem II.1 *Assume that* $\varphi_1(t), \varphi_2(t) \in C^{0,\alpha}([0, T]), \alpha > \frac{1}{2},\ \lambda(t), g_1(t), g_2(t) \in C[0, T],\ f(x) \in C^2[\varphi_1(t), \varphi_2(t)]$ *and* $F(x, t) \in C[[\varphi_1(t), \varphi_2(t)] \times [0, T]]$. *The problem (1)-(3) has an unique solution and following form:*

$$u(x,t) = e^{-\int_0^t \lambda(s)ds}\left\{\omega(\mu_1(\tau)) + \omega(\mu_2(\tau)) + \int_0^t d\tau \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} \rho(\xi,\tau)G(x,t;\xi,\tau)d\xi + f(x)\right\}, \tag{19}$$

where $\omega(\mu_i(\tau)) = \int_0^t \mu_i(\tau)\frac{\partial G}{\partial \xi}(x,t;\varphi_i(\tau),\tau)d\tau, i = 1,2.$

Proof II.1 *From (11),(16) and assumptions, we can obtain $\hat{v_1}(t)$ and $\hat{v_2}(t)$ are continuous in the interval $[0,T]$. And because $\varphi_1(t), \varphi_2(t) \in C^{0,\alpha}([0,T]), \alpha > \frac{1}{2}$ we get that (18) has an unique solution. Substituting (18) into (17), we can obtain the problem (14)-(16) has at least one solution. Suppose there were two solution $v_1(x,t)$ and $v_2(x,t)$ of our initial boundary problem (14)-(16). Then the function $\hat{v}(x,t) = v_1(x,t) - v_2(x,t)$ would satisfy PDE (14),the initial conditions (15) and the boundary condition $\hat{v}(\varphi_i(t),t) = 0, i = 1,2$. By the maximum principle, $\hat{v}(x,t) = v_1(x,t) - v_2(x,t) \equiv 0$. Hence, problem (14)-(16) has unique solution. From (4),(8), (13) and (17), we have the initial boundary problem (1)-(3) has unique solution as (19).*

C The Mathematics of Volterra Procedure

In this section we solve (18) numerically. If we can get the numerical solution of (18), then consequently $u(x,t)$ of our direct problem $(1)-(3)$ can be obtained. The (18) is a system of Volterra equations and there are many computer programs that numerically solve that when the kernel function is without singularity. Among them is Becker L, Wheeler M [2] and so on. For our case, the kernel function is weakly singular, so we give the following modified algorithm to avoid the singularity.

Recall the Definition II.2 of system of second kind of volterra equations, we have that (18) is the case with $m = 2, f_1(t) = 2\hat{v_1}(t), f_2(t) = -2\hat{v_2}(t)$,

$$\mathrm{k}_1 = -2\int_0^t \mu_1(\tau)\frac{\partial G}{\partial \xi}(\varphi_1(t),t;\varphi_1(\tau),\tau)d\tau$$

$$= -2\int_0^t [\mu_1(\tau)\frac{\varphi_1(t)-\varphi_1(\tau)}{4\sqrt{\pi(t-\tau)^3}}e^{\frac{-(\varphi_1(t)-\varphi_1(\tau))^2}{4(t-\tau)}} + \mu_2(\tau)\frac{\varphi_1(t)-\varphi_2(\tau)}{4\sqrt{\pi(t-\tau)^3}}e^{\frac{-(\varphi_1(t)-\varphi_2(\tau))^2}{4(t-\tau)}}]d\tau,$$

$k_2 = 2\int_0^t \mu_2(\tau)\frac{\partial G}{\partial \xi}(\varphi_2(t),t;\varphi_2(\tau),\tau)d\tau$

$$= 2\int_0^t [\mu_2(\tau)\frac{\varphi_2(t)-\varphi_2(\tau)}{4\sqrt{\pi(t-\tau)^3}}e^{\frac{-(\varphi_2(t)-\varphi_2(\tau))^2}{4(t-\tau)}} + \mu_1(\tau)\frac{\varphi_2(t)-\varphi_1(\tau)}{4\sqrt{\pi(t-\tau)^3}}e^{\frac{-(\varphi_2(t)-\varphi_1(\tau))^2}{4(t-\tau)}}]d\tau$$

Now, we build Volterra procedure to approximate the solution $\mu(t)$ of

$$\mu(t) = f(t) + \int_0^t k(t,s,\mu(s))ds \tag{20}$$

at the equally spaced points

$$t_n = t_0 + nh, n = 1, ..., N \tag{21}$$

where $t_0 = 0$ and N is the total number of steps of size h. X_n denotes the approximation of $\mu(t)$ at $t = t_n$.

Let $t = t_n$, then (20) becomes

$$\mu(t_n) = f(t_n) + \int_0^{t_n} k(t_n,t,\mu(t))dt, \quad \mu(0) = f(0) \tag{22}$$

To avoid the singularity i.e. $t = \tau$ in $k(t,\tau,\mu(\tau))$, we get an approximation of the integral in (22) by modifying the composite trapezoidal rule,

$$k(t_n,t,\mu(t)) = \frac{h}{2}[k(t_n,t_0,\mu(t_0)) + 2\sum_{j=1}^{n-1} k(t_n,t_j,\mu(t_j)) + k(t_n,t_{n-1},\mu(t_n))] \tag{23}$$

Replacing $\mu(t_n)$ by X_n in (22) and (23) , we obtain

$$X_n = f(t_n) + h[\frac{k(t_n,t_0,\mu(t_0))}{2} + \sum_{j=1}^{n-1} k(t_n,t_j,\mu(t_j)) + \frac{k(t_n,t_{n-1},\mu(t_n))}{2}], \quad X_0 = f(0) \tag{24}$$

Let

$$\Theta_n = f(t_n) + h[\frac{k(t_n,t_0,\mu(t_0))}{2} + \sum_{j=1}^{n-1} k(t_n,t_j,\mu(t_j))],$$

then (24) becomes

$$X_n - \frac{1}{2}hk(t_n,t_n,X_n) - \Theta_n = 0 \tag{25}$$

Now we see that X_n is the solution of the vector equation

$$\Phi(u) = u - \frac{1}{2}hk(t_n,t_n,u) - \Theta_n = 0 \tag{26}$$

Next, we will get an approximation to the solution X_n of (26) by way of the matrix-valued function M as following:

$$M(u) := u - A(u)\Phi(u) \tag{27}$$

where $A(u)$ is an m by m matrix-valued function that is invertible in a neighborhood of X_n, then X_n is the fixed point of (27).

The Jacobian matrix of Φ is the m by m matrix $J(u)$ with element as following:

$$J(u)_{i,j} = \frac{\partial \Phi(u)}{\partial u_j} = \delta_{i,j} - \frac{1}{2}h\frac{\partial k_i(t_n, t_n, u)}{\partial u_j} \tag{28}$$

where $\delta_{i,j}$ is the Kronecker delta.

It can be shown that if $A(u)$ is equal to the Jacobian matrix of Φ, the iterates X_n^p defined by following will converge to X_n by Newton's method

$$X_n^p = G(X_n^{p-1}) = X_n^{p-1} - J^{-1}(X_n^{p-1})\Phi(X_n^{p-1}) \quad p = 1, 2, 3... \tag{29}$$

Let z be the solution of the matrix equation

$$J^{-1}(X_n^{p-1})z = \Phi(X_n^{p-1}) \tag{30}$$

Thus the iteration formula becomes

$$X_n^p = X_n^{p-1} - z \tag{31}$$

The iterates X_n^p are computed until the infinity norm of vector z is not more than a given tolerance. Then X_n is assigned the value of the last iteration.

Example 2.1

Now use one example to compare numerical result and exact result.

$$\mu_1(t) = t - 2\int_0^t \frac{se^{-\frac{1}{4(t-s)}}}{4\sqrt{\pi(t-s)^3}}ds + \int_0^t 2\mu_2(s)\frac{e^{\frac{-1}{4(t-s)}}}{4\sqrt{\pi(t-s)^3}}$$

$$\mu_2(t) = t - 2\int_0^t \frac{se^{-\frac{1}{4(t-s)}}}{4\sqrt{\pi(t-s)^3}}ds + \int_0^t 2\mu_1(s)\frac{e^{\frac{-1}{4(t-s)}}}{4\sqrt{\pi(t-s)^3}}$$

t	$\hat{\mu_1}(t)$	$\hat{\mu_2}(t)$	exact $\mu_1(t)$	exact $\mu_2(t)$
0.0	0.00007	0.00007	0.0	0.0
0.1	0.10055	0.10055	0.1	0.1
0.2	0.19979	0.19979	0.2	0.2
0.3	0.30027	0.30027	0.3	0.3
0.4	0.40134	0.40134	0.4	0.4
0.5	0.50243	0.50243	0.5	0.5
0.6	0.60326	0.60326	0.6	0.6
0.7	0.70382	0.70382	0.7	0.7
0.8	0.80429	0.80429	0.8	0.8
0.9	0.90502	0.90502	0.9	0.9
1.0	1.00648	1.00648	1.0	1.0

TABLE 1

Numerical result and exact solution of systems of Volterra integral equations for Example 2.1.

Here $f_i(t) = t - 2\int_0^t \frac{se^{-\frac{1}{4(t-s)}}}{4\sqrt{\pi(t-s)^3}}ds, \qquad i = 1, 2$ have singularity, so we approximate that by polynomial function. Even so we get the numerical result which is close to the exact result $\mu_1(t) = \mu_2(t) = t$. (See Table 1)

D Numerical Evidence

Here we illustrate our method and volterra algorithm to our direct problem, which include 4 types of moving boundary. The computations, associated with the examples, are performed by MAPLE.

Case I Boundaries are linear functions.

Example 2.2

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - tu(x,t), \quad \frac{1}{3} - t < x < \frac{1}{2} + t, \quad 0 < t < 3;$$

with initial condition

$$u(x,0) = x, \quad \frac{1}{3} < x < \frac{1}{2}; \tag{32}$$

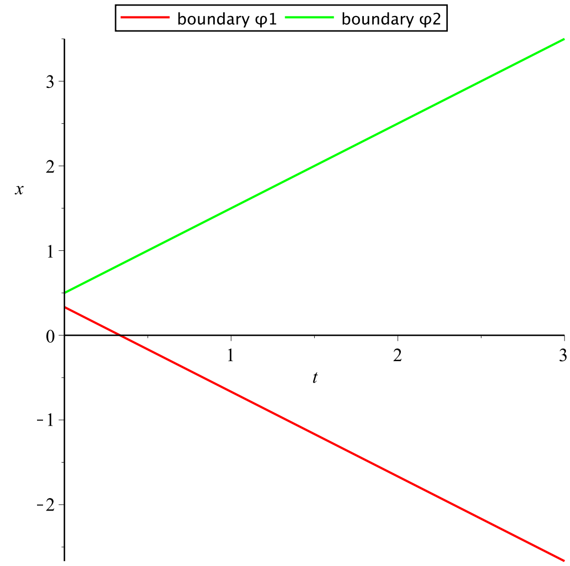

Figure 1. Boundaries for case I

and boundary conditions

$$u(\frac{1}{3}-t,t)=\frac{1}{3}-t,\quad 0<t<3$$

$$u(\frac{1}{2}+t,t)=\frac{1}{2}+t,\quad 0<t<3$$

Then we can get corresponding 3-D plot for $u(x,t)$. (See Figure 2)

Case II Boundaries are smooth curves.

Example 2.3

$$\frac{\partial u}{\partial t}=\frac{\partial^2 u}{\partial x^2}-tu(x,t),\quad \frac{1}{4}-t^2<x<\frac{1}{2}+t^2,\quad 0<t<3;$$

with initial condition

$$u(x,0)=x,\quad \frac{1}{4}<x<\frac{1}{2}; \tag{33}$$

and boundary conditions

$$u(\frac{1}{4}-t^2,t)=\frac{1}{4}-t^2,\quad 0<t<3$$

$$u(\frac{1}{2}+t^2,t)=\frac{1}{2}+t^2,\quad 0<t<3$$

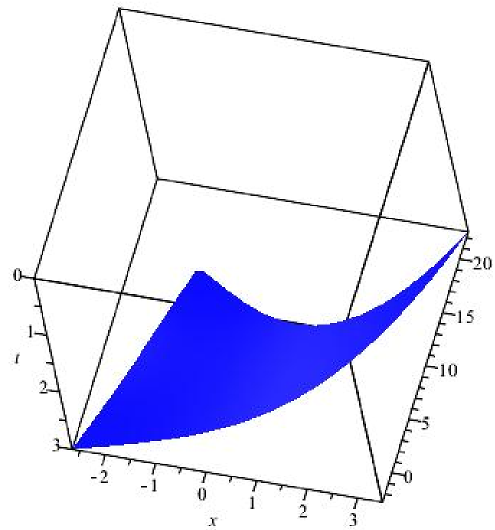

Figure 2. u(x,t)for case I

Then we can get corresponding 3-D graph for $u(x,t)$.(See Figure 4)

Case III Boundaries are periodical functions.

Example 2.4

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - tu(x,t), \quad -sin(4t) - 2 < x < sin(4t) + 2, \quad 0 < t < 3;$$

with initial condition

$$u(x,0) = x, \quad -2 < x < 2; \tag{34}$$

and boundary conditions

$$u(-sin(4t) - 2, t) = -sin(4t) - 2, \quad 0 < t < 3$$

$$u(sin(4t) + 2, t) = sin(4t) + 2, \quad 0 < t < 3$$

Then we can get corresponding contour plot for $u(x,t)$. (See Figure 6)

Case IV Boundaries are smooth functions with asymptotes.

Example 2.5

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - tu(x,t), \quad -\frac{1}{t+1} - 2 < x < \frac{1}{t+1} + 1, \quad 0 < t < 3;$$

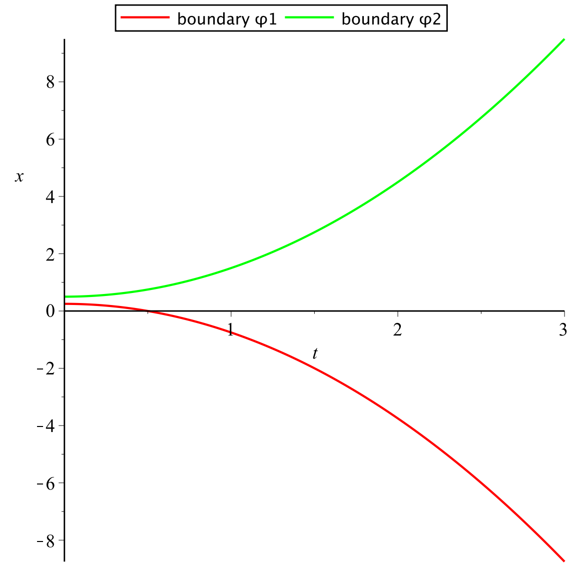

Figure 3. Boundaries for case II

with initial condition

$$u(x,0) = x, \quad -3 < x < 2; \tag{35}$$

and boundary conditions

$$u(-\frac{1}{t+1} - 2, t) = -\frac{1}{t+1} - 2, \quad 0 < t < 3$$

$$u(\frac{1}{t+1} + 1, t) = \frac{1}{t+1} + 1, \quad 0 < t < 3$$

Then we can get corresponding 3-D graph for $u(x,t)$. (See Figure 8)

E Conclusion

In this chapter, we present existence and uniqueness of model 1–parabolic partial differential equation with initial and known moving boundaries in one dimensional case. And we give the integral form of solution of problem by applying heat potential theory and necessary transformations. To validate our findings, results of numerical experiments and simulation are also presented.

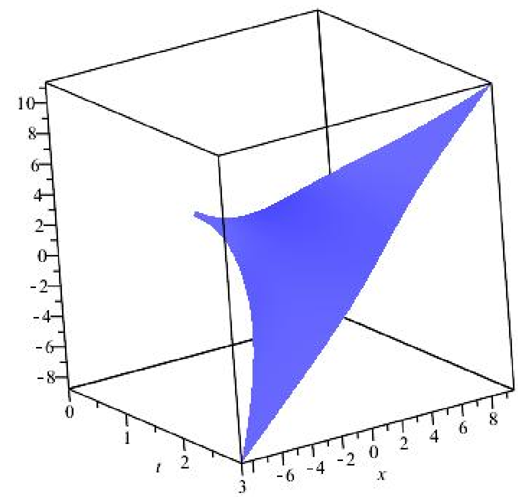

Figure 4. u(x,t)for case II

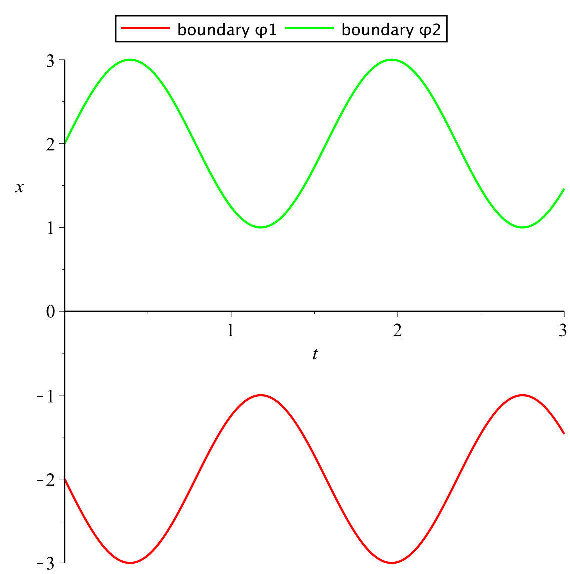

Figure 5. Boundaries for case III

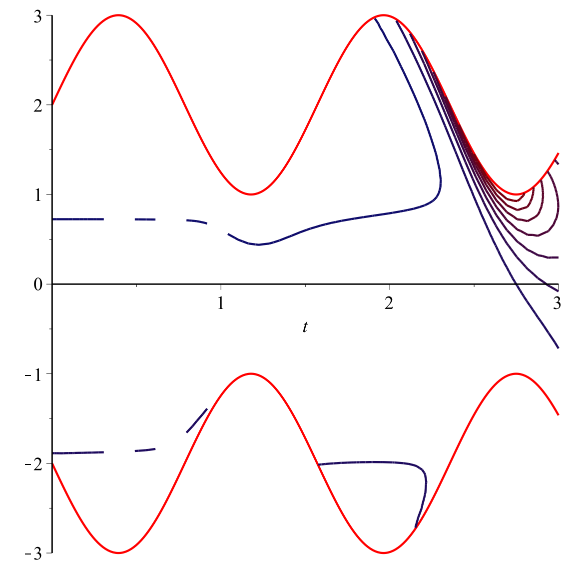

Figure 6. u(x,t)for case III

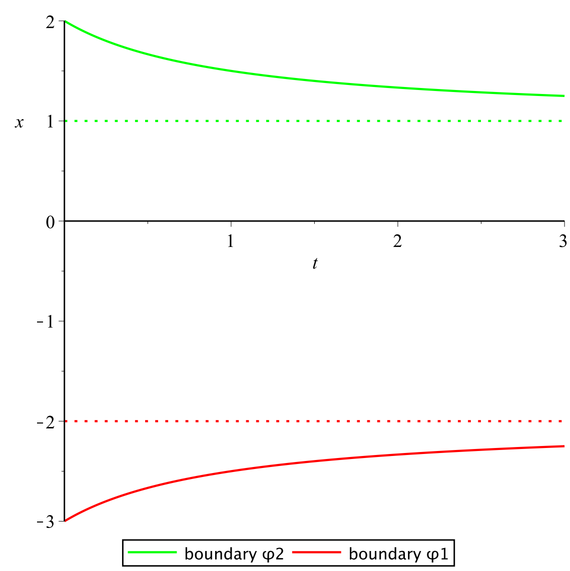

Figure 7. Boundaries for case IV

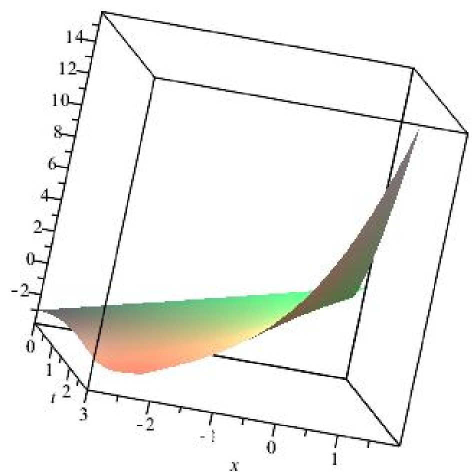

Figure 8. u(x,t)for case IV

CHAPTER III

DIRECT PROBLEM FOR MODEL 2

A Introduction

In this chapter, we consider the direct problem of model 2 as following parabolic equation:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda(x)u(x,t), \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < 1, \tag{36}$$

with initial condition

$$u(x,0) = f(x), \quad \varphi_1(0) < x < \varphi_2(0), \tag{37}$$

boundary conditions

$$u(\varphi_1(t),t) = g_1(t), \quad 0 < t < 1, \tag{38}$$

$$u(\varphi_2(t),t) = g_2(t), \quad 0 < t < 1, \tag{39}$$

and compatibility conditions

$$g_1(0) = f(\varphi_1(0)), \tag{40}$$

$$g_2(0) = f(\varphi_2(0)), \tag{41}$$

Here $\lambda(x)u(x,t)$ denotes the nutrient consumption rate at the location x at time t. The problem is to determine $u(x,t)$ for given $\lambda(x), f(x), \varphi_1(t)$, $\varphi_2(t), g_1(t)$ and $g_2(t)$.

