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ANALYSIS OF STRUCTURES SUBJECTED TO DYNAMIC LOADING

By

Jose Enrique Carrasco
B.S., University of Louisville, 1977

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ANALYSIS OF STRUCTURES
SUBJECTED TO DYNAMIC LOADING

Submitted by: José Enrique Carrasco

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Date

by the Following Reading and Examination Committee:

Thesis Director, Mario M. Paz

Edmond Miller

Michael A. Cassaro

Gilbert A. Ramirez

ABSTRACT

The objective of this thesis is to develop computer programs for the dynamic analysis of structures. For a shear building two computer programs were developed: (1) Dynamic Analysis of a Shear Building within the Elastic Range and (2) the Dynamic Analysis of a Shear Building with Elasto-Plastic Behavior.

Parallel to this computer work a study was performed to investigate the error due to static condensation applied to dynamic problems. In the development of computer programs the stiffness method and the consistent mass matrix were used; and viscous damping was assumed.

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NOMENCLATURE

Roman Alphabet

$P_i(t)$	the normal force at function of time acting on ith level
X_{imax}	the maximum response from the spectrum at ith
x_i	the displacement at ith
$\dot{x}_i$	the velocity at ith
$\ddot{x}_i$	the acceleration at ith
g	the constant of gravity
C_i	the damping at ith
K_i	the stiffness in column i in the lower floor level
$F_i(t)$	the forcing function at ith in function of time
m_i	mass concentrated at level i
$[C]$	the damping matrix
$[K]$	the stiffness matrix
$\{F\}$	the forcing vector
$\{X\}$	the displacement vector
$\{\dot{X}\}$	the velocity vector
$\{\ddot{X}\}$	the acceleration vector
a_i	amplitude of motion of ith coordinate
a_{ij}	amplitude of the mode shape at coordinate i mode n (before normalization)
$[I]$	unit matrix
$Z_i(t)$	factor which will uncouple a set of coupled equations
$[T]$	transformation matrix
$\{x_p\}$	the vector corresponding to the p degrees of freedom to be reduced

$\{x_q\}$	the vector corresponding to the remaining q independent degrees of freedom
$[\overline{K}]$	the reduced stiffness matrix
$[\overline{M}]$	the reduced mass matrix
$[\overline{C}]$	the reduced damping matrix
V	potential energy
K.E.	kinetic energy
$F_I(t)$	inertial force at nonlinear systems
$F_D(t)$	damping force at nonlinear systems
$F_S(t)$	spring force
F(t)	excitation force, function of time

Greek Alphabet

ω	natural frequency
ω_i	the i-th natural frequency
ϕ_{in}	amplitude of mode shape at coordinate i mode n (after normalization)
$[\Phi]$	square modal matrix
Γ	participation factor
ξ	damping factor
Δ	increment
θ	Wilson constant equal to 1.38 taken as 1.4
τ	the product of Wilson and the time increment
$\hat{\Delta}$	increment associated with extended time step

I. INTRODUCTION

Almost any type of structure may be subjected to dynamic loading in one form or another during its existence. From the analytical point of view, it is convenient to divide the dynamic loading condition into two basic categories; periodic and nonperiodic. Periodic loadings are repetitive loads which exhibit the same time variation successively for a large number of cycles. A typical case for periodic motion is rotating machinary in a building. On the other hand nonperiodic loadings may be either short-duration, impulsive loadings or long duration, general forms of loads. A typical nonperiodic motion is a nuclear blast or an earthquake excitation.

In recent years considerable emphasis has been given to the problems of blast and earthquakes. The earthquake problem is rather old, but most of the knowledge on this subject was developed in the last two decades. The blast problem is rather new and information is made available mostly through publications of the Army Corps of Engineers, Department of Defense Agency, and other federal agencies. It is very important to mention the fact that in the last decade the rapid expansion in number and size of nuclear power plants in regions close to large populated centers requires very careful structural consideration.

As an effort toward developing better techniques in the field of structural dynamics, the main objective of this thesis is to develop computer programs for structures modeled as a shear building subjected to dynamic loading conditions and the investigation of error, due to static condensation.

II. FREE VIBRATION OF A SHEAR BUILDING

A. Concept of a Shear Building. A shear building may be defined as a structure in which there is no rotation of a horizontal section at the level of the floors. In this respect, the deflected building will have many of the features of a cantilever beam that is deflected by shear forces only; hence, the name shear building. To accomplish such deflection in a building, it must be assumed that (1) the total mass of the structure is concentrated at the levels of the floors; (2) the girders on the floors are infinitely rigid as compared to the columns; and (3) the deformation of the structure is independent of the axial forces present in the columns.

B. Free Vibration. When free vibration is under consideration, the structure is not subjected to any external excitation (force or support motion) and its motion is governed only by the initial conditions. There are occasionally circumstances for which it is necessary to determine the motion of the structure under conditions of free vibration, but this is seldom the case. Nevertheless, the analysis of the structure in free motion provides the most important dynamic properties of the structure which are the natural frequencies and the corresponding normal modes.

Figure 1(a) shows the possible displacements of a two-story shear building and figure 1(b) shows the two possible modes of vibration.

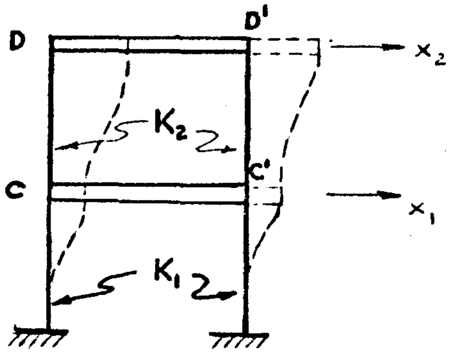

FIGURE 1(a) - Possible Displacements of a Two Story-Shear Building

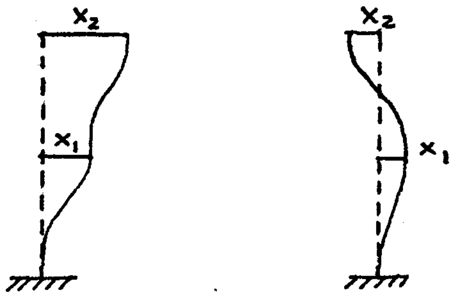

FIGURE 1(b) - First and Second Mode of Vibration

Any displacement x_1 of member C-C' is resisted by the restoring forces of the columns. If K_1 is the stiffness of the first story then the force on C-C' will be $-K_1x_1$. If K_2 is the stiffness of the second story then the forces on C-C' and D-D' are $-K_2(x_1-x_2)$ and $K_2(x_2-x_1)$ respectively. The equations of motion are then obtained from the corresponding free body diagram as is shown in Figure 2.

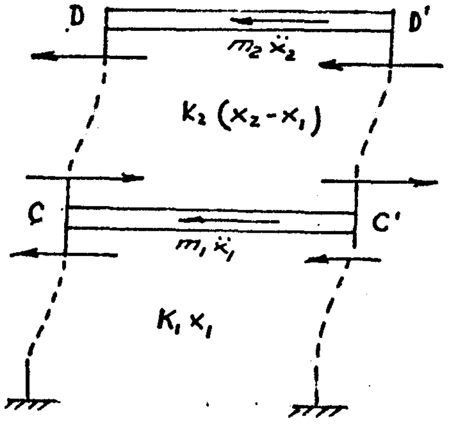

FIGURE 2 - Free Body Diagram of a Two-Story Shear Building

Hence, equating to zero the sum of forces in x direction for bodies C-C' and D-D' results in

$$m_1\ddot{x}_1 + K_1x_1 - K_2(x_2-x_1) = 0 \tag{1}$$

$$m_2\ddot{x}_2 + K_2(x_2-x_1) = 0 \tag{2}$$

and rearranging these equations gives

$$m_1\ddot{x}_1 + (K_1+K_2)x_1 - K_2x_2 = 0 \tag{3}$$

$$m_2\ddot{x}_2 + K_2x_2 - K_2x_1 = 0 \tag{4}$$

where $\ddot{x}_1$, $\ddot{x}_2$ are the accelerations and x_1 , x_2 represent the displacements. Equations (3) and (4) may be written as

$$\begin{bmatrix} m_1 & 0 \\ 0 & m_2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} K_1+K_2 & -K_2 \\ -K_2 & K_2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} . \tag{5}$$

or in a condensed form as

$$[M]\ \{\ddot{x}\} + [K]\ \{x\} = \{0\} \tag{6}$$

in which

$[M]$ is the mass matrix,

$[K]$ is the stiffness matrix,

$\{\ddot{x}\}$ is the acceleration vector, and

$\{x\}$ is the vector displacement.

The system of equation (5) is linear and homogeneous, and its solution can be expressed as

$$
\begin{aligned}
x_1 &= a_1\, e^{i\omega t} \\
x_2 &= a_2\, e^{i\omega t}
\end{aligned}
\qquad (7)
$$

where a_1 and a_2 are constants, and ω is a parameter to be determined.

Substituting (7) into (5) results in

$$
\begin{aligned}
\{-m_1\omega^2 a_1 + (K_1+K_2)a_1 - K_2 a_2\}e^{i\omega t} &= 0 \\
\{-m_2\omega^2 a_2 + K_2 a_2 - K_1 a_1\}e^{i\omega t} &= 0
\end{aligned}
\qquad (8)
$$

which upon simplification gives

$$
\begin{aligned}
\{(K_1+K_2) - \omega^2 m_1\}a_1 - K_2 a_2 &= 0 \\
-K_2 a_1 + (K_2-\omega^2 m_2)a_2 &= 0
\end{aligned}
\qquad (9)
$$

or in matrix form

$$
\begin{bmatrix} (K_1+K_2)-\omega^2 m_1 & -K_2 \\ -K_2 & K_2-\omega^2 m_2 \end{bmatrix}
\begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}
\qquad (10)
$$

and in condensed notation

$$
\{[K] - \omega^2 [M]\}\,\{a\} = \{0\} \qquad (11)
$$

Equation (9) is satisfied for the trivial solution, that is, $a_1=a_2=0$; however this solution would indicate no motion of the structure and therefore will not satisfy the initial conditions of the problem.

In order to find the nontrivial solution for this homogeneous system of equations, the determinant of the coefficient matrix has to be equal to zero, that is

$$\begin{vmatrix} (K_1+K_2)-m_1\omega^2 & -K_2 \\ -K_2 & K_2-m_2\omega^2 \end{vmatrix} = 0 \tag{12}$$

The expansion of the determinant results in a quadratic equation in ω^2, namely

$$m_1m_2\omega^4 - [(K_1+K_2)m_2 + m_1K_2]\omega^2 + K_1K_2 = 0 \tag{13}$$

After the roots of (13), ω_1 and ω_2 (natural frequencies) are determined and substituting back into equation (11) the relative amplitudes of motion (normal modes) can be found.

C. Orthogonality Property of the Normal Modes. This property constitutes the basis of one of the most attractive methods for solving dynamic problems of multi-degree-of-freedom systems. For a system of two-degree-of-freedom equations (11) may be written as

$$\begin{aligned} (K_1+K_2)a_1 - K_2a_2 &= m_1\omega^2a_1 \\ -K_2a_1 + K_2a_2 &= m_2\omega^2a_2 \end{aligned} \tag{14}$$

The normal modes may then be considered as the static deflections resulting from the forces on the right of (14) for any of the two modes. For the following two systems of loading and corresponding displacement

System I:

Forces: $\omega_1^2\, a_{11}\, m_1$, $\omega_1^2\, a_{21}\, m_2$

Displacements: a_{12} a_{22}

System II:

Forces: $\omega_2^2\, a_{12}\, m_1$, $\omega_2^2\, a_{22}\, m_2$

Displacements: a_{11} a_{21}

The application of Betti's theorem yields:

$$\omega_1^2 m_1 a_{11} a_{12} + \omega_1^2 m_2 a_{21} a_{22} = \omega_2^2 m_1 a_{12} a_{11} + \omega_2^2 m_2 a_{22} a_{21} \qquad (15)$$

or

$$(\omega_1^2 - \omega_2^2)\,(m_1 a_{11} a_{12} + m_2 a_{21} a_{22}) = 0 \qquad (16)$$

If the natural frequences are different ($\omega_1 \neq \omega_2$), it follows from (16) that

$$m_1 a_{11} a_{12} + m_2 a_{21} a_{22} = 0 \qquad (17)$$

Equation (17) is the orthogonality relation between the normal modes of a two-degree-of-freedom system. The modes are conveniently normalized to satisfy the following relation:

$$m_1\phi_{11}^2 + m_2\phi_{21}^2 = 1$$

$$m_1\phi_{12}^2 + m_2\phi_{22}^2 = 1$$

where

$$\phi_{11} = \frac{a_{11}}{\sqrt{m_1 a_{11}^2 + m_2 a_{21}^2}} \qquad \phi_{12} = \frac{a_{12}}{\sqrt{m_1 a_{12}^2 + m_2 a_{22}^2}} \qquad (18)$$

$$\phi_{21} = \frac{a_{21}}{\sqrt{m_1 a_{11}^2 + m_2 a_{21}^2}} \qquad \phi_{22} = \frac{a_{22}}{\sqrt{m_1 a_{12}^2 + m_2 a_{22}^2}}$$

D. Numerical Example. To illustrate the steps of the procedure for the determination of the natural frequencies and normal modes, consider the two-degrees-of-freedom system shown in Figure 3, in which the initial conditions are the following: $x_{01}=0$, $x_{02}=1.0$ in , $\dot{x}_{01}=0$, $\dot{x}_{02}=0$

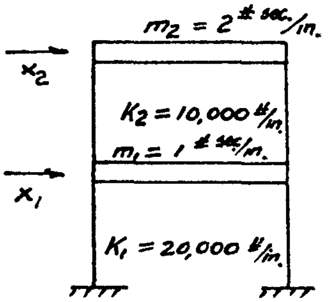

FIGURE 3 - Example of a Two Story Shear Building

Substituting numerical values in (3) and (4) gives

$$1\ \ddot{x}_1 + 30{,}000\ x_1 - 10{,}000\ x_2 = 0$$

$$2\ \ddot{x}_2 - 10{,}000\ x_1 + 10{,}000\ x_2 = 0$$

or in matrix notation

$$\begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} \ddot{x}_1 \\ \ddot{x}_2 \end{bmatrix} + \begin{bmatrix} 30{,}000 & -10{,}000 \\ -10{,}000 & 10{,}000 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0$$

assuming solution given by (7) results in

$$\begin{bmatrix} 30{,}000-\omega^2 & -10{,}000 \\ -10{,}000 & 10{,}000-2\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Then, the characteristic equation is

$$\begin{vmatrix} 30{,}000-\omega^2 & -10{,}000 \\ -10{,}000 & 10{,}000-2\omega^2 \end{vmatrix} = 0$$

and in expanded form

$$(\omega^2)^2 - 35{,}000\omega^2 + (100\text{x}10^6) = 0$$

which has the following roots

$$\omega_1^2 = 31{,}861.4$$

$$\omega_2^2 = 3{,}138.6$$

Then, the natural frequencies for this structure are

$$\omega_1 = 178.49 \text{ rad/sec}$$

$$\omega_2 = 56.02 \text{ rad/sec}$$

Consider the first equation of (10) and substituting the first natural frequency, ω_1 = 178.49 rad/sec results in

$$-1861.4\, a_{11} - 10{,}000\, a_{21} = 0$$

A second subindex was introduced in a_1 and a_2 to indicate that the value a_1 has been used in this equation. Since in this case there are two unknowns and only one independent equation it is possible to solve for the relative value of a_{21} and a_{11}. This relative value is known as the normal mode or modal shape corresponding to the first frequency. For this example, the first normal mode is

$$\frac{a_{21}}{a_{11}} = -0.18614$$

It is customary to describe the normal modes by assigning a unit value to one of the amplitudes; thus, for the first mode setting a_{11} equal to unity

$$a_{11} = 1.00$$

$$a_{21} = \sout{1.263}\ -0.18614$$

Similarly, substituting the second natural frequency, ω_2=56.02 rad/sec into (10), gives the second normal mode as

$$a_{12} = 1.00$$

$$a_{22} = 2.6861$$

The normal modes are conveniently arranged in the column of the modal matrix as

$$[a] = \begin{bmatrix} 1 & 1 \\ -0.18614 & 2.6861 \end{bmatrix}$$

The general solution to the equations of motion for free vibration in terms of constant of integration A_1 , A_2 , A_3 and A_4 takes the following form:

$$x_1(t)=a_{11}A_1 \sin \omega_1 t+a_{11}A_2 \cos \omega_1 t+a_{21}A_3 \sin \omega_2 t+a_{12}A_4 \cos \omega_2 t$$

$$x_2(t)=a_{21}A_1 \sin \omega_1 t+a_{21}A_2 \cos \omega_1 t+a_{22}A_3 \sin \omega_2 t+a_{22}A_4 \cos \omega_2 t$$

which upon numerical substitution yields

$$x_1(t)=A_1 \sin \omega_1 t+A_2 \cos \omega_1 t+A_3 \sin \omega_2 t+A_4 \cos \omega_2 t$$

$$x_2(t)=-0.18614\ A_1 \sin \omega_1 t-0.18614\ A_2 \cos \omega_1 t+2.086\ A_3 \sin \omega_2 t +2.686\ A_4 \cos \omega_2 t$$

Evaluation of the constants of integration is performed by using the initial conditions which for this example are

$$x_{01}=0 \quad x_{02}=1.0 \text{ in} \quad \dot{x}_{01}=0 \quad \dot{x}_{02}=0$$

Performing all the necessary algebra and solving for the constants of integration, gives

$$A_1=0 \qquad A_2=-0.34817$$

$$A_2=0 \qquad A_4= 0.34817$$

Then, the general solution may be expressed as

$$x_1 = -0.34817 \cos 178.5t + 0.34817 \cos 56.02t$$

$$x_2 = 0.0648 \cos 178.5t + 0.9353 \cos 56.02t$$

and finally the normalized vectors are calculated by using equation (18) as

$$\phi_{11} = \frac{1}{\sqrt{1(1)^2 + 2(-0.18614)^2}} = 0.9670$$

$$\phi_{12} = \frac{1}{\sqrt{1(1)^2 + 2(2.6861)^2}} = 0.2545$$

$$\phi_{21} = \frac{-0.18614}{\sqrt{1(1)^2 + 2(-0.8614)^2}} = -0.18$$

Similarly for

$$\phi_{22} = 0.6838$$

In matrix form, the normal modes can be represented as

$$\Phi = \begin{bmatrix} 0.9670 & 0.2545 \\ -0.180 & 0.6838 \end{bmatrix}$$

On free vibration of a shear building the eigenproblem was solved to determine the natural frequencies and normal modes of vibration. For a system of many degrees of freedom, the algebraic and numerical work required for the solution of an eigenproblem became a tedious task. For the purpose of solving an eigenproblem, the Jacobi Method was selected among several numerical methods.

E. Subroutine Jacobi. This subroutine program developed by Professor Wilson is used throughout this thesis to solve the eigenproblem. The description of the symbols utilized in this program are listed as follows:

Variables	Symbol in Thesis	Description
A(I,I)	[K]	Stiffness matrix
B(I,I)	[M]	Mass matrix
X(I,I)	[Φ]	Modal matrix
EIGV(I)	ω_1^2	Eigenvalues
D(I)		Working Vector
N		Order of matrices A and B
RTOL		Converge Tolerance (Set to $10.^{-12}$)
NSMAX		Maximum number of sweeps (Set to 15)
ISPR		Index for printing during iteration 1=Print;0=Do not Print

And the input data is subjected to the following formats

Formats	Variables
2I10	N , IFPR
8F10.4	A(I,J) (read by rows)
8F10.4	B(I,J) (read by rows)

III. FORCED VIBRATION OF SHEAR BUILDINGS

In the preceding chapter, it was shown that the free motion of a dynamic system may be expressed in terms of the normal modes in free vibration. The objective of this chapter is to show that the normal modes may also be used to transform the system of coupled differential equations into a set of uncoupled differential equations in which each equation contains only one dependent variable. Thus, the modal superposition method reduces the problem of finding the response of a multi-degree-of-freedom system to the determination of the response of a single degree-of-freedom systems.

