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<https://doi.org/10.18297/etd/1237>

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A NEW PROCEDURE FOR CALCULATING
DIPOLAR SOURCES OF EEG DISCHARGES

By

Joseph Harvey Rothweiler

A Thesis
Submitted to the Faculty of the
University of Louisville
Speed Scientific School
as Partial Fulfillment of the Requirements
for the Professional Degree

MASTER OF ENGINEERING

Department of Electrical Engineering

December 1977

A New Procedure for Calculating
Dipolar Sources of EEG Discharges

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A Thesis Approved on

November 22, 1977

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ABSTRACT

A new procedure has been developed for determining the location of the source of certain EEG discharges given the measured surface potentials. The source is modeled as a single current dipole with arbitrary position and orientation, while the head is modeled either as a single homogeneous sphere or as three concentric spheres.

This procedure is developed by extending the matched filter concept of communications theory to functions defined on a spherical surface. In addition, simplifications are developed which reduce the amount of computation required. Simulations have been performed using computer generated data to show the validity of the locating procedure and to investigate its sensitivity to changes in source location.

It is shown that the new procedure has approximately the same accuracy as previously used techniques, and possible computational advantages of the new procedure are discussed.

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I. BACKGROUND AND SCOPE OF THESIS

A general problem which exists in electroencephalography and electrocardiography is to determine the relationship between the potentials on the surface of a volume conductor and the internal sources. The forward problem, finding the surface potentials given the source distribution, can be solved by applying electromagnetic theory to an electrical model of the biological system under study.¹

A difficulty in the inverse problem, determining the sources given the surface potentials, is the lack of uniqueness. It has been shown that the surface potentials do not provide enough information to completely specify the source distribution.² A way of dealing with this problem is to use an electrical model of the biological system in which the source structure is restricted in such a way that it is uniquely specified by the surface potentials. Some approximations which have been shown to be valid for most models of the volume conductor are that the medium is linear and that capacitive and inductive effects and propagation delays are negligible at the frequencies of interest.³ As a result, problems may be treated as if steady state conditions existed at any instant; and the static, rather than time varying, forms of the electro-

magnetic equations may be used.

Two models of the head are widely used in studies of the EEG. A simple model is a homogeneous sphere. A more complex model consists of three concentric spheres with different conductivities, representing the scalp, skull, and brain. It has been shown that the three spheres model is quite accurate in most cases.^{4,5}

It is desirable that the restrictions placed on the source structure be related to the physiological characteristics of the system being studied. In EEG investigations it has been shown that the source of the spike discharges observed in certain forms of epilepsy can be modeled as a single current dipole located in the brain.⁶ Studies of artificially induced epilepsy in animals have observed the formation of secondary sources of epileptic activity in the brain.⁷⁻¹⁰ Whether these secondary sources may also be modeled as current dipoles is not known.

The approach to the inverse problem used by Rogers and Pilkington^{11,12} for electrocardiographic applications is to model the source as a number of current dipoles which are fixed in position and orientation, but have variable magnitudes. Since the volume conductor is linear, the surface potential is a linear function of the dipole magnitudes. A dipole with random orientation may then be represented by resolving it into three component dipoles.

The relationship between the surface potentials at M locations and N dipole magnitudes is described as

follows. The M-dimensional column vector Φ consists of the observed electrode voltages. The electrode potentials due to the i'th dipole are modeled as $C_i p_i$, where p_i is the dipole magnitude, and the M-dimensional vector C_i is computed based on an electrical model of the system under study. The surface potential is then modeled as (eq. 7 of ref. 12):

$$\Phi = \sum_{j=1}^N C_j p_j + R \quad (I-1)$$

where the vector R represents an error term.

To estimate the dipole magnitudes, M is chosen greater than N; and a set of values of the p's is found which provides a best fit to the measured surface potentials using a least squares criterion. The p's are found by solving the set of equations (eq. 11 of ref. 12):

$$C_k^t \Phi = \sum_{j=1}^N C_k^t C_j p_j \quad (k=1, \dots, N), \quad (I-2)$$

where superscript (t) denotes the transpose. An algorithm to solve this set of equations is presented in reference 12.

The above equations may be written more concisely by defining a vector p to represent the set of dipole magnitudes, and by defining an M by N matrix C such that the columns of C are the vectors $C_1 - C_N$. The above set of equations may now be rewritten as

$$C^t C p = C^t \Phi \quad (I-3)$$

Assuming the columns of C are linearly independent,

the matrix $C^t C$ has an inverse; and the solution to this equation is

$$p = (C^t C)^{-1} C^t \bar{p} \quad . \quad (I-4)$$

The matrix $(C^t C)^{-1} C^t$ is the Penrose generalized inverse of C .¹³ According to Pierce,¹⁴ the generalized inverse gives an optimum estimate of p under the following conditions: the elements of the noise vector R have equal variance, the covariance of any two different elements is zero, and the noise variance is vanishingly small compared to the signal (Cp).