In present chapter, we obtain existence and uniqueness theorem of the model 2. Furthermore, this is the first time the Ritz-Galerkin method in Bernstein polynomials basis is applied to solve approximation solution of parabolic equation with known moving boundaries.

The Ritz-Galerkin method in Bernstein polynomials basis essentially transfer equation to a weak formulation by converting a continuous operator problem to a discrete problem, and then characterize the space with a finite set of basis functions in use of some constraints on the function space. It has been widely used in many areas and provide powerful numerical simulation, especially in the field of differential equation [3, 5, 13, 21, 33, 44, 45].

The chapter is divided as follows. In Section B, we present equivalent forms of original problem. Section C is devoted to existence and uniqueness of solution. The properties of Bernstein polynomials are presented in Section D. The numerical schemes for the solution of equations (36)-(41) are described in Section E. Section F presents two test examples to support the new method. Finally, conclusions are made in Section G.

B Equivalent Problems

In this section, we introduce two transformations to convert our problem (36) – (41) to two equivalent forms. Therefore, we can apply the Ritz-Galerkin method to the second equivalent form to get approximation solution of problem.

Introduce first transformation:

$$\xi = \frac{x - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)}. \tag{42}$$

By doing so, variable $x \in [\varphi_1(t), \varphi_2(t)]$ makes $\xi \in [0, 1]$.

Let

$$v(x,t) = u((\varphi_2(t) - \varphi_1(t))x + \varphi_1(t), t), \quad 0 \le x \le 1,\ t \ge 0. \tag{43}$$

Then

$$u(x,t) = v\left(\frac{x - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)}, t\right) = v(\xi, t), \tag{44}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi} \cdot \frac{1}{\varphi_2(t) - \varphi_1(t)}, \tag{45}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2} \cdot \frac{1}{[\varphi_2(t) - \varphi_1(t)]^2}, \tag{46}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \xi} \cdot B(x,t), \tag{47}$$

where

$$B(x,t) = \frac{\varphi_1'(t)(\varphi_2(t) - \varphi_1(t)) + (x - \varphi_1(t))[(\varphi_2'(t) - \varphi_1'(t)]}{((\varphi_2(t) - \varphi_1(t))^2}. \tag{48}$$

Under this transformation (42), the problem (36)-(41) becomes the first equivalent form as following:

$$\frac{\partial v}{\partial t} = \frac{1}{[\varphi_2(t) - \varphi_1(t)]^2} \cdot \frac{\partial^2 v}{\partial x^2} + \widetilde{B}(x,t)\frac{\partial v}{\partial x} - \widetilde{\lambda}(x,t)v(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{49}$$

with initial condition

$$v(x,0) = \widetilde{f}(x), \quad 0 < x < 1, \tag{50}$$

boundary conditions

$$v(0,t) = g(t), \quad 0 < t < 1, \tag{51}$$

$$\int_0^1 v(x,t)dx = \frac{E(t)}{\varphi_2(t) - \varphi_1(t)}, \quad 0 < t < 1, \tag{52}$$

and compatibility conditions

$$g(0) = f(\varphi_1(0)) = \widetilde{f}(0), \tag{53}$$

$$\int_0^1 \widetilde{f}(x)dx = \frac{E(0)}{\varphi_2(0) - \varphi_1(0)}, \tag{54}$$

where

$$\widetilde{B}(x,t) = B(\varphi_1(t) + (\varphi_2(t) - \varphi_1(t))x, t) = \frac{\varphi_1'(t) + [\varphi_2'(t) - \varphi_1'(t)]x}{\varphi_2(t) - \varphi_1(t)}, \tag{55}$$

$$\widetilde{\lambda}(x,t) = \lambda(\varphi_1(t) + (\varphi_2(t) - \varphi_1(t))x), \tag{56}$$

$$\widetilde{f}(x) = f((\varphi_2(0) - \varphi_1(0))x + \varphi_1(0)). \tag{57}$$

In order to facilitate the application of the Ritz-Galerkin methods, we introduce second transformation:

$$H(x,t) = v(x,t) - (1-x)g_1(t) - xg_2(t), \tag{58}$$

then

$$\frac{\partial H}{\partial t} = \frac{\partial v}{\partial t} - (1-x)\frac{d(g_1(t))}{\partial t} - x\frac{d(g_2(t))}{\partial t}, \tag{59}$$

$$\frac{\partial H}{\partial x} = \frac{\partial v}{\partial x} + g_1(t) - g_2(t), \tag{60}$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}, \tag{61}$$

According to the transformation (58) and equations (59)-(61), we have

$$\frac{\partial H}{\partial t} = \frac{1}{[\varphi_2(t)-\varphi_1(t)]^2}\cdot\frac{\partial^2 H}{\partial x^2} + \widetilde{B}(x,t)\frac{\partial H}{\partial x} - \widetilde{\lambda}(x,t)H(x,t) + K(x,t), \quad 0<x<1, \quad 0<t<1, \tag{62}$$

with initial condition

$$H(x,0) = \widetilde{f}(x) - (1-x)g_1(0) - xg_2(0), \quad 0<x<1, \tag{63}$$

boundary conditions

$$H(0,t) = 0, \quad 0<t<1, \tag{64}$$

$$H(1,t) = 0, \quad 0<t<1, \tag{65}$$

and compatibility conditions

$$H(0,0) = 0, \tag{66}$$

$$H(1,0) = 0, \tag{67}$$

where

$$K(x,t) = \widetilde{B}(x,t)(g_2(t)-g_1(t)) - \widetilde{\lambda}(x,t)((1-x)g_1(t)+xg_2(t)) - (1-x)\frac{d(g_1(t))}{dt} - x\frac{d(g_2(t))}{dt}. \tag{68}$$

From (43), (44) and (58), we can obtain

$$H(x,t) = u((\varphi_2(t)-\varphi_1(t))x + \varphi_1(t), t) - (1-x)g_1(t) - xg_2(t), \tag{69}$$

and

$$u(x,t) = H\left(\frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}, t\right) + \left(1 - \frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}\right)g_1(t) + \frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}g_2(t). \tag{70}$$

C Existence and Uniqueness

In this section the existence and uniqueness of the problem (36)-(41) are discussed.

In order to facilitate the deduction of the problem, we need to make another transformation. Let

$$\eta = \int_0^t [\varphi_2(\tau) - \varphi_1(\tau)]^{-2} d\tau := A(t), \tag{71}$$

and

$$t = \psi(\eta), \tag{72}$$

where ψ is the inverse of the mapping $\eta = A(t)$.

Let

$$v(x,t) = w(x,\eta), \tag{73}$$

it follows from the Chain Rule that

$$\frac{\partial v}{\partial t} = \frac{\partial w}{\partial \eta} \cdot \frac{1}{[\varphi_2(t) - \varphi_1(t)]^2}, \tag{74}$$

$$\frac{\partial v}{\partial x} = \frac{\partial w}{\partial x}, \tag{75}$$

$$\frac{\partial^2 v}{\partial x^2} = \frac{\partial^2 w}{\partial x^2}. \tag{76}$$

Thus, the problem (49)-(54) can be deduced to the following form:

$$\frac{\partial w(x,\eta)}{\partial \eta} = \frac{\partial^2 w(x,\eta)}{\partial x^2} + \hat{B}(x,\eta)\frac{\partial w(x,\eta)}{\partial x} - \hat{\lambda}(x,\eta) w(x,\eta), \quad 0 < x < 1, \quad 0 < t < T, \tag{77}$$

with initial condition

$$w(x,0) = \widetilde{f}(x), \quad 0 < x < 1, \tag{78}$$

boundary conditions

$$w(0,\eta) = g_1(\psi(\eta)), \quad 0 < \eta < T, \tag{79}$$

$$w(1,\eta) = g_2(\psi(\eta)), \quad 0 < \eta < T, \tag{80}$$

and compatibility conditions

$$w(0,0) = g_1(0) = \widetilde{f}(0) = f(\varphi_1(0)), \tag{81}$$

$$w(1,0) = g_2(0) = \widetilde{f}(1) = f(\varphi_2(0)), \tag{82}$$

where

$$\hat{B}(x,\eta) = [\varphi_2(\psi(\eta)) - \varphi_1(\psi(\eta))]^2 \tilde{B}(x,\psi(\eta)), \tag{83}$$

$$\hat{\lambda}(x,\eta) = [\varphi_2(\psi(\eta)) - \varphi_1(\psi(\eta))]^2 \tilde{\lambda}(x,\psi(\eta)), \tag{84}$$

$$T = \int_0^1 [\varphi_2(\tau) - \varphi_1(\tau)]^{-2} d\tau. \tag{85}$$

Assumption:

For the function $F(x,\eta,w,p)$, we assume it satisfy the following conditions:

(a) The function $F(x,\eta,w,p)$ is defined and continuous on the set

$$\Omega = \{(x,\eta,w,p) | (x,\eta) \in [0,1] \times [0,1], -\infty < w < \infty, -\infty < p < \infty\}$$

(b) For each $C > 0$ and for $|w|, |p| < C$, the function $F(x,\eta,w,p)$ is uniformly Hölder continuous in x and η for each compact subset of
$D_T = \{(x,\eta) | (x,\eta) \in (0,1) \times (0,1]\}$.
(c) There exist a constant C_F such that

$$|F(x,\eta,w_1,p_1)| - |F(x,\eta,w_2,p_2)| \leq C_F[|w_1 - w_2| + |p_1 - p_2|]$$

holds for all $(w_i,p_i), i = 1,2$.

Apply the results of [10] [Page 351, Theorem 20.3.3] to the initial boundary value problem given by equation (77)-(82), we have the following existence and uniqueness theorem.

Theorem III.1 *If function*

$$F(x,\eta,w,p) = \hat{B}(x,\eta)p - \hat{\lambda}(x,\eta)w \tag{86}$$

satisfies above assumption, $\widetilde{f}(x)$ is continuously differentiable such that $\widetilde{f}(x)$ and $\widetilde{f}'(x)$ are bounded,and $g_1(\psi(\eta))$ is continuously differentiable, $g_2(\psi(\eta))$ is continuously differentiable, then there exists a unique bounded solution $w = w(x, \eta)$ of initial boundary value problem (77)-(82). Moreover, this unique solution has a bounded continuous derivative with respect to x.

According to the relationship of functions $u(x,t), v(x,t)$ and $w(x,\eta)$, we can easily get the following existence and uniqueness theorem of original problem.

Theorem III.2 *Assume that*

$$\lambda(x) \in C[0,1],\ f(x) \in C^1[\varphi_1(0), \varphi_2(0)],\ g_1(t),\ g_2(t),\ \varphi_1(t),\ \varphi_2(t) \in C^1[0,1], \quad (87)$$

then there exists a unique bounded solution $u = u(x,t)$ of initial boundary value problem (36)-(41),moreover, this unique solution is with bounded continuous derivative respect to x.

D Bernstein Polynomials and Properties

The general form of the Bernstein polynomials of mth degree proposed by Bhatti and Bracken [4] is defined on the interval $[0, 1]$ as

$$B_{i,m}(x) = \frac{m!}{i!(m-i)!} x^i (1-x)^{m-i}, \quad 0 \le i \le m. \quad (88)$$

It can be easily shown that each of the Bernstein polynomials is positive and also the sum of all the Bernstein polynomials is unity for all real $x \in [0, 1]$, that is,

$$\sum_{i=0}^{m} B_{i,m}(x) = 1, \quad x \in [0,1]. \quad (89)$$

Moreover, the Bernstein polynomials have the following properties:

$$B_{i,m}(x) = (1-x)B_{i,m-1}(x) + xB_{i-1,m-1}(x), \quad (90)$$

$$B_{i,m-1}(x) = \frac{m-i}{m} B_{i,m}(x) + \frac{i+1}{m} B_{i+1,m}(x), \quad (91)$$

$$B'_{i,m}(x) = m(B_{i-1,m-1}(x) - B_{i,m-1}(x)), \tag{92}$$

$$\int_0^1 B_{i,m}(x)dx = \frac{1}{m+1}, \quad i = 0, 1, \cdots m. \tag{93}$$

Each kth degree Bernstein basis function can be written in the mth degree Bernstein basis as (see [24])

$$B_{i,k}(x) = \sum_{j=i}^{m-k+i} \frac{k!(m-k)!j!(m-j)!}{i!(k-i)!(j-i)!(m-k-j+i)!m!} B_{j,m}(x), \ (i = 0, 1, \cdots k), \ as\ k \leq m. \tag{94}$$

A set of Legendre polynomials, denoted by $\{L_k(x)\}$ for $k = 0, 1, \cdots$, are orthogonal with respect to the weighting function $\omega(x) = 1$ over the interval $[0, 1]$. These polynomials satisfy the recurrence relation [14]

$$(k+1)L_{k+1}(x) = (2k+1)(2x-1)L_k(x) - kL_{k-1}(x), \quad k = 1, 2, \cdots, \tag{95}$$

with

$$L_0(x) = 1, \quad L_1(x) = 2x - 1. \tag{96}$$

It can be shown [31] that the Legendre polynomial $L_m(x)$ can be written in the mth degree Bernstein basis $B_{0,m}(x), B_{1,m}(x), \cdots, B_{m,m}(x)$ as

$$L_m(x) = \sum_{i=0}^{m} (-1)^{m+i} \frac{m!}{i!(m-i)!} B_{i,m}(x). \tag{97}$$

Thus, from (94) and (97), we can obtain that any given polynomial $P_m(x)$ of degree m can be expanded in the mth degree Legendre and Bernstein base on $x \in [0, 1]$

$$P_m(x) = \sum_{k=0}^{m} l_k L_k(x) = \sum_{i=0}^{m} c_i B_{i,m}(x). \tag{98}$$

Let $V = L^2[0, 1]$ is the vector space of real functions whose domain is the close interval $[0, 1]$ and all functions in $V = L^2[0, 1]$ are assumed to be square integrable. We define the inner product of $f(x)$ and $g(x)$ as follows

$$< f(x), g(x) >= \int_0^1 f(x)g(x)dx. \tag{99}$$

Remarks:

(1) Space

$Span\{L_0(x), L_1(x), \cdots, L_m(x)\} = Span\{B_{0,m}(x), B_{1,m}(x), \cdots, B_{m,m}(x)\} := Y \subset V$ and $B_{1,m}(x), B_{2,m}(x), \cdots, B_{m,m}(x)$ are basis of subspace Y of V.

(2) Suppose $f(x) \in V = L^2[0,1]$, then there exist a unique best approximation to $f(x)$ out of Y such as $y_0(x) \in Y$; that is, if $y(x) \in Y$,

$$\| y_0(x) - f(x) \| \leq \| y(x) - f(x) \|, \tag{100}$$

moreover

$$y_0(x) = \sum_{k=0}^{m} c_k B_{k,m} = (c_0, c_1, \cdots, c_m)(B_{0,m}(x), B_{1,m}(x), \cdots, B_{m,m}(x))^T := C^T \phi, \tag{101}$$

where coefficient matrix C^T can be obtained by

$$C^T =< f, \phi^T >< \phi, \phi^T >^{-1} . \tag{102}$$

E Bernstein Ritz-Galerkin Method

In this section, we apply Ritz-Galerkin method to the second equivalent problem (62)-(67) in section B, then by (70) we can easily obtain the approximate solution of original problem (62)-(67).

Consider the parabolic equation

$$\frac{\partial H}{\partial t} = \frac{1}{[\varphi_2(t) - \varphi_1(t)]^2} \cdot \frac{\partial^2 H}{\partial x^2} + \widetilde{B}(x,t)\frac{\partial H}{\partial x} - \widetilde{\lambda}(x,t)H(x,t) + K(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{103}$$

with initial condition

$$H(x,0) = \widetilde{f}(x) - (1-x)g_1(0) - xg_2(0), \quad 0 < x < 1, \tag{104}$$

boundary conditions

$$H(0,t) = 0, \quad 0 < t < 1, \tag{105}$$

$$H(1,t) = 0, \quad 0 < t < 1, \tag{106}$$

and compatibility conditions

$$H(0,0)=0, \tag{107}$$

$$H(1,0)=0, \tag{108}$$

where

$$\widetilde{\lambda}(x,t)=\lambda(\varphi_1(t)+(\varphi_2(t)-\varphi_1(t))x), \tag{109}$$

$$\widetilde{f}(x)=f((\varphi_2(0)-\varphi_1(0))x+\varphi_1(0)), \tag{110}$$

$$\widetilde{B}(x,t)=B(\varphi_1(t)+(\varphi_2(t)-\varphi_1(t))x,t)=\frac{\varphi_1'(t)+[\varphi_2'(t)-\varphi_1'(t)]x}{\varphi_2(t)-\varphi_1(t)}, \tag{111}$$

$$K(x,t)=\widetilde{B}(x,t)(g_2(t)-g_1(t))-\widetilde{\lambda}(x,t)((1-x)g_1(t)+xg_2(t))-(1-x)\frac{d(g_1(t))}{dt}-x\frac{d(g_2(t))}{dt}. \tag{112}$$

Let

$$F(H)=\frac{\partial H}{\partial t}-\frac{1}{[\varphi_2(t)-\varphi_1(t)]^2}\cdot\frac{\partial^2 H}{\partial x^2}-\widetilde{B}(x,t)\frac{\partial H}{\partial x}+\widetilde{\lambda}(x,t)H(x,t)-K(x,t)=0, \tag{113}$$

We construct Ritz-Galerkin approximation to (113) as following. The approximation solution $\widetilde{H}(x,t)$ is sought in the form of the truncated series

$$\widetilde{H}(x,t)=H(x,0)\cdot\left(\sum_{i=0}^{N}\sum_{j=0}^{M}k_{i,j}\ t\ B_{i,N}(x)B_{j,M}(t)+1\right), \tag{114}$$

where $B_{i,N}(x), B_{j,M}(t)$ are Bernstein polynomials. By compatibility conditions (107)-(108), it is easy to check that our approximation solution $\widetilde{H}(x,t)$ satisfies the initial condition (104) and the boundary conditions (105) and (106).