A. Modal Superposition Method

Considering the equation of motion for a two story building subjected to forced vibration.

$$\begin{aligned} m_1\ddot{x}_1 + (K_1+K_2)x_1 - K_2x_2 &= F_1(t) \\ m_2\ddot{x}_2 - K_2x_1 + K_2x_2 &= F_2(t) \end{aligned} \qquad (19)$$

In seeking the transformation from a coupled system into an uncoupled system of equations in which each equation contains only one unknown, it is necessary to express the solution in terms of the normal modes multiplied by some factors determining the contribution of each mode. Hence, the solution of (19) is assumed to be of the form:

$$\begin{aligned} x_1(t) &= a_{11}z_1(t) + a_{12}z_2(t) \\ x_2(t) &= a_{21}z_1(t) + a_{22}z_2(t) \end{aligned} \qquad (20)$$

Substituting (20) into (19) gives

$$m_1a_{11}\ddot{z}_1 + (K_1+K_2)a_{11}z_1 - K_2a_{21}z_1 + m_1a_{12}\ddot{z}_2 + (K_1+K_2)a_{12}z_2 - K_2a_{11}z_2 = F_1(t) \qquad (21)$$

$$m_2a_{21}\ddot{z}_1 - K_2a_{11}z_1 + K_2a_{21}z_1 + m_2a_{22}\ddot{z}_2 - K_2a_{12}z_2 + K_2a_{22}z_2 = F_2(t)$$

To determine the appropriate factors $z_1(t)$ and $z_2(t)$ which will uncouple (21) it is advantageous to make use of the orthogonality relations to separate the modes. This is accomplished by multiplying the first of the equations (21) by a_{11} and the second by a_{21}. The addition of these equations after all the necessary algebra is performed, equation (21) yields:

$$(m_1a_{11}^2+m_2a_{21}^2)\ddot{z}_1 + \omega_1^2(m_1a_{11}^2+m_2a_{21}^2)z_1 = a_{11}F_1(t) + a_{21}F_2(t) \qquad (22)a$$

Similarly, multiplying the first of (21) by a_{12} and the second by a_{22}, yields

$$(m_1a_{12}^2+m_2a_{22}^2)\ddot{z}_2 + \omega_2^2(m_1a_{12}^2+m_2a_{22}^2)z_2 = a_{12}F_1(t) + a_{22}F_2(t) \qquad (22)b$$

Therefore, equations (22)a and (22)b correspond to a single degree-of-freedom system which may be written as

$$M_1\ddot{Z}_1 + K_1Z_1 = P_1(t)$$
$$M_2\ddot{Z}_2 + K_2Z_2 = P_2(t) \qquad (23)$$

in which, $M_1=m_1a_{11}^2+m_2a_{22}^2$ and $M_2=m_1a_{12}^2+m_2a_{22}^2$ are the modal masses; $K_1=\omega_1^2M_1$ and $K_2=\omega_2^2M_2$, the modal spring constants and $P_1(t)=a_{11}F_1(t)+a_{21}F_2(t)$ and $P_2(t)=a_{12}F_1(t)+a_{22}F_2(t)$ are the modal forces. When the modal shapes are normalized, equation (23) can be written as

$$\begin{aligned} \ddot{Z}_1 + \omega_1^2 Z_1 &= P_1(t) \\ \ddot{Z}_2 + \omega_2^2 Z_2 &= P_2(t) \end{aligned} \qquad (24)$$

in which, P_1 and P_2 are given by

$$\begin{aligned} P_1 &= \phi_{11}F_1(t) + \phi_{21}F_2(t) \\ P_2 &= \phi_{12}F_1(t) + \phi_{22}F_2(t) \end{aligned} \qquad (25)$$

The solution of the uncoupled equation (23) or (24) can be found by the application of Duhamel's integral as will be shown in a numerical example.

B. Numerical Example

Consider the structure of the numerical example of chapter one shown in Figure 3 with the only difference that, this time the first and the second story are subjected to constant loading applied suddenly at t=0; as is shown in Figure 4.

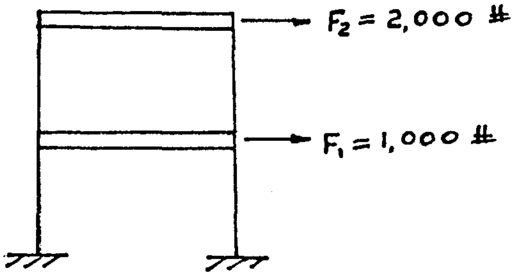

FIGURE 4 - Building Subjected to Constant Loading

The values of natural frequencies, and the modes are known by solving the building as free vibration. This was shown in a numerical example in the preceding chapter. These values are:.

ω_1 = 178.5 rad/sec ϕ_{11} = 0.9670 ϕ_{21} = -0.18
ω_2 = 56.02 rad/sec ϕ_{12} = 0.2545 ϕ_{22} = 0.6838

To determine the appropriate functions $Z_1(t)$ and $Z_2(t)$, which will enable to uncouple equation (21), it is necessary to use equation (23), by substituting into (25) the numerical values found in the preceding chapter, gives

$$P_1 - 0.967(1000) + (-0.18)(2{,}000) = 607$$
$$P_2 = 0.254(1000) + (0.6838)(2{,}000) = 1{,}621.6$$

Performing the numerical substitution in equation (23) yields,

$$\ddot{Z}_1 + (178.5)^2 Z_1 = 607$$
$$\ddot{Z}_2 + (56.02)^2 Z_2 = 1{,}621.6$$

Since it was assumed that $F_1(t)$ and $F_2(t)$ are constant loading applied suddenly at time equal zero the solution of the above equations is given by

$$Z_1(t) = \frac{P_1}{\omega_1}(1-\cos \omega_1 t) = \frac{607}{31{,}862.25}(1-\cos 178.5t)$$

$$Z_2(t) = \frac{P_2}{\omega_2}(1-\cos \omega_2 t) = \frac{1{,}621.6}{3{,}138.24}(1-\cos 56.02t)$$

and the maximum displacement by

$$Z_{1max} = (2)\frac{P_1(t)}{\omega_1^2} = (2)\frac{607}{31,862.25} = 0.038$$

$$Z_{2max} = (2)\frac{P_2(t)}{\omega_2^2} = (2)\frac{1,621.6}{3,138.24} = 1.032$$

A method which is widely accepted and which gives a good estimation of the maximum response from the spectrum values is the square root of the sum of the squares of the modal contributions. This calculation is given by

$$\begin{aligned} X_{1max} &= \sqrt{(\phi_{11}Z_{1max})^2 + (\phi_{12}Z_{2max})^2} \\ X_{2max} &= \sqrt{(\phi_{12}Z_{1max})^2 + (\phi_{22}Z_{2max})^2} \end{aligned} \tag{26}$$

which upon substitution gives,

$$X_{1max} = \sqrt{(0.9670x0.038)^2 + (0.2545x1.032)^2} = 0.2652$$

$$X_{2max} = \sqrt{(-0.180x0.038)^2 + (0.6838x1.032)^2} = 0.7057$$

C. Response of a Shear-Building to Ground Motion

The response of a shear building to the base or foundation motion is conveniently obtained in terms of relative displacements with respect to the base motion.

For a two-story shear building shown in Figure 5a which has its mathematical model shown in Figure 5b, the equations of motion are obtained by applying Newton's second law to Figure 5b as follows,

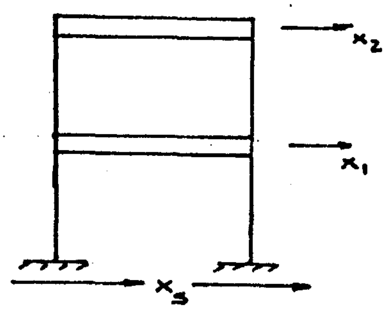

FIGURE 5(a) - Shear Building Subjected to Ground Motion

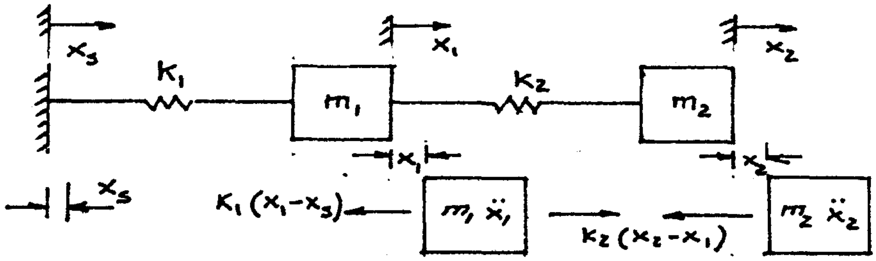

FIGURE 5(b) - Mathematical Model and its Free Body Diagram

$$m_1\ddot{x}_1 + K_1(x_1-x_s) - K_2(x_2-x_1) = 0$$
$$m_2\ddot{x}_2 + K_2(x_2-x_1) = 0 \qquad (27)$$

where $x_s=x_s(t)$ is the displacement imposed to the base of the structure. Expressing the displacements in terms of relative displacements,

$$u_1 = x_1-x_s$$
$$u_2 = x_2-x_s \qquad (28)$$

and derivading (28) twice with respect to time yields,

$$\ddot{x}_1 = \ddot{u}_1+\ddot{x}_s$$
$$\ddot{x}_2 = \ddot{u}_2+\ddot{x}_s \qquad (29)$$

By substituting (28) and (29) into (27) gives,

$$
\begin{aligned}
m_1\ddot{u}_1 + (K_1+K_2)u_1 - K_2u_2 &= -m_1\ddot{x}_s \\
m_2\ddot{u}_2 - K_2u_1 + K_2u_2 &= -m_2\ddot{x}_s
\end{aligned}
\tag{30}
$$

For a base motion of shear building equations (29) may be written as,

$$
\begin{aligned}
\ddot{Z}_1 + \omega_1^2 Z_1 &= \frac{-m_1a_{11} + m_2a_{21}}{m_1a_{11}^2 + m_2a_{21}^2}\,\ddot{X}_s(t) \\
\ddot{Z}_2 + \omega_2^2 Z_2 &= \frac{-m_1a_{12} + m_2a_{22}}{m_1a_{12}^2 + m_2a_{22}^2}\,\ddot{X}_s(t)
\end{aligned}
\tag{31}
$$

in a compact form gives,

$$
\begin{aligned}
\ddot{Z}_1 + \omega_1^2 Z_1 &= \Gamma_1\,\ddot{X}_s(t) \\
\ddot{Z}_2 + \omega_2^2 Z_2 &= \Gamma_2\,\ddot{X}_s(t)
\end{aligned}
\tag{32}
$$

where Γ_1 and Γ_2 are called the participation factors which are represented by

$$
\Gamma_1 = -\frac{m_1a_{11} + m_2a_{21}}{m_1a_{11}^2 + m_2a_{21}^2} \quad \text{and} \quad \Gamma_2 = -\frac{m_1a_{12} + m_2a_{22}}{m_1a_{12}^2 + m_2a_{22}^2}
\tag{33}
$$

The relation between the modal displacement Z_1 , Z_2 and the relative displacement u_1 , u_2 is given in equation (20) as

$$
\begin{aligned}
u_1 &= a_{11}Z_1 + a_{12}Z_2 \\
u_2 &= a_{21}Z_1 + a_{22}Z_2
\end{aligned}
\tag{34}
$$

The change of variable to make the second member of equation (32) equal $X_s(t)$, take the form of

$$
\begin{aligned}
Z_1 &= \Gamma_1 \, g_1 \\
Z_2 &= \Gamma_2 \, g_2
\end{aligned}
\qquad (35)
$$

substituting (35) into (32) gives

$$
\begin{aligned}
\ddot{g}_1 + \omega_1^2 g_1 &= \ddot{X}_s(t) \\
\ddot{g}_2 + \omega_2^2 g_2 &= \ddot{X}_s(t)
\end{aligned}
\qquad (36)
$$

Finally, solving for $g_1(t)$ and $g_2(t)$ the uncoupled equation (36) and substituting this solution into (34) and (35) gives

$$
\begin{aligned}
u_1(t) &= \Gamma_1 a_{11} g_1(t) + \Gamma_2 a_{12} g_2(t) \\
u_2(t) &= \Gamma_1 a_{21} g_1(t) + \Gamma_2 a_{22} g_2(t)
\end{aligned}
\qquad (37)
$$

Whenever the maximum modal response g_{1max} and g_{2max} are obtained from spectral charts, the maximum values of u_{1max} and u_{2max} can be obtained by using (26) in the following form:

$$
\begin{aligned}
u_{1max} &= \sqrt{(\Gamma_1 a_{11} g_{1max})^2 + (\Gamma_2 a_{21} g_{2max})^2} \\
u_{2max} &= \sqrt{(\Gamma_1 a_{12} g_{1max})^2 + (\Gamma_2 a_{22} g_{2max})^2}
\end{aligned}
\qquad (38)
$$

D. <u>Subroutine Modal</u>

This modal is utilized to obtain the response of multiple degree of freedom system by using the superposition method. The theory and the manipulation was shown throughout this chapter. The symbols for this subroutine are shown below.

Variables	Symbols in Thesis	Description
ND	N	Number of degrees of freedom
GR	g	Excitation index: For support excitation, g-acceleration of gravity. For forced excitation, g=0.
EIGEN(I)	ω_i^2	Square of natural frequencies (eigenvalues)
X(I,J)	$\|\Phi\|$	Modal matrix (eigen-vectors)
DT		Time step of integration
TMAX		Maximum time of integration
NQ(L)		Number of points defining the excitation at coordinate L
M(I,J)		Mass matrix
T(I)	t_i	Time at point i
P(I)	$P(t_i)$	Force or acceleration at time t_i
XIS(I)	ξ_i	Damping ratios

The input data are subjected to the following formats.

Format	Variables
(I10,F10.0)	ND, GR
(8F10.4)	M(I,J) (read by rows)
(8F10.4)	EIGEN(I),(I = 1, ND)
(8F10.4)	X(I,J) (read by rows)
(2F10.4,1215)	DT, TMAX, NQ(L) (L=1....NG), where NG=ND when forces are at coordinates or NG=1 when acceleration is at support
(8F10.2)	T(I), P(I) (I=1,NQ(L)) (one card per forcing function)
(8F10.3)	2SI(I), (I=1,ND)

IV. DAMPED MOTION OF SHEAR BUILDING

In the previous chapter the analysis of a shear building was based upon undamped system of motion; the techniques to determine the response of the shear building were discussed, giving special emphasis on the tranformation from coupled systems to uncoupled systems, by means of a transformation of coordinates which incorporate the property known as orthogonality of the modal shapes.

In the consideration of damping forces in the dynamic analysis of shear building presented in this chapter, the system of equations of motion became more complicated, not only because the system will contain one more forcing factor, but the procedure to uncouple the system will also become difficult. One way to avoid this difficulty is by introducing some restrictions or conditions on the functional expression for the coefficients of damping.

For practical purposes, damping is neglected for the calculation of natural frequencies and modal shapes of the system. Consequently for the solution of the Eigenvalue problem the system is reduced to an undamped and free vibration system.

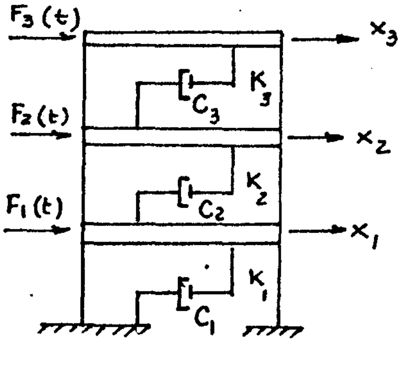

FIGURE 6(a) - Shear Building Subjected to Damped Motion

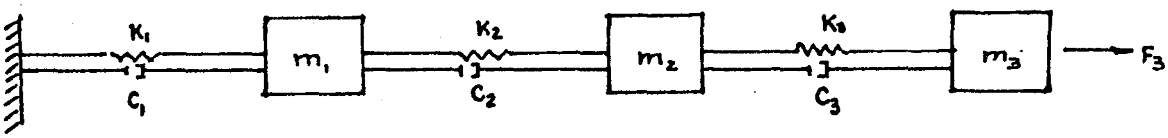

FIGURE 6(b) - Mathematical Model of Shear Building

A. Equation of Motion for Damped System

For a viscously damped three-story shear building shown in Figure 6(a) the equation of motion can be obtained by applying Newton's second law to the free body diagram of the mathematical model shown in Figure 6(b); these equations are,

$$m_1\ddot{x}_1 + c_1\dot{x}_1 + K_1x_1 - c_2(\dot{x}_2-\dot{x}_1) - K_2(x_2-x_1) = F_1(t)$$

$$m_2\ddot{x}_2 + c_2(\dot{x}_2-\dot{x}_1) + K_2(x_2-x_1) - c_3(\dot{x}_3-\dot{x}_2) - K_3(x_3-x_2) = F_2(t) \qquad (39)$$

$$m_3\ddot{x}_3 + c_3(\dot{x}_3-\dot{x}_2) + K_3(x_3-x_2) = F_3(t)$$

in matrix form

$$[M]\{\ddot{x}\} + [c]\{\dot{x}\} + [K]\{x\} = \{F(t)\} \qquad (40)$$

where the only new factor introduced is the damping matrix [c] which is given by

$$[C] = \begin{bmatrix} C_1+C_2 & -C_2 & 0 \\ -C_2 & C_2+C_3 & -C_3 \\ 0 & -C_3 & C_3 \end{bmatrix}$$

Since, equation (40) is obviously a coupled system of equations, then it is convenient to uncouple by introducing the following transformation of coordinates:

$$\{x\} = [\Phi]\{Z\} \qquad (41)$$

where $[\Phi]$ is the modal matrix obtained by solving the system as undamped free vibration, substituting (41) into (40) gives,

$$[n][\Phi]\{\ddot{Z}\} \quad [c][\Phi]\{\dot{Z}\} \quad [K][\Phi]\{Z\} \quad \{F(t)\} \qquad (42)$$

Premultiplying (42) by the transpose of the nth modal vector $\{\Phi\}_n^T$ yields

$$\{\Phi\}_n^T[M][\Phi]\{\ddot{Z}\} + \{\Phi\}_n^T[C][\Phi]\{\dot{Z}\} + \{\Phi\}_n^T[K][\Phi]\{Z\} = \{\Phi\}_n^T\{F(t)\} \qquad (43)$$

It is noticed that the orthogonality property of the modal shapes, is given by

$$\begin{aligned} \{\Phi\}_n^T[M]\{\Phi\}_m &= 0 \\ \{\Phi\}_n^T[K]\{\Phi\}_m &= 0 \ , \ m \neq n \end{aligned} \qquad (44)$$

Causing all components except the nth mode in the first two terms of (43) to vanish. A similar reduction is assumed to apply to the damping term in (43) that is

$$\{\Phi\}_n^T[C]\{\Phi\}_m = 0 \quad n \neq m \tag{45}$$

then the coefficient of the damping term in (43) will reduce to $\{\Phi\}_n^T[C]\{\Phi\}_n$; therefore (43) gives

$$M_n \ddot{Z}_n + C_n \dot{Z}_n + K_n Z_n = F_n(t)$$

or (46)

$$\ddot{Z}_n + Z_n \omega_n \dot{Z}_n + \omega_n^2 Z_n = \frac{F_n(t)}{M_n}$$

in which

$$\begin{aligned} M_n &= \{\Phi\}_n^T[M]\{\Phi\}_n \\ K_n &= \{\Phi\}_n^T[K]\{\Phi\}_n = \omega_n^2 M_n \\ C_n &= \{\Phi\}_n^T[C]\{\Phi\}_n = 2\xi\omega_n M_n \\ F_n(t) &= \{\Phi\}_n^T\{F(t)\} \end{aligned} \tag{47}$$

The normalization that was presented previously

$$\{\Phi\}_n^T[M]\{\Phi\}_n = 1 \tag{48}$$

will give $M_n=1$, so that (46) will reduce to

$$\ddot{Z}_n + 2\xi\omega_n \dot{Z} + \omega_n^2 Z_n = F_n(t) \tag{49}$$

which is a set of uncoupled differential equations.

B. Conditions to Uncoupled Equations in Damped Systems

The derivation of equation (49) was based upon the assumption that damping can also be uncoupled by using the normal coordinate transformation utilized to uncouple the inertial and elastic forces.

It is crucial, at this point to explain the condition under which this uncoupling will occur, that is, the form of the damping matrix [C] to which (45) applies.

Rayleigh showed that in damping matrix of the form

$$[C] = a_o[M] + a_1[K] \tag{50}$$

in which a_o and a_1 are proportionality factors, the orthogonality condition will be satisfied, that is, premultiplying both sides of (50) by the transpose of nth mode $\{\Phi\}_n^T$ and postmultiplying by the modal matrix $[\Phi]$ gives equation (51) as follows:

$$\{\Phi\}_n^T[C][\Phi] = a_o\{\Phi\}_n^T[M][\Phi] + a_1\{\Phi\}_n^T[K][\Phi] \tag{51}$$

with the orthogonality condition (44) equation (51) reduces to

$$\{\Phi\}_n^T[C][\Phi] = a_o\{\Phi\}_n^T[M][\Phi] + a_1\{\Phi\}_n^T[K][\Phi]$$

or by (47) equation (51) takes the following form

$$\begin{aligned} \{\Phi\}_n^T[C][\Phi] &= a_o\, M_n + a_1\, M_n\, \omega_n^2 \\ \{\Phi\}_n^T[C][\Phi] &= (a_o + a_1\, \omega_n^2)\, M_n \end{aligned} \tag{52}$$

which shows that, when the damping matrix [C] is of the form (50), the damping is coupled with equation (41). It can also be shown that [M] and [K] satisfy the orthogonality condition. In general, it takes the form

$$[C] = [M] \sum_i ai \; ([M]^{-1}[K])^i \qquad (53)$$

in which as many terms may be included as desired.

Rayleigh damping equation (50) obviously is contained in equation (53); however, by including additional terms in this equation it is possible to obtain a greater degree of control over the modal damping ratios resulting from damping matrix. With this type of damping matrix it is possible to compute the damping influence coefficients necessary to provide a decouple system having any desired damping ratios in any specified number of modes. For each mode n, the generalized damping is given by equation (54) of the following form

$$C_n = \{\Phi\}_n^T[C]\{\Phi\}_n = 2\Sigma_n \; \omega_n \; M_n \qquad (54)$$

But if [C] as given by equation (53) is substituted in the expression for C_n, the series of generalized damping is

$$C_n = \{\Phi\}_n^T[M]_i \; \Sigma \; a_i \; ([M]^{-1}[K]^i\{\Phi\}_n) \qquad (55)$$

Now, by using the equation of motion as free vibration $[K]\{a\}=\omega^2[M]\{a\}$ after normalized $K\{\Phi\}_n=\omega^2M\{\Phi\}_n$ and performing the necessary algebra it is possible to show that the damping coefficient associated with any mode n may be written as

$$C_n = \sum_i a_i \, \omega_n^{2i} \, M_n = 2 \, \xi_n \, \omega_n \, M_n \tag{56}$$

from which the damping ratio can be given as

$$\xi_n = \frac{1}{2\omega_n} \Sigma a_i \, \omega_n^{2i} \tag{57}$$

Equation (57) may be used to determine the constants a_i for any desired values of modal damping ratios corresponding to any specified numbers of modes. For instance, to evaluate the first four damping ratios ξ_1, ξ_2, ξ_3, and ξ_4 in this case (57) gives the following equation

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = 1/2 \begin{bmatrix} \omega_1 & \omega_1^3 & \omega_1^5 & \omega_1^7 \\ \omega_2 & \omega_2^3 & \omega_2^5 & \omega_2^7 \\ \omega_3 & \omega_3^3 & \omega_3^5 & \omega_3^7 \\ \omega_4 & \omega_4^3 & \omega_4^5 & \omega_4^7 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \end{bmatrix} \tag{58}$$

In general (58) may be expressed symbolically and in condensed form as follows

$$\{\xi\} = 1/2 \; [Q]^{-1}\{a\} \tag{59}$$

from which it is possible to get the constant {a} as

$$\{a\} = 2[Q]^{-1}\{\xi\} \tag{60}$$

Finally, the damping matrix is obtained after the substitution of equation (60) into (53).

It is interesting to observe from equation (57) that in the special case when the damping matrix is proportional to the mass $\{C\}=a_0\ [M]$ when i=0, the damping ratios are inversely proportional to the natural frequencies; thus the higher modes of the structure will be given very little damping.

There is yet a second method for evaluating the damping matrix corresponding to any set of specified modal damping ratio. This method is presented starting with the following relationship

$$[A] = [\Phi]^T[C][\Phi] = \begin{bmatrix} 2\xi_1\omega_1M_1 & 0 & 0 & \\ 0 & 2\xi_2\omega_2M_2 & 0 & \\ 0 & 0 & 2\xi_3\omega_3M_3 & \\ \vdots & \cdots & \cdots & \cdots \end{bmatrix} \qquad (61)$$

It is evident that the damping matrix [C] may be evaluated by pre- and post-multiplying (61) by the inverse of the modal matrix and its inverse transpose, such that

$$[C] = [\Phi]^{-T}[A][\Phi]^{-1} \qquad (62)$$

Therefore, for any specified set of modal damping ratios $\{\xi\}$, matrix [A] can be evaluated from (61) and damping matrix [C] from (62). However, in practice, the inversion of modal matrix is a tedious task. But taking advantage of orthogonality properties of the mode shapes, the following expression can be deduced.

$$[C] = [M] \left(\sum_{n=1}^{N} \frac{2\xi_n \omega_n}{M_n} \{\Phi\}_n \{\Phi\}_n^T \right) [M] \quad (63)$$

The damping matrix [C] obtained from (63) will satisfy the property of orthogonality and therefore, the damping term in equation (40) will be uncoupled with the same transformation (41) which serves to uncouple the inertial and elastic forces.

C. Subroutine Damp

This subroutine developed by Professor Paz calculates the system damping [C] using (63) from specified modal damping ratios. The main program gives the values of [Φ] and [M] to the subroutine, but, the damping ratio should be given, with the following input format.

Variable	Symbol in Text	Format	Description
x(I) (I=1,NL)	ξ	8F10.2	Damping ratio for modes 1 to NL

The past experience indicates that values for the modal damping ratios in structures are generally in the range of 2% to 10%, probably no more than 20%. Therefore for all practical purposes in a design of a dynamic structure the engineer takes 10% as a typical figure.

D. Seismic Response of an Elastic Shear Building

The computer program that is presented in this section, calculates the dynamic response of a shear building, within the linear-elastic range and subjected to excitation at its foundation. The modal superposition method of analysis is utilized to uncouple the system of differential equations. Subroutine Jacobi, developed by Professor Wilson, is called to solve the eigenproblem resulting in eigenvalues (ω_i^2) and the eigenvectors which form the modal matrix [Φ]. Subroutine Modal, which is called next, solves the resulting modal equations using Duhamel's integral described by Professor Paz in Chapter 4 of Structural Dynamics. Finally at each step, the solution of the modal equations are combined in equation (41) to obtain the response in terms of the original coordinates of the shear building.

The variables and input formats used in this program are shown in tabular form below.

Variable	Symbol in Thesis	Description
DT	Δt	Time increment
E	E	Modules of elasticity
GR	g	Acceleration of gravity
TMAX		Maximum time response
NEQ		Number of points of the excitation function
ND		Number of degrees of freedom
IFPR		Index for intermediate printing in Jacobi; 1=Print, 0=do not print
SI	I	Moment of inertia of story i
SL	L	Height of story i
SM(I,I)	M_i	Mass at floor level i
TC(I)	t_i	Time at point i
P(I)	$\ddot{Y}_s$	Support acceleration at time t_i

These variables are subjected to the following input formats.

Formats	Variables
(4F10.2, 255)	DT, E, GR, TMAX, NEQ, ND
(3F10.2)	SI, SL, SM(I,I) (one card for each story)
(8F10.2)	TC(1), P(1), TC(2), P(2)···TC(NEQ), P(NEQ)

E. Computer Program #1

```
$JOB                 ,PAGES=5,TIME=5,LINES=400
C
C       SEISMIC RESPONSE ELASTIC SHEAR BUILDING
C
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION SK(30,30),SM(30,30),SC(30,30),F(30),X(30,30),
      1 DUA(30),UD(30),UV(30),UA(30),TC(30),P(30),S(30),EIGEN(30)
C
C       READ INPUT DATA AND INITIALIZE
C
        READ(5,100) THETA,DT,E,GR,TMAX,NEQ,ND,IFPR
        WRITE(6,100)THETA,DT,E,GR,TMAX,NEQ,ND,IFPR
    100 FORMAT(2F10.2,2F10.0,F10.2,3I5)
        NX=TMAX/DT+2
        DO 1 I=1,NX
      1 F(I)=0.0
        DO 2 I=1,ND
        DO 2 J=1,ND
        SM(I,J)=0.0
        SC(I,J)=0.0
        X(I,J)=0.0
      2 SK(I,J)=0.0
        ND1=ND+1
        TU=THETA*DT
        A1=3./TU
        A2=6./TU
        A3=TU/2.
        A4=A2/TU
        DO 7 I=1,ND
        READ(5,110) SI,SL,SM(I,I)
        WRITE(6,110)SI,SL,SM(I,I)
    110 FORMAT(3F10.2,F10.0)
        S(I)=12.0*E*SI/SL**3
        SC(I,I)=SM(I,I)
        UD(I)=0.0
      7 UV(I)=0.0
C
C       ASSEMBLE STIFFNESS MATRIX
C
        S(ND+1)=0.0
        DO 19 I=1,ND
        IF(I.EQ.1) GO TO 19
        SK(I,I-1)=-S(I)
        SK(I-1,I)=-S(I)
     19 SK(I,I)=S(I)+S(I+1)
C
C       DETERMINE NATURAL FREQUENCIES AND MODE SHAPES
C
        CALL JACOBI(SK,SC,X,EIGEN,TC,ND,IFPR)
C
C       DETERMINE DAMPING MATRIX
C
        CALL DAMP(ND,X,SM,SC,EIGEN)
C
C       INTERPOLATION BETWEEN DATA POINTS
C
        READ(5,120) (TC(L),P(L),L=1,NEQ)
        WRITE(6,120)(TC(L),P(L),L=1,NEQ)
    120 FORMAT(4F10.2)
        DO 4 I=1,NEQ
      4 P(I)=P(I)*GR
```

```
      NT=TC(NEQ)/DT
      IF (NT.GT.TMAX/DT) NT=TMAX/DT
      NT1=NT+1
      F(1)=P(1)
      ANN=0.0
      II=1
      DO 10 I=2,NT1
      AI=I-1
      T=AI*DT
      IF(T.GT.TC(NEQ)) GO TO 16
      IF(T.LE.TC(II+1)) GO TO 9
      ANN=-TC(II+1)+T-DT
      II=II+1
    9 ANN=ANN+DT
      F(I)=P(II)+(P(II+1)-P(II))*ANN/(TC(II+1)-TC(II))
C     WRITE(6,110) T,F(I)
   10 CONTINUE
   16 CONTINUE
C
C     CALCULATE INITIAL ACCELERATION
C
      NT=TMAX/DT
      DO 22 I=1.ND
      X(I,ND1)=-F(1)*SM(I,I)
      DO 22 J=1.ND
   22 X(I,J)=SM(I,J)
C     DO 301 LI=1,ND
C 301 WRITE(6,210) (X(LI,LJ),LJ=1,ND1)
      CALL SOLVE(ND,X)
C     WRITE(6,210) (X(LI,ND1).LI=1,ND)
      DO 23 I=1,ND
   23 UA(I)=X(I,ND1)
  251 FORMAT (1H1,6X,'TIME',9X,'DISPL.',9X,'VELOC.',11X,'ACC.'/)
      WRITE(6,251)
C
C     STEP BY STEP LOOP TO CALCULATE RESPONSE
C
      DO 90 L=1,NT
      AL = L
      T=DT*AL
      DO 20 I=1,ND
      IF(I.EQ.1) GO TO 20
      SK(I,I-1) = -S (I)
      SK((I-1),I)=-S (I)
   20 SK(I,I)=S (I)+S (I+1)
      DO 25 I=1,ND
      DO 25 J=1,ND
   25 X(I,J)=SK(I,J)+A4*SM(I,J)+A1*SC(I,J)
      DO 35 I=1,ND
      X(I,ND1)=(F(L+1)+(F(L+2)-F(L+1))*(THETA-1.0)-F(L))*(-SM(I,I))
      DO 30 J=1,ND
   30 X(I,ND1)=X(I,ND1)+(SM(I,J)*A2+SC(I,J)*3.0)*UV(J)
     1 +(SM(I,J)*3.0+A3*SC(I,J))*UA(J)
   35 CONTINUE
C     DO 302 LI=1,ND
C 302 WRITE(6,210) (X(LI,LJ),LJ=1,ND1)
      CALL SOLVE(ND,X)
C     WRITE(6,210) (X(LI,ND1),LI=1,ND)
      DO 36 I=1,ND
      DUA(I)=A4*X(I,ND1)-A2*UV(I)-3.0*UA(I)
      DUA(I)=DUA(I)/THETA
```

```
      DUV=DT*UA(I)+DT*DUA(I)/2.0
      UD(I)=   UD(I)+DT*UV(I)+DT*DT*UA(I)/2.0+DT*DT*DUA(I)/6.0
      UV(I)=UV(I)+DUV
   38 CONTINUE
      DO 50 I=1,ND
      X(I,ND1)=F(L+1)*(-SM(I,I))
      DO 45 J=1,ND
      X(I,ND1)=X(I,ND1)-SC(I,J)*UV(J)-SK(I,J)*UD(J)
   45 X(I,J)=SM(I,J)
   50 CONTINUE
C     DO 303 LI=1,ND
C 303 WRITE(6,210) (X(LI,LJ),LJ=1,ND1)
      CALL SOLVE (ND,X)
C     WRITE(6,210) (X(LI,ND1),LI=1,ND)
      DO 60 I=1,ND
      UA(I)=X(I,ND1)
   60 WRITE(6,250) T,UD(I),UV(I),UA(I)
  250 FORMAT(F10.3,3F15.4)
   90 CONTINUE
      STOP
      END

      SUBROUTINE SOLVE (N,A)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION              A(30,30)
      M=1
      EPS=1.0E-10
      NPLUSM=N+M
      DET=1.0
      DO 9 K=1,N
      DET=DET*A(K,K)
      IF(DABS(A(K,K)).GT.EPS) GO TO 5
      WRITE(6,202)
      GO TO99
    5 KP1=K+1
      DO 6 J=KP1,  NPLUSM
    6 A(K,J)=A(K,J)/A(K,K)
      A(K,K)=1.
      DO 9 I=1,N
      IF (I.EQ.K.OR.A(I,K).EQ.0.) GO TO 9
      DO 8 J=KP1,NPLUSM
    8 A(I,J)=A(I,J)-A(I,K)*A(K,J)
      A(I,K)=0.D00
    9 CONTINUE
  202 FORMAT(37H0SMALL PIVOT -MATRIX MAY BE SINGULAR )
   99 RETURN
      END

      SUBROUTINE JACOBI (A,B,X,EIGV,D,N,IFPR)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(30,30),B(30,30),X(30,30),EIGV(30),D(30)
C
C      INITIALIZE EIGENVALUE AND EIGENVECTOR MATRICES
C
      NSMAX = 15
      RTOL = 1.D-12
      IOUT=6
      DO 10 I=1,N
      IF(A(I,I).GT.0. .AND. B(I,I).GT.0.)GO TO 4
      WRITE(IOUT,2020)
      STOP
```

```
    4 D(I)=A(I,I)/B(I,I)
   10 EIGV(I)=D(I)
      DO 30 I=1,N
      DO 20 J=1,N
   20 X(I,J)=0.
   30 X(I,I)=1.
      IF(N.EQ.1) RETURN
C
C     INITIALIZE SWEEP COUNTER AND BEGIN ITERATION
C
      NSWEEP=0
      NR=N-1
   40 NSWEEP=NSWEEP+1
      IF(IFPR.EQ.1)WRITE(IOUT,2000)NSWEEP
C
C     CHECK IF PRESENT OFF-DIAGONAL ELEMENT IS LARGE
C
      EPS=(.01**NSWEEP)**2
      DO 210 J=1,NR
      JJ=J+1
      DO 210 K=JJ,N
      EPTOLA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPTOLB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPTOLA.LT.EPS).AND.(EPTOLB.LT.EPS))GO TO210
C
C     IF ZEROING IS REQUIRED,CALCULATE THE ROTATION MATRIX ELEMENT CA,CG
C
      AKK=A(K,K)*B(J,K)-B(K,K)*A(J,K)
      AJJ=A(J,J)*B(J,K)-B(J,J)*A(J,K)
      AB=A(J,J)*B(K,K)-A(K,K)*B(J,J)
      CHECK=(AB*AB+4.*AKK*AJJ)/4.
      IF(CHECK)50,60,60
   50 WRITE(IOUT,2020)
      STOP
   60 SQCH=DSQRT(CHECK)
      D1=AB/2.+SQCH
      D2=AB/2.-SQCH
      DEN=D1
      IF(DABS(D2).GT.DABS(D1))DEN=D2
      IF(DEN)80,70,80
   70 CA=0.
      CG=-A(J,K)/A(K,K)
      CG=-A(J,K)/A(K,K)
      GO TO 90
   80 CA=AKK/DEN
      CG=-AJJ/DEN
C
C     GENERALIZED ROTATION TO ZERO THE PRESENT OFF-DIAGONAL ELEMENT
C
   90 IF(N-2)100,190,100
  100 JP1=J+1
      JM1=J-1
      KP1=K+1
      KM1=K-1
      IF(JM1-1) 130,110,110
  110 DO 120 I=1,JM1
      AJ=A(I,J)
      BJ=B(I,J)
      AK=A(I,K)
      BK=B(I,K)
      A(I,J)=AJ+CG*AK
```

```
      B(I,J)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  120 B(I,K)=BK+CA*BJ
  130 IF (KP1-N)140,140,160
  140 DO 150 I=KP1,N
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(K,I)
      BK=B(K,I)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(K,I)=AK+CA*AJ
  150 B(K,I)=BK+CA*BJ
  160 IF(JP1-KM1)170,170,190
  170 DO 180 I=JP1,KM1
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(I,K)
      BK=B(I,K)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  180 B(I,K)=BK+CA*BJ
  190 AK=A(K,K)
      BK=B(K,K)
      A(K,K)=AK+2.*CA*A(J,K)+CA*CA*A(J,J)
      B(K,K)=BK+2.*CA*B(J,K)+CA*CA*B(J,J)
      A(J,J)=A(J,J)+2.*CG*A(J,K)+CG*CG*AK
      B(J,J)=B(J,J)+2.*CG*B(J,K)+CG*CG*BK
      A(J,K)=0.
      B(J,K)=0.
C
C     UPDATE THE EIGENVECTOR MATRIX AFTER EACH ROTATION
C
      DO 200 I=1,N
      XJ=X(I,J)
      XK=X(I,K)
      X(I,J)=XJ+CG*XK
  200 X(I,K)=XK+CA*XJ
  210 CONTINUE
C
C     UPDATE THE EIGENVALUES AFTER EACH SWEEP
C
      DO 220 I=1,N
      IF (A(I,I).GT.0. .AND. B(I,I).GT.0.) GO TO 220
      WRITE(IOUT,2020)
      STOP
  220 EIGV(I)=A(I,I)/B(I,I)
      IF(IFPR.EQ.0)GO TO 230
      WRITE(IOUT,2030)
      WRITE(IOUT,2010) (EIGV(I),I=1,N)
C
C     CHECK FOR CONVERGENCE
C
  230 DO 240 I=1,N
      TOL=RTOL*D(I)
      DIF=DABS(EIGV(I)-D(I))
      IF(DIF.GT.TOL)GO TO 280
  240 CONTINUE
C
C     CHECK ALL OFF-DIAGONAL ELEMENTS  TO SEE IF ANOTHER SWEEP IS
```

```
C       REQUIRED
C
        EPS=RTOL**2
        DO 250 J=1,NR
        JJ=J+1
         DO 250 K=JJ,N
        EPSA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
        EPSB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
        IF((EPSA.LT.EPS).AND.(EPSB.LT.EPS))GO TO 250
         GO TO 280
  250 CONTINUE
C
C       FILL OUT BOTTOM TRIANGLE OF RESULTANT MATRICES
C       AND SCALE EIGENVECTORS
C
  255 DO 260 I=1,N
        DO 260 J=1,N
        A(J,I)=A(I,J)
  260 B(J,I)=B(I,J)
        DO 270 J=1,N
        BB=DSQRT(B(J,J))
        DO 270 K=1,N
  270 X(K,J)=X(K,J)/BB
C
C       UPDATE MATRIX AND START NEW SWEEP,IF ALLOWED
C
        WRITE(6,2010) ((X(LI,LJ),LJ=1,N),LI=1,N)
        RETURN
  280 DO 290 I=1,N
  290 D(I)=EIGV(I)
        IF(NSWEEP.LT.NSMAX)GO TO 40
        GO TO 255
 2000 FORMAT(/,27H0SWEEP NUMBER IN *JACOBI* = ,I4)
 2010 FORMAT(1H0,6E20.12)
 2020 FORMAT (25H0*** ERROR  SOLUTION STOP  /
     1          30H MATRICES NOT POSITVE DEFINITE)
 2030 FORMAT(36H0CURRENT EIGENVALUES IN *JACOBI*ARE,/)
        END

        SUBROUTINE DAMP (NL,X,SM,SC,EIGEN)
        IMPLICIT REAL*8(A-H,O-Z)
        DIMENSION X(30,30),T(30,30),SM(30,30),SC(30,30),EIGEN(30),XIS(30)
        READ (5,110) (XIS(L),L=1,NL)
        WRITE(6,110) (XIS(L),L=1,NL)
        DO 10 I=1,NL
        EIGEN(I)=DSQRT(EIGEN(I))
        DO 10 J=1,NL
   10 SC(I,J) =0.0
        DO 20 II=1,NL
        DA = 2.*XIS(II)*EIGEN(II)
        DO 20 I=1,NL
        DO20 J=1,NL
   20 SC(I,J)=SC(I,J)+X(I,II)*X(J,II)*DA
        DO 30 I=1,NL
        DO 30 J=1,NL
        T(I,J)=0.0
        DO 30 K = 1,NL
   30 T(I,J) = T(I,J)+SM(I,K)*SC(K,J)
        DO 40 I=1,NL
        DO 40 J=1,NL
        SC(I,J)=0.0
```

```
          DO 40 K=1,NL
       40 SC(I,J) = SC(I,J)+T(I,K)*SM(K,J)
          DO 50 I=1,NL
       50 WRITE(6,120) (SC(I,J),J=1,NL)
      110 FORMAT(3F10.2)
      120 FORMAT (6D14.4)
          RETURN
          END
```

```
     $ENTRY
  1.40        0.01 30000000.       386.        0.20     2     2     1
497.20      180.00     136.00
212.60      120.00      66.00

SWEEP NUMBER IN *JACOBI* =    1
CURRENT EIGENVALUES IN *JACOBI*ARE,

 0.139898881239D 03  0.108253274714D 04

SWEEP NUMBER IN *JACOBI* =    2
CURRENT EIGENVALUES IN *JACOBI*ARE,

 0.139898881239D 03  0.108253274714D 04
 0.643696854346D-01 -0.566520875662D-01  0.813230024006D-01  0.924017555681D-0
   0.00        0.00
  0.0000D 00     0.0000D 00
  0.0000D 00     0.0000D 00
   0.00        0.28        1.00        0.28
```