Now the expansion coefficients $k_{i,j}$ are determined by the Galerkin equations

$$< F(\widetilde{H}(x,t)), B_{i,N}(x)B_{j,M}(t) >= 0, \quad (i=0,1,\cdots,N,\ j=0,1,\cdots,M), \tag{115}$$

where $< . >$ denotes the inner product defined by

$$< F(\widetilde{H}(x,t)), B_{i,N}(x)B_{j,M}(t) >= \int_0^1\int_0^1 F(\widetilde{H}(x,t))B_{i,N}(x)B_{j,M}(t)dtdx. \tag{116}$$

Here (115) is a system of $(N+1)(M+1)$ linear equations and we can solve it for the elements $k_{i,j}$.

F Numerical Application

In this section, two test examples using the Ritz-Galerkin methods are described in previous sections. The validity and efficiency of our numerical scheme are demonstrated by providing absolute error.

Example 3.1:

Consider (36)-(41) with

$$\lambda(x) = x, \tag{117}$$

$$\varphi_1(t) = 0, \quad 0 \leq t \leq 1, \tag{118}$$

$$\varphi_2(t) = \frac{1}{2-t}, \quad 0 \leq t \leq 1, \tag{119}$$

$$f(x) = e^{2x}, \quad 0 = \varphi_1(0) \leq x \leq \varphi_2(0) = \frac{1}{2}, \tag{120}$$

$$g_1(t) = e^{\frac{1}{3}t^3-2t^2+4t}, \quad 0 \leq t \leq 1, \tag{121}$$

$$g_2(t) = e^{\frac{1}{3}t^3-2t^2+4t+1}, \quad 0 \leq t \leq 1, \tag{122}$$

which has the exact solution

$$u(x,t) = e^{x(2-t)+\frac{1}{3}t^3-2t^2+4t}, \tag{123}$$

From (62)-(67), we can obtain the following its equivalent problem

$$\frac{\partial H}{\partial t} = (2-t)^2 \cdot \frac{\partial^2 H}{\partial x^2} + \frac{x}{2-t}\frac{\partial H}{\partial x} - \frac{x}{2-t}H(x,t) + K(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{124}$$

with initial condition

$$H(x,0) = e^{x} - 1 + x(1-e), \quad 0 < x < 1, \tag{125}$$

boundary conditions

$$H(0,t) = 0, \quad 0 < t < 1, \tag{126}$$

$$H(1,t) = 0, \quad 0 < t < 1, \tag{127}$$

where

$$K(x,t) = e^{\frac{1}{3}t^3-2t^2+4t}\left(\frac{x(e-2+x-ex)}{2-t}-(t-2)^2(1-x+ex)\right). \tag{128}$$

From (43), (58) and (123), we can deduce that the problem (124)-(127) has the exact solution

$$H(x,t) = e^{\frac{1}{3}t^3-2t^2+4t}\left(e^x-1+x(1-e)\right). \tag{129}$$

We applied the method presented in this chapter with $N = 2, M = 4$ and solved equation (124).

From Galerkin equations (115), we have

$$\begin{cases} k_{0,0} = 4.026,\ k_{0,1} = 5.411,\ k_{0,2} = 7.706,\ k_{0,3} = 9.082,\ k_{0,4} = 9.311, \\ k_{1,0} = 4.027,\ k_{1,1} = 5.406,\ k_{1,2} = 7.711,\ k_{1,3} = 9.077,\ k_{1,4} = 9.313, \\ k_{2,0} = 4.024,\ k_{2,1} = 5.414,\ k_{2,2} = 7.701,\ k_{2,3} = 9.085,\ k_{2,4} = 9.310. \end{cases} \tag{130}$$

From equations (114), we can obtain the approximate solution $\widetilde{H}(x,t)$ of the problem (124)-(127) as following

$$\widetilde{H}(x,t) = H(x,0)\cdot\left(\sum_{i=0}^{N=2}\sum_{j=0}^{M=4} k_{i,j}\ t\ B_{i,N}(x)B_{j,M}(t)+1\right), \tag{131}$$

According to (70), we can get following corresponding approximate solution $\widetilde{u}(x,t)$ of the problem (36)-(41).

$$\widetilde{u}(x,t) = \widetilde{H}((2-t)x,t)+(1-(2-t)x)e^{\frac{1}{3}t^3-2t^2+4t}+(2-t)xe^{\frac{1}{3}t^3-2t^2+4t+1}. \tag{132}$$

Similarly, we can get approximate solutions of the problem (124)-(127) and (36)-(41) for different value of N and M.

In Figure 9, the exact and approximate solutions of $H(x,t)$ with $N = 2, M = 4$ are plotted.

In Figure 10, the exact and approximate solutions of $u(x,t)$ with $N = 2, M = 4$ are plotted.

Table 2 and Table 3 present respectively absolute error for $H(x,t)$ and $u(x,t)$ in Example 3.1 after using the same method with different N and M.

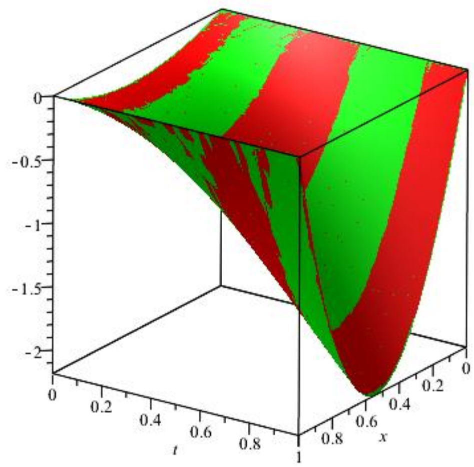

Figure 9. Exact (red) and approximate (green) solutions of $H(x,t)$ in Example 3.1

Table 4 present L^2 norm error for functions $H(x,t)-\widetilde{H}(x,t)$ and $u(x,t)-\widetilde{u}(x,t)$ in Example 3.1 with different N and M.

Example 3.2:

In this example, we solve (36)-(41) with

$$\lambda(x) = 0, \tag{133}$$

$$\varphi_1(t) = \sin(\frac{\pi}{2}t), \quad 0 \le t \le 1, \tag{134}$$

$$\varphi_2(t) = 1 + \sin(\frac{\pi}{2}t), \quad 0 \le t \le 1, \tag{135}$$

$$f(x) = e^x, \quad 0 = \varphi_1(0) \le x \le \varphi_2(0) = 1, \tag{136}$$

$$g_1(t) = e^{t+\sin(\frac{\pi}{2}t)}, \quad 0 \le t \le 1, \tag{137}$$

$$g_2(t) = e^{t+\sin(\frac{\pi}{2}t)+1}, \quad 0 \le t \le 1, \tag{138}$$

which has the exact solution

$$u(x,t) = e^{t+x}, \tag{139}$$

From (62)-(67), we can obtain the following its equivalent problem

$$\frac{\partial H}{\partial t} = \frac{\partial^2 H}{\partial x^2} + \frac{\pi}{2}\cos(\frac{\pi}{2}t)\frac{\partial H}{\partial x} + K(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{140}$$

TABLE 2

The absolute error for $H(x,t)$ in Example 3.1.

(x,t)	$N=2,\ M=4$	$N=2,\ M=6$	$N=2,\ M=8$
(0,0)	0	0	0
(0.1,0.1)	4.51×10^{-6}	-9.23×10^{-7}	9.09×10^{-9}
(0.2,0.2)	8.98×10^{-5}	1.00×10^{-6}	-3.19×10^{-8}
(0.3,0.3)	-7.04×10^{-6}	2.68×10^{-6}	6.72×10^{-8}
(0.4,0.4)	-1.97×10^{-4}	-3.49×10^{-6}	-3.04×10^{-8}
(0.5,0.5)	-1.60×10^{-4}	-4.19×10^{-6}	-8.21×10^{-8}
(0.6,0.6)	1.23×10^{-4}	4.37×10^{-6}	1.02×10^{-7}
(0.7,0.7)	2.80×10^{-4}	4.62×10^{-6}	9.62×10^{-9}
(0.8,0.8)	5.62×10^{-5}	-4.14×10^{-6}	-7.34×10^{-8}
(0.9,0.9)	-1.46×10^{-4}	-4.27×10^{-7}	5.06×10^{-8}
(1,1)	0	0	0

TABLE 3

The absolute error for $u\left(\frac{x}{2-t}, t\right)$ in Example 3.1.

(x,t)	$N=2,\ M=4$	$N=2,\ M=6$	$N=2,\ M=8$
(0,0)	0	0	0
(0.1,0.1)	7.80×10^{-6}	-1.64×10^{-7}	1.62×10^{-8}
(0.2,0.2)	1.37×10^{-4}	1.52×10^{-6}	-4.89×10^{-8}
(0.3,0.3)	-9.19×10^{-6}	3.39×10^{-6}	8.52×10^{-8}
(0.4,0.4)	-2.02×10^{-4}	-3.61×10^{-6}	-3.25×10^{-8}
(0.5,0.5)	-1.23×10^{-4}	-3.19×10^{-6}	-6.21×10^{-8}
(0.6,0.6)	7.91×10^{-5}	2.70×10^{-6}	6.13×10^{-8}
(0.7,0.7)	1.12×10^{-4}	1.69×10^{-6}	-8.15×10^{-10}
(0.8,0.8)	8.81×10^{-6}	-1.10×10^{-6}	-1.74×10^{-8}
(0.9,0.9)	-1.68×10^{-5}	-1.12×10^{-8}	5.72×10^{-9}
(1,1)	0	0	0

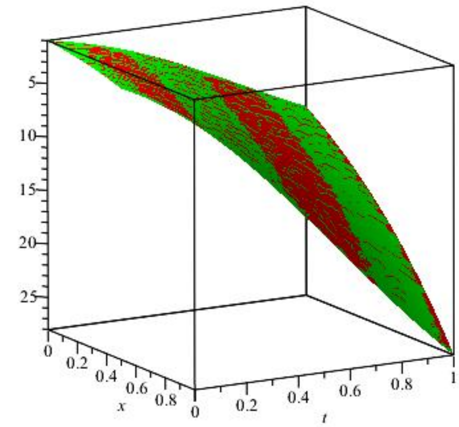

Figure 10. Exact (red) and approximate (green) solutions of $u(x,t)$ in Example 3.1

TABLE 4

The L^2 norm error for functions $H(x,t) - \widetilde{H}(x,t)$ and $u(x,t) - \tilde{u}(x,t)$ in Example 3.1.

(N, M)	$\|\|H(x,t) - \widetilde{H}(x,t)\|\|_{L^2([0,1]\times[0,1])}$	$\|\|u(x,t) - \tilde{u}(x,t)\|\|_{L^2([\varphi_1(t),\varphi_2(t)]\times[0,1])}$
(N=2,M=3)	3.34×10^{-7}	2.74×10^{-7}
(N=2,M=4)	1.92×10^{-8}	1.51×10^{-8}
(N=2,M=5)	2.00×10^{-10}	1.61×10^{-10}
(N=2,M=6)	9.25×10^{-12}	7.07×10^{-12}
(N=2,M=7)	1.06×10^{-13}	8.30×10^{-14}
(N=2,M=8)	2.81×10^{-15}	2.09×10^{-15}

with initial condition

$$H(x,0) = e^x - 1 + x(1-e), \quad 0 < x < 1, \tag{141}$$

boundary conditions

$$H(0,t) = 0, \quad 0 < t < 1, \tag{142}$$

$$H(1,t) = 0, \quad 0 < t < 1, \tag{143}$$

where

$$K(x,t) = e^{t+\sin(\frac{\pi}{2}t)} \left(\frac{\pi}{2}\cos(\frac{\pi}{2}t)(e-1) - (1 + \frac{\pi}{2}\cos(\frac{\pi}{2}t))(1 - x + xe) \right). \tag{144}$$

From (43), (58) and (139), we can deduce that the problem (140)-(143) has the exact solution

$$H(x,t) = e^{t+\sin(\frac{\pi}{2}t)}\left(e^x - 1 + x(1-e)\right). \tag{145}$$

Now apply our Galerkin method with $N = 2, M = 4$ and solve equation (140).

From Galerkin equations (115), we have

$$\begin{cases} k_{0,0} = 2.600,\ k_{0,1} = 3.321,\ k_{0,2} = 4.689,\ k_{0,3} = 6.159,\ k_{0,4} = 6.389, \\ k_{1,0} = 2.600,\ k_{1,1} = 3.321,\ k_{1,2} = 4.690,\ k_{1,3} = 6.158,\ k_{1,4} = 6.390, \\ k_{2,0} = 2.593,\ k_{2,1} = 3.336,\ k_{2,2} = 4.674,\ k_{2,3} = 6.167,\ k_{2,4} = 6.387. \end{cases} \tag{146}$$

From equation (114), we can obtain the approximate solution $\widetilde{H}(x,t)$ of the problem (140)-(143) as following

$$\widetilde{H}(x,t) = H(x,0) \cdot \left(\sum_{i=0}^{N=2} \sum_{j=0}^{M=4} k_{i,j}\ t\ B_{i,N}(x) B_{j,M}(t) + 1 \right), \tag{147}$$

According to (70), we can get corresponding approximate solution $\widetilde{u}(x,t)$ of the problem (36)-(41).

$$\widetilde{u}(x,t) = \widetilde{H}(x - \sin(\frac{\pi t}{2}), t) + (1 - (x - \sin(\frac{\pi t}{2})))e^{t+\sin(\frac{\pi t}{2})} + (x - \sin(\frac{\pi t}{2}))e^{t+\sin(\frac{\pi t}{2})+1}. \tag{148}$$

Similarly, we can get approximate solutions of the problem (124)-(127) and (36)-(41) for different value of N and M.

In Figure 11, exact and approximate solution of $H(x,t)$ with $N = 2, M = 4$ are presented.

In Figure 12, the exact and approximate solution of $u(x,t)$ with $N = 2, M = 4$ are plotted.

Table 5 and Table 6 present absolute error for $H(x,t)$ and $u(x,t)$ respectively in Example 3.2 for different N and M.

Table 7 shows L^2 norm error for functions $H(x,t) - \widetilde{H}(x,t)$ and $u(x,t) - \widetilde{u}(x,t)$ respectively in Example 3.2 for different N and M.

TABLE 5

The absolute error for $H(x,t)$ in Example 3.2.

(x,t)	$N=2,\ M=2$	$N=2,\ M=4$	$N=2,\ M=6$
(0,0)	0	0	0
(0.1,0.1)	1.69×10^{-3}	-1.36×10^{-5}	-3.93×10^{-7}
(0.2,0.2)	1.40×10^{-3}	1.14×10^{-4}	-5.00×10^{-7}
(0.3,0.3)	-2.74×10^{-3}	9.58×10^{-5}	2.47×10^{-6}
(0.4,0.4)	-7.35×10^{-3}	-1.48×10^{-4}	4.35×10^{-7}
(0.5,0.5)	-7.96×10^{-3}	-2.81×10^{-4}	-4.25×10^{-6}
(0.6,0.6)	-2.92×10^{-3}	-5.10×10^{-5}	-5.68×10^{-7}
(0.7,0.7)	4.58×10^{-3}	2.65×10^{-4}	4.70×10^{-6}
(0.8,0.8)	7.91×10^{-3}	1.70×10^{-4}	-9.47×10^{-7}
(0.9,0.9)	3.18×10^{-3}	-1.25×10^{-4}	-1.45×10^{-6}
(1,1)	0	0	0

TABLE 6

The absolute error for $u(x+sin(\frac{\pi}{2}t),t)$ in Example 3.2.

(x,t)	$N=2,\ M=2$	$N=2,\ M=4$	$N=2,\ M=6$
(0,0)	0	0	0
(0.1,0.1)	-1.10×10^{-3}	8.91×10^{-6}	2.48×10^{-7}
(0.2,0.2)	-1.05×10^{-3}	-7.88×10^{-5}	3.76×10^{-7}
(0.3,0.3)	1.87×10^{-3}	-7.79×10^{-5}	-1.83×10^{-6}
(0.4,0.4)	5.66×10^{-3}	1.04×10^{-4}	-5.49×10^{-7}
(0.5,0.5)	6.76×10^{-3}	2.31×10^{-4}	3.37×10^{-6}
(0.6,0.6)	3.35×10^{-3}	7.52×10^{-5}	9.43×10^{-7}
(0.7,0.7)	-2.45×10^{-3}	-1.85×10^{-4}	-3.68×10^{-6}
(0.8,0.8)	-5.56×10^{-3}	-1.56×10^{-4}	1.55×10^{-7}
(0.9,0.9)	-2.82×10^{-3}	6.30×10^{-5}	1.34×10^{-6}
(1,1)	0	0	0

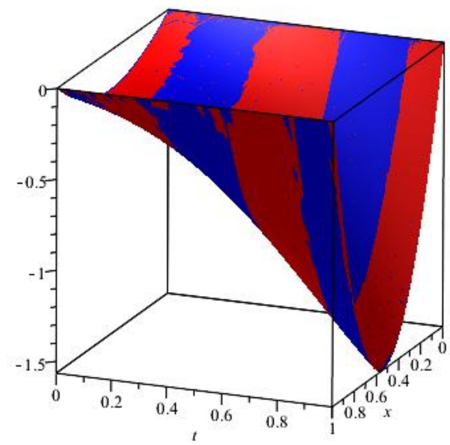

Figure 11. Exact (red) and approximate (green) solutions of $H(x,t)$ in Example 3.2

TABLE 7

The L^2 norm error for functions $H(x,t) - \widetilde{H}(x,t)$ and $u(x,t) - \widetilde{u}(x,t)$ in Example 3.2.

(N, M)	$\|\|H(x,t) - \widetilde{H}(x,t)\|\|_{L^2([0,1]\times[0,1])}$	$\|\|u(x,t) - \widetilde{u}(x,t)\|\|_{L^2([\varphi_1(t),\varphi_2(t)]\times[0,1])}$
(N=2,M=1)	1.12×10^{-4}	1.12×10^{-4}
(N=2,M=2)	2.18×10^{-5}	2.18×10^{-5}
(N=2,M=3)	7.24×10^{-8}	7.24×10^{-8}
(N=2,M=4)	2.24×10^{-8}	2.24×10^{-8}
(N=2,M=5)	5.96×10^{-10}	5.96×10^{-10}
(N=2,M=6)	4.95×10^{-12}	4.95×10^{-12}

Note: In table 7, we notice that L^2 norm error for functions $H(x,t) - \widetilde{H}(x,t)$ and $u(x,t) - \widetilde{u}(x,t)$ are same. Actually, we can verify this fact theoretically:

$$||H(x,t) - \widetilde{H}(x,t)||_{L^2(\Omega_1)} = ||u(x,t) - \widetilde{u}(x,t)||_{L^2(\Omega_2)}, \tag{149}$$

where

$$\Omega_1 := [0,1]\times[0,1], \quad \Omega_2 := [\varphi_1(t), \varphi_2(t)]\times[0,1] = [\sin(\frac{\pi t}{2}), 1+\sin(\frac{\pi t}{2})]\times[0,1]. \tag{150}$$

In fact, we have

$$||u(x,t) - \widetilde{u}(x,t)||_{L^2(\Omega_2)} \;=\; \int_0^1 \int_{\varphi_1(t)}^{\varphi_2(t)} (u(x,t) - \widetilde{u}(x,t))^2 dxdt \quad (applying\ equation\ (70))$$

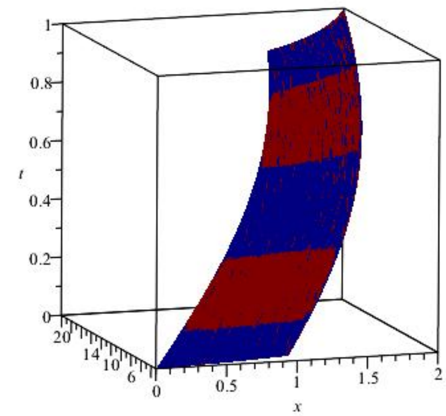

Figure 12. Exact (red) and approximate (green) solutions of $u(x,t)$ in Example 3.2

$$
\begin{aligned}
&= \int_0^1 \int_{\varphi_1(t)}^{\varphi_2(t)} \left(H\left(\frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}, t \right) - \widetilde{H}\left(\frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}, t \right) \right)^2 dxdt \\
&= \int_0^1 \int_0^1 \frac{(H(x,t)-\widetilde{H}(x,t))^2}{\varphi_2(t)-\varphi_1(t)} dxdt \quad (\textit{noting } \varphi_2(t)-\varphi_1(t)=1) \\
&= ||H(x,t)-\widetilde{H}(x,t)||_{L^2(\Omega_1)}.
\end{aligned}
$$

G Conclusion

In this part, we deal with model 2, the existence and uniqueness are discussed. Also, we use the Ritz-Galerkin method in Bernstein polynomial basis to obtain an approximate solution of our problem. And the parabolic equation with known moving boundaries are finally converted to algebraic equations, which can be solved quickly.