TIME	DISPL.	VELOC.	ACC.
0.010	-0.0054	-1.0762	-107.2574
0.010	-0.0054	-1.0845	-108.4476
0.020	-0.0215	-2.1298	-103.7003
0.020	-0.0217	-2.1680	-108.3095
0.030	-0.0478	-3.1353	-97.9815
0.030	-0.0486	-3.2491	-107.8307
0.040	-0.0839	-4.0756	-90.3353
0.040	-0.0866	-4.3196	-106.6702
0.050	-0.1291	-4.9329	-81.3074
0.050	-0.1351	-5.3735	-104.4110
0.060	-0.1823	-5.6944	-71.1409
0.060	-0.1940	-6.3967	-100.6104
0.070	-0.2426	-6.3514	-60.3210
0.070	-0.2629	-7.3719	-94.8560
0.080	-0.3090	-6.8993	-49.2481
0.080	-0.3412	-8.2773	-86.8207
0.090	-0.3802	-7.3372	-38.2644
0.090	-0.4281	-9.0909	-76.3102
0.100	-0.4553	-7.6671	-27.6294
0.100	-0.5226	-9.7966	-63.2974
0.110	-0.5332	-7.8934	-17.5048
0.110	-0.6234	-10.3407	-47.9417
0.120	-0.6129	-8.0212	-7.9510
0.120	-0.7289	-10.7317	-30.5889
0.130	-0.6933	-8.0561	1.0654
0.130	-0.8374	-10.9425	-11.7523
0.140	-0.7737	-9.0028	9.6535
0.140	-0.9471	-10.9613	7.9227
0.150	-0.8531	-7.8648	17.9719
0.150	-1.0560	-10.7837	27.7078
0.160	-0.9307	-7.6439	26.1969
0.160	-1.1622	-10.4121	46.8502
0.170	-1.0057	-7.3403	34.4903
0.170	-1.2636	-9.8566	64.6328
0.180	-1.0772	-6.9528	42.9669
0.180	-1.3597	-9.1336	80.4318
0.190	-1.1445	-6.4794	51.6692
0.190	-1.4458	-8.2654	93.7649
0.200	-1.2060	-5.7644	60.4352
0.200	-1.5231	-7.1229	104.3233

F. Computer Program #2

```
$JOB              ,PAGES=5,TIME=5,LINES=400
C
C     SEISMIC RESPONSE ELASTIC SHEAR BUILDING
C
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION SK(40,40),SM(40,40),SC(40,40),X(40,40),
     1 DUA(40),UD(40),UV(40),UA(40),S(40),EIGEN(40)
C
C     READ INPUT DATA AND INITIALIZE
C
      READ (5,100) E,GR,ND,IFPR
      WRITE (6,100)E,GR,ND,IFPR
  100 FORMAT (2F10.0,2I5)
      DO 2 I=1,ND
      DO 2 J=1,ND
      SM(I,J)=0.0
      SC(I,J)=0.0
      X(I,J)=0.0
    2 SK(I,J)=0.0
      ND1=ND+1
      DO 7 I=1,ND
      READ(5,110) SI,SL,SM(I,I)
      WRITE(6,110)SI,SL,SM(I,I)
  110 FORMAT(3F10.2,F10.0)
      S(I)=12.0*E*SI/SL**3
      SC(I,I)=SM(I,I)
      UD(I)=0.0
    7 UV(I)=0.0
C
C     ASSEMBLE STIFFNESS MATRIX
C
      S(ND+1)=0.0
      DO 19 I=1,ND
      IF(I.EQ.1) GO TO 19
      SK(I,I-1)=-S(I)
      SK(I-1,I)=-S(I)
   19 SK(I,I)=S(I)+S(I+1)
C
C     DETERMINE NATURAL FREQUENCIES AND MODE SHAPES
      CALL JACOBI (SK,SC,X,EIGEN,S,ND,IFPR)
C
C     RESPONSE USING MODAL SUPERPOSITION
C
      CALL MODAL(ND,EIGEN,X,SC,GR,SM)
C
      STOP
      END
C
C     SOLVE EIGENPROBLEM USING JACOBI METHOD
C

      SUBROUTINE JACOBI (A,B,X,EIGV,D,N,IFPR)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(40,40),B(40,40),X(40,40),EIGV(40),D(40)
C
C      INITIALIZE EIGENVALUE AND EIGENVECTOR MATRICES
C
      NSMAX = 15
      RTOL = 1.D-12
      IOUT=6
      DO 10 I=1,N
```

```
      IF(A(I,I).GT.0. .AND. B(I,I).GT.0.)GO TO 4
      WRITE(IOUT,2020)
      STOP
    4 D(I)=A(I,I)/B(I,I)
   10 EIGV(I)=D(I)
      DO 30 I=1,N
      DO 20 J=1,N
   20 X(I,J)=0.
   30 X(I,I)=1.
      IF(N.EQ.1) RETURN
C
C     INITIALIZE SWEEP COUNTER AND BEGIN ITERATION
C
      NSWEEP=0
      NR=N-1
   40 NSWEEP=NSWEEP+1
      IF(IFPR.EQ.1)WRITE(IOUT,2000)NSWEEP
C
C     CHECK IF PRESENT OFF-DIAGONAL ELEMENT IS LARGE
C
      EPS=(.01**NSWEEP)**2
      DO 210 J=1,NR
      JJ=J+1
      DO 210 K=JJ,N
      EPTOLA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPTOLB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPTOLA.LT.EPS).AND.(EPTOLB.LT.EPS))GO TO 210
C
C     IF ZEROING IS REQUIRED,CALCULATE THE ROTATION MATRIX ELEMENT CA,CG
C
      AKK=A(K,K)*B(J,K)-B(K,K)*A(J,K)
      AJJ=A(J,J)*B(J,K)-B(J,J)*A(J,K)
      AB=A(J,J)*B(K,K)-A(K,K)*B(J,J)
      CHECK=(AB*AB+4.*AKK*AJJ)/4.
      IF(CHECK)50,60,60
   50 WRITE(IOUT,2020)
      STOP
   60 SQCH=DSQRT(CHECK)
      D1=AB/2.+SQCH
      D2=AB/2.-SQCH
      DEN=D1
      IF(DABS(D2).GT.DABS(D1))DEN=D2
      IF(DEN)80,70,80
   70 CA=0.
      CG=-A(J,K)/A(K,K)
      CG=-A(J,K)/A(K,K)
      GO TO 90
   80 CA=AKK/DEN
      CG=-AJJ/DEN
C
C     GENERALIZED ROTATION TO ZERO THE PRESENT OFF-DIAGONAL ELEMENT
C
   90 IF(N-2)100,190,100
  100 JP1=J+1
      JM1=J-1
      KP1=K+1
      KM1=K-1
      IF(JM1-1)130,110,110
  110 DO 120 I=1,JM1
      AJ=A(I,J)
```

```
          AK=A(I,K)
          BK=B(I,K)
          A(I,J)=AJ+CG*AK
          B(I,J)=BJ+CG*BK
          A(I,K)=AK+CA*AJ
      120 B(I,K)=BK+CA*BJ
      130 IF (KP1-N)140,140,160
      140 DO 150 I=KP1,N
          AJ=A(J,I)
          BJ=B(J,I)
          AK=A(K,I)
          BK=B(K,I)
          A(J,I)=AJ+CG*AK
          B(J,I)=BJ+CG*BK
          A(K,I)=AK+CA*AJ
      150 B(K,I)=BK+CA*BJ
      160 IF(JP1-KM1)170,170,190
      170 DO 180 I=JP1,KM1
          AJ=A(J,I)
          BJ=B(J,I)
          AK=A(I,K)
          BK=B(I,K)
          A(J,I)=AJ+CG*AK
          B(J,I)=BJ+CG*BK
          A(I,K)=AK+CA*AJ
      180 B(I,K)=BK+CA*BJ
      190 AK=A(K,K)
          BK=B(K,K)
          A(K,K)=AK+2.*CA*A(J,K)+CA*CA*A(J,J)
          B(K,K)=BK+2.*CA*B(J,K)+CA*CA*B(J,J)
          A(J,J)=A(J,J)+2.*CG*A(J,K)+CG*CG*AK
          B(J,J)=B(J,J)+2.*CG*B(J,K)+CG*CG*BK
          A(J,K)=0.
          B(J,K)=0.
C
C         UPDATE THE EIGENVECTOR MATRIX AFTER EACH ROTATION
C
          DO 200 I=1,N
          XJ=X(I,J)
          XK=X(I,K)
          X(I,J)=XJ+CG*XK
      200 X(I,K)=XK+CA*XJ
      210 CONTINUE
C
C         UPDATE THE EIGENVALUES AFTER EACH SWEEP
C
          DO 220 I=1,N
          IF (A(I,I).GT.0. .AND. B(I,I).GT.0.) GO TO 220
          WRITE(IOUT,2020)
          STOP
      220 EIGV(I)=A(I,I)/B(I,I)
          IF(IFPR.EQ.0)GO TO 230
          WRITE(IOUT,2030)
          WRITE(IOUT,2010) (EIGV(I),I=1,N)
C
C         CHECK FOR CONVERGENCE
C
      230 DO 240 I=1,N
          TOL=RTOL*D(I)
          DIF=DABS(EIGV(I)-D(I))
          IF(DIF.GT.TOL)GO TO 280
```

```
  240 CONTINUE
C
C       CHECK ALL OFF-DIAGONAL ELEMENTS  TO SEE IF ANOTHER SWEEP IS
C        REQUIRED.
C
        EPS=RTOL**2
        DO 250 J=1,NR
        JJ=J+1
         DO 250 K=JJ,N
        EPSA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
        EPSB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
        IF((EPSA.LT.EPS).AND.(EPSB.LT.EPS))GO TO 250
         GO TO 280
  250 CONTINUE
C
C       FILL OUT BOTTOM TRIANGLE OF RESULTANT MATRICES
C       AND SCALE EIGENVECTORS
C
  255 DO 260 I=1,N
        DO 260 J=1,N
        A(J,I)=A(I,J)
  260 B(J,I)=B(I,J)
        DO 270 J=1,N
        BB=DSQRT(B(J,J))
        DO 270 K=1,N
  270 X(K,J)=X(K,J)/BB
C
C       UPDATE MATRIX AND START NEW SWEEP,IF ALLOWED
C
        WRITE(6,1990)
        DO 1991 LI=1,N
 1991 WRITE(6,2010) (X(LI,LJ),LJ=1,N)
 1990 FORMAT(/10X,'EIGENVECTORS',/)
        RETURN
  280 DO 290 I=1,N
  290 D(I)=EIGV(I)
        IF(NSWEEP.LT.NSMAX)GO TO 40
        GO TO 255
 2000 FORMAT(/,27H0SWEEP NUMBER IN *JACOBI* = ,I4)
 2010 FORMAT(1H0,6E14.5/)
 2020 FORMAT (25H0*** ERROR  SOLUTION STOP  /
     1          30H MATRICES NOT POSITVE DEFINITE)
 2030 FORMAT(36H0CURRENT EIGENVALUES IN *JACOBI*ARE,/)
        END
C
C       RESPONSE USING MODAL SUPERPOSITION METHOD
C

        SUBROUTINE MODAL(ND,EIGEN,X,F,GR,SM)
        IMPLICIT REAL*8(A-H,O-Z)
        REAL*8INT1,INT2,INT3,INT4,K,M
        DIMENSION EIGEN(40),X(40,40),XIS(40),F(40,40),F(40),T(40),Y(40,40)
     1 ,UD(40),FF(40),NO(40),SM(40,40)
C
C STEMENT FUNCTIONS
C
        INT1(TAU)=DEXP(XIWD*TAU)*(XIWD*DCOS(WD*TAU)+WD*DSIN(WD*TAU))/DWSQ
        INT2(TAU)=DEXP(XIWD*TAU)*(XIWD*DSIN(WD*TAU)-WD*DCOS(WD*TAU))/DWSQ
        INT3(TAU)=TAU*INT2(TAU)-XIWD*INT2(TAU)/DWSQ+WD*INT1(TAU)/DWSQ
        INT4(TAU)=TAU*INT1(TAU)-XIWD*INT1(TAU)/DWSQ-WD*INT2(TAU)/DWSQ
C
```

```
C      READ FORCING FUNTIONS AND INTERPOLATE
C
       NG=ND
       IF(GR.NE.0.) NG=1
       NNN=40
       READ(5,110) DT,TMAX,(NG(L),L=1,NG)
       WRITE(6,110)DT,TMAX,(NG(L),L=1,NG)
   110 FORMAT(2F10.4,12I5)
       DO 76 I=1,NNN
       FF(I)=0.0
       DO 76 J=1,NNN
    76 F(I,J)=0.0
       DO 77 ID=1,NG
       NEQ=NG(ID)
       IF(NEQ.EQ.0) GO TO 77
       READ(5,120) (T(L),P(L),L=1,NEQ)
       WRITE(6,120)( T(L),P(L),L=1,NEQ)
   120 FORMAT(4F10.2)
       NT= T(NEQ)/DT
       IF (NT.GT.TMAX/DT) NT=TMAX/DT
       NT1=NT+1
       FF(1)=P(1)
       ANN=0.0
       II=1
       DO 19 I=2,NT1
       AI=I-1
       TA=AI*DT
       IF(TA.GT.T (NEQ)) GO TO 160
       IF(TA.LE.T (II+1)) GO TO 9
       ANN= -T(II+1)+TA-DT
       II=II+1
     9 ANN=ANN+DT
       FF(I)=P(II)+(P(II+1)-P(II))*ANN/( T(II+1)- T(II))
        F(ID,I)=FF(I)
    19 CONTINUE
   160 CONTINUE
    77 CONTINUE
C
C      DETERMINE TIME AND EQUIVALENT FORCES
C
       NT=TMAX/DT
       DO 17 L=1,NNN
       AL=L-1
        T(L)= T(1)+AL*DT
       IF(GR.EQ.0.) GO TO 17
       DO 18 ID=1,ND
   18  F(ID,L)=-FF(L)*SM(ID,ID)
   17  CONTINUE
C
C      READ DAMPING RATIOS AND SET INITIAL VALUES
C
       READ(5,100) (XIS(L),L=1,ND)
       WRITE(6,100)(XIS(L),L=1,ND)
   100 FORMAT(8F10.3)
C
C      WRITE HEADINGS
C
       WRITE (6,700)
   700 FORMAT(1H1,6X,'SEISMIC RESPONSE OF ELASTIC SHEAR BUILDING',//,
      16X,'TIME',6X,'DISPLACEMENTS',/)
       NT1=NT+1
```

```
      DO 50 ID=1,ND
      DO 10 IT=1,NT1
      P(IT)=0.0
      DO 10 I=1,ND
   10 P(IT)=P(IT)+F(I,IT) *X(I,ID)
      M=1.0
      K=EIGEN(ID)
      XI=XIS(ID)
    6 FIM1=P(1)
      TIM1=T(1)
      ATI=0.0
      BTI=0.0
      DAT=0.0
      DBT=0.0
      Y(ID,1)=0.0
      OMEGA=DSQRT(K/M)
      CRIT=2*DSQRT(K*M)
      C=XI*CRIT
      WD=OMEGA*DSQRT(1.-(XI**2))
      XIWD=XI*OMEGA
      DWSQ=XIWD**2+WD**2
C
C     LOOP OVER TIME AND SOLVE FOR MODAL DISPLACEMENTS
C
      NM1=NT-1
      DO 1 I=1,NM1
      FI=P(I+1)
      TI=T(I+1)
      DFTI=FI-FIM1
      DTI=TI-TIM1
      FT=DFTI/DTI
      G=FIM1-TIM1*FT
      AI=INT1(TI)-INT1(TIM1)
      BI=INT2(TI)-INT2(TIM1)
      VS=INT3(TI)-INT3(TIM1)
      VC=INT4(TI)-INT4(TIM1)
      AI=AI*G
      AI=AI+FT*VC
      ATI=ATI+AI
      BI=BI*G
      BI=BI+FT*VS
      BTI=BTI+BI
      Y(ID,I+1) =DEXP(-XIWD*TI)*(ATI*DSIN(WD*TI)-BTI*DCOS(WD*TI))/(M*WD)
      TIM1=TI
      FIM1=FI
    1 CONTINUE
   50 CONTINUE
      DO 53 IT=1,NT
      DO 52 I=1,ND
      UD(I)=0.0
      DO 52 J=1,ND
   52 UD(I)=UD(I)+X(I,J)*Y (J,IT)
   53 WRITE(6,301) T(IT),(UD(L),L=1,ND)
  301 FORMAT(F10.3,6F14.4)
      RETURN
      END
```

V. ERROR INVESTIGATION DUE TO STATIC CONDENSATION

Due to different loading conditions, and changes in geometry; it is sometimes necessary to divide the structure into a large number of elements. When the elements of the entire structure are assembled, the number of unknown displacements, or in dynamical terms, the number of degrees-of-freedom become very large. Therefore, the stiffness, the mass and the damping matrices become very large.

In such cases the solution of the eigenproblem to determine natural frequencies and mode shapes will be difficult and tedious. For this reason it is convenient to reduce the size of matrices in order to make the solution easier and manageable.

A. Static Condensation

A practical method of accomplishing the reduction of these matrices is to identify those degrees-of-freedom to be reduced as dependent coordinates and to express them in terms of the remaining independent degrees-of-freedom. The relation between the dependent and independent degrees-of-freedom is found by establishing the static relation between them, hence, the name static condensation method. This relation provides the means to reduce the stiffness matrix.

In order to reduce the mass and the damping matrices, it is assumed that the same static relation between dependent and independent degrees-of-freedom remains valid in the dynamic problem. Hence the same transformation based on static condensation for the reduction of the stiffness matrix is also used in reducing the mass and damping matrices.

In general this method of reducing the dynamic problem is not exact and introduces errors in the results. The magnitude of these errors depends on the relative numbers of degrees-of-freedom reduced as well as on the specific selection of these degrees-of-freedom for a given structure. No error is introduced in reducing massless degrees-of-freedom, that is, degrees-of-freedom for which there is no mass allocated. The procedure of static condensation also is used in static problems to eliminate unwanted degrees-of-freedom such as the internal degrees-of-freedom of an element used with the finite element method of analysis. Initially the stiffness matrix is represented by a partition matrix as follows:

$$\left[\begin{array}{c|c} Kpp & Kpq \\ \hline Kqp & Kqq \end{array}\right] \begin{bmatrix} \{Xp\} \\ \{Xq\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{Fq\} \end{bmatrix} \qquad (61)$$

which can be reduced or condensed by using the gauss elimination for the first p unknown displacement. At this stage of the elimination process, the stiffness equation for the structure may be arranged in partition matrices as follows:

$$\begin{bmatrix} [I] & -[\bar{T}] \\ 0 & [\bar{K}] \end{bmatrix} \begin{bmatrix} \{X_p\} \\ \{X_q\} \end{bmatrix} = \begin{bmatrix} \{0\} \\ \{F_q\} \end{bmatrix} \qquad (62)$$

where $\{x_p\}$ is the vector corresponding to the p degrees-of-freedom to be reduced and $\{x_p\}$ the vector corresponding to the remaining q independent degrees of freedom. It should be noted that in (62) it was assumed that the external forces were zero at the dependent degree-of-freedom $\{x_\rho\}$. Equation (62) is equivalent to the following relations:

$$\{x_p\} = [\bar{T}]\ \{x_q\}\ , \tag{63}$$

$$[\bar{K}]\ \{x_q\} = [F_q]\ . \tag{64}$$

Equation (63) which expresses the static relation between coordinates $\{x_p\}$ and $\{x_q\}$ may also be written as

$$\begin{bmatrix} \{x_p\} \\ \{x_q\} \end{bmatrix} = \begin{bmatrix} [\bar{T}] \\ [I] \end{bmatrix} \{x_q\} \tag{65}$$

or

$$\{x\} = [T]\ \{x_q\} \tag{66}$$

where

$$\{x\} = \begin{bmatrix} \{x_p\} \\ \{x_q\} \end{bmatrix}\ ,\quad [T] = \begin{bmatrix} [\bar{T}] \\ [I] \end{bmatrix} \tag{67}$$

Equation (64) which establishes the relation between coordinates $\{x_q\}$ and forces $\{F_q\}$ is the reduced stiffness equation and $[\bar{K}]$ the reduced stiffness matrix of the system, which may also be expressed as a transformation of the system stiffness matrix [K] as

$$[\bar{K}] = [T]^T[K][T] \tag{68}$$

B. Static Condensation Applied to Dynamic Problems

In a previous section a case was considered in which the discretization of the mass has left a number of massless degrees-of-freedom. For this case it is only necessary to condense the stiffness matrix and delete from the mass matrix the rows and columns corresponding to the massless degrees-of-freedom. In this case the methods used do not alter the original problem, thus the results are equivalent eigenproblems.

In cases when the discretization process has allocated mass to the system, the procedure commonly used is to apply the transformation shown in equation (68) not only to the stiffness matrix, but also to the mass and to the damping matrix of the system, analytically that is:

$$[\overline{M}] = [T]^T[M][T] \tag{69}$$

and the reduced damping matrix is

$$[\overline{C}] = [T]^T[C][T] \tag{70}$$

where the transformation matrix [T] is defined in (67). The justification of the mass and damping matrices reduction is shown as follows:

$$V = 1/2 \; \{x\}^T[K]\{x\} \tag{71}$$

$$K.E. = 1/2 \; \{\dot{x}\}^T[M]\{\dot{x}\} \tag{72}$$

where V is the potential energy and the kinetic energy is represented by K.E. in equations (71) and (72) respectively.

Analogously, the work δw_d done by the damping forces $F_d=[C]\{\dot{x}\}$ corresponding to displacements $\{\delta x\}$ may be expressed as:

$$\delta w_d = \{\delta x\}^T[C]\{\dot{x}\} \tag{73}$$

By using the transformation (67) in equations (71), (72) and (73) gives the following results

$$V = 1/2\ \{x_q\}^T[T]^T[K][T]\{x_q\} \tag{74}$$

$$K.E. = 1/2\ \{\dot{x}_q\}^T[T]^T[M][T]\{\dot{x}_q\} \tag{75}$$

$$\delta w_d = \{\delta x_q\}^T[T]^T[C][T]\{\dot{x}\} \tag{76}$$

The respective substitution of $[K]$, $[\overline{M}]$ and $[\overline{C}]$ from (68), (69) and (70) for the product of the three matrices in (74), (75) and (76) yields:

$$V = 1/2\ \{x_q\}^T[K]\{x_q\} \tag{77}$$

$$K.E. = 1/2\ \{\dot{x}_q\}^T[M]\{\dot{x}_q\} \tag{78}$$

$$\delta w_d = \{\delta x_{\bar{q}}\}[\overline{C}]\{\dot{x}_q\} \tag{79}$$

These last three expressions represent the potential, the kinetic energy and the virtual work of the damping forces in terms of independent coordinates $\{x_p\}$.

C. Numerical Example

To illustrate the theory, consider a three degree-of-freedom shear building shown in Figure 7, and find the natural frequencies and

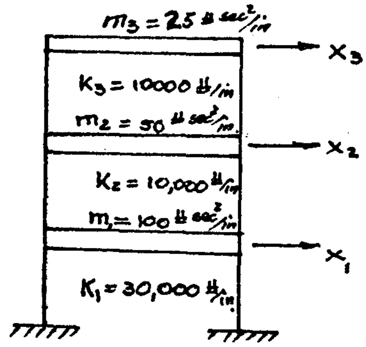

FIGURE 7.- Shear Building of Numerical Example

modal shapes; also condense one degree-of-freedom and compare the resulting values obtained for natural frequencies and mode shapes.

The equation of motion is given as free vibration in the following form:

$$[M]\{\ddot{x}\} + [K]\{x\} = [0]$$

Substituting the corresponding numerical values in this equation yields

$$\begin{bmatrix} 100 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + 10^3 \begin{bmatrix} 40 & -10 & 0 \\ -10 & 20 & -10 \\ 0 & -10 & 10 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

assuming a solution $x_i = a_i \sin \omega t$, and substituting into the equation of motion yields,

$$\begin{bmatrix} 40{,}000-100\omega^2 & -10{,}000 & 0 \\ -10{,}000 & 20{,}000-50\omega^2 & 10{,}000 \\ 0 & 10{,}000 & 10{,}000-25\omega^2 \end{bmatrix} \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad \text{(a)}$$

from which the characteristic determinant of the system can easily be deducted, such as

$$\begin{vmatrix} 40{,}000-100\omega^2 & -10{,}000 & 0 \\ -10{,}000 & 20{,}000-50\omega^2 & 10{,}000 \\ 0 & 10{,}000 & 10{,}000-25\omega^2 \end{vmatrix} = 0 \qquad \text{(b)}$$

expanding the determinant and solving gives

$$\omega_1^2 = 84.64 \text{ rad/sec}$$
$$\omega_2^2 = 400$$
$$\omega_3^2 = 536$$

The natural frequencies are calculated by $f=\omega/2\pi$, so that

$$f_1 = 1.464 \text{ CPS}$$
$$f_2 = 3.183$$
$$f_3 = 3.685$$

The modal shapes are determined by substituting each value of natural frequencies into equation (a) deleting one of the equations and solving the remaining two equations for two of the unknowns in terms of the third unknown, setting the unknown equal to one. Performing the operation gives,

$a_{11}=1.00 \quad a_{12}= 1.00 \quad a_{13}= 1.00$

$a_{21}=3.18 \quad a_{22}= 0 \quad a_{23}=-2.88$

$a_{31}=4.00 \quad a_{32}=-1.00 \quad a_{33}= 4.00$

Since the stiffness for this structure is

$$\begin{bmatrix} 40{,}000 & -10{,}000 & 0 \\ -10{,}000 & 20{,}000 & -10{,}000 \\ 0 & -10{,}000 & 10{,}000 \end{bmatrix}$$

By the use of gauss elimination of the first unknown gives

$$\begin{bmatrix} 1 & -0.25 & 0 \\ 0 & 17{,}500 & -10{,}000 \\ 0 & -10{,}000 & 10{,}000 \end{bmatrix} \qquad (c)$$

Comparing (c) with (62) indicates that

$$[\overline{T}] = [0.25 \quad 0]$$

$$[\overline{K}] = \begin{bmatrix} 17{,}500 & -10{,}000 \\ -10{,}000 & 10{,}000 \end{bmatrix} \qquad (d)$$

also from (67)

$$[T] = \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} \tag{e}$$

The condensed mass matrix is calculated by substituting matrix [T] and its transpose from (e) into equation (69).

$$[\overline{M}] = \begin{bmatrix} 0.25 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 100 & 0 & 0 \\ 0 & 50 & 0 \\ 0 & 0 & 25 \end{bmatrix} \begin{bmatrix} 0.25 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

which results in

$$[\overline{M}] = \begin{bmatrix} 56.25 & 0 \\ 0 & 25 \end{bmatrix}$$

Substituting the reduced stiffness and reducing mass into the equation of motion gives

$$\begin{bmatrix} 56.25 & 0 \\ 0 & 25 \end{bmatrix} \begin{Bmatrix} \ddot{x}_2 \\ \ddot{x}_3 \end{Bmatrix} + \begin{bmatrix} 17{,}500 & -10{,}000 \\ -10{,}000 & 10{,}000 \end{bmatrix} \begin{Bmatrix} x_2 \\ x_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}$$

The natural frequencies and mode shapes are then determined by solving the eigenvalue problem.

$$\begin{bmatrix} 17{,}500-56.25\omega^2 & -10{,}000 \\ -10{,}000 & 10{,}000-25\omega^2 \end{bmatrix} \begin{Bmatrix} a_2 \\ a_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{f}$$

equating the characteristic determinant to zero yields

$$\begin{vmatrix} 17{,}500-56.25\omega^2 & -10{,}000 \\ -10{,}000 & 10{,}000-25\omega^2 \end{vmatrix} = 0$$

expanding the determinant and solving for the natural frequencies gives

$$\omega_1 = 9.2304 \text{ rad/sec}$$
$$\omega_2 = 25.018 \quad \downarrow$$

Then

$$f_1 = \frac{9.2304}{2\pi} = 1.47 \text{ CPS}$$
$$f_2 = \frac{25.018}{2\pi} = 3.98 \quad \downarrow$$

The corresponding mode shapes are obtained by substituting the frequencies into equation (f) gives,

$$a_{21} = 1 \qquad a_{22} = 1$$
$$a_{31} = 1.27 \qquad a_{32} = 1.77$$

For this system of only three degrees-of-freedom, the reduction of one coordinate gives results that compare well only for the first mode. Experiencing with different numbers of degrees-of-freedom, it is clear that the condensation process results in an eigenproblem, which provides only about half of its natural frequencies and modal shapes within acceptable approximate values.

D. Computer Program For Investigation of Error

This program to investigate the error due to static condensation, eliminates rows or degrees-of-freedom by using a subroutine program called CONDE. This subroutine calculates the reduced stiffness matrix $[\bar{K}]$, the reduced mass matrix $[\bar{M}]$, and the transformation matrix [T]; with these reduced values, the program proceeds to solve for the natural frequencies and modal shapes, giving enough values to compare with the results of a non reduced system.

The subroutine CONDE, in order to perform the condensation of degrees-of-freedom uses the following variables.

Variable	Symbol in Thesis	Description
ND	N	Total number of degrees-of-freedom
NCR	P	Number of dependent modal coordinates
NL	ND-NCR	Number of degrees-of-freedom minus number of dependent coordinates
SM(I,J)	[M]	Mass matrix
SK(I,J)	[K]	Stiffness matrix
T(I,J)	[T]	Transformation matrix

The elimination of degrees-of-freedom can be done in an organized fashion. For this purpose this thesis introduces the subroutine ORDER. Therefore the programer has the freedom to choose the desired row to eliminate this and proceed to solve for the remaining degrees-of-freedom.

After experimenting with this program, it is obvious that the static condensation approach provides only about half of its eigenvalues and eigenvectors within acceptable approximate values.

E. Computer Program #3

```
C
C      EVALUATION OF ERROR FOR MULTI - DEGREE OF FREEDOM STRUCTURE
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION SM(50,50),SK(50,50),SC(50,50),T(50,50),TT(50),EIGV(50)
C
C      READ INPUT DATA AND INITIALIZE
C
       READ(5,100) ND,IFPR
       WRITE(6,100)ND,IFPR
   100 FORMAT(2I10)
       NL=ND
       LOC=1
       NM1=ND-1
       DO 2 I=1,ND
       DO 2 J=1,ND
       SM(I,J)=0.0
       SM(I,I)=1.0
       SC(I,J)=0.0
       SC(I,I)=1.0
     2 SK(I,J)=0.0
       DO 19 I=1,ND
       IF (I.EQ.1) GO TO 19
       SK(I,I-1)=-12.
       SK(I-1,I)=-12.
    19 SK(I,I)=24.
       SK(ND,ND)=12.
       DO 90 IC=1,ND
       IF(IC.EQ.1) GO TO 80
       NL=ND-IC+1
       NCR=ND-NL
       CALL CONDE (ND,NCR,LOC,SK,SM,SC,T)
    80 CALL JACOBI(SK,SC,T,EIGV,TT,NL,IFPR)
    90 CONTINUE
       STOP
       END
C      STATIC CONDENSATION OF STIFFNESS AND MASS MATRICES

       SUBROUTINE CONDE (ND,NCR,LOC,SK,SM,SC,T)
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION SK(50,50),SM(50,50),T(50,50),TT(50),SC(50,50)
C
C      CALCULATE THE REDUCED STIFFNESS MATRIX AND THE TRANSFORMATION MATR
C
       NL=ND-NCR
       DO 9 K=1,NCR
       IF (DABS(SK(K,K)).GT.1.D-10) GO TO 5
       WRITE (6,202) K
   202 FORMAT ('                    PIVOT TOO SMALL',I10)
       GO TO 99
     5 KP1 = K+1
       DO 6 J=KP1,ND
     6 SK(K,J) = SK(K,J)/SK(K,K)
       SK(K,K) =1.
       DO 9 I = 1,ND
       IF (I.EQ.K.OR. SK(I,K) .EQ.0) GO TO 9
       DO 8 J=KP1,ND
     8 SK(I,J) = SK(I,J) - SK(I,K)* SK(K,J)
       SK(I,K) = 0.0
     9 CONTINUE
```

```
      DO 30 I = 1,NCR
      DO 30 J = 1,NL
      JJ = J+NCR
   30 T(I,J) = -SK(I,JJ)
      DO 40 I=1,NL
      II = I + NCR
      DO 50 J = 1,NL
   50 T(II,J) = 0.0
      T(II,I) = 1.0
   40 CCNTINUE
      DO 20 I= 1,NL
      DO 20 J = 1,NL
      II = I + NCR
      JJ = J+NCR
   20 SK(I,J) = SK(II,JJ)
      WRITE (6,169)
  169 FORMAT(1H1,5X,'THE REDUCED STIFFNESS MATRIX IS'/)
      DO 80 I=1,NL
   80 WRITE (6,190) (SK(I,J),J=1,NL)
      WRITE(6,170)
  170 FORMAT(/6X,'THE TRANSFORMATION MATRIX IS'/)
      DO 81 I = 1,ND
   81 WRITE(6,190) (T(I,J),J = 1,NL)
  190 FORMAT (6E14.4)
      IF(LOC.EQ.0) GO TO 99
C
C     CALCULATE THE REDUCED MASS MATRIX
C
      READ(5,100) KEY
  100 FORMAT(I5)
      IF(KEY.EQ.0) GO TO 12
      CALL ORDER(ND,SK,SC)
      CALL ORDER(ND,SM,SC)
   12 CONTINUE
   99 RETURN
      END

      SUBROUTINE ORDER (N,A,B)
      IMPLICIT REAL *8(A-H,O-Z)
      DIMENSION A(50,50),B (50,50),M(50)
C
C     READ INPUT DATA AND INITIALIZE
C
      READ(5,100) (M(L),L=1,N)
      WRITE(6,100)(M(L),L=1,N)
  100 FORMAT(16I5)
      DO 30 II=1,N
      III=N-II+1
      I=M(III)
      DO 30 JJ=1,N
      JJJ=N-JJ+1
      J=M(JJJ)
   30 B(II,JJ)=A(I,J)
      DO 40 I=1,N
      DO 40 J=1,N
   40 A(I,J)=B(I,J)
   99 RETURN
      END
C
C     SOLVE EIGENPROBLEM USING JACOBI METHOD
C
```

```
      SUBROUTINE JACOBI (A,B,X,EIGV,D,N,IFPR)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(50,50),B(50,50),X(50,50),EIGV(50),D(50)
C
C       INITIALIZE EIGENVALUE AND EIGENVECTOR MATRICES
C
      NSMAX = 15
      RTCL = 1.D-12
      IOUT=6
      DO 10 I=1,N
      IF(A(I,I).GT.0. .AND. B(I,I).GT.0.)GO TO 4
      WRITE(IOUT,2020)
      STOP
    4 D(I)=A(I,I)/B(I,I)
   10 EIGV(I)=D(I)
      DO 30 I=1,N
      DO 20 J=1,N
   20 X(I,J)=0.
   30 X(I,I)=1.
      IF(N.EQ.1) RETURN
C
C       INITIALIZE SWEEP COUNTER AND EEGIN ITERATION
C
      NSWEEP=0
      NR=N-1
   40 NSWEEP=NSWEEP+1
      IF(IFPR.EQ.1)WRITE(IOUT,2000)NSWEEP
C
C       CHECK IF PRESENT OFF-DIAGONAL ELEMENT IS LARGE
C
      EPS=(.01**NSWEEP)**2
      DO 210 J=1,NR
      JJ=J+1
      DO 210 K=JJ,N
      EPTOLA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPTOLB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPTOLA.LT.EPS).AND.(EPTOLB.LT.EPS))GO TO 210
C
C       IF ZERCING IS REQUIRED,CALCULATE THE ROTATION MATRIX ELEMENT CA,C(
C
      AKK=A(K,K)*B(J,K)-B(K,K)*A(J,K)
      AJJ=A(J,J)*B(J,K)-B(J,J)*A(J,K)
      AB=A(J,J)*B(K,K)-A(K,K)*B(J,J)
      CHECK=(AB*AB+4.*AKK*AJJ)/4.
      IF(CHECK)50,60,60
   50 WRITE(IOUT,2020)
      STOP
   60 SQCH=DSQRT(CHECK)
      D1=AB/2.+SQCH
      D2=AB/2.-SQCH
      DEN=D1
      IF(DABS(D2).GT.DABS(D1))DEN=D2
      IF(DEN)80,70,80
   70 CA=0.
      CG=-A(J,K)/A(K,K)
      CG=-A(J,K)/A(K,K)
      GO TO 90
   80 CA=AKK/DEN
      CG=-AJJ/DEN
C
C       GENERALIZED ROTATION TO ZERO THE PRESENT OFF-DIAGONAL ELEMENT
```

```
   90 IF(N-2)100,190,100
  100 JP1=J+1
      JM1=J-1
      KP1=K+1
      KM1=K-1
      IF(JM1-1)130,110,110
  110 DO 120 I=1,JM1
      AJ=A(I,J)
      BJ=B(I,J)
      AK=A(I,K)
      BK=B(I,K)
      A(I,J)=AJ+CG*AK
      B(I,J)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  120 B(I,K)=BK+CA*BJ
  130 IF (KP1-N)140,140,160
  140 DO 150 I=KP1,N
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(K,I)
      BK=B(K,I)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(K,I)=AK+CA*AJ
  150 B(K,I)=BK+CA*BJ
  160 IF(JP1-KM1)170,170,190
  170 DO 180 I=JP1,KM1
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(I,K)
      BK=B(I,K)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  180 B(I,K)=BK+CA*BJ
  190 AK=A(K,K)
      BK=B(K,K)
      A(K,K)=AK+2.*CA*A(J,K)+CA*CA*A(J,J)
      B(K,K)=BK+2.*CA*B(J,K)+CA*CA*B(J,J)
      A(J,J)=A(J,J)+2.*CG*A(J,K)+CG*CG*AK
      B(J,J)=B(J,J)+2.*CG*B(J,K)+CG*CG*BK
      A(J,K)=0.
      B(J,K)=0.
C
C     UPDATE THE EIGENVECTOR MATRIX AFTER EACH ROTATION
C
      DO 200 I=1,N
      XJ=X(I,J)
      XK=X(I,K)
      X(I,J)=XJ+CG*XK
  200 X(I,K)=XK+CA*XJ
  210 CONTINUE
C
C     UPDATE THE EIGENVALUES AFTER EACH SWEEP
C
      DO 220 I=1,N
      IF (A(I,I).GT.0. .AND. B(I,I).GT.0.) GO TO 220
      WRITE(IOUT,2020)
      STOP
  220 EIGV(I)=A(I,I)/B(I,I)
```

```
      IF(IFPR.EQ.0)GO TO 230
      WRITE(IOUT,2030)
      WRITE(IOUT,2010) (EIGV(I),I=1,N)
C
C     CHECK FOR CONVERGENCE
C
  230 DO 240 I=1,N
      TOL=RTCL*D(I)
      DIF=DABS(EIGV(I)-D(I))
      IF(DIF.GT.TOL)GO TO 280
  240 CONTINUE
C
C     CHECK ALL OFF-DIAGONAL ELEMENTS  TO SEE IF ANOTHER SWEEP IS
C      REQUIRED
C
      EPS=RTCL**2
      DO 250 J=1,NR
      JJ=J+1
       DO 250 K=JJ,N
      EPSA=(A(J,K)*A(J.K))/(A(J,J)*A(K,K))
      EPSB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPSA.LT.EPS).AND.(EPSB.LT.EPS))GO TO 250
       GO TO 280
  250 CONTINUE
C
C     FILL OUT BOTTOM TRIANGLE OF RESULTANT MATRICES
C     AND SCALE EIGENVECTORS
C
  255 DO 260 I=1,N
      DO 260 J=1,N
      A(J,I)=A(I,J)
  260 B(J,I)=B(I,J)
      DO 270 J=1,N
      BB=DSQRT(B(J,J))
      DO 270 K=1,N
  270 X(K.J)=X(K,J)/BB
C
C     UPDATE MATRIX AND START NEW SWEEP,IF ALLOWED
C
      WRITE(6,2010) ((X(LI,LJ),LJ=1,N),LI=1,N)
      RETURN
  280 DO 290 I=1,N
  290 D(I)=EIGV(I)
      IF(NSWEEP.LT.NSMAX)GO TO 40
      GO TO 255
 2000 FORMAT(/,27HOSWEEP NUMBER IN *JACOBI* = ,I4)
 2010 FORMAT(1HO,3E20.12/)
 2020 FORMAT (25HO*** ERROR  SOLUTION STOP  /
     1       30H MATRICES NOT POSITVE DEFINITE)
 2030 FORMAT(36HOCURRENT EIGENVALUES IN *JACOBI*ARE,/)
      END
$ENTRY
 3          1

WEEP NUMBER IN *JACOBI* =    1
URRENT EIGENVALUES IN *JACOBI*ARE,

 0.366969384567D 02  0.188498382905D 02  0.245322325283D 01
```

```
SWEEP NUMBER IN *JACOBI* =    2
CURRENT EIGENVALUES IN *JACOBI*ARE,

  0.389637526057D 02  0.186595002239D 02  0.237674717049D 01


SWEEP NUMBER IN *JACOBI* =    3
CURRENT EIGENVALUES IN *JACOBI*ARE,

  0.389637552446D 02  0.186594975850D 02  0.237674717034D 01


SWEEP NUMBER IN *JACOBI* =    4
CURRENT EIGENVALUES IN *JACOBI*ARE,

  0.389637552446D 02  0.186594975850D 02  0.237674717034D 01


SWEEP NUMBER IN *JACOBI* =    5
CURRENT EIGENVALUES IN *JACOBI*ARE,

  0.389637552446D 02  0.186594975850D 02  0.237674717034D 01

  0.591009048506D 00  0.736976229100D 00  0.327985277606D 00

 -0.736976229100D 00  0.327985277606D 00  0.591009048506D 00

  0.327985277606D 00 -0.591009048506D 00  0.736976229100D 00
```