CHAPTER IV

INVERSE PROBLEM FOR MODEL 1

A Mathematical Problem

In this section, we discuss a inverse problem where determine the source parameter of our parabolic equation for model 1 and corresponding theorem for existence and uniqueness are given in section A. To support our findings, we also present results of numerical experiments in Section B. Section C concludes this chapter with a brief summary.

First, we introduce the inverse problem of parabolic equation with initial value and known moving boundaries as following:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda(t)u(x,t) + F(x,t), \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < T; \tag{151}$$

with initial condition

$$u(x,0) = f(x), \quad \varphi_1(0) < x < \varphi_2(0); \tag{152}$$

boundary conditions

$$u(\varphi_i(t),t) = g_i(t), \quad 0 < t < T, \quad i = 1,2, \tag{153}$$

and with the overspecification at a point in the domain

$$u(\bar{x},t) = E(t), \quad \varphi_1(t) < \bar{x} < \varphi_2(t) \tag{154}$$

where $f(x), g_i(t)\ and\ \varphi_i(t)(i = 1,2), E(t)$ are known functions, we need to find function $\lambda(t)$ and $u(x,t)$.

Without loss of generality, we assume $F(x,t)=0$ and let $\hat{\lambda}(t) := e^{\int_0^t \lambda(s)ds}$. From (19) and (154), we have

$$
\begin{aligned}
\hat{\lambda}(t) &= \frac{\bar{\omega}(\mu_1(\tau)) + \bar{\omega}(\mu_2(\tau)) + \int_0^t d\tau \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} \rho(\xi,\tau) G(x,t;\xi,\tau) d\xi + f(\bar{x})}{E(t)} \\
&= \frac{\bar{\omega}(\mu_1(\tau)) + \bar{\omega}(\mu_2(\tau)) + K(t) + f(\bar{x})}{E(t)} \qquad (155)
\end{aligned}
$$

where

$$
\bar{\omega}(\mu_i(\tau)) = \int_0^t \mu_i(\tau) \frac{\partial G}{\partial \xi}(\bar{x},t;\varphi_i(\tau),\tau) d\tau, i=1,2.
$$

$$
K(t) = \int_0^t d\tau \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} \rho(\xi,\tau) G(x,t;\xi,\tau) d\xi
$$

then plug $\hat{\lambda}(t)$ into (16), we have

$$
\begin{aligned}
\hat{v}_i(t) &= \hat{\lambda}(t) g_i(t) - f(\varphi_i(t)) - K(\varphi_i(t)) \\
&= \frac{\bar{\omega}(\mu_1(\tau)) + \bar{\omega}(\mu_2(\tau)) + K(t) + f(\bar{x})}{E(t)} - f(\varphi_i(t)) - K(\varphi_i(t)) \qquad (156)
\end{aligned}
$$

where

$$
K(\varphi_i(t)) := \int_0^t d\tau \int_{\varphi_1(\tau)}^{\varphi_2(\tau)} \rho(\xi,\tau) G(\varphi_i(t),t;\xi,\tau) d\xi
$$

finally plug $\hat{v}_i(t)$ into (18), we have

$$
\begin{cases}
\dfrac{\mu_1(t)}{2} + \displaystyle\int_0^t M_1 d\tau = \dfrac{g_1(t)}{E(t)} f(\bar{x}) + K(t)\dfrac{g_1(t)}{E(t)} - K(\varphi_1(t)) - f(\varphi_1(t)), \\
\\
\dfrac{-\mu_2(t)}{2} + \displaystyle\int_0^t M_2 d\tau = \dfrac{g_2(t)}{E(t)} f(\bar{x}) + K(t)\dfrac{g_2(t)}{E(t)} - K(\varphi_2(t)) - f(\varphi_2(t)).
\end{cases} \qquad (157)
$$

where

$$
\begin{aligned}
M_1 &= \mu_1(\tau)(\frac{\partial G}{\partial \xi}(\varphi_1(t),t;\varphi_1(\tau),\tau) - \frac{g_1(t)}{E(t)} \frac{\partial G}{\partial \xi}(\bar{x},t;\varphi_1(\tau),\tau)) \\
&+ \mu_2(\tau)(\frac{\partial G}{\partial \xi}(\varphi_1(t),t;\varphi_2(\tau),\tau) - \frac{g_1(t)}{E(t)} \frac{\partial G}{\partial \xi}(\bar{x},t;\varphi_2(\tau),\tau)); \\
M_2 &= \mu_2(\tau)(\frac{\partial G}{\partial \xi}(\varphi_2(t),t;\varphi_2(\tau),\tau) - \frac{g_2(t)}{E(t)} \frac{\partial G}{\partial \xi}(\bar{x},t;\varphi_2(\tau),\tau)) \\
&+ \mu_1(\tau)(\frac{\partial G}{\partial \xi}(\varphi_2(t),t;\varphi_1(\tau),\tau) - \frac{g_2(t)}{E(t)} \frac{\partial G}{\partial \xi}(\bar{x},t;\varphi_1(\tau),\tau));
\end{aligned}
$$

Note (157) is the system of volterra equations of second kind with singular kernel, we can get $\mu_i(t), i = 1, 2$ by applying volterra algorithm in chapter II, section C. Thus $\lambda(t) = (\ln \hat{\lambda}(t))'$. Moreover, we can get $u(x,t)$ by applying (19).

According by Lemma II.3, we have the following existence and uniqueness of theorem for our inverse problem.

Theorem IV.1 *Assume that $\varphi_1(t), \varphi_2(t) \in C^{0,\alpha}([0,T]), \alpha > \frac{1}{2}$, $g_1(t), g_2(t) \in C[0,T]$, $f(x) \in C^2[\varphi_1(t), \varphi_2(t)]$ and $F(x,t) \in C[[\varphi_1(t), \varphi_2(t)] \times [0,T]]$. The problem (151)-(154) has an unique solution in form of (155), (19) for $\lambda, u(x,t)$ respectively.*

B Numerical Evidence

To test the efficiency of the our method on a parabolic partial differential equation with an unknown time-dependent parameter, we present two examples.

Example 4.1 Consider the problem $(151) - (154)$ with

$$F(x,t) = 0;$$

$$f(x) = e^{\frac{x}{100} - \frac{1}{100}}$$

$$g_1(t) = e^{\frac{t}{10000}}$$

$$g_2(t) = e^{1 + \frac{t}{10000}}$$

$$\varphi_1(t) = cos(t)$$

$$\varphi_2(t) = 100 + cos(t)$$

$$E(t) = e^{\frac{1}{2} - \frac{cos(t)}{100} + \frac{t}{10000}}$$

and $\bar{x} = 50$, for which the exact solution is

$$\lambda(t) = -\frac{sin(t)}{100}$$

Now applying our method and volterra algorithm we have comparison of exact λ and numerical λ (See Figure 13)

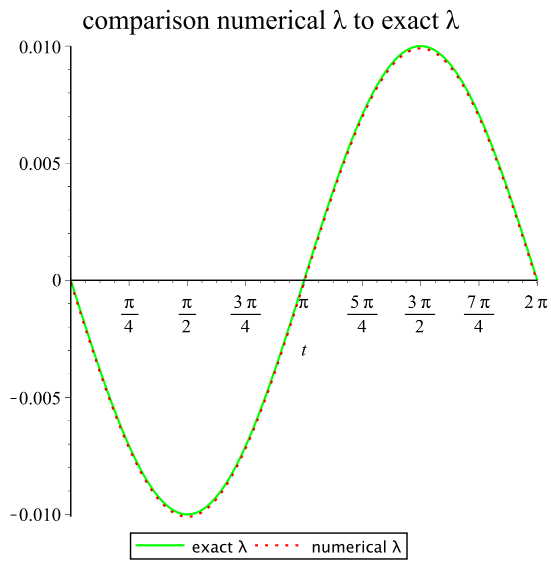

Figure 13. Comparison of exact λ and numerical λ in Example 4.1

And we can get the numerical $u(x,t) = e^{x/100 - cost/100 + t/10000}$ in $[0, 2\pi]$, which is same as our exact $u(x,t)$. The graph for $u(x,t)$ is presented in Figure 14.

Example 4.2 Consider the problem $(151) - (154)$ with

$$F(x,t) = 0;$$

$$f(x) = x$$
$$g_1(t) = -t + \frac{1}{3}$$
$$g_2(t) = t + \frac{1}{2}$$
$$\varphi_1(t) = -t + \frac{1}{3}$$
$$\varphi_2(t) = t + \frac{1}{2}$$
$$E(t) = e^{\frac{-t^2}{2}}(\int_0^t \frac{\tilde{\mu}_1(\tau)(\frac{1}{15} + \tau)}{4\sqrt{\pi(t-\tau)^3}} e^{-\frac{(\frac{1}{15}+\tau)^2}{4(t-\tau)}} + \int_0^t \frac{\tilde{\mu}_2(\tau)(\frac{-1}{10} - \tau)}{4\sqrt{\pi(t-\tau)^3}} e^{-\frac{(\frac{-1}{10}-\tau)^2}{4(t-\tau)}})$$

where $\tilde{\mu}_1(\tau) = 0.000697 - 0.17166\tau + 1.37976\tau^2 - 2.4062\tau^3 and \quad \tilde{\mu}_2(\tau) = 0.00333 - 0.2499\tau + 0.73867\tau^2 - 2.62058\tau^3$ and $\bar{x} = \frac{2}{5}$.

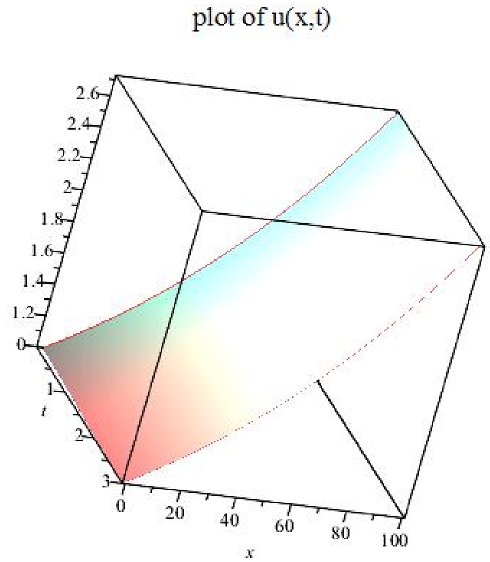

Figure 14. Plot of exact(numerical) u(x,t)in Example 4.1

The exact solution that we show in Chapter II, Example 2.2 is $\lambda(t) = t$. Now apply our method and volterra algorithm in Chapter II, we obtain graphs for comparison of λ and $u(x,t)$. (See Figure 15 and 16)

C Conclusion

In this part, we consider one inverse problem of model 1. Similar to the idea of Chapter II, mathematically we can determine the coefficient $\lambda(t)$ and nutrient concentration $u(x,t)$ of model 1 after obtaining incisional biopsy information. Also, two numerical examples are presented to validate our method and finding.

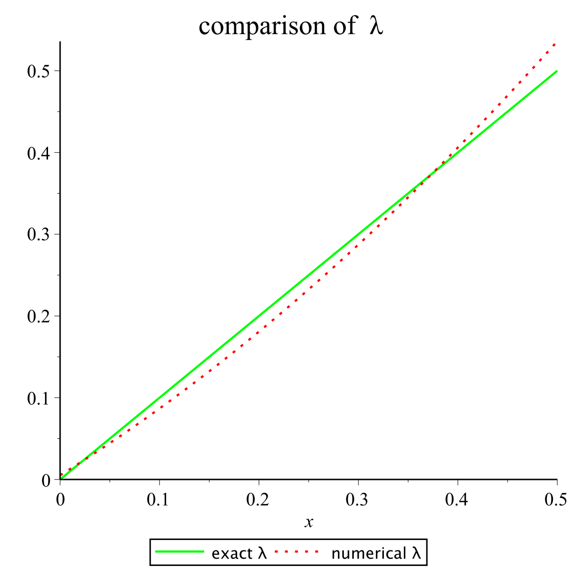

Figure 15. Comparison of exact λ and numerical λ in Example 4.2

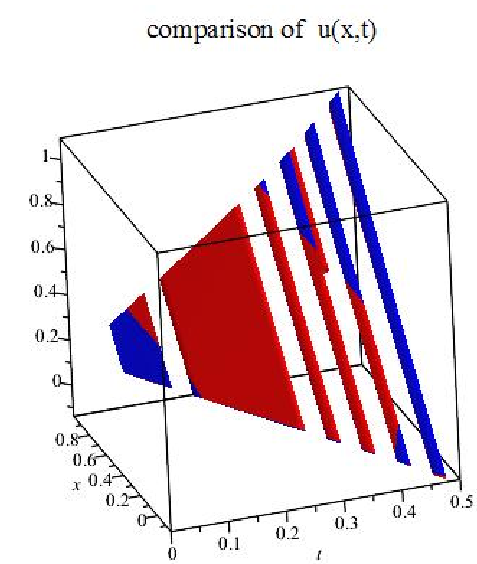

Figure 16. Numerical(blue) and exact (red) solution of u(x,t) in Example 4.2

CHAPTER V

NONLOCAL PROBLEM FOR MODEL 2

A Introduction

In this chapter, we discuss nonlocal problem for model 2 in form of non-classical parabolic equation with nonlocal and time-dependent boundaries as following:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda(x)u(x,t), \quad \varphi_1(t) < x < \varphi_2(t), \quad 0 < t < 1, \tag{158}$$

with initial condition

$$u(x,0) = f(x), \quad \varphi_1(0) < x < \varphi_2(0), \tag{159}$$

time-dependent boundary condition

$$u(\varphi_1(t),t) = g(t), \quad 0 < t < 1, \tag{160}$$

non-local boundary condition

$$\int_{\varphi_1(t)}^{\varphi_2(t)} u(x,t)dx = E(t), \quad 0 < t < 1, \tag{161}$$

and compatibility conditions

$$f(\varphi_1(0)) = u(\varphi_1(0),0) = g(0), \tag{162}$$

$$\int_{\varphi_1(0)}^{\varphi_2(0)} f(x)dx = E(0) \tag{163}$$

Here $\lambda(x)u(x,t)$ denotes the nutrient consumption rate at the location x at time t. The problem is to determine $u(x,t)$ for given $\lambda(x), f(x), \varphi_1(t), \varphi_2(t)$ and $E(t)$.

The main difficulty for this problem is integral term in non-local boundary condition, which can greatly complicate the application of standard numerical techniques such as finite-difference procedures, finite-element methods, spectral techniques, boundary integral equation schemes, etc. Therefore converting nonlocal boundary value problems to a more desirable equivalent form is the most important task, which is usually tough [3, 13, 48].

In this part, to overcome this difficulty and obtain approximate solution of this problem, we first introduce several transformations and transition function $G(x,t)$ to convert nonlocal boundary to non-classical boundary and then implement the Ritz-Galerkin method to solve it efficiently. This chapter is organized as follows. In section B, we obtain several equivalent forms of our problem. Existence and uniqueness theorem of problem are presented in Section C. The Ritz-Galerkin scheme is described in section D. Section E shows two examples to support and validate our numerical scheme. Finally, conclusions are made in section F.

B Equivalent Problems

In this section, we obtain three equivalent forms of original problem (158) – (163) by introducing two transformations and an transition function $G(x,t)$. Later, the second equivalent form is used to prove existence and uniqueness in section C and third equivalent form is used to get approximation solution of problem by applying the Ritz-Gelerkin method.

Introduce first transformation:

$$\xi = \frac{x - \varphi_1(t)}{\varphi_2(t) - \varphi_1(t)}. \tag{164}$$

then variable $x \in [\varphi_1(t), \varphi_2(t)]$ makes $\xi \in [0,1]$.

Let

$$v(x,t) = u((\varphi_2(t) - \varphi_1(t))x + \varphi_1(t), t), \quad 0 \le x \le 1,\ t \ge 0. \tag{165}$$

Then

$$u(x,t) = v\left(\frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}, t\right) = v(\xi,t), \tag{166}$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial \xi} \cdot \frac{1}{\varphi_2(t)-\varphi_1(t)}, \tag{167}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 v}{\partial \xi^2} \cdot \frac{1}{[\varphi_2(t)-\varphi_1(t)]^2}, \tag{168}$$

$$\frac{\partial u}{\partial t} = \frac{\partial v}{\partial t} - \frac{\partial v}{\partial \xi} \cdot B(x,t), \tag{169}$$

where

$$B(x,t) = \frac{\varphi_1'(t)(\varphi_2(t)-\varphi_1(t)) + (x-\varphi_1(t))[(\varphi_2'(t)-\varphi_1'(t)]}{((\varphi_2(t)-\varphi_1(t))^2}. \tag{170}$$

Under the first transformations (164), the first equivalent form of problem (158)-(163)is obtained as following:

$$\frac{\partial v}{\partial t} = \frac{1}{[\varphi_2(t)-\varphi_1(t)]^2} \cdot \frac{\partial^2 v}{\partial x^2} + \widetilde{B}(x,t)\frac{\partial v}{\partial x} - \widetilde{\lambda}(x,t)v(x,t), \quad 0<x<1, \quad 0<t<1, \tag{171}$$

with initial condition

$$v(x,0) = \widetilde{f}(x), \quad 0<x<1, \tag{172}$$

boundary conditions

$$v(0,t) = g(t), \quad 0<t<1, \tag{173}$$

$$\int_0^1 v(x,t)dx = \frac{E(t)}{\varphi_2(t)-\varphi_1(t)}, \quad 0<t<1, \tag{174}$$

and compatibility conditions

$$g(0) = f(\varphi_1(0)) = \widetilde{f}(0), \tag{175}$$

$$\int_0^1 \widetilde{f}(x)dx = \frac{E(0)}{\varphi_2(0)-\varphi_1(0)}, \tag{176}$$

where

$$\widetilde{B}(x,t) = B(\varphi_1(t) + (\varphi_2(t)-\varphi_1(t))x, t) = \frac{\varphi_1'(t) + [\varphi_2'(t)-\varphi_1'(t)]x}{\varphi_2(t)-\varphi_1(t)}, \tag{177}$$

$$\widetilde{\lambda}(x,t) = \lambda(\varphi_1(t) + (\varphi_2(t) - \varphi_1(t))x), \tag{178}$$

$$\widetilde{f}(x) = f((\varphi_2(0) - \varphi_1(0))x + \varphi_1(0)). \tag{179}$$

Next, introduce second transformation:

$$w(x,t) = v(x,t) - F(x,t), \tag{180}$$

where

$$F(x,t) = (1-2x)g(t) + \frac{2xE(t)}{\varphi_2(t) - \varphi_1(t)}, \tag{181}$$

Under this transformation (180), second equivalent form is acquired as following:

$$\frac{\partial w}{\partial t} = \frac{1}{[\varphi_2(t) - \varphi_1(t)]^2} \cdot \frac{\partial^2 w}{\partial x^2} + \widetilde{B}(x,t)\frac{\partial w}{\partial x} - \widetilde{\lambda}(x,t)w(x,t) + K(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{182}$$

with initial condition

$$w(x,0) = \widetilde{f}(x) - (1-2x)g(0) - \frac{2xE(0)}{\varphi_2(0) - \varphi_1(0)} := w_0(x), \quad 0 < x < 1, \tag{183}$$

boundary conditions

$$w(0,t) = 0, \quad 0 < t < 1, \tag{184}$$

$$\int_0^1 w(x,t)dx = 0, \quad 0 < t < 1, \tag{185}$$

and compatibility conditions

$$w(0,0) = 0, \tag{186}$$

$$\int_0^1 w(x,0)dx = 0, \tag{187}$$

where K(x,t) $= \widetilde{B}(x,t)\frac{\partial F(x,t)}{\partial x} - \widetilde{\lambda}(x,t)F(x,t) - \frac{\partial F(x,t)}{\partial t}$

$=2\widetilde{B}(x,t)\left(\frac{E(t)}{\varphi_2(t)-\varphi_1(t)} - g(t)\right) - \widetilde{\lambda}(x,t)\left(\frac{2xE(t)}{\varphi_2(t)-\varphi_1(t)} + (1-2x)g(t)\right)$

-$\left((1-2x)g'(t) + \frac{2x[E'(t)(\varphi_2(t)-\varphi_1(t)) - E(t)(\varphi_2'(t)-\varphi_1'(t))]}{(\varphi_2(t)-\varphi_1(t))^2}\right)$.