THE REDUCED STIFFNESS MATRIX IS

```
 0.6472D 02    -0.2867D-36
-0.2867D-36     0.4504D 01
```

THE TRANSFORMATION MATRIX IS

```
0.1667D-14     0.1982D-23
0.1000D 01     0.0000D 00
0.0000D 00     0.1000D 01
3    2    1
3    2    1
```

```
SWEEP NUMBER IN *JACOBI* =    1
CURRENT EIGENVALUES IN *JACOBI*ARE,

 0.647157982544D 02  0.450410305819D 01
 0.100000000000D 01  0.000000000000D 00  0.000000000000D 00

 0.100000000000D 01
```

THE REDUCED STIFFNESS MATRIX IS

```
0.4504D 01
```

THE TRANSFORMATION MATRIX IS

```
0.3063D-25
0.6366D-37
0.1000D 01
```

VI. ANALYSIS OF NONLINEAR STRUCTURAL RESPONSE

In the analysis of linear structures subjected to any arbitrary dynamic loadings, the Duhamel integral provides the most convenient approach for the solution of the systems. However, it must be emphasized that the Principle of Superposition that was employed in the derivation of Duhamel integral, can only be used with linear systems, that is, systems for which the properties remain constant during the response.

There are however, physical situations for which this linear model does not represent adequately the dynamic characteristics of the structure, such as the response of a building to an earthquake motion severe enough to cause structural damages. Consequently, it is necessary to develop another method of analysis suitable to use with non-linear systems.

A. Incremental Equation of Equilibrium

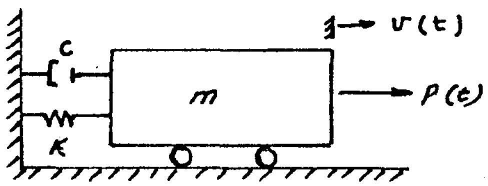

FIGURE 8(a) - Mathematical Model for Nonlinear Structural Response

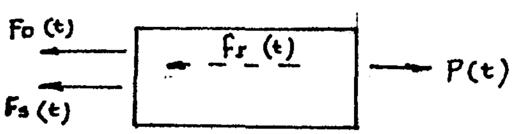

FIGURE 8(b) - Free Body Diagram

The structure to be considered in this section is a single degree-of-freedom shown in Figure 8(a). The dynamic equilibrium in the system is established by equating to zero the forces acting on the mass of the system indicated in Figure 8(b). This summation at any instant of time t in equilibrium of forces acting on the mass m requires

$$F_I(t) + F_D(t) + F_S(t) = F(t) \qquad (80)a$$

or

$$m\ddot{x}(t_i) + C_i\ \dot{x}(t_i) + K_i\ x(t_i) = F(t_i) \qquad (80)b$$

In equation (80)b the coefficient C_i and K_i are calculated for values of velocity and displacement at time t_i.

For an increment Δt later the equation (80)a takes the following form:

$$F_I(t+\Delta t) + F_D(t+\Delta t) + F_S(t+\Delta t) = F(t+\Delta t) \qquad (81)a$$

and equation (80)b takes the form of

$$m\ddot{x}(t_i+\Delta t) + C_i\ \dot{x}(t_i+\Delta t) + K_i\ x(t_i+\Delta t) = F(t_i+\Delta t) \qquad (81)b$$

Subtracting (81)b from (80)b gives the following convenient form of differential equation in terms of increments, namely

$$\Delta F_I(t) + \Delta F_D(t) + \Delta F_S(t) = \Delta F(t) \qquad (82)a$$

or

$$m\Delta\ddot{x}_i + C_i\Delta\dot{x}_i + K_i\Delta x_i = \Delta F_i \qquad (82)b$$

where the incremental forces in (82)a may be expressed as follows:

$$\begin{aligned} \Delta F_I(t) &= F_I(t+\Delta t) - F_I(t) && (a) \\ \Delta F_D(t) &= F_D(t+\Delta t) - F_D(t) && (b) \\ \Delta F_S(t) &= F_S(t+\Delta t) - F_S(t) && (c) \\ \Delta F\ (t) &= F\ (t+\Delta t) - F\ (t) && (d) \end{aligned} \qquad (83)$$

and from equation (82)b the incremental displacement, velocity, acceleration and force are

$$\begin{aligned} \Delta x_i &= x(t_i+\Delta t) - x(t_i) && (a) \\ \Delta\dot{x}_i &= \dot{x}(t_i+\Delta t) - \dot{x}(t_i) && (b) \\ \Delta\ddot{x}_i &= \ddot{x}(t_i+\Delta t) - \ddot{x}(t_i) && (c) \\ \Delta F_i &= F(t_i+\Delta t) - \Delta F_i && (d) \end{aligned} \qquad (84)$$

The general nonlinear characteristics of spring and damping forces are shown in Figure (9)a,b.

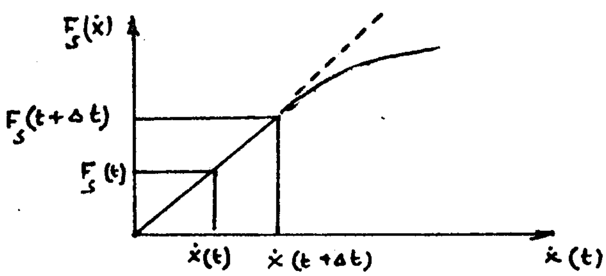

FIGURE 9(a) - Nonlinear Characteristic of Spring

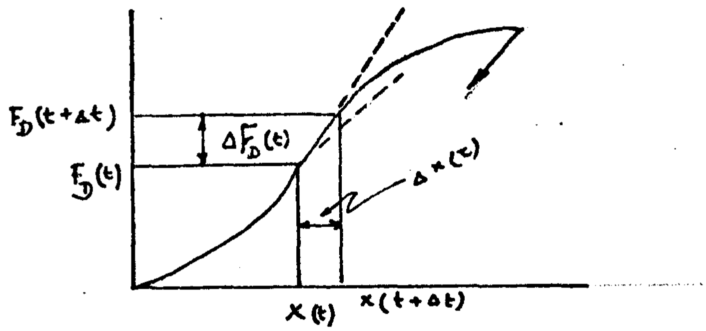

FIGURE 9(b) - Nonlinear Characteristic of Damping Force

In practice, the secant slope indicated could be evaluated only by iteration because the velocity and displacement at the end of the time increment depends on the damping and stiffness properties, corresponding to the velocity and displacement existing during the time interval. For this reason the tangent slope defined at the beginning of the time intervals are used instead.

$$C(t) = \frac{dF_D}{dx} \quad , \qquad K(t) = \frac{dF_S}{dx} \tag{85}$$

Among the methods available for the solution of equation (82)b, the most effective is the step by step integration method. In this method, the response is calculated at successive increments of time, usually taken at equal time intervals. At the beginning of each interval, the condition of dynamic equilibrium is established. Then the response of a time increment Δt is evaluated approximately on the basis that the coefficients K(x) and C(x) remain constant during the interval Δt. The nonlinear characteristic of these coefficients are found at the beginning of each time increment. The response is then obtained using the displacement and velocity calculated at the end of the time interval as the initial condition for the next time step.

There are several procedures available for performing the step by step integration of (82)b. Two of the most common used are the constant acceleration method. As may be expected the linear acceleration method will be presented here in detail.

B. Step By Step Integration (Linear Acceleration Method)

In this method, it is assumed that the acceleration may be expressed by a linear function of time during the time interval Δt. When the acceleration is assumed to be linear function of time the interval of time t_i to $t_{i+1} = t_i+\Delta t$, then the acceleration should be expressed as

$$\ddot{x}(t) = \ddot{x}_i + \frac{\Delta\ddot{x}_i}{\Delta t}(t-t_i) \quad (86)$$

where $\Delta\ddot{x}_i = \ddot{x}(t_i+\Delta t) - \ddot{x}(t_i)$ as shown before; integrating (86) twice between the limits t_i and t yields

$$\dot{x}(t) = \dot{x}_i + \ddot{x}(t-t_i) + 1/2\ \frac{\Delta\ddot{x}}{\Delta t}(t-t_i)^2 \quad (87)$$

and

$$x(t) = x_i + \dot{x}_i(t-t_i) + 1/2\ \ddot{x}_i(t-t_i)^2 + 1/6\ \frac{\Delta\ddot{x}_i}{\Delta t}(t-t_i)^3 \quad (88)$$

The evaluation of (87) and (88) at time $t=t_i+\Delta t$ gives

$$\Delta\dot{x}_i = \ddot{x}_i\ \Delta t + 1/2\ \ddot{x}_i\ \Delta t \quad (89)$$

and

$$\Delta x_i = \dot{x}_i \Delta t + 1/2\, \ddot{x}_i \Delta t^2 + 1/6\, \Delta\ddot{x}_i \Delta t^2 \qquad (90)$$

where Δx_i and $\Delta\dot{x}_i$ are defined in (84).

Now it will be convenient to use the incremental displacement as the basic variable of the analysis. (89) is solved for the incremental acceleration $\Delta\ddot{x}_i$, and is substituted into equation (90) to obtain:

$$\Delta\ddot{x}_i = \frac{6}{\Delta t^2} \Delta x_i - \frac{6}{\Delta t} \dot{x}_i - 3\, \ddot{x}_i \qquad (91)$$

and

$$\Delta\dot{x}_i = \frac{3}{\Delta t} \Delta x_i - 3\dot{x}_i - \frac{\Delta t}{2} \ddot{x}_i \qquad (92)$$

Substituting (90) and (91) into equation (82)b leads to the following form of equation of motion:

$$m\left\{\frac{6}{\Delta t} \Delta x_i - \frac{6}{\Delta t} \dot{x}_i - 3\dddot{x}_i\right\} + C_i \left\{\frac{3}{\Delta t} \Delta x_i - 3\dot{x}_i - \frac{\Delta t}{2} \ddot{x}_i\right\} + K_i \Delta x_i = \Delta F_i \qquad (93)$$

Finally transferring all terms associated with containing the unknown incremental displacement Δx_i to the left side gives,

$$\overline{K}_i \Delta x_i = \Delta\overline{F}_i \qquad (94)$$

in which

$$\overline{K}_i = K_i + \frac{6m}{\Delta t^2} + \frac{3C_i}{\Delta t} \tag{95}$$

and

$$\Delta\overline{F}_i = \Delta F_i + m\left\{\frac{6}{\Delta t}\,\dot{x}_i + 3\ddot{x}_i\right\} + C_i\left\{3\dot{x}_i + \frac{\Delta t}{2}\,\ddot{x}_i\right\} \tag{96}$$

It should be noted that (94) is equivalent to the static incremental-equilibrium equation, and may be solved for the incremental displacement by simply dividing the equivalent incremental load $\Delta\overline{F}_i$ by the equivalent spring constant $\overline{K}_i$, that is,

$$x_i = \frac{\Delta\overline{F}_i}{\overline{K}_i} \tag{97}$$

To obtain the displacement at time $t_{i+1}=t_i+\Delta t$, this value of Δx_i is substituted into (84)a yielding

$$x_{i+1} = x_i + \Delta x_i \tag{98}$$

Then the incremental velocity $\Delta\dot{x}_i$ is obtained from (92) and the velocity $t_{i+1}=t_i+\Delta t$ from (84)b as

$$\dot{x}_{i+1} = \dot{x}_i + \Delta\dot{x}_i \tag{99}$$

Finally, the acceleration $\ddot{x}_{i+1}$ at the end of the time step is obtained directly from the differential equation of motion (80)b where the equation is written for time $t_{i+1}=t_i+\Delta t$. Hence from (80)b it follows that

$$\ddot{x}_{i+1} = \frac{1}{m}\{F(t_{i+1}) - C_{i+1}\,\dot{x}_{i+1} - K_{i+1}\,x_{i+1}\} \qquad (100)$$

After the displacement, velocity and acceleration have been determined at time $t_{i+1}=t_i+\Delta t$, the outlined procedure is repeated to calculate these quantities at the following time step $t_{i+2}=t_{i+1}+\Delta t$ and the process is continued to any desired final value of time.

This numerical procedure involves two significant approximations: 1) the acceleration is assumed to vary linearly during the time increment Δt; and 2) the damping and stiffness properties of the system are evaluated at the initiation of each time increment and assumed to remain constant during the time interval.

This concludes the background analysis of a single degree-of-freedom system using step by step linear acceleration. It was necessary to include this analysis in this chapter to present a modification of the extension of this method known as the Wilson-θ method, for the solution of the structures with elasto-plastic behavior.

The modification introduced by Wilson is utilized to assure the numerical stability of the solution process regardless of the magnitude selected for the time step; for this reason, such a method is said to be unconditionally stable.

C. Incremental Equation of Motion

The basic assumption of the Wilson-θ method is that the acceleration varies linearly over the time interval from t to $t+\theta\Delta t$ where $\theta \geq 1.0$. The value of the factor θ is determined to obtain optimum stability of the numerical process and accuracy of the solution. It has been shown by Wilson that, for $\theta \geq 1.38$, the method becomes unconditionally stable.

The equations expressing the incremental equilibrium conditions for a multidegree-of-freedom system can be derived as the matrix equivalent of the incremental equation of motion of the single degree-of-freedom system (82)b. Thus taking the difference between dynamic equilibrium conditions defined at times t_i and $t_{i+\tau}$, where $\tau=\theta\Delta t$; then the following incremental equations are obtained.

$$\underset{\sim}{M}\hat{\Delta}\ddot{\underset{\sim}{x}}_i + C(\dot{x})\hat{\Delta}\dot{\underset{\sim}{x}}_i + \underset{\sim}{K}(x)\hat{\Delta}\underset{\sim}{x}_i = \hat{\Delta}\underset{\sim}{F}_i \qquad (101)$$

in which the symbol over $\hat{\Delta}$ indicates that the increments are associated with the extended time step $\tau=\theta\Delta t$. Thus

$$\begin{aligned} \hat{\Delta}\underset{\sim}{x}_i &= \underset{\sim}{x}(t_i+\tau) - \underset{\sim}{x}(t_i) \quad , \qquad (a) \\ \hat{\Delta}\dot{\underset{\sim}{x}}_i &= \dot{\underset{\sim}{x}}(t_i+\tau) - \dot{\underset{\sim}{x}}(t_i) \quad , \qquad (b) \\ \hat{\Delta}\ddot{\underset{\sim}{x}}_i &= \ddot{\underset{\sim}{x}}(t_i+\tau) - \ddot{\underset{\sim}{x}}(t_i) \quad , \qquad (c) \end{aligned} \qquad (102)$$

and

$$\hat{\Delta}\underset{\sim}{F}_i = \underset{\sim}{F}(t_i+\tau) - \underset{\sim}{F}(t_i) \qquad (103)$$

In writing (101), it was assumed that the stiffness and damping are obtained for each time step as the initial values of tangent of the corresponding curves, as shown in Figure 8, rather than the slope of the secant line which requires iteration. Hence the stiffness coefficient is defined as

$$K_{ij} = \frac{dF_{si}}{dx_j} \tag{104}$$

and the damping coefficient as

$$C_{ij} = \frac{dF_{Di}}{dx_j} \tag{105}$$

in which F_{si} and F_{Di} are respectively the elastic and damping forces at modal coordinate i; x_j and $\dot{x}_j$ are respectively the displacement and velocity at modal coordinate j.

D. The Wilson-θ Method

At this point it is necessary to consider the detailed performance and efficiency of this unconditionally stable method of time integration, as it has already been mentioned, on the assumption that acceleration may be represented by a linear function during the time step $\tau=\theta\Delta t$ as is shown in Figure 10.

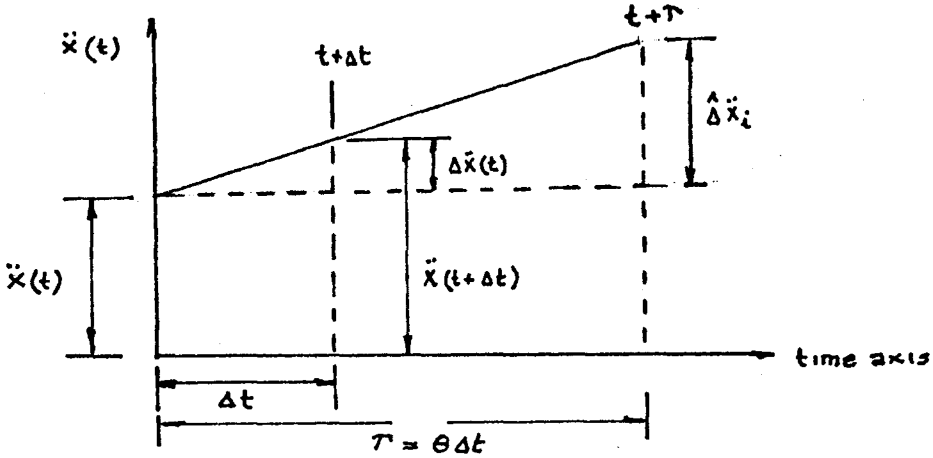

FIGURE 10 - Linear Acceleration; Normal and Extended Time Steps

From this figure can be written the linear expression for the acceleration during the extended time step as

$$\ddot{\underset{\sim}{x}}(t) = \ddot{\underset{\sim}{x}}_i + \frac{\hat{\Delta}\ddot{\underset{\sim}{x}}_i}{\tau}(t-t_i) \tag{106}$$

in which $\hat{\Delta}\ddot{\underset{\sim}{x}}_i$ is given by (102)c. Integrating (106) twice yields

$$\dot{\underset{\sim}{x}}(t) = \dot{\underset{\sim}{x}}_i + \ddot{\underset{\sim}{x}}_i(t-t_i) + 1/2\,\frac{\hat{\Delta}\ddot{\underset{\sim}{x}}_i}{\tau}(t-t_i)^2 \tag{107}$$

and

$$\underset{\sim}{x}(t) = \underset{\sim}{x}_i + \dot{\underset{\sim}{x}}(t-t_i) + 1/2\,\ddot{\underset{\sim}{x}}_i(t-t_i)^2 + 1/6\,\frac{\hat{\Delta}\ddot{\underset{\sim}{x}}_i}{\tau}(t-t_i)^3 \tag{108}$$

Evaluation of (107) and (108) at the end of the extended interval $t=t_i+\tau$ gives

$$\hat{\Delta}\dot{\underset{\sim}{x}}_i = \ddot{\underset{\sim}{x}}_i\,\tau + 1/2\,\hat{\Delta}\ddot{\underset{\sim}{x}}_i\,\tau \tag{109}$$

and

$$\hat{\Delta}\underset{\sim}{x}_i \ \tau + 1/2 \ \underset{\sim}{\ddot{x}}_i \ \tau^2 + 1/6 \ \hat{\Delta}\underset{\sim}{\ddot{x}}_i \ \tau^2 \qquad (110)$$

in which $\hat{\Delta}\underset{\sim}{x}_i$ and $\hat{\Delta}\underset{\sim}{\dot{x}}_i$ are defined by (84)b,c respectively. Then (110) is solved for incremental acceleration $\hat{\Delta}\underset{\sim}{\ddot{x}}_i$ and substituted in (109) yields

$$\hat{\Delta}\underset{\sim}{\ddot{x}}_i = \frac{6}{\tau^2} \hat{\Delta}x_i - \frac{6}{\tau} \underset{\sim}{\dot{x}}_i - 3\underset{\sim}{\ddot{x}}_i \qquad (111)$$

and

$$\hat{\Delta}\underset{\sim}{\dot{x}}_i = \frac{3}{\tau} \hat{\Delta}\underset{\sim}{x}_i - 3\underset{\sim}{\dot{x}}_i = \frac{\tau}{2} \underset{\sim}{\ddot{x}}_i \qquad (112)$$

Finally, substituting (111) and (112) into the incremental equation of motion (82)b results in an equation for incremental displacement $\hat{\Delta}\underset{\sim}{x}_i$ which may be conveniently written as

$$\overline{\underset{\sim}{K}}_i \ \hat{\Delta}\underset{\sim}{x}_i = \overline{\hat{\Delta} \ F_i} \qquad (113)$$

in which

$$\overline{\underset{\sim}{K}}_i = \underset{\sim}{K}_i + \frac{6}{\tau^2} \underset{\sim}{M} + \frac{3}{\tau} \underset{\sim}{C}_i \qquad (114)$$

and

$$\overline{\hat{\Delta}\underset{\sim}{F}_i} = \hat{\Delta}\underset{\sim}{F}_i + \underset{\sim}{M}(\frac{6}{\tau}\underset{\sim}{\dot{x}}_i + 3\underset{\sim}{\ddot{x}}_i) + \underset{\sim}{C}_i(3\underset{\sim}{\dot{x}}_i + \frac{\tau}{2}\underset{\sim}{\ddot{x}}_i) \qquad (115)$$

Equation (113) has the same form as the static incremental equilibrium equation and may be solved for the incremental displacement $\hat{\Delta}\underset{\sim}{x}_i$ by solving a system of linear equations.

To obtain the incremental acceleration $\hat{\Delta}\dot{\underset{\sim}{x}}_i$ for the extended time interval, the value of $\hat{\Delta}\underset{\sim}{x}_i$ obtained from the solution of (113) is substituted into (111). The incremental acceleration $\hat{\Delta}\ddot{\underset{\sim}{x}}_i$ for the normal time interval Δt is then obtained by a simple linear interpolation. Hence

$$\Delta\ddot{\underset{\sim}{x}} = \frac{\hat{\Delta}\ddot{x}}{\theta} \tag{116}$$

To calculate the incremental velocity $\Delta\dot{\underset{\sim}{x}}_i$ and incremental displacement $\Delta\dot{\underset{\sim}{x}}_i$ and incremental displacement $\Delta\underset{\sim}{x}_i$ corresponding to the normal interval Δt, use is made of (109) and (110) with the extended time interval parameter τ substituted for Δt, that is

$$\Delta\dot{\underset{\sim}{x}}_i = \ddot{\underset{\sim}{x}}_i \; \Delta t + 1/2 \; \Delta\ddot{\underset{\sim}{x}}_i \; \Delta t \tag{117}$$

and

$$\Delta\underset{\sim}{x}_i = \dot{\underset{\sim}{x}}_i \; \Delta t + 1/2 \; \ddot{\underset{\sim}{x}}_i \; \Delta t^2 + 1/6 \; \Delta\ddot{x}_i \; \Delta t^2 \tag{118}$$

Finally, the displacement $\underset{\sim}{x}_{i+1}$ and velocity $\dot{\underset{\sim}{x}}_{i+1}$ at the end of the normal time interval are calculated by

$$\underset{\sim}{x}_{i+1} = \underset{\sim}{x}_i + \Delta\underset{\sim}{x}_i \tag{119}$$

and

$$\dot{\underset{\sim}{x}}_{i+1} = \dot{\underset{\sim}{x}}_i + \Delta\dot{\underset{\sim}{x}}_i \tag{120}$$

As mentioned in the section dealing with single degree-of-freedom, the initial acceleration for the next step should be calculated from the condition of dynamic equilibrium at time t+Δt; thus

$$\ddot{x}_{i+1} = \underset{\sim}{M}^{-1}[\underset{\sim}{F}_{i+1}-\underset{\sim}{C}_{i+1}\,\dot{\underset{\sim}{x}}_{i+1}-\underset{\sim}{K}_{i+1}\,\underset{\sim}{x}_{i+1}] \tag{121}$$

in which the products $\underset{\sim}{C}_{i+1}\,\underset{\sim}{x}_{i+1}$ and $\underset{\sim}{K}_{i+1}\,\underset{\sim}{x}_{i+1}$ represent respectively the damping force and the stiffness force vectors evaluated at the end of the time step $t_{i+1}=t_{i+\Delta t}$. Once the displacement, velocity and acceleration vectors at time $t_{i+1}=t_{i+\Delta t}$, then the outline procedure is repeated to calculate these quantities at the next step $t_{i+2}=t_{i+1}+\Delta t$ and the process is continued until the desired final time.

E. Algorithm for Step-by-Step Solution of a Linear System, Using the Wilson-θ Integration Method

Initiation of Values:

1. Assemble system stiffness matrix $\underset{\sim}{K}$, mass matrix $\underset{\sim}{M}$, and damping matrix $\underset{\sim}{C}$.
2. Set initial values for displacement $\underset{\sim}{x}_0$, velocity $\dot{\underset{\sim}{x}}_0$ and forces $\underset{\sim}{F}_0$.
3. Calculate initial acceleration $\ddot{\underset{\sim}{x}}_0$ from

$$\underset{\sim}{M}\,\ddot{x}_0 = \underset{\sim}{F}_0 - \underset{\sim}{C}\,\dot{\underset{\sim}{x}}_0 - \underset{\sim}{K}\,\underset{\sim}{x}_0$$

4. Select time step Δt, the factor θ (for all practical purposes taken as 1.4) and calculate the constants, τ, a_1, a_2, a_3 and a_4 for the following relation

$$\tau=\theta\Delta t \; ; \; a_1=\frac{3}{\tau} \; , \; a_2=\frac{6}{\tau} \; , \; a_3=\frac{\tau}{3} \; , \; a_4=\frac{6}{\tau^2}$$

5. From the effective stiffness matrix $\bar{\underset{\sim}{K}}$, namely

$$\bar{\underset{\sim}{K}} = \underset{\sim}{K} + a_4 \underset{\sim}{M} + a_1 \underset{\sim}{C}$$

For Time Intervals (one at the time):

1. Calculate by linear interpolation the incremental load $\hat{\Delta}\underset{\sim}{F}_i$ for the time interval t_i to $t_i+\tau$, from the relation

$$\hat{\Delta}\underset{\sim}{F}_i = \underset{\sim}{F}_{i+1} + (\underset{\sim}{F}_{i+2}-\underset{\sim}{F}_{i+1})\,(\theta-1) - \underset{\sim}{F}_i$$

2. Calculate the effective incremental load $\overline{\hat{\Delta}F_i}$ for the time interval t_i to $t_{i+\tau}$, from the relation

$$\overline{\hat{\Delta}\underset{\sim}{F}_i} = \hat{\Delta}\underset{\sim}{F}_i + (a_2\underset{\sim}{M}+3\underset{\sim}{C})\,\dot{\underset{\sim}{x}}_i + (3\underset{\sim}{M}+a_3\underset{\sim}{C})\,\ddot{\underset{\sim}{x}}_i$$

3. Solve for incremental displacement $\hat{\Delta}\underset{\sim}{x}_i$ from

$$\bar{\underset{\sim}{K}}\,\hat{\Delta}\,\underset{\sim}{x}_i = \overline{\hat{\Delta}\underset{\sim}{F}_i}$$

4. Calculate the incremental acceleration for the extended time interval τ, from the relation

$$\hat{\Delta} \ddot{\underset{\sim}{x}}_i = \frac{6}{\tau^2} \hat{\Delta} \underset{\sim}{x}_i - \frac{6}{\tau} \dot{\underset{\sim}{x}}_i - 3 \ddot{\underset{\sim}{x}}_i$$

5. Calculate the incremental acceleration for the normal interval from

$$\Delta \ddot{\underset{\sim}{x}} = \frac{\hat{\Delta} \ddot{\underset{\sim}{x}}}{\theta}$$

6. Calculate the incremental velocity $\Delta \dot{\underset{\sim}{x}}_i$ and the incremental displacement $\Delta \underset{\sim}{x}_i$ from time t_i to $t_i + \Delta t$ from the following relations

$$\Delta \dot{\underset{\sim}{x}}_i = \ddot{\underset{\sim}{x}}_i \Delta t + 1/2 \Delta \ddot{\underset{\sim}{x}}_i \Delta t$$

$$\Delta \underset{\sim}{x}_i = \dot{\underset{\sim}{x}}_i \Delta t + 1/2 \ddot{\underset{\sim}{x}}_i \Delta t^2 + 1/6 \Delta \ddot{\underset{\sim}{x}}_i \Delta t$$

7. Calculate the displacement and velocity at time $t_{i+1} = t_i + \Delta t$ using

$$\Delta \underset{\sim}{x}_{i+1} = \underset{\sim}{x}_i + \Delta \underset{\sim}{x}_i$$

$$\Delta \dot{\underset{\sim}{x}}_{i+1} = \dot{\underset{\sim}{x}}_i + \Delta \dot{\underset{\sim}{x}}_i$$

8. Calculate the acceleration $\ddot{\underset{\sim}{x}}_{i+1}$ at time $t_{i+1} = t_i + \Delta t$ directly from the equilibrium equation of motion, namely

$$\underset{\sim}{M} \ddot{\underset{\sim}{x}}_{i+1} = \underset{\sim}{F}_{i+1} - \underset{\sim}{C} \dot{\underset{\sim}{x}}_{i+1} - \underset{\sim}{K} \underset{\sim}{x}_{i+1}$$

F. Subroutine Step

This is used for a type of dynamic loading of irregular behavior such as an earthquake. This subroutine will find the response for each modal coordinate at each increment of time up to the maximum specified by programer. The list of operational variables are shown in a tabular form, below.

Variable	Symbol in Thesis	Description
SK(I,J)	[K]	System stiffness matrix
SM(I,J)	[M]	System mass matrix
SC(I,J)	[C]	System damping matrix
ND	N	Number of degrees-of-freedom
THETA	θ	Wilson-θ factor
DT	Δt	Time step of integration
TMAX		Maximum time of integration
NEQ(L)		Number of data points for excitation at modal coordinates (L-1,ND)
TC(I),P(I)	$t_i, F_i(t)$	Time-force values

G. Program 4 - Seismic Response of Shear Buildings

A computer program for the analysis of a multidegree-of-freedom shear building with elastoplastic behavior, linear viscous damping, subjected to an arbitrary acceleration at the foundation, is presented in this section. This program may be conceived as a combination of three computer programs already presented: (1) the elastoplastic single degree-of-freedom system; (2) the seismic response of elastic shear buildings using modal superposition method; and (3) the subroutine

STEP using the Wilson-θ integration method for linear systems in this chapter.

The listing of Program 4 is given on page 89. The program calls subroutine JACOBI to solve the eigenproblem of the system in the linear range and then calls subroutine DAMP to determine from specified modal damping ratios, the damping matrix of the system. A listing of the principal variables used in the program are given below. Input data cards and corresponding formats are indicated following the list of variables.

Variables	Symbols in Thesis	Description
SK (I,J)	[K]	Stiffness matrix
SM (I,J)	[M]	Mass matrix
SC (I,J)	[C]	Damping matrix
THETA	θ	Wilson-θ factor
DT	Δt	Time step
E	E	Modules of elasticity
GR	g	Acceleration of gravity
TMAX		Maximum time of calculation
NEQ	NT	Number of data points for the excitation
ND	N	Number of degrees-of-freedom
IFPR		Printing index of subroutine JACOBI: 1=Print eigenvalues during iteration; 0=Do not print
SI	I	Moment of inertia of story columns
SL	L	Height of story
SM (I,I)	M	Mass at floor level
PM	M_p	Plastic moment of story
TC(I),P(I)	t_i, F_i	Time-Acceleration values (acceleration in g's)
XIS (I)	ξ_i	Modal damping ratios

Formats	Variables
(2F10.2,3F10.0,3I5)	THETA DT E GR TMAX NEQ ND IFPR
(8F10.0)	SI SL SM(I,I) PM (one card per degree of freedom)
(8F10.2)	TC(L) P(L) (L=1,NEQ)
(8F10.3)	XIS(L) (L=1,ND)

H. Illustrative Example

Use Program 4 to determine the response of the two-story shear-building of the example subjected to a constant acceleration of 0.28 g applied suddenly at the foundation. The plastic moment for the columns on the first or second story is M_p = 15,000 lb-in.

The listing of the input data followed by the computer results are shown on the following page.

Input Data and Computer Results

Input Data

```
  1.40      0.05 30000000.      386.         1.     2     2     1
497.20    180.00    136.00    15000.
212.60    120.00      6.00    15000.
```

EIGENVALUES

```
SWEEP NUMBER IN *JACOBI* =    1
   0.21490D 03    0.70473D 04

SWEEP NUMBER IN *JACOBI* =    2
   0.21490D 03    0.70473D 04
   0.21490D 03    0.70473D 04
```

EIGENVECTORS

```
   0.83611D-01   -0.19028D-01
   0.86377D-01    0.37950D 00
```

THE DAMPING MATRIX IS

```
   0.0000D 00    0.0000D 00
   0.0000D 00    0.0000D 00
```

THE RESPONSE IS

CORD.	TIME	DISPL.	VELOC.	ACC.
1	0.050	-0.1210	-4.5493	-107.9725
2	0.050	-0.1233	-4.6842	-93.3823
1	0.100	-0.4831	-9.9253	-106.8883
2	0.100	-0.4820	-9.8206	-115.7234
1	0.150	-1.1132	-15.2820	-107.3327
2	0.150	-1.1134	-15.3525	-106.5670
1	0.200	-2.0115	-20.6513	-107.8588
2	0.200	-2.0134	-20.6266	-95.7245
1	0.250	-3.1784	-26.9171	-105.6968
2	0.250	-3.1737	-25.9716	-140.2763
1	0.300	-4.6125	-31.3723	-109.0787
2	0.300	-4.6274	-31.5372	-70.5872
1	0.350	-6.3175	-36.8262	-109.0787
2	0.350	-6.2885	-35.0665	-70.5872
1	0.400	-8.2052	-42.2801	-109.0787
2	0.400	-8.1301	-38.5959	-70.5872
1	0.450	-10.5455	-47.7341	-109.0787
2	0.450	-10.1481	-42.1252	-70.5872
1	0.500	-13.0686	-53.1880	-109.0787
2	0.500	-12.3426	-45.6546	-70.5872
1	0.550	-15.8643	-58.6419	-109.0787
2	0.550	-14.7136	-49.1840	-70.5872
1	0.600	-18.9328	-64.0959	-109.0787
2	0.600	-17.2610	-52.7133	-70.5872
1	0.650	-22.2739	-69.5498	-109.0787
2	0.650	-19.9049	-56.2427	-70.5872
1	0.700	-25.8870	-75.0038	-109.0787
2	0.700	-22.8853	-59.7720	-70.5872
1	0.750	-29.7743	-80.4577	-109.0787
2	0.750	-25.9621	-63.3014	-70.5872
1	0.800	-33.9335	-85.9116	-109.0787
2	0.800	-29.2154	-66.8308	-70.5872
1	0.850	-38.3654	-91.3656	-109.0787
2	0.850	-32.6452	-70.3601	-70.5872
1	0.900	-43.0761	-96.8195	-109.0787
2	0.900	-36.2514	-73.8895	-70.5872
1	0.950	-48.0345	-101.4987	-109.0787
2	0.950	-40.0212	-76.6441	-70.5872