From (165),(166) and (180), we can obtain

$$w(x,t) = u((\varphi_2(t) - \varphi_1(t))x + \varphi_1(t), t) - (1-2x)g(t) - \frac{2xE(t)}{\varphi_2(t) - \varphi_1(t)}, \tag{188}$$

and

$$u(x,t) = w\left(\frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}, t\right) + \left(1 - \frac{2(x-\varphi_1(t))}{\varphi_2(t)-\varphi_1(t)}\right) g(t) + \frac{2E(t)(x-\varphi_1(t))}{(\varphi_2(t)-\varphi_1(t))^2}. \quad (189)$$

To obtain desirable form of non-local boundary condition which could be applied by Ritz-Galerkin method, the transition function is introduced:

$$G(x,t) = \int_0^x w(s,t)ds, \quad (190)$$

then we easily obtain

$$\frac{\partial G(x,t)}{\partial x} = w(x,t), \ \frac{\partial^2 G(x,t)}{\partial x^2} = \frac{\partial w(x,t)}{\partial x}, \ \frac{\partial^3 G(x,t)}{\partial x^3} = \frac{\partial^2 w(x,t)}{\partial x^2}, \ \frac{\partial w(x,t)}{\partial t} = \frac{\partial^2 G(x,t)}{\partial x \partial t}. \quad (191)$$

thus the problem becomes the third equivalent form as following:

$$\frac{\partial G^2}{\partial x \partial t} = \frac{1}{[\varphi_2(t)-\varphi_1(t)]^2} \cdot \frac{\partial^3 G}{\partial x^3} + \widetilde{B}(x,t)\frac{\partial^2 G}{\partial x^2} - \widetilde{\lambda}(x,t)\frac{\partial G}{\partial x} + K(x,t), \ 0<x<1, \ 0<t<1, \quad (192)$$

with initial condition

$$G(x,0) = \int_0^x w(s,0)ds = \int_0^x w_0(s)ds, \quad 0<x<1, \quad (193)$$

boundary conditions

$$\frac{\partial G}{\partial x}(0,t) = w(0,t) = 0, \quad 0<t<1, \quad (194)$$

$$G(1,t) = \int_0^1 w(x,t)dx = 0, \quad 0<t<1, \quad (195)$$

and compatibility conditions

$$\frac{\partial G}{\partial x}(0,0) = w(0,0) = 0, \quad (196)$$

$$G(1,0) = \int_0^1 w(x,0)dx = 0, \quad (197)$$

Furthermore, we can obtain the relationship between $G(x,t)$ and $u(x,t)$ as following:

$$G(x,t) = \int_0^x u((\varphi_2(t)-\varphi_1(t))s + \varphi_1(t), t)ds - x(1-x)g(t) - \frac{x^2 E(t)}{\varphi_2(t)-\varphi_1(t)}, \quad (198)$$

and

$$u(x,t) = \frac{\partial G}{\partial x}\left(\frac{x-\varphi_1(t)}{\varphi_2(t)-\varphi_1(t)}, t\right) + \left(1 - \frac{2(x-\varphi_1(t))}{\varphi_2(t)-\varphi_1(t)}\right) g(t) + \frac{2E(t)(x-\varphi_1(t))}{(\varphi_2(t)-\varphi_1(t))^2}, \tag{199}$$

C Existence and Uniqueness

In this section the existence and uniqueness theorem of problem (158)-(163) are discussed.

A. Bouziani [6] investigated solvability of the following parabolic equation with a nonlocal boundary condition.

$$\frac{\partial y}{\partial t} - \frac{\partial}{\partial x}\left(q_1(x,t)\frac{\partial y}{\partial x}\right) + q_2(x,t)\frac{\partial y}{\partial x} + q_3(x,t)y(x,t) = \hat{f}(x,t),\ 0 < x < 1,\ 0 < t < T, \tag{200}$$

with initial condition

$$y(x,0) = y^0(x), \tag{201}$$

non-local boundary conditions

$$y(1,t) = 0, \tag{202}$$

$$\int_0^b y(x,t)dx = 0, \tag{203}$$

where $0 \leq b \leq 1$.

The main results of [6] are Theorem 2.8 and Corollary 2.10, we summary them as following lemma:

Lemma V.1 ([6]) *Suppose* $0 < c_0 \leq q_1(x,t) \leq c_1,\ |\frac{\partial q_1(x,t)}{\partial t}| \leq c_2,\ |\frac{\partial q_1(x,t)}{\partial x}| \leq c_3,\ |q_2(x,t)| \leq c_4,\ |q_3(x,t)| \leq c_5$, *for all* $0 \leq x \leq 1,\ 0 \leq t \leq T$; *If* $y^0(x) \in H^1_\rho(0,1)$ *and* $\hat{f}(x,t) \in L^2_\rho(I, L^2_\rho(0,1))$, *then there exists a unique weak solution* $y(x,t)$ *of non-local boundary value problem (200)-(203), where*

$$\rho(x) = \begin{cases} x^2, & 0 \leq x \leq b; \\ b^2, & b^2 \leq x \leq 1. \end{cases}$$

.

Apply this lemma to our second equivalent form of problem (182)-(187), we can easily obtain the following theorem.

Theorem V.1 *Assume that*

$$\lambda(x) \in C^0[\varphi_1(t), \varphi_2(t)],\ f(x) \in C^1[\varphi_1(0), \varphi_2(0)] \text{ and } g(t),\ E(t),\ \varphi_1(t),\ \varphi_2(t) \in C^1[0,1], \tag{204}$$

then there exists a unique weak solution $w(x,t)$ of non-local boundary value problem (182)-(187).

According to the relationship of functions $u(x,t)$ and $w(x,t)$, we can easily get the following existence and uniqueness theorem.

Theorem V.2 *Assume*

$$\lambda(x) \in C^0[\varphi_1(t), \varphi_2(t)],\ f(x) \in C^1[\varphi_1(0), \varphi_2(0)] \text{ and } g(t),\ E(t),\ \varphi_1(t),\ \varphi_2(t) \in C^1[0,1], \tag{205}$$

then there exists a unique weak solution $u(x,t)$ of non-local and time-dependent boundary value problem (158)-(163).

D Numerical Scheme for Nonlocal Problem

In this section, we apply Ritz-Galerkin method to the third equivalent form of problem (192)-(197) in section B, then by (198)-(199) we can obtain the approximate solution of original problem easily.

Rewrite the third equivalent form as following:

$$\frac{\partial^2 G}{\partial x \partial t} = \frac{1}{[\varphi_2(t)-\varphi_1(t)]^2} \cdot \frac{\partial^3 G}{\partial x^3} + \widetilde{B}(x,t)\frac{\partial^2 G}{\partial x^2} - \widetilde{\lambda}(x,t)\frac{\partial G}{\partial x} + K(x,t),\ 0<x<1,\ 0<t<1, \tag{206}$$

with initial condition

$$G(x,0) = \int_0^x w(s,0)ds = \int_0^x w_0(s)ds = \int_0^x \widetilde{f}(s)ds - g(0)x(1-x) - \frac{E(0)\ x^2}{\varphi_2(0)-\varphi_1(0)}, \quad 0<x<1, \tag{207}$$

boundary conditions

$$\frac{\partial G}{\partial x}(0,t) = w(0,t) = 0, \quad 0 < t < 1, \tag{208}$$

$$G(1,t) = \int_0^1 w(x,t)dx = 0, \quad 0 < t < 1, \tag{209}$$

and compatibility conditions

$$\frac{\partial G}{\partial x}(0,0) = w(0,0) = 0, \tag{210}$$

$$G(1,0) = \int_0^1 w(x,0)dx = 0, \tag{211}$$

where

$$\widetilde{\lambda}(x,t) = \lambda(\varphi_1(t) + (\varphi_2(t) - \varphi_1(t))x), \tag{212}$$

$$\widetilde{f}(x) = f((\varphi_2(0) - \varphi_1(0))x + \varphi_1(0)), \tag{213}$$

$$\widetilde{B}(x,t) = B(\varphi_1(t) + (\varphi_2(t) - \varphi_1(t))x, t) = \frac{\varphi_1'(t) + [\varphi_2'(t) - \varphi_1'(t)]x}{\varphi_2(t) - \varphi_1(t)}, \tag{214}$$

$$\begin{aligned}\text{K(x,t)} \ &= 2\widetilde{B}(x,t)\left(\tfrac{E(t)}{\varphi_2(t)-\varphi_1(t)} - g(t)\right) - \widetilde{\lambda}(x,t)\left(\tfrac{2xE(t)}{\varphi_2(t)-\varphi_1(t)} + (1-2x)g(t)\right)\\ &\text{-}\left((1-2x)g'(t) + \tfrac{2x[E'(t)(\varphi_2(t)-\varphi_1(t))-E(t)(\varphi_2'(t)-\varphi_1'(t))]}{(\varphi_2(t)-\varphi_1(t))^2}\right).\end{aligned}$$

Let

$$W(G) = \frac{\partial G^2}{\partial x \partial t} - \frac{1}{[\varphi_2(t) - \varphi_1(t)]^2} \cdot \frac{\partial^3 G}{\partial x^3} - \widetilde{B}(x,t)\frac{\partial^2 G}{\partial x^2} + \widetilde{\lambda}(x,t)\frac{\partial G}{\partial x} - K(x,t) = 0, \tag{215}$$

A Ritz-Galerkin approximation to (215) is constructed as follows. The approximation solution $\widetilde{G}(x,t)$ is sought in the form of the truncated series

$$\widetilde{G}(x,t) = G(x,0) \cdot \left(\sum_{i=0}^{N}\sum_{j=0}^{M} c_{i,j}\ t\ B_{i,N}(x)B_{j,M}(t) + 1\right), \tag{216}$$

where $B_{i,N}(x), B_{j,M}(t)$ are Bernstein polynomials. From compatibility conditions (210)-(211), it is easy to see that the approximation solution $\widetilde{G}(x,t)$ satisfies the initial condition (207) and the boundary conditions (208) and (209).

Now the expansion coefficients $c_{i,j}$ are determined by the Galerkin equations

$$< W(\widetilde{G}(x,t)), B_{i,N}(x)B_{j,M}(t) >= 0, \quad (i = 0,1,\cdots,N,\ j = 0,1,\cdots,M), \tag{217}$$

where $< . >$ denotes the inner product defined by

$$< W(\widetilde{G}(x,t)), B_{i,N}(x)B_{j,M}(t) >= \int_0^1 \int_0^1 W(\widetilde{G}(x,t))B_{i,N}(x)B_{j,M}(t)dtdx. \tag{218}$$

Galerkin equations (217) gives a system of $(N+1)(M+1)$ linear equations which can be solved for the elements $c_{i,j}$ using mathematical software.

E Numerical Application

In this section, two numerical examples by using the Ritz-Galerkin methods are performed. Also, by providing absolute error of exact and numerical solution,the validity and efficiency of our numerical scheme are presented.

Example 5.1:

Consider (158)-(163) with

$$\lambda(x) = x, \tag{219}$$

$$\varphi_1(t) = 0, \quad 0 \le t \le 1, \tag{220}$$

$$\varphi_2(t) = \frac{1}{2-t}, \quad 0 \le t \le 1, \tag{221}$$

$$f(x) = e^{2x}, \quad 0 = \varphi_1(0) \le x \le \varphi_2(0) = \frac{1}{2}, \tag{222}$$

$$g(t) = e^{\frac{1}{3}t^3 - 2t^2 + 4t}, \quad 0 \le t \le 1, \tag{223}$$

$$E(t) = \frac{e-1}{2-t} e^{\frac{1}{3}t^3 - 2t^2 + 4t}, \quad 0 \le t \le 1, \tag{224}$$

which has the exact solution

$$u(x,t) = e^{x(2-t) + \frac{1}{3}t^3 - 2t^2 + 4t}, \tag{225}$$

From (192)-(197), we can obtain its equivalent problem as following:

$$\frac{\partial^2 G}{\partial x \partial t} = (2-t)^2 \cdot \frac{\partial^3 G}{\partial x^3} + \frac{x}{2-t}\frac{\partial^2 G}{\partial x^2} - \frac{x}{2-t}\frac{\partial G}{\partial x} + K(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{226}$$

with initial condition

$$G(x,0) = e^x + (2-e)x^2 - x - 1, \quad 0 < x < 1, \tag{227}$$

boundary conditions

$$\frac{\partial G}{\partial x}(0,t) = 0, \quad 0 < t < 1, \tag{228}$$

$$G(1,t) = 0, \quad 0 < t < 1, \tag{229}$$

where

$$K(x,t) = e^{\frac{1}{3}t^3 - 2t^2 + 4t} \left(\frac{(4-2e)x^2 + (2e-5)x}{2-t} - (1-2x)(t-2)^2 - 2x(e-1)(2-t)^2 \right). \tag{230}$$

From (165), (180) and (225), we can deduce that the problem (226)-(229) has the exact solution

$$G(x,t) = e^{\frac{1}{3}t^3 - 2t^2 + 4t} \left(e^x + (2-e)x^2 - x - 1 \right). \tag{231}$$

Now apply our numerical scheme with $N = 2, M = 4$ and solve equation (226).

From Galerkin equations (217), we have

$$\begin{cases} c_{0,0} = 4.0227,\ c_{0,1} = 5.4178,\ c_{0,2} = 7.6963,\ c_{0,3} = 9.0875,\ c_{0,4} = 9.3101, \\ c_{1,0} = 4.0235,\ c_{1,1} = 5.4156,\ c_{1,2} = 7.6996,\ c_{1,3} = 9.0847,\ c_{1,4} = 9.3114, \\ c_{2,0} = 4.0215,\ c_{2,1} = 5.4211,\ c_{2,2} = 7.6914,\ c_{2,3} = 9.0915,\ c_{2,4} = 9.3086. \end{cases} \tag{232}$$

From equations (216), we can obtain the approximate solution $\widetilde{H}(x,t)$ of the problem (226)-(229) as following

$$\widetilde{G}(x,t) = G(x,0) \cdot \left(\sum_{i=0}^{N=2} \sum_{j=0}^{M=4} c_{i,j}\ t\ B_{i,N}(x) B_{j,M}(t) + 1 \right), \tag{233}$$

According to (199), corresponding approximate solution $\widetilde{u}(x,t)$ of the problem (158)-(163) is obtained.

$$\widetilde{u}(x,t) = \frac{\partial \widetilde{G}}{\partial x}((2-t)x, t) + e^{\frac{1}{3}t^3 - 2t^2 + 4t}(1 - 2(e-2)x(2-t)). \tag{234}$$

Similarly, approximate solutions of the problem (226)-(229) and (158)-(163) for different value of N and M can be obtained.

In Figure 17, the exact and approximate solutions of $H(x,t)$ with $N = 2, M = 4$ are plotted.

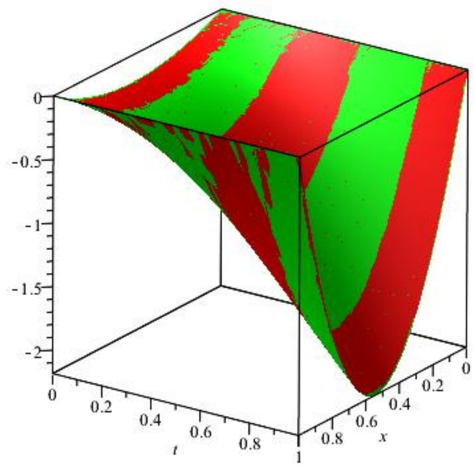

Figure 17. Exact (red) and approximate (green) solutions of $H(x,t)$ in Example 5.1

In Figure 18, the exact and approximate solutions of $u(x,t)$ with $N = 2, M = 4$ are plotted.

Table 8 and Table 9 present respectively absolute error for $H(x,t)$ and $u(x,t)$ with different N and M in Example 5.1.

Table 10 present L^2 norm error for functions $H(x,t) - \widetilde{H}(x,t)$ and $u(x,t) - \widetilde{u}(x,t)$ with different N and M in Example 5.1.