I. Program Listing

```
$JOB             ,PAGES=5,TIME=5,LINES=400
C
C      SEISMIC RESPONSE ELASTOPLASTIC SHEAR BUILDING
C
       IMPLICIT REAL*8(A-H,O-Z)
       DIMENSION SK(30,30),SM(30,30),SC(30,30),F(30),X(30,30),DD(30),
     1 DUA(30),UD(30),UV(30),UA(30),TC(30),P(30),SKP(30),RT(30),
     1 R(30),YT(30),YC(30),S(30),SP(30),KEY(30),EIGEN(30)
C
C      READ INPUT DATA AND INITIALIZE
C
       READ(5,100) THETA,DT,E,GR,TMAX,NEQ,ND,IFPR
       WRITE(6,100)THETA,DT,E,GR,TMAX,NEQ,ND,IFPR
   100 FORMAT(2F10.2,3F10.0,3I5)
       NX=TMAX/DT+2
       DO 1 I=1,NX
     1 F(I)=0.0
       DO 2 I=1,ND
       DO 2 J=1,ND
       SM(I,J)=0.0
       SC(I,J)=0.0
       X(I,J)=0.0
     2 SK(I,J)=0.0
       ND1=ND+1
       TU=THETA*DT
       A1=3./TU
       A2=6./TU
       A3=TU/2.
       A4=A2/TU
       DO 7 I=1,ND
       READ(5,110) SI,SL,SM(I,I),PM
       WRITE(6,110)SI,SL,SM(I,I),PM
   110 FORMAT(3F10.2,F10.0)
       S(I)=12.0*E*SI/SL**3
       SP(I)=S(I)
       RT(I)=2*PM/SL
       SC(I,I)=SM(I,I)
       UD(I)=0.0
       UV(I)=0.0
       YT(I)=RT(I)/S(I)
       YC(I)=-RT(I)/S(I)
       KEY(I)=0
     7 SP(I)=S(I)
C
C      ASSEMBLE STIFFNESS MATRIX
C
       S(ND+1)=0.0
       DO 19 I=1,ND
       IF(I.EQ.1) GO TO 19
       SK(I,I-1)=-S(I)
       SK(I-1,I)=-S(I)
    19 SK(I,I)=S(I)+S(I+1)
C
C      DETERMINE NATURAL FREQUENCIES AND MODE SHAPES
C
       CALL JACOBI(SK,SC,X,EIGEN,TC,ND,IFPR)
C
C      DETERMINE DAMPING MATRIX
C
       CALL DAMP(ND,X,SM,SC,EIGEN)
       READ(5,120) (TC(L),P(L),L=1,NEQ)
```

```
          WRITE(6,120)(TC(L),P(L),L=1,NEQ)
      120 FORMAT(4F10.2)
          DO 4 I=1,NEQ
        4 P(I)=P(I)*GR
C
C         INTERPOLATION BETWEEN DATA POINTS
C
          NT=TC(NEQ)/DT
          NT1=NT+1
          F(1)=P(1)
          ANN=0.0
          II=1
          DO 10 I=2,NT1 ,
          AI=I-1
          T=AI*DT
          IF(T.GT.TC(NEQ)) GO TO 16
          IF(T.LE.TC(II+1)) GO TO 9
          ANN=-TC(II+1)+T-DT
          II=II+1
        9 ANN=ANN+DT
          F(I)=P(II)+(P(II+1)-P(II))*ANN/(TC(II+1)-TC(II))
       10 CONTINUE
       16 CONTINUE
C
C         INITIALIZE AND DETERMINE INITIAL ACCELERATION
C
          NT=TMAX/DT
          DO 22 I=1,ND
          X(I,ND1)=-F(1)*SM(I,I)
          DO 22 J=1,ND
       22 X(I,J)=SM(I,J)
          CALL SOLVE(ND,X)
          DO 23 I=1,ND
       23 UA(I)=X(I,ND1)
          SP(ND+1)=0.0
          R(ND+1)=0.0
C
C         LOOP OVER TIME CALCULATING RESPONSE
C
          WRITE (6,170)
          DO 50 L=1,NT
          AL = L
          T=DT*AL
          DO 20 I=1,ND
          IF(I.EQ.1) GO TO 20
          SK(I,I-1) = -SP(I)
          SK((I-1),I)=-SP(I)
       20 SK(I,I)=SP(I)+SP(I+1)
          DO 25 I=1,ND
          DO 25 J=1,ND
       25 X(I,J)=SK(I,J)+A4*SM(I,J)+A1*SC(I,J)
          DO 35 I=1,ND
          X(I,ND1)=(F(L+1)+(F(L+2)-F(L+1))*(THETA-1.0)-F(L))*(-SM(I,I))
          DO 30 J=1,ND
       30 X(I,ND1)=X(I,ND1)+(SM(I,J)*A2+SC(I,J)*3.0)*UV(J)
         1 +(SM(I,J)*3.0+A3*SC(I,J))*UA(J)
       35 CONTINUE
          CALL SOLVE(ND,X)
          DO 38 I=1,ND
          DUA(I)=A4*X(I,ND1)-A2*UV(I)-3.0*UA(I)
          DUA(I)=DUA(I)/THETA
```

```
      DUV=DT*UA(I)+DT*DUA(I)/2.0
      UD(I)=   UD(I)+DT*UV(I)+DT*DT*UA(I)/2.0+DT*DT*DUA(I)/6.0
   38 UV(I)=UV(I)+DUV
      DD(1)=UD(1)
      DO 39 I=2,ND
   39 DD(I)=UD(I)-UD(I-1)
      DO 40 I=1,ND
      IF(KEY(I)) 11,12,13
   12 R(I)=RT(I)-(YT(I)-DD(I))*S(I)
      SP(I)=S(I)
      IF (DD(I).GT.YC(I).AND.DD(I).LT.YT(I)) GO TO 40
      IF(DD(I).LT.YC(I)) GO TO 15
      KEY(I)=1
      SP(I)=0.0
      R(I)=RT(I)
      GO TO 40
   13 IF(UV(I).GT.0.) GO TO 40
      KEY(I)=0
      SP(I)=S(I)
      YT(I)=DD(I)
      YC(I)=DD(I)-2.0*RT(I)/S(I)
      R(I)=RT(I)-(YT(I)-DD(I))*S(I)
      GO TO 40
   11 IF(UV(I).LT.0) GO TO 40
      KEY(I)=0
      SP(I)=S(I)
      YC(I)=DD(I)
      YT(I)=DD(I)+2.*RT(I)/S(I)
      R(I)=RT(I)-(YT(I)-DD(I))*S(I)
      GO TO 40
   15 KEY(I)=-1
      R(I)=-RT(I)
      SP(I)=0.0
   40 CONTINUE
      DO 50 I=1,ND
      X(I,ND1)=F(L+1)*(-SM(I,I))-R(I)+R(I+1)
      DO 45 J=1,ND
      X(I,ND1)=X(I,ND1)-SC(I,J)*UV(J)
   45 X(I,J)=SM(I,J)
   50 CONTINUE
      CALL SOLVE (ND,X)
      DO 60 I=1,ND
      UA(I)=X(I,ND1)
   60 WRITE(6,250) I,T,UD(I),UV(I),UA(I)
   90 CONTINUE
  170 FORMAT(1H1,5X,'THE RESPONSE IS',/,5X,'CORD.',6X,'TIME',9X,
     1  'DISPL.',9X,'VELOC.',11X,'ACC.'/)
  250 FORMAT(I10,F10.3,3F15.4)
      STOP
      END

      SUBROUTINE SOLVE (N,A)
      IMPLICIT REAL * 8 (A-H,O-Z)
      DIMENSION            A(30,30)
      M=1
      EPS=1.0E-10
      NPLUSM=N+M
      DET=1.0
      DO 9 K=1,N
      DET=DET*A(K,K)
      IF(DABS(A(K,K)).GT.EPS) GO TO 5
```

```
      WRITE(6,202)
      GO TO99
    5 KP1=K+1
      DO 6 J=KP1,  NPLUSM
    6 A(K,J)=A(K,J)/A(K,K)
      A(K,K)=1.
      DO 9 I=1,N
      IF (I.EQ.K.OR.A(I,K).EQ.0.) GO TO 9
      DO 8 J=KP1,NPLUSM
    8 A(I,J)=A(I,J)-A(I,K)*A(K,J)
      A(I,K)=0.D00
    9 CONTINUE
  202 FORMAT(37HOSMALL PIVOT -MATRIX MAY BE SINGULAR )
   99 RETURN
      END

      SUBROUTINE JACOBI (A,B,X,EIGV,D,N,IFPR)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION A(30,30),B(30,30),X(30,30),EIGV(30),D(30)
C
C      INITIALIZE EIGENVALUE AND EIGENVECTOR MATRICES
C
      WRITE (6,1980)
      NSMAX = 15
      RTCL = 1.D-12
      IOUT=6
      DO 10 I=1,N
      IF(A(I,I).GT.0. .AND. B(I,I).GT.0.)GO TO 4
      WRITE(IOUT,2020)
      STOP
    4 D(I)=A(I,I)/B(I,I)
   10 EIGV(I)=D(I)
      DO 30 I=1,N
      DO 20 J=1,N
   20 X(I,J)=0.
   30 X(I,I)=1.
      IF(N.EQ.1) RETURN
C
C     INITIALIZE SWEEP COUNTER AND EEGIN ITERATION
C
      NSWEEP=0
      NR=N-1
   40 NSWEEP=NSWEEP+1
      IF(IFPR.EQ.1)WRITE(IOUT,2000)NSWEEP
C
C     CHECK IF PRESENT OFF-DIAGONAL ELEMENT IS LARGE
C
      EPS=(.01**NSWEEP)**2
      DO 210 J=1,NR
      JJ=J+1
      DO 210 K=JJ,N
      EPTOLA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPTOLB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPTOLA.LT.EPS).AND.(EPTOLB.LT.EPS))GO TO 210
C
C     IF ZEROING IS REQUIRED,CALCULATE THE ROTATION MATRIX ELEMENT CA,CG
C
      AKK=A(K,K)*B(J,K)-B(K,K)*A(J,K)
      AJJ=A(J,J)*B(J,K)-B(J,J)*A(J,K)
      AB=A(J,J)*B(K,K)-A(K,K)*B(J,J)
      CHECK=(AB*AB+4.*AKK*AJJ)/4.
```

```
      IF(CHECK)50,60,60
   50 WRITE(ICUT,2020)
      STOP
   60 SQCH=DSQRT(CHECK)
      D1=AB/2.+SQCH
      D2=AB/2.-SQCH
      DEN=D1
      IF(DABS(D2).GT.DABS(D1))DEN=D2
      IF(DEN)80,70,80
   70 CA=0.
      CG=-A(J,K)/A(K,K)
      CG=-A(J,K)/A(K,K)
      GO TO 90
   80 CA=AKK/DEN
      CG=-AJJ/DEN
C
C     GENERALIZED ROTATION TO ZERO THE PRESENT OFF-DIAGONAL ELEMENT
C
   90 IF(N-2)100,190,100
  100 JP1=J+1
      JM1=J-1
      KP1=K+1
      KM1=K-1
      IF(JM1-1)130,110,110
  110 DO 120 I=1,JM1
      AJ=A(I,J)
      BJ=B(I,J)
      AK=A(I,K)
      BK=B(I,K)
      A(I,J)=AJ+CG*AK
      B(I,J)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  120 B(I,K)=BK+CA*BJ
  130 IF (KP1-N)140,140,160
  140 DO 150 I=KP1,N
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(K,I)
      BK=B(K,I)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(K,I)=AK+CA*AJ
  150 B(K,I)=BK+CA*BJ
  160 IF(JP1-KM1)170,170,190
  170 DO 180 I=JP1,KM1
      AJ=A(J,I)
      BJ=B(J,I)
      AK=A(I,K)
      BK=B(I,K)
      A(J,I)=AJ+CG*AK
      B(J,I)=BJ+CG*BK
      A(I,K)=AK+CA*AJ
  180 B(I,K)=BK+CA*BJ
  190 AK=A(K,K)
      BK=B(K,K)
      A(K,K)=AK+2.*CA*A(J,K)+CA*CA*A(J,J)
      B(K,K)=BK+2.*CA*B(J,K)+CA*CA*B(J,J)
      A(J,J)=A(J,J)+2.*CG*A(J,K)+CG*CG*AK
      B(J,J)=B(J,J)+2.*CG*B(J,K)+CG*CG*BK
      A(J,K)=0.
      B(J,K)=0.
```

```
C
C     UPDATE THE EIGENVECTOR MATRIX AFTER EACH ROTATION
C
      DO 200 I=1,N
      XJ=X(I,J)
      XK=X(I,K)
      X(I,J)=XJ+CG*XK
  200 X(I,K)=XK+CA*XJ
  210 CONTINUE
C
C     UPDATE THE EIGENVALUES AFTER EACH SWEEP
C
      DO 220 I=1,N
      IF (A(I,I).GT.0. .AND. B(I,I).GT.0.) GO TO 220
      WRITE(IOUT,2020)
      STOP
  220 EIGV(I)=A(I,I)/B(I,I)
      IF(IFPR.EQ.0)GO TO 230
      WRITE(IOUT,2010) (EIGV(I),I=1,N)
C
C     CHECK FOR CONVERGENCE
C
  230 DO 240 I=1,N
      TOL=RTCL*D(I)
      DIF=DABS(EIGV(I)-D(I))
      IF(DIF.GT.TOL)GO TO 280
  240 CONTINUE
C
C     CHECK ALL OFF-DIAGONAL ELEMENTS  TO SEE IF ANOTHER SWEEP I
C      REQUIRED
C
      EPS=RTCL**2
      DO 250 J=1,NR
      JJ=J+1
       DO 250 K=JJ,N
      EPSA=(A(J,K)*A(J,K))/(A(J,J)*A(K,K))
      EPSB=(B(J,K)*B(J,K))/(B(J,J)*B(K,K))
      IF((EPSA.LT.EPS).AND.(EPSB.LT.EPS))GO TO 250
       GO TO 280
  250 CONTINUE
C
C     FILL OUT BOTTOM TRIANGLE OF RESULTANT MATRICES
C     AND SCALE EIGENVECTORS
C
  255 DO 260 I=1,N
      DO 260 J=1,N
      A(J,I)=A(I,J)
  260 B(J,I)=B(I,J)
      DO 270 J=1,N
      BB=DSQRT(B(J,J))
      DO 270 K=1,N
  270 X(K,J)=X(K,J)/BB
C
C     UPDATE MATRIX AND START NEW SWEEP,IF ALLOWED
C
      WRITE (6,2010) (EIGV(IL),IL=1,N)
      WRITE(6,1990)
      DO 1991 LI=1,N
 1991 WRITE(6,2010) (X(LI,LJ),LJ=1,N)
 1980 FORMAT (//,10X,'EIGENVALUES',/)
 1990 FORMAT(/10X,'EIGENVECTORS',/)
```

```
      RETURN
  280 DO 290 I=1,N
  290 D(I)=EIGV(I)
      IF(NSWEEP.LT.NSMAX)GO TO 40
      GO TO 255
 2000 FORMAT(/,27H0SWEEP NUMBER IN *JACOBI* = ,I4)
 2010 FORMAT(1H0,6E14.5/)
 2020 FORMAT (25H0*** ERROR  SOLUTION STOP  /
     1        30H MATRICES NOT POSITVE DEFINITE)
      END
C
C     DETERMINATION OF DAMPING MATRIX FROM MODAL DAMPING RATIOS
C

      SUBROUTINE DAMP (NL,X,SM,SC,EIGEN)
      IMPLICIT REAL*8(A-H,O-Z)
      DIMENSION X(30,30),T(30,30),SM(30,30),SC(30,30),EIGEN(30),XIS(30)
      READ (5,110) (XIS(L),L=1,NL)
      DO 10 I=1,NL
      EIGEN(I)=DSQRT(EIGEN(I))
      DO 10 J=1,NL
   10 SC(I,J) =0.0
      DO 20 II=1,NL
      DA = 2.*XIS(II)*EIGEN(II)
      DO 20 I=1,NL
      DO20 J=1,NL
   20 SC(I,J)=SC(I,J)+X(I,II)*X(J,II)*DA
      DO 30 I=1,NL
      DO 30 J=1,NL
      T(I,J)=0.0
      DO 30 K = 1,NL
   30 T(I,J) = T(I,J)+SM(I,K)*SC(K,J)
      DO 40 I=1,NL
      DO 40 J=1,NL
```

```
      SC(I,J)=0.0
      DO 40 K=1,NL
   40 SC(I,J) = SC(I,J)+T(I,K)*SM(K,J)
      WRITE(6,170)
  170 FORMAT(//,5X,'THE DAMPING MATRIX IS',/)
      DO 50 I=1,NL
   50 WRITE(6,120) (SC(I,J),J=1,NL)
  110 FORMAT(3F10.2)
  120 FORMAT (6D14.4)
      RETURN
      END
```

```
      $ENTRY
  1.40       0.05 30000000.        386.           1.     2     2     1
497.20     180.00     136.00  1000000.
212.60     120.00      66.00  1000000.


      EIGENVALUES


SWEEP NUMBER IN *JACOBI* =    1
 0.13990D 03   0.10825D 04

SWEEP NUMBER IN *JACOBI* =    2
 0.13990D 03   0.10825D 04
 0.13990D 03   0.10825D 04

      EIGENVECTORS

 0.64370D-01  -0.56652D-01
 0.81323D-01   0.92402D-01


    THE DAMPING MATRIX IS

 0.00000D 00    0.00000D 00
 0.00000D 00    0.00000D 00
   0.00       0.28       1.00       0.28
```

THE RESPONSE IS

CORD.	TIME	DISPL.	VELOC.	ACC.
1	0.050	-0.1224	-4.6336	-83.6079
2	0.050	-0.1309	-5.1437	-102.7605
1	0.100	-0.4362	-7.4704	-46.8782
2	0.100	-0.4980	-9.1664	-67.0241
1	0.150	-0.8773	-10.3531	-73.7455
2	0.150	-1.0216	-11.4072	-11.6610
1	0.200	-1.4830	-13.7893	-69.8076
2	0.200	-1.6151	-12.5077	-19.7756
1	0.250	-2.2483	-16.5991	-39.8756
2	0.250	-2.2886	-14.8989	-81.4515
1	0.300	-3.1272	-18.5379	-30.0295
2	0.300	-3.1373	-19.0848	-101.7425
1	0.350	-4.1031	-20.7254	-56.7322
2	0.350	-4.1951	-22.7584	-46.7187
1	0.400	-5.2166	-23.9396	-79.4874
2	0.400	-5.3785	-24.3163	0.1706
1	0.450	-6.5039	-27.3728	-62.1527
2	0.450	-6.6126	-25.4228	-35.4875
1	0.500	-7.9395	-29.8347	-30.3152
2	0.500	-7.9504	-28.5310	-101.1538
1	0.550	-9.4735	-31.6143	-33.5414
2	0.550	-9.4943	-33.0450	-94.5060
1	0.600	-11.1095	-34.0913	-68.3686
2	0.600	-11.2372	-36.1220	-22.7409
1	0.650	-12.9011	-37.6070	-81.2444
2	0.650	-13.0684	-37.0584	3.7911
1	0.700	-14.8698	-40.8782	-50.8350
2	0.700	-14.9437	-38.4989	-58.8705
1	0.750	-16.9694	-42.9491	-23.1636
2	0.750	-16.9584	-42.4126	-115.8493
1	0.800	-19.1560	-44.7157	-42.0796
2	0.800	-19.2030	-46.9534	-76.9121
1	0.850	-21.4573	-47.5955	-79.8093
2	0.850	-21.6202	-49.2005	0.8340
1	0.900	-23.9323	-51.3127	-77.6543
2	0.900	-24.0886	-49.7218	-3.6065
1	0.950	-26.5665	-53.4820	-37.1971
2	0.950	-26.5985	-51.0615	-86.9729

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VITA

The author, Jose Enrique Carrasco, was born in La Paz, Bolivia on March 20, 1950. He is the son of Mario Carrasco Gumucio, Hydraulic Engineer and the late Victoria Valdivieso Guzman de Carrasco, Kindergarden Principal. He graduated from Israeli High School in La Paz, Bolivia in May, 1968. The same year he enrolled in the University "Tomas Frias" in Potosi, Bolivia. In 1973 he transferred to the University of Louisville, where he received the Bachelor of Science in May 1977. He was a project manager for a construction company in Sellersburg, Indiana and then in 1978 he completed his Master of Engineering with specialty in Civil Engineering (Structural Dynamics).

He is married to the former Gayle Jo Senger from Devils Lake, North Dakota. They have a daughter named Alexandra Victoria born on October 25, 1978.