Example 5.2:

In this example, we solve (158)-(163) with

$$\lambda(x) = 0, \tag{235}$$

$$\varphi_1(t) = \sin(\frac{\pi}{2}t), \quad 0 \leq t \leq 1, \tag{236}$$

$$\varphi_2(t) = 1 + \sin(\frac{\pi}{2}t), \quad 0 \leq t \leq 1, \tag{237}$$

$$f(x) = e^x, \quad 0 = \varphi_1(0) \leq x \leq \varphi_2(0) = 1, \tag{238}$$

$$g(t) = e^{t+\sin(\frac{\pi}{2}t)}, \quad 0 \leq t \leq 1, \tag{239}$$

TABLE 8

The absolute error for $H(x,t)$ in Example 5.1

(x,t)	$N=2,\ M=4$	$N=2,\ M=6$	$N=2,\ M=8$
(0,0)	0	0	0
(0.1,0.1)	1.78×10^{-7}	-2.40×10^{-8}	1.94×10^{-10}
(0.2,0.2)	4.55×10^{-6}	6.29×10^{-8}	-1.41×10^{-9}
(0.3,0.3)	-2.02×10^{-6}	1.74×10^{-7}	5.18×10^{-9}
(0.4,0.4)	-2.12×10^{-5}	-4.22×10^{-7}	-5.12×10^{-9}
(0.5,0.5)	-1.56×10^{-5}	-3.97×10^{-7}	-7.78×10^{-9}
(0.6,0.6)	2.88×10^{-5}	8.99×10^{-7}	1.90×10^{-8}
(0.7,0.7)	5.07×10^{-5}	5.62×10^{-7}	-6.90×10^{-9}
(0.8,0.8)	-6.04×10^{-6}	-1.25×10^{-6}	-1.33×10^{-8}
(0.9,0.9)	-4.42×10^{-5}	3.04×10^{-7}	1.54×10^{-8}
(1,1)	0	0	0

TABLE 9

The absolute error for $u\left(\frac{x}{2-t},t\right)$ in Example 5.1

(x,t)	$N=2,\ M=4$	$N=2,\ M=6$	$N=2,\ M=8$
(0,0)	0	0	0
(0.1,0.1)	5.62×10^{-6}	-7.64×10^{-7}	6.23×10^{-9}
(0.2,0.2)	5.18×10^{-5}	7.19×10^{-7}	-1.59×10^{-8}
(0.3,0.3)	-1.01×10^{-5}	7.21×10^{-7}	2.22×10^{-8}
(0.4,0.4)	-1.75×10^{-5}	-3.69×10^{-7}	-5.51×10^{-9}
(0.5,0.5)	2.59×10^{-5}	6.96×10^{-7}	1.42×10^{-8}
(0.6,0.6)	-8.59×10^{-5}	-2.72×10^{-6}	-5.84×10^{-8}
(0.7,0.7)	-2.31×10^{-4}	-2.43×10^{-6}	3.45×10^{-8}
(0.8,0.8)	5.81×10^{-5}	8.40×10^{-6}	7.76×10^{-8}
(0.9,0.9)	5.26×10^{-4}	-4.53×10^{-6}	-1.73×10^{-7}
(1,1)	-1.03×10^{-3}	2.13×10^{-5}	-3.37×10^{-7}

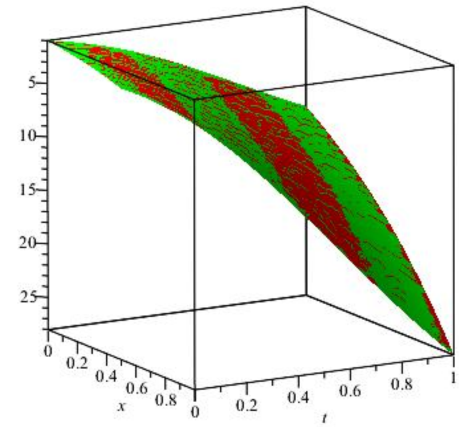

Figure 18. Exact (red) and approximate (green) solutions of $u(x,t)$ in Example 5.1

TABLE 10

The L^2 norm error for functions $H(x,t) - \widetilde{H}(x,t)$ and $u(x,t) - \tilde{u}(x,t)$ in Example 5.1

(N,M)	$\|\|H(x,t) - \widetilde{H}(x,t)\|\|_{L^2([0,1]\times[0,1])}$	$\|\|u(x,t) - \tilde{u}(x,t)\|\|_{L^2([\varphi_1(t),\varphi_2(t)]\times[0,1])}$
(N=2,M=3)	7.535×10^{-9}	9.670×10^{-8}
(N=2,M=4)	5.088×10^{-10}	6.302×10^{-9}
(N=2,M=5)	4.979×10^{-12}	6.266×10^{-11}
(N=2,M=6)	2.754×10^{-13}	3.386×10^{-12}
(N=2,M=7)	2.972×10^{-15}	3.711×10^{-14}
(N=2,M=8)	8.966×10^{-17}	1.099×10^{-15}

$$E(t) = e^{t+\sin(\frac{\pi}{2}t)}(e-1), \quad 0 \le t \le 1, \tag{240}$$

which has the exact solution

$$u(x,t) = e^{t+x}, \tag{241}$$

From (192)-(197), we can obtain the following its equivalent problem

$$\frac{\partial^2 G}{\partial x \partial t} = \frac{\partial^3 G}{\partial x^3} + \frac{\pi}{2}\cos(\frac{\pi t}{2})\frac{\partial^2 G}{\partial x^2} + K(x,t), \quad 0 < x < 1, \quad 0 < t < 1, \tag{242}$$

with initial condition

$$G(x,0) = e^x + (2-e)x^2 - x - 1, \quad 0 < x < 1, \tag{243}$$

boundary conditions

$$\frac{\partial G}{\partial x}(0,t) = 0, \quad 0 < t < 1, \tag{244}$$

$$G(1,t) = 0, \quad 0 < t < 1, \tag{245}$$

where

$$K(x,t) = e^{t+\sin(\frac{\pi t}{2})}\left((e-2)\pi\cos(\frac{\pi t}{2}) - \left(1+\frac{\pi}{2}\cos(\frac{\pi t}{2})\right)(1+(2e-4)x)\right). \tag{246}$$

From (165), (180) and (241), we can deduce that the problem (242)-(245) has the exact solution

$$G(x,t) = e^{t+\sin(\frac{\pi t}{2})}\left(e^x - (e-2)x^2 - x - 1\right). \tag{247}$$

Now apply our numerical scheme with $N = 2, M = 4$ and solve equation (242).

From Galerkin equations (217), we have

$$\begin{cases} c_{0,0} = 2.5937,\ c_{0,1} = 3.3350,\ c_{0,2} = 4.6750,\ c_{0,3} = 6.1663,\ c_{0,4} = 6.3877, \\ c_{1,0} = 2.5937,\ c_{1,1} = 3.3342,\ c_{1,2} = 4.6772,\ c_{1,3} = 6.1636,\ c_{1,4} = 6.3891, \\ c_{2,0} = 2.5888,\ c_{2,1} = 3.3461,\ c_{2,2} = 4.6623,\ c_{2,3} = 6.1738,\ c_{2,4} = 6.3858. \end{cases} \tag{248}$$

From equations (216), the approximate solution $\widetilde{H}(x,t)$ of the problem (242)-(245) can be obtained as following

$$\widetilde{G}(x,t) = G(x,0)\cdot\left(\sum_{i=0}^{N=2}\sum_{j=0}^{M=4} c_{i,j}\ t\ B_{i,N}(x)B_{j,M}(t) + 1\right), \tag{249}$$

According to (199), we can get following corresponding approximate solution $\widetilde{u}(x,t)$ of the problem (158)-(163).

$$\widetilde{u}(x,t) = \frac{\partial\widetilde{G}}{\partial x}\left(x-\sin(\frac{\pi t}{2}),t\right) + \left(1+(2e-4)\left(x-\sin(\frac{\pi t}{2})\right)\right)e^{t+\sin(\frac{\pi t}{2})}. \tag{250}$$

Similarly, we can get approximate solutions of the problem (242)-(245) and (158)-(163) for different value of N and M.

In Figure 19, the exact and approximate solutions of $H(x,t)$ with $N = 2, M = 4$ are plotted.

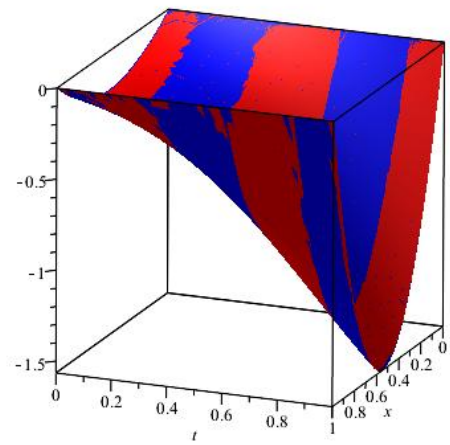

Figure 19. Exact (red) and approximate (green) solutions of $H(x,t)$ in Example 5.2

In Figure 20, the exact and approximate solutions of $u(x,t)$ with $N=2, M=4$ are plotted.

Table 11 and Table 12 present respectively absolute error for $H(x,t)$ and $u(x,t)$ with different N and M in Example 5.2.

Table 13 present L^2 norm error for functions $H(x,t)-\widetilde{H}(x,t)$ and $u(x,t)-\widetilde{u}(x,t)$ with different N and M in Example 5.2.

F Conclusion

In this chapter, the existence and uniqueness theorem is presented. Then the Ritz-Galerkin method in Bernstein polynomial basis is implemented to obtain an approximate solution of non-classical parabolic equation subject to given initial and nonlocal time-dependent boundary conditions by converting the problem to a system of algebraic equations. The nonlocal problem builds a solid foundation for us to solve the free boundary problem model in next chapter.

TABLE 11

The absolute error for $H(x,t)$ in Example 5.2

(x,t)	$N=2,\ M=2$	$N=2,\ M=4$	$N=2,\ M=6$
(0,0)	0	0	0
(0.1,0.1)	4.12×10^{-5}	-2.17×10^{-7}	-7.99×10^{-9}
(0.2,0.2)	5.20×10^{-5}	5.44×10^{-6}	-1.69×10^{-8}
(0.3,0.3)	-2.55×10^{-4}	4.98×10^{-6}	1.62×10^{-7}
(0.4,0.4)	-7.74×10^{-4}	-1.76×10^{-5}	-9.25×10^{-9}
(0.5,0.5)	-9.17×10^{-4}	-3.20×10^{-5}	-4.96×10^{-7}
(0.6,0.6)	-1.60×10^{-4}	4.19×10^{-6}	8.07×10^{-8}
(0.7,0.7)	1.19×10^{-3}	5.46×10^{-5}	8.05×10^{-7}
(0.8,0.8)	1.78×10^{-3}	2.22×10^{-5}	-4.60×10^{-7}
(0.9,0.9)	4.75×10^{-4}	-4.21×10^{-5}	-1.75×10^{-7}
(1,1)	0	0	0

TABLE 12

The absolute error for $u(x+sin(\frac{\pi}{2}t),t)$ in Example 5.2

(x,t)	$N=2,\ M=2$	$N=2,\ M=4$	$N=2,\ M=6$
(0,0)	0	0	0
(0.1,0.1)	-5.42×10^{-4}	2.94×10^{-6}	1.02×10^{-7}
(0.2,0.2)	-4.09×10^{-4}	-3.95×10^{-5}	1.37×10^{-7}
(0.3,0.3)	1.30×10^{-3}	-2.90×10^{-5}	-8.45×10^{-7}
(0.4,0.4)	3.28×10^{-3}	6.94×10^{-5}	-7.46×10^{-8}
(0.5,0.5)	3.44×10^{-3}	1.15×10^{-4}	1.71×10^{-6}
(0.6,0.6)	9.19×10^{-4}	3.26×10^{-6}	3.58×10^{-8}
(0.7,0.7)	-2.56×10^{-3}	-1.28×10^{-4}	-2.03×10^{-6}
(0.8,0.8)	-3.72×10^{-3}	-6.40×10^{-5}	6.31×10^{-7}
(0.9,0.9)	-1.22×10^{-3}	6.7×10^{-5}	6.02×10^{-7}
(1,1)	0	0	0

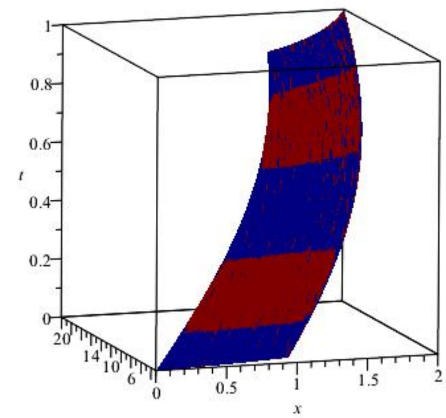

Figure 20. Exact (red) and approximate (green) solutions of $u(x,t)$ in Example 5.2

TABLE 13

The L^2 norm error for functions $H(x,t)-\widetilde{H}(x,t)$ and $u(x,t)-\tilde{u}(x,t)$ in Example 5.2

(N,M)	$\|\|H(x,t)-\widetilde{H}(x,t)\|\|_{L^2([0,1]\times[0,1])}$	$\|\|u(x,t)-\tilde{u}(x,t)\|\|_{L^2([\varphi_1(t),\varphi_2(t)]\times[0,1])}$
(N=2,M=1)	2.545×10^{-6}	3.858×10^{-5}
(N=2,M=2)	5.220×10^{-7}	7.972×10^{-6}
(N=2,M=3)	2.068×10^{-9}	3.183×10^{-8}
(N=2,M=4)	5.286×10^{-10}	7.881×10^{-9}
(N=2,M=5)	1.611×10^{-11}	2.431×10^{-10}
(N=2,M=6)	1.162×10^{-13}	1.686×10^{-12}

CHAPTER VI

FREE BOUNDARY PROBLEM MODEL

A Introduction

In this chapter, we focus on the solid DCIS model by considering an one-dimensional case, which is in form of free boundary problem. The free boundary problem is a tough problem. Though many publications has been done toward it, the problem is far from well understood.

The presentation of this part is as follows:

In section B, we introduce mathematical model of DCIS in form of free boundary problem. An equivalent problem and important preliminary results are stated in section C. An iteration algorithm and well-posedness theorem of this free boundary problem are presented in section D. In section E, numerical examples and simulation are discussed. Section F concludes this article with a brief summary.

B Mathematical Model

We model tumor growth pattern by using dimensionless nutrient concentration $u(x, t)$ which satisfies a reaction - diffusion equation.

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda u(x, t) + F(x, t), \quad \varphi_1(t) < x < \varphi_2(t), t > 0; \tag{251}$$

Here λ in condition (251) could be a function $\lambda(x), \lambda(t)$ or $\lambda(x, t)$. $\lambda u(x, t)$ denotes the nutrient consumption rate; $F(x, t)$ is the transfer of nutrient from or to the neighborhood, which may be positive or negative.

Also, we assume $u(x,t)$ satisfies the following initial and boundary and free boundary conditions:

$$u(x,0) = f(x), \quad \varphi_1(0) < x < \varphi_2(0); \tag{252}$$

$$u(\varphi_1(t),t) = g_1(t), \quad 0 < t < T, \tag{253}$$

$$u(\varphi_2(t),t) = g_2(t), \quad 0 < t < T, \tag{254}$$

Here $f(x), \varphi_1(t), g_1(t), and\ g_2(t)$ are given.

Finally, the mass conservation consideration implies following relationship:

$$\mu \int_{\varphi_1(t)}^{\varphi_2(t)} (u(x,t) - u_0)dx = \frac{\partial \varphi_2(t)}{\partial t}, \quad \varphi_2(0) = s_0 > 0 \tag{255}$$

Here $\mu, u_0, and\ s_0$ are known constant.

Our purpose is to determine $(u(x,t), \varphi_2(t))$ which satisfy (251)-(255) for given $\lambda, F(x,t),\ f(x), \varphi_1(t),\ g_1(t), g_2(t), \mu, u_0$ and s_0.

To avoid tedious formal complications, we consider the following simplified version:

(In fact, by introducing several transformations, we am able to get this version easily, see [46])

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - \lambda u(x,t), \quad (x,t) \in \mathcal{D}, \tag{256}$$

$$u(x,0) = f(x), \quad 0 < x < s(0), \tag{257}$$

$$u(0,t) = g(t), \quad 0 < t < T, \tag{258}$$

$$u(s(t),t) = 0, \quad 0 < t < T, \tag{259}$$

$$\int_0^{s(t)} u(x,t)dx = \frac{\partial s(t)}{\partial t}, \quad s(0) = s_0 > 0 \tag{260}$$

Here, $\mathcal{D} := \{(x,t)|0 < x < s(t), 0 < t < T\}$. The problem is to determine $(u(x,t), s(t))$ for given $T, \lambda, f(x), g(t)$ and s_0.

C Equivalent Problem

In this section, we obtain equivalent problem of free boundary model— nonlinear parabolic equation with initial-boundary conditions.

First, let's introduce some necessary notations and preliminary results.

Define

$$L^p(0,T;X) = \{f(t) : (\int_0^T \parallel f \parallel_X^p dt)^{1/p} < \infty\}, 1 \le p \le \infty$$

$$L^\infty(0,T;X) = \{f(t) : \parallel f \parallel_\infty := ess \sup_{0\le t \le T} \parallel f \parallel_X < \infty\}, 1 \le p \le \infty$$

$$W^{i,p}(0,T;X) = \{f(t) : f \in L^p(0,T;X), \frac{df}{dt}, ... \frac{d^i f}{dt^i} \in L^p(0,T;X)\},$$

$$H^i(0,T;X) = W^{i,2}(0,T;X), 1 \le p \le \infty$$

Here X be a Banach space, $L(0,T;X)$ be the space of all weakly measurable function from $[0,T]$ into X.

Now consider the parabolic equation with initial and boundary conditions:

$$\frac{\partial u}{\partial t} + Au + F(u) = 0, \quad (x,t) \in (0,1) \times (0,T), \tag{261}$$

$$u = 0, \quad (x,t) \in \{0,1\} \times [0,T], \tag{262}$$

$$u = f, \quad (x,t) \in [0,1] \times \{0\}, \tag{263}$$

where $Au = -a(t)\frac{\partial^2 u}{\partial x^2}, F(u) = b(x,t)\frac{\partial u}{\partial x} + \lambda u - p(x,t), u(x,t)$ is the unknown function.

Motivated by ([35], Ch 15) we construct semigroup solution operator for (261)-(263):

Let $e^{t\triangle}$ be semigroup solution operator to the heat equation with initial-boundary condition (262)-(263). Then we have

$$e^{t\triangle} f(x) = \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n t} \phi_n(x), \tag{264}$$

where

$$\phi_n(x) = \sqrt{2}\sin(\sqrt{\lambda_n x}), \quad \sqrt{\lambda_n} = n\pi, \quad \alpha_n =< f, \phi_n > \tag{265}$$

Now if we define

$$e^{\alpha(t,\tau)\triangle} f(x) := \sum_{n=1}^{\infty} \alpha_n e^{-\lambda_n \alpha(t,\tau)} \phi_n(x), \tag{266}$$

Then $e^{\alpha(t,\tau)\triangle}$ is the semigroup solution operator to

$$\frac{\partial u}{\partial t} - a(t)\frac{\partial^2 u}{\partial x^2} = 0 \tag{267}$$

with initial-boundary condition (262)-(263), where $\alpha(t,\tau) = \int_\tau^t a(\eta)d\eta$

Then problem (261)-(263) is equivalent to solve the following integral equation:

$$u(t) = e^{\alpha(t,0)\triangle} f(x) + \int_0^t e^{\alpha(t,\tau)\triangle} F(u(\tau))d\tau, \tag{268}$$

From ([35], Ch. 15), some important properties of solution operator and its estimates are as following:

$$e^{\alpha\triangle}(\partial f) = \partial_x e_N^{\alpha\triangle} f \tag{269}$$

where $e_N^{\alpha\triangle}$ is the solution operator to the heat equation on $[0,\infty]\times[0,1]$ with Neumann boundary condition.

$$e_N^{\alpha(t,\tau)\triangle} f(x) := \sum_{n=1}^{\infty} \beta_n e^{-\lambda_n \alpha(t,\tau)} \psi_n(x), \tag{270}$$

where

$$\psi_n(x) = \sqrt{2}\cos(\sqrt{\lambda_n x}), \quad \sqrt{\lambda_n} = n\pi, \quad \beta_n =< f, \psi_n > \tag{271}$$

Moreover, we have following estimates:

$$\| e^{\alpha(t,0)\triangle} f \|_{C^r} \leq C t^{-r/2} \| f \|_\infty, \quad \| e_N^{\alpha(t,0)\triangle} f \|_{C^r} \leq C t^{-r/2} \| f \|_\infty, \tag{272}$$

where $f \in L^\infty(0,1), r \geq 0, t \in (0,T], with C = C(r,T)$

And

$$\| e^{\alpha(t,0)\triangle} f \|_{C^{\mu+r}} \leq Ct^{-r/2} \| f \|_{C^{\mu}}, \tag{273}$$

where $f \in C_b^r[0,1], \mu \in [0,2], r \geq 0, t \in (0,T]$

$$\| e_N^{\alpha(t,0)\triangle} f \|_{C^{\mu+r}} \leq Ct^{-r/2} \| f \|_{C^{\mu}}, \tag{274}$$

where $f \in C^r[0,1], \mu \in [0,1], r \geq 0, t \in (0,T]$

Now, let's go back to consider (256)-(260), we change variable $x \in [0, s(t)]$ to $\xi \in [0,1]$ by setting

$$\xi = \frac{x}{s(t)}. \tag{275}$$

Let

$$H(x,t) = u((s(t))x, t), \quad 0 \leq x \leq 1,\ 0 \leq t \leq T. \tag{276}$$

Then

$$u(x,t) = H\left(\frac{x}{s(t)}, t\right) = H(\xi, t), \tag{277}$$

$$\frac{\partial u}{\partial x} = \frac{\partial H}{\partial \xi} \cdot \frac{1}{s(t)}, \tag{278}$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{\partial^2 H}{\partial \xi^2} \cdot \frac{1}{s^2(t)}, \tag{279}$$

$$\frac{\partial u}{\partial t} = \frac{\partial H}{\partial t} - \frac{\partial H}{\partial \xi} \cdot \frac{xs'(t)}{s^2(t)} \tag{280}$$

Also, from (260) we have

$$\frac{s'(t)}{s(t)} = \int_0^1 u(s(t)\xi, t) d\xi \tag{281}$$

$$\ln(s(t)) = \int_0^t \int_0^1 u(s(t)\xi, t) d\xi d\tau + C \tag{282}$$

$$s(t) = Ce^{\int_0^t \int_0^1 u(s(t)\xi, t) d\xi d\tau} \tag{283}$$

Since $s(0) = s_0$, then

$$s(t) = s_0 e^{\int_0^t \int_0^1 u(s(t)\xi, t) d\xi d\tau} \tag{284}$$

then problem (256)-(260) becomes

$$\frac{\partial H}{\partial t} = \frac{1}{s^2(t)}\frac{\partial^2 H}{\partial x^2} + B(x,t)\frac{\partial H}{\partial x} - \lambda H(x,t), \quad (x,t) \in \mathcal{D}, \tag{285}$$

with initial condition

$$H(x,0) = f(s(0)x), \quad 0 < x < 1, \tag{286}$$

boundary conditions

$$H(0,t) = g(t), \quad 0 < t < T, \tag{287}$$

$$H(1,t) = 0, \quad 0 < t < T, \tag{288}$$

$$s(t) = s_0 e^{\int_0^t \int_0^1 H(\xi,t) d\xi d\tau} \tag{289}$$

where

$$\mathcal{D} := \{(x,t)|0 < x < 1, 0 < t < T\} \tag{290}$$

$$B(x,t) = \frac{xs'(t)}{s(t)}, \quad \widetilde{f}(x) = f(s(0)x). \tag{291}$$

Now we introduce another transformation.

Set

$$v(x,t) = H(x,t) - (1-x)g(t), \tag{292}$$

then

$$\frac{\partial v}{\partial t} = \frac{\partial H}{\partial t} - (1-x)\frac{d(g(t))}{\partial t}, \tag{293}$$

$$\frac{\partial v}{\partial x} = \frac{\partial H}{\partial x} + g(t), \tag{294}$$

$$\frac{\partial^2 H}{\partial x^2} = \frac{\partial^2 v}{\partial x^2}, \tag{295}$$

Under this transformation (292) and equations (293)-(295), we have

$$\frac{\partial v}{\partial t} = \frac{1}{s^2(t)} \cdot \frac{\partial^2 v}{\partial x^2} + B(x,t)\frac{\partial v}{\partial x} - \lambda v(x,t) + K(x,t), \quad (x,t) \in \mathcal{D}, \tag{296}$$

with initial condition

$$v(x,0) = \widetilde{f}(x), \quad 0 < x < 1, \tag{297}$$

boundary conditions

$$v(0,t) = 0, \quad 0 < t < T, \tag{298}$$

$$v(1,t) = 0, \quad 0 < t < T, \tag{299}$$

$$s(t) = s_0 e^{\int_0^t \int_0^1 [v(\xi,t)+(1-x)g(t)]d\xi d\tau} \tag{300}$$

where

$$K(x,t) = B(x,t)g(t) - \lambda(1-x)g(t) - (1-x)g'(t). \tag{301}$$

$$\widetilde{f}(x) = f(s(0)x) - (1-x)g(0). \tag{302}$$

Now by using the arguments in ([22] Ch.7) the following lemma is obtained:

Lemma VI.1 *(a) Assume that* $b, \lambda \in L^\infty([0,1]\times[0,T]); p(x,t) \in L^2([0,1]\times[0,T])$ *and* $f \in L^2(0,1)$. *Then there exists a unique weak solution of (261)-(263).*

(b) If $p(x,t) \in L^2(0,T;L^2(0,1))$*and*$f \in H_0^1(0,1)$. *Also, suppose* $u \in L^2(0,T;H^{-1}$ $(0,1))$ *is a weak solution of the nonlinear parabolic initial boundary value problem (261)-(263). Then* $u \in L^2(0,T;H^2(0;1)) \cap L^\infty(0,T;H_0^1(0,1)), u' \in L^2(0,T;L^2(0,1))$. *By Sobolev embedding theorem, if* $u \in L^2(0,T;H^2(0,1))$, *with* $u' \in L^2(0,T;L^2(0,1))$, *then* $u \in C(0,T;H^1(0,1))$.

Now go back to our free boundary problem and we have

Lemma VI.2 *Suppose that* $\lambda \in L^\infty(0,\infty)$ *and nonnegative,* $f \in C^1, g \in C^1$ *with bounded* $g', 0 \le T \le \infty$ *and* $\overline{u} = \max_{0\le x\le s(0)}\{f,g\}$. *If* $u(x,t)$ *satisfies (256)-(260), then we have following:*

(a) $0 \le u(x,t) \le \overline{u}$ *for* $0 \le x \le s(t), 0 \le t \le T$

(b) $0 \le s'(t) \le \overline{u}s(t) for\ 0 \le t \le T$

(c) $0 \le s(t) \le s(0)e^{\overline{u}t} for\ 0 \le t \le T$

Note (a) is directly from maximum principle. (b) and (c) can be easily obtained from (281) and (300).

Theorem VI.1 *Suppose $\widetilde{f} \in H_0^1(0,1)$, then the problem (296)-(300) has a unique solution.*

Proof VI.1 *Let $(v_1, s_1), (v_2, s_2)$ are two solutions of (296)-(300), then we have for $i = 1, 2$*

$$\frac{\partial v_i}{\partial t} - \frac{1}{s_i^2(t)} \cdot \frac{\partial^2 v_i}{\partial x^2} - \frac{x s_i'(t)}{s_i(t)} \frac{\partial v_i}{\partial x} + \lambda v_i - K(x,t) = 0, \quad (x,t) \in \mathcal{D}, \tag{303}$$

with initial condition

$$v_i(x,0) = \widetilde{f}(x), \quad 0 < x < 1, \tag{304}$$

boundary conditions

$$v_i(0,t) = 0, \quad 0 < t < T, \tag{305}$$

$$v_i(1,t) = 0, \quad 0 < t < T, \tag{306}$$

$$s_i(t) = s_0 e^{\int_0^t \int_0^1 [v_i(\xi,t) + (1-x) g(t)] d\xi d\tau} \tag{307}$$

Now consider the difference $w = v_1 - v_2$, and $\delta = s_1 - s_2$, we have

$$\frac{\partial w}{\partial t} - \frac{1}{s_1^2(t)} \cdot \frac{\partial^2 w}{\partial x^2} - \frac{x s_1'(t)}{s_1(t)} \frac{\partial w}{\partial x} - \left(\frac{1}{s_1^2(t)} - \frac{1}{s_2^2(t)}\right) \frac{\partial^2 v_2}{\partial x^2} - x\left(\frac{s_1'}{s_1} - \frac{s_2'}{s_2}\right) \frac{\partial v_2}{\partial x} + \lambda w, \quad (x,t) \in \mathcal{D}, \tag{308}$$

with initial condition

$$w(x,0) = 0, \quad 0 < x < 1, \tag{309}$$

boundary conditions

$$w(0,t) = 0, \quad 0 < t < T, \tag{310}$$

$$w(1,t) = 0, \quad 0 < t < T, \tag{311}$$

$$\delta(t) = s_0 e^{\int_0^t \int_0^1 [v_1(\xi,t) + (1-x) g(t)] d\xi d\tau} \left[1 - e^{-\int_0^t \int_0^1 [w(\xi,t) + (1-x) g(t)] d\xi d\tau}\right] \tag{312}$$

By Lemma VI.1 (a), we have

$$\| v_i \|_\infty \leq C and \| w \|_\infty \leq C. \tag{313}$$

And by above equation, we have

$$|\delta| \leq Ct \parallel w \parallel_{L^\infty([0,1]\times[0,t])}, \quad 0 \leq t \leq T \tag{314}$$

Then

$$\left|\frac{1}{s_1^2(t)} - \frac{1}{s_2^2(t)}\right| = \left|\frac{s_1+s_2}{s_1^2 s_2^2}\right| |\delta(t)| \leq C|\delta| \leq Ct \parallel w \parallel_{L^\infty([0,1]\times[0,t])} \tag{315}$$

$$\begin{aligned}
\left|\frac{s_1^{'}}{s_1} - \frac{s_2^{'}}{s_2}\right| &= \left|\int_0^1 [v_1(\xi,t) + (1-x)g(t)]d\xi - \int_0^1 [v_2(\xi,t) + (1-x)g(t)]d\xi\right| \\
&= \left|\int_0^1 [w(\xi,t) + 2(1-x)g(t)]d\xi\right| \\
&\leq \parallel w \parallel_{L^\infty(0,1)} + C.
\end{aligned} \tag{316}$$

Moreover, because $\widetilde{f} \in H_0^1(0,1)$, then there exist a constant C such that

$$\parallel v_i(t) \parallel_{H_0^1(0,1)} \leq C \tag{317}$$

By using (268), our problem (308)-(312) is equivalent to following:

$$w(t) = 0 - \int_0^t e^{\alpha(t,\tau)\triangle} \frac{x s_1^{'}(t)}{s_1(t)} \frac{\partial w}{\partial x} d\tau - \int_0^t e^{\alpha(t,\tau)\triangle} F(\tau, s_1, s_2, v_2, w) d\tau, \tag{318}$$

where

$$F(\tau, s_1, s_2, v_2, w) = -\left(\frac{1}{s_1^2(t)} - \frac{1}{s_2^2(t)}\right)\frac{\partial^2 v_2}{\partial x^2} - x\left(\frac{s_1^{'}}{s_1} - \frac{s_2^{'}}{s_2}\right)\frac{\partial v_2}{\partial x} + \lambda w \tag{319}$$

By above estimates, we have

$$\begin{aligned}
|F(\tau, s_1, s_2, v_2, w)| &\leq \left|\frac{1}{s_1^2(t)} - \frac{1}{s_2^2(t)}\right| \parallel v_2(t) \parallel_{H^2(0,1)} - \left|\frac{s_1^{'}}{s_1} - \frac{s_2^{'}}{s_2}\right| \parallel v_2(t) \parallel_{H^1(0,1)} + |\lambda w| \\
&\leq C|\delta| + \parallel w \parallel_{L^\infty(0,1)} + C + C \\
&\leq C\left[t \parallel w \parallel_{L^\infty([0,1]\times[0,t])} + 2\right] + \parallel w \parallel_{L^\infty(0,1)}
\end{aligned} \tag{320}$$

By (269)-(274), (318) becomes

$$w(t) = -\int_0^t \left[e^{\alpha(t,\tau)\triangle} \frac{s_1^{'}(t)}{s_1(t)} + \partial_x e_N^{\alpha(t,\tau)\triangle} \frac{x s_1^{'}(t)}{s_1(t)}\right] w(\tau) d\tau - \int_0^t e^{\alpha(t,\tau)\triangle} F(\tau, s_1, s_2, v_1, v_2) d\tau, \tag{321}$$

and then

$$\begin{aligned}\| w \|_{L^\infty(0,1)} &\leq C\int_0^t (t-\tau)^{-r/2} \| w \|_{L^\infty([0,1]\times[0,t])} \, d\tau \\ &+ \int_0^t (t-\tau)^{-r/2} \left[C \left(t \| w \|_{L^\infty([0,1]\times[0,t])} + 2 \right) + \| w \|_{L^\infty(0,1)} \right] d\tau \end{aligned} \tag{322}$$

For $r < 2$ and $t > 0$ small enough, we have

$$\| w \|_{L^\infty([0,1]\times[0,t])} \leq t^{(2-r)/2} \left[C \| w \|_{L^\infty([0,1]\times[0,t])} + 2 \right] \tag{323}$$

Then we have $v_1(\cdot,t) = v_2(\cdot,t)$ in $[0,t]$. Therefore, repeat the above argument we finish the proof.

D Mathematical Algorithm and Well Posedness Theorem

In this section, we design an iteration algorithm to construct the solution of (296)-(300). Then well-posedness theorem are proved.

Now suppose $s_0 = s(0)$ and v^{n+1} is the solution of following problem:

$$\frac{\partial v^{n+1}}{\partial t} = A_n(t)\frac{\partial^2 v^{n+1}}{\partial x^2} + B_n(x,t)\frac{\partial v^{n+1}}{\partial x} - \lambda v^{n+1} + K(x,t), \quad (x,t) \in \mathcal{D}, \tag{324}$$

with initial condition

$$v^{n+1}(x,0) = \widetilde{f}(x), \quad 0 < x < 1, \tag{325}$$

boundary conditions

$$v^{n+1}(0,t) = 0, \quad 0 < t < T, \tag{326}$$

$$v^{n+1}(1,t) = 0, \quad 0 < t < T, \tag{327}$$

and $s^{n+1}(t)$ is determined by

$$s_{n+1}(t) = s_0 e^{\int_0^t \int_0^1 [v^{n+1}(\xi,t)+(1-x)g(t)]d\xi d\tau} \tag{328}$$

where

$$A_n(t) = \frac{1}{s_n^2(t)}, \quad B_n(x,t) = \frac{x s_n^{'}(t)}{s_n(t)}, \quad n = 0,1,2,3,... \tag{329}$$

Theorem VI.2 *Suppose* $\widetilde{f} \in H_0^1(0,1), \lambda \in L^\infty[0,1], g \in L^\infty[0,1] with bounded g'$ *then there exists a unique solution* (v^{n+1}, s_{n+1}) *of the problem (324)-(328) with* $v^{n+1} \in L^2(0,T;H^2[0,1]) \cap L^2(0,T;H_0^1(0,T;H_0^1[0,1])) and s_{n+1} \in C^1(0,T)$.

Proof VI.2 *We use induction method to prove it. When* $n = 0, s_0 = s(0), A_0(t) = \frac{1}{s_0^2}$*is a positive constant,* $B_0(x,t) = 0$*, then by Lemma VI.1, there is a unique solution the problem (324)-(328)* $v^1 \in L^2(0,T;H^2[0,1]) \cap L^2(0,T;H_0^1(0,T;H_0^1[0,1]))$ *and then* $v^1 \in C(0,T;H^1[0,1])$ *by Sobolev embedding theorem, hence by (4.5)* $s_1 \in C^1(0,T)$.

Now assume it is true for $n = k$*, then* $A_k, B_k \in L^\infty(0,T;L^\infty(0,1))$*. Moreover, by Lemma VI.2 there exist positive constants* $\delta 1, \delta 2$ *and* N *such that*

$$0 < \delta 1 \leq A_k \leq \delta 2, \quad |B_k| \leq N \tag{330}$$

then there exists a unique solution (v^{k+1}, s_{k+1}) *of the problem (324)-(328) with* $v^{k+1} \in L^2(0,T;H^2\,[0,1]) \cap L^2(0,T;H_0^1(0,T;H_0^1[0,1])) and s_{k+1} \in C^1(0,T)$*. Moreover,* A_{k+1}, B_{k+1} *satisfy the condition (328), Therefore, we finish the proof.*

Next, we prove the uniform boundedness for v^n and s_n.

Theorem VI.3 *Under the assumption of Theorem 2, we have:*

(a) $\| v^n \|_L^\infty$ *uniformly for all n;*

(b) v^n *is bounded uniformly in* $C([0,T], C^r([0,1]), r < 2$;

(c) s_n *is bounded uniformly in* $C^1[0,T]$.

Proof VI.3 *(a) follows directly from maximum principle and Lemma VI.2.*

(b) By (268), we can express v^{n+1} *as following:*

$$\begin{aligned} v^{n+1} &= e^{\alpha_n(t,0)\triangle}\widetilde{f} - \int_0^t e^{\alpha_n(t,\tau)\triangle}\frac{xs_n^{'}(t)}{s_n(t)}\frac{\partial v^n}{\partial x}d\tau \\ &+ \int_0^t e^{\alpha(t,\tau)\triangle}\Big[\lambda v^{n+1} - K(x,t)\Big]\,d\tau, \end{aligned} \tag{331}$$

where

$$\alpha_n(t,\tau) = \int_\tau^t \frac{1}{s_n^2(t)} \tag{332}$$

By (269)-(272), (331) becomes

$$v^{n+1} = e^{\alpha_n(t,0)\triangle}\widetilde{f} - \int_0^t \left[e^{\alpha_n(t,\tau)\triangle}\frac{s_n^{'}(t)}{s_n(t)} + \partial_x e_N^{\alpha_n(t,\tau)\triangle}\frac{xs_n^{'}(t)}{s_n(t)}\right] v^{n+1}(\tau)d\tau \quad (333)$$

$$+ \int_0^t e^{\alpha(t,\tau)\triangle}\left[\lambda v^{n+1} - K(x,t)\right] d\tau, \quad (334)$$

and then by (273) and this theorem (a), we have

$$\begin{aligned}\| v^{n+1} \|_{C^r} &\leq \| e^{\alpha_n(t,0)\triangle}\widetilde{f} \|_{C^r} + \int_0^t \left[e^{\alpha_n(t,\tau)\triangle}\frac{s_n^{'}(t)}{s_n(t)} + \partial_x e_N^{\alpha_n(t,\tau)\triangle}\frac{xs_n^{'}(t)}{s_n(t)}\right] v^{n+1}(\tau)_{C^r} \, d\tau \\ &+ \int_0^t \| e^{\alpha(t,\tau)\triangle}\left[\lambda v^{n+1} - K(x,t)\right] \|_{C^r} \, d\tau, \quad (335) \\ &\leq \| e^{\alpha_n(t,0)\triangle}\widetilde{f} \|_{C^r} + C\int_0^t (t-\tau)^{-r/2} \| v^{n+1} \|_{L^\infty} d\tau + C\int_0^t (t-\tau)^{-r/2} d\tau \\ &\leq \| e^{\alpha_n(t,0)\triangle}\widetilde{f} \|_{C^r} + C\int_0^t (t-\tau)^{-r/2} \| \widetilde{f} \|_{L^\infty} d\tau + C\int_0^t (t-\tau)^{-r/2} d\tau \\ &\leq \| e^{\alpha_n(t,0)\triangle}\widetilde{f} \|_{C^r} + Ct^{(2-r)/2} \| \widetilde{f} \|_{L^\infty} d\tau + Ct^{(2-r)/2}\end{aligned} \quad (336)$$

By Lemma 2, there exist positive constants δ_1, δ_2 and N such that

$$0 < \delta_1 T \leq \alpha_n(t,0) \leq \delta_2 T, \quad (337)$$

Therefore, (2) are true for $r \leq 2$.

(c) can be obtained by (b), lemma VI.2 and (328).

Now we prove v^n, s_n are Cauchy sequence and global existence of problem as following:

Theorem VI.4 *Under the assumption of Theorem 2, the problem (296)-(300) has a unique solution (v,s) such that $v \in C([0,T], C^r([0,1])), r < 2$, and $s \in C^1([0,T]$, where*

$$\lim_{n\to\infty} \| v^n(t) - v(t) \|_{C^r(0,1)} = 0 \text{ and } \lim_{n\to\infty} s_n(t) = s(t).$$

Proof VI.4 *Now consider the difference $w^n = v^{n+1} - v^n$, and $\delta_n = s_{n+1} - s_n$, we have*

$$\frac{\partial w^n}{\partial t} - \frac{1}{s_n^2(t)}\cdot\frac{\partial^2 w^n}{\partial x^2} - \frac{xs_n^{'}(t)}{s_n(t)}\frac{\partial w}{\partial x} - \left(\frac{1}{s_n^2(t)} - \frac{1}{s_{n-1}^2(t)}\right)\frac{\partial^2 v_n}{\partial x^2} - x\left(\frac{s_n^{'}}{s_n} - \frac{s_{n-1}^{'}}{s_{n-1}}\right)\frac{\partial v^{n-1}}{\partial x} + \lambda w^n, \quad (x,t) \in \mathcal{D} \quad (338)$$

with initial condition

$$w^n(x,0) = 0, \quad 0 < x < 1, \tag{339}$$

boundary conditions

$$w^n(0,t) = 0, \quad 0 < t < T, \tag{340}$$

$$w^n(1,t) = 0, \quad 0 < t < T, \tag{341}$$

$$\delta^n(t) = s_0 e^{\int_0^t \int_0^1 [v^{n+1}(\xi,t)+(1-x)g(t)]d\xi d\tau} \left[1 - e^{-\int_0^t \int_0^1 [w^n(\xi,t)+(1-x)g(t)]d\xi d\tau}\right] \tag{342}$$

By using Theorem VI.1, we have

$$\begin{aligned} \| w^n \|_{L^\infty([0,1]\times[0,t])} &\leq t^{(2-r)/2} \left[C \| w^{n-1} \|_{L^\infty([0,1]\times[0,t])} + 2\right] \qquad (343) \\ &\leq t^{(2-r)/2n} \left[C \| w^0 \|_{L^\infty([0,1]\times[0,t])} + 2\right], r \leq 2, 0 \leq t \leq 1 \end{aligned}$$

then for any $n > m > 0$,

$$\begin{aligned} \| v^n - v^m \|_{L^\infty([0,1]\times[0,t])} &\leq \sum_{k=m}^{n} \| w^k \|_{L^\infty([0,1]\times[0,t])} \qquad (344) \\ &\leq \sum_{k=m}^{n} (t^{(2-r)/2})^k \left[C \| w^0 \|_{L^\infty([0,1]\times[0,t])} + 2\right] \qquad (345) \\ &\leq t^{(2-r)m/2} \frac{1}{1 - t^{(2-r)/2}} \left[C \| w^0 \|_{L^\infty([0,1]\times[0,t])} + 2\right] \end{aligned}$$

so we conclude v^n *is a Cauchy sequence. By (328),* s_n *is a Cauchy sequence too. Let* $v = \lim_{n\to\infty} v^n, ands = \lim_{n\to\infty} s_n$, *hence by last theorem the global existence of problem (296)-(300) are proved.*

Next, we get the continuous dependence theorem of problem (256)-(260).

Theorem VI.5 *Under assumptions of theorem 2 solution* $(u(x,t), s(t))$ *of problem (256)-(260) depends upon the data and coefficients continuously.*

Proof VI.5 *Suppose* $(u^{[i]}, s^{[i]})$ *are two solution of problem (251)-(255) based on the data and coefficients* $f^{[i]}, g^{[i]}, i = 1, 2;$

Similarly, change variable $x \in [0, s(t)]$ *to* $y \in [0, 1]$ *by setting* $x = ys(t)$, *then*

$$\frac{\partial v^{[i]}}{\partial t} = \frac{1}{(s^{[i]})^2}\frac{\partial^2 v^{[i]}}{\partial y^2} + \frac{y}{s^{[i]}}\frac{\partial s^{[i]}}{\partial t}\frac{\partial v^{[i]}}{\partial y} - \lambda^{[i]} v^{[i]}, \quad (y,t) \in \mathcal{D}, \tag{346}$$

$$v^{[i]}(y,0) = h^{[i]}(y), \quad 0 < y < 1, \tag{347}$$

$$v^{[i]}(0,t) = g^{[i]}(t), \quad 0 < t < T, \tag{348}$$

$$v^{[i]}(1,t) = 0, \quad 0 < t < T, \tag{349}$$

$$\int_0^1 v^{[i]}(y,t) s^{[i]} dy = \frac{\partial s^{[i]}}{\partial t}, \quad s(0) = s_0 \tag{350}$$

Choose approximate sequence $\{f_k^{[i]}\}$ *and* $\{g_k^{[i]}\}$ *of smooth functions which converge to* $f^{[i]}$ *and* $g^{[i]}$ *in norms of corresponding spaces.The same argument of proof in theorem for* $v_k^{[i]}$ *and* $s_k^{[i]}$ *follows that* $v_k^{[i]}$ *converges to* $v^{[i]}$ *uniformly on* $[0,1] \times [0,T]$ *and* $s_k^{[i]}$ *converges to* $s^{[i]}$ *uniformly in* $C^1[0,T], T > 0$*. Moreover, the estimates of proof in theorem for* $v_k^{[i]}$ *and* $s_k^{[i]}$ *are also valid.*

Now, consider the difference $d(y,t) = v^{[2]}(y,t) - v^{[1]}(y,t)$ *and* $\delta = s^{[2]}(t) - s^{[1]}(t)$*, then (346)-(350) becomes*

$$\frac{\partial d(y,t)}{\partial t} = \frac{1}{(s^{[1]})^2}\frac{\partial^2 d(y,t)}{\partial y^2} + \frac{y}{s^{[1]}}\frac{\partial s^{[1]}}{\partial t}\frac{\partial d(y,t)}{\partial y} - \lambda d(y,t) + F(y,t), \quad (y,t) \in \mathcal{D}, \tag{351}$$

where

$$F(y,t) = \delta\{\frac{-(s^{[2]} + s^{[1]})}{(s^{[2]}s^{[1]})^2}\frac{\partial^2 v^{[2]}}{\partial y^2} - \frac{y}{s^{[1]}s^{[2]}}\frac{\partial s^{[2]}}{\partial t}\frac{\partial v^{[2]}}{\partial y} + \frac{\delta' y}{\delta s^{[1]}}\frac{\partial v^{[2]}}{\partial y}\} \tag{352}$$

Also, we have

$$d(1,t) = 0, \quad t \in (0,T) \tag{353}$$

$$d(y,0) = \widetilde{h(y)}, \quad \widetilde{h(y)} := h^{[2]} - h^{[1]}, \quad y \in (0,1) \tag{354}$$

$$d(0,t) = \widetilde{g(t)}, \quad \widetilde{g(t)} := g^{[2]}(t) - g^{[1]}(t), \quad t \in (0,T) \tag{355}$$

Now we can split $d(y,t)$ *into following:*

$$d(y,t) = U + V + \widetilde{g(t)}(1-y)^2, \tag{356}$$

where U *is the solution of*

$$\frac{\partial U(y,t)}{\partial t} = \frac{1}{(s^{[1]}(t))^2}\frac{\partial^2 U}{\partial y^2} + \frac{y}{s^{[1]}}\frac{\partial s^{[1]}}{\partial t}\frac{\partial U}{\partial y} - \lambda U(y,t) + F(y,t) - \widetilde{g'(t)}(1-y)^2 - \frac{2}{(s^{[1]}(t))^2}\widetilde{g(t)} \tag{357}$$

$$U(0,t) = U(1,t) = U(y,0) = 0 \tag{358}$$

and V is the solution of

$$\mathcal{L}V := \frac{\partial V(y,t)}{\partial t} - \frac{1}{(s^{[1]}(t))^2}\frac{\partial^2 V(y,t)}{\partial y^2} = 0, \tag{359}$$

$$V(0,t) = V(1,t) = 0 \tag{360}$$

$$V(y,0) = \widetilde{h(y)} - \widetilde{g(0)}(1-y)^2 := \widehat{h(y)} \tag{361}$$

By [23], we have

$$\max_{y\in(0,1)} |U(y,t)| \le Kt \parallel \delta \parallel_t + \parallel \widetilde{h} \parallel_1 + \parallel \widetilde{g} \parallel_1 \tag{362}$$

For $V(y,t)$, by ([27], Page 4-9),we can write

$$V(y,t) = \int_{-\infty}^{+\infty} \widehat{h(\eta)}\Gamma(y,t;\eta,0)d\eta \tag{363}$$

where $\Gamma(y,t;\eta,\tau)$ is the fundamental solution of operator $\mathcal{L}$ and can be constructed by parametrix method:

$$\Gamma(y,t;\eta,\tau) = \Gamma_0(y,t;\eta,\tau) + \int_\tau^t \int_{-\infty}^{+\infty} \Gamma_0(y,t;\xi,\sigma)\Phi(\xi,\sigma;\eta,\tau)d\xi d\sigma \tag{364}$$

$$\Gamma_0(y,t;\eta,\tau) = \frac{1}{2\sqrt{\pi\frac{1}{(s^{[1]}(t))^2}(t-\tau)}} e^{\frac{-(y-\eta)^2}{4(t-\tau)\frac{1}{(s^{[1]}(t))^2}}} \tag{365}$$

and Φ is determined by $\mathcal{L}\Gamma = 0$

Therefore, we can rewrite $V(y,t)$ as

$$V(y,t) = \int_{-\infty}^{+\infty} \widehat{h(\eta)}\Gamma_0(y,t;\eta,0)d\eta + \int_{-\infty}^{+\infty} \widehat{h(\eta)} \int_\tau^t \int_{-\infty}^{+\infty} \Gamma_0(y,t;\xi,0)\Phi(\xi,\sigma;\eta,0)d\xi d\sigma d\eta \tag{366}$$

The above integral is well defined because $\widetilde{h}$ and $\frac{1}{(s^{[1]}(t))^2}$ are continuous almost everywhere and bounded, and $\widehat{h(0)} = \widehat{h(1)} = 0$.

The first term of integral is dominated by

$$\sup_{y\in(0,1)} |h'(y)| \int_{-\infty}^{+\infty} \Gamma_0 d\eta \le K \parallel h \parallel \tag{367}$$

To study the 2nd term of integral, recall estimate ([27], page 9),

$$|\Gamma_0(y,t;\xi,\tau)| \le K(t-\tau)^{-n}|\xi - y|^{2n-1} e^{\frac{\kappa(y-\xi)^2}{t-\tau}} \tag{368}$$

where κ is a positive constant.

and

$$|\Phi(\xi,\sigma;\eta,\tau)| \le K(\sigma-\tau)^{-n}|\xi-\eta|^{2n-3+\gamma} e^{\frac{\kappa(\xi-\eta)^2}{\sigma-\tau}} \tag{369}$$

where γ stands the Hölder constant of coefficient $\frac{1}{(s^{[1]})^2}$ and $n \in [0, \frac{3-\gamma}{2}]$

After we choose appropriate γ, n, the 2nd term of integral can be dominated by $Lt^{\frac{\gamma}{2}}$.

Summate the above estimate, we have

$$|V(y,t))| \le K(\parallel h \parallel_1 + t^{\frac{\gamma}{2}}) \tag{370}$$

Therefore, by (356), (362) and (370) we have

$$|v^{[1]} - v^{[2]}| \le Kt \parallel \delta \parallel_t + \parallel h^{[1]} - h^{[2]} \parallel_1 + \parallel g^{[1]} - g^{[2]} \parallel_1 + \sup |\widetilde{Q}| \tag{371}$$

Finally, by (350) and (371), we have similar estimate for $|s^{[1]} - s^{[2]}|$.

Hence, continuous dependence of (u, s) is proved.

Note: Reverse the transformations we can conclude the solution (u, s) of (256)-(260) is well-posed under assumption $\widetilde{f} \in H_0^1(0,1), \lambda \in L^\infty[0,1], g \in L^\infty[0,1]$ *with bounded* g'.

E Numerical Evidence and Graphical Illustrations

In this section, we perform two numerical examples to test the validity and efficiency of our numerical scheme.

Example 6.1:

Consider (251)-(255) with

$$F(x,t) = 0, \tag{372}$$

$$\lambda = 1 - \frac{1}{t+1}, \tag{373}$$

$$\varphi_1(t) = 0, \tag{374}$$

$$f(x) = 1 + x, \tag{375}$$

$$g_1(t) = e^{-t}(t+1), \quad 0 \le t \le 1, \tag{376}$$

$$g_2(t) = e^{-t}(t+1)(1 + \frac{2}{2e^{(t+2)e^{-t}} - 1}), \quad 0 \le t \le 1, \tag{377}$$

$$\mu = 0, u_0 = 0 \tag{378}$$

which has the exact solution

$$u(x,t) = e^{-t}(1+x)(1+t), \quad \varphi_2(t) = \frac{2}{2e^{(t+2)e^{-t}} - 1} \tag{379}$$

Now apply our algorithm to this problem, we have numerical solution

$$\widehat{u}(x,t) = e^{-t}(1+x)(1+t)$$

and by (255) numerical solution

$$\widehat{\varphi_2}(t) = \frac{2}{2e^{(t+2)e^{-t}} - 1},$$

which are same as our exact solution. The graph for $u(x,t)$ *and* $\varphi_2(t)$ are as following(Figure 21, Figure 22):

Example 6.2:

Consider (251)-(255) with

$$F(x,t) = 0, \tag{380}$$

$$\lambda = 1, \tag{381}$$

$$\varphi_1(t) = 0, \tag{382}$$

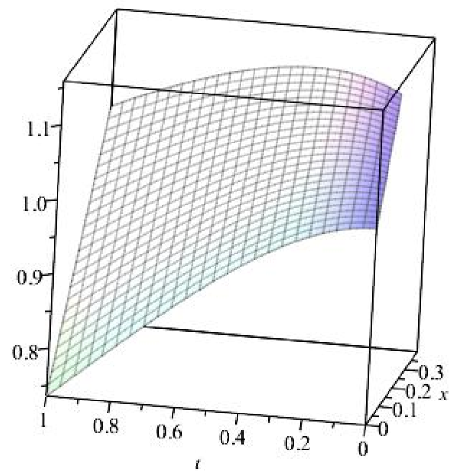

Figure 21. Graph for u(x,t) in Example 6.1

$$f(x) = 0, \tag{383}$$

$$g_1(t) = sin(t), \quad 0 \le t \le 1, \tag{384}$$

$$g_2(t) = sin(t)(1 + \frac{2}{2e^{cos(t)} - 1}), \quad 0 \le t \le 1, \tag{385}$$

$$\mu = 0, u_0 = 0 \tag{386}$$

which has the exact solution

$$u(x,t) = (1+x)sin(t), \quad \varphi_2(t) = \frac{2}{2e^{cos(t)} - 1} \tag{387}$$

Now apply our algorithm to this problem, we can get numerical solution $\widehat{u}(x,t)$ (see Table 14 about absolute error of approximate solution and exact solution for u and Figure 23 about graph of approximate u).

and then by (255), we have $\widehat{\varphi_2}(t)$, see Figure 24 about comparison of exact and approximate s(t).

F Conclusion

In this chapter, we discuss the nonlinear parabolic partial differential equation with initial, boundary and free boundary conditions which comes from one dimen-

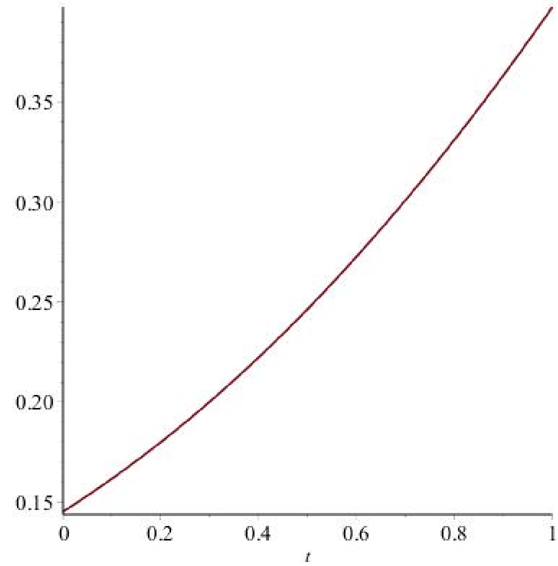

Figure 22. Graph for $\varphi_2(t)$ in Example 6.1

sional case of DCIS. Well-posedness theorems of this problem is obtained by using semigroup operator theory. Results of numerical experiments and simulation are also presented to validate our findings.

x	t	Approximate $\widehat{u}(x,t)$	Exact $u(x,t)$	Absolute error for $u(x,t)$
0.05	0.1	0.09777	0.1048	0.00705
	0.2	0.20303	0.2086	0.00558
	0.3	0.30459	0.3103	0.00571
	0.4	0.40072	0.4089	0.00817
0.45	0.1	0.14774	0.1448	0.00298
	0.2	0.29094	0.2881	0.00287
	0.3	0.42976	0.4285	0.00126
	0.4	0.56254	0.5647	0.00212

TABLE 14

Absolute error of $u(x,t)$ in Example 6.2

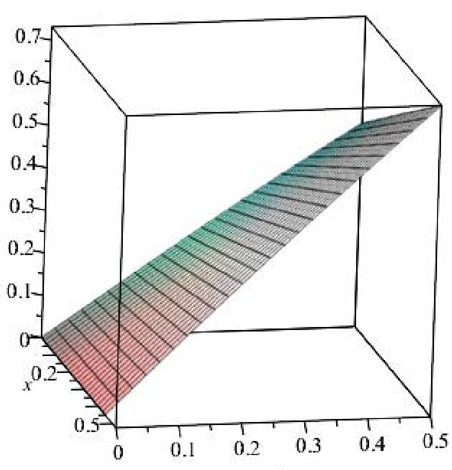

Figure 23. Graph for approximate solution $\widehat{u}(x,t)$ in Example 6.2

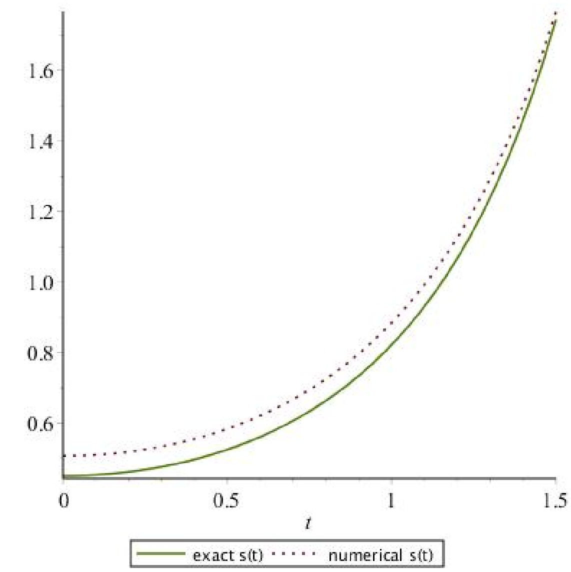

Figure 24. Comparison of exact and approximate $\varphi_2(t)$ in Example 6.2

CHAPTER VII

FUTURE DIRECTIONS

The future research could be conducted in following directions:

The nonlinear PDE with free boundary condition is a very complicated problem, these is almost no result combining clinical data into the model. Our main direction is to develop a number of inverse problems relating clinical diagnoses of cancer, which involve mathematical analysis, numerical simulation and clinical data. After the breast cancer is possibly detected, there are several options to do:

(1) Do an incisional biopsy to find DCIS development along with changing speed at the moment.

(2) Conduct a sequence of needle biopsies over a certain time to find the DCIS development change.

(3) Do a sequence of screening over a certain time to find DCIS change.

These may lead to the inverse problem of finding u, the coefficients λ and free boundary $s(t)$ in mathematics.

Since we only consider problems of DCIS model in one dimensional case, it is important to carry out analysis and numerical simulations for higher dimensional model, in that way, we are able to compare our simulation result with clinical data. Further investigation of the relation among the coefficients and solution patterns of the higher dimensional model may provide more revealing results and help us better understand the tumor growth and diagnoses procedures of DCIS.

Future research based on the findings of this dissertation, as has been shown, could take multiple directions and thus the method represented in this work holds fruitful possibilities for further exploration.

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