The effect of noise on the results is discussed in ref. 12, although not exhaustively. The main conclusion is that if the columns of C are nearly linearly dependent, the solution will be more sensitive to noise.

Other investigators have used similar models which include additional constraints. Barr, et al. used an on-off model in which the magnitude of each dipole is constrained to be either a fixed positive value or zero.^{15,16} Lynn, et al. used a model in which the dipole magnitude is constrained to be nonnegative.¹⁷ While both models have been applied experimentally, neither paper contains an analysis of the effects of noise on the results.

In studying epileptic EEG discharges, Schneider⁶ modeled the source as a single dipole with random position and orientation. The head was modeled as a homogeneous sphere. The procedure used to determine the dipole para-

meters is as follows: An initial guess is made for the three position and three magnitude components. A parameter to measure the degree of mismatch is defined, which may be written as

$$S = |W_m - W_c|^2 / |W_m|^2 \quad (I-5)$$

W_m is a column vector consisting of the measured electrode potentials, and W_c is a column vector of the theoretical potentials which would exist at the electrodes due to the assumed dipole.

W_m is constant for a particular experiment, and W_c may be calculated using an equation originally derived by Wilson and Bailey,¹⁸ so S may be written as a function of the six parameters of the assumed dipole. The differential of this function may be computed, and a new estimate of the dipole parameters is determined by adjusting them so as to decrease S . This iteration is repeated until S becomes less than a predetermined maximum.

Accuracy of the procedure was studied experimentally by taking several readings on the same patient, and by simulating a source with added noise. Schneider concludes that the process can locate a dipole within a volume of about two cubic centimeters.

Schneider reaches the following conclusions about the physical significance of the calculated dipole: EEG anomalies having a narrow distribution on the scalp usually result from a small anatomic lesion which can be found

surgically or with intracerebral recordings. In the case of anomalies with large distributions, a single dipole near the center of the brain provides a good representation in most cases, but this is probably only an electrical equivalent since high intensities have not been observed deep inside the brain. In some cases the error, S , does not approach zero after many iterations. Possible reasons for this lack of convergence are that a single dipole is not a good model of the actual source in these cases, or that there are conditions under which the algorithm does not converge.

In a later paper,¹⁹ Schneider demonstrates that the single sphere model of the head leads to significant errors, and that for better accuracy the three spheres model should be used.

SCOPE OF THESIS

In this thesis, a new procedure is presented for estimating the location and orientation of a current dipole source given the potentials on the surface of a spherical volume conductor. Original work to be described includes an extension of the theory of the matched filter to functions defined on a spherical surface and the subsequent application of this result to the problem of locating a dipole current source.

Simplifications of this locating procedure are also presented which result from resolving the dipole source into three components. The procedure, including simplifications, can be applied equally well to either the single sphere or three spheres model. The filtering procedure is developed assuming a single dipole is present, but it may be possible to apply the procedure with reduced accuracy if several are present.

Certain notational conventions have been adopted for use in this thesis. For simplicity, the integral of a function $f(\theta, \phi)$ over the surface of a sphere will be written as $\int f(\theta, \phi) dq$. It is to be understood that $dq = r_p^2 \sin\theta d\theta d\phi$, where r_p is the radius of the sphere, and that the limits of integration are $0 < \theta < \pi$ and $0 < \phi < 2\pi$. To distinguish between scalars and vectors in abstract

spaces, the former will be represented by lower case letters and the latter by capitals. Vectors in a three dimensional coordinate system will be represented by lower case letters with a bar ($\bar{}$) above them. Estimated values of parameters will be indicated by placing a caret ($\hat{}$) above them.

II. THE MATCHED FILTER DETECTION PROCESS

The basic theory of the matched filter will now be described and it will be shown that a matched filter can also be applied to two dimensional functions. It will then be shown that the matched filter can be applied to the problem of determining the location of a current dipole. Finally, some simplifications to make this locating procedure easier to implement will be described.

A problem which arises in communications is to design a filter which will maximize the probability of detecting a pulse in the presence of noise. If the pulse waveform is $s(t)$, the filter input may be written as $v(t) = s(t) + n(t)$, where n is random noise. If the impulse response of the filter is^{*} $h(-t)$, the filter output is $b(t) = \int_{-\infty}^{\infty} h(\tau - t)v(\tau)d\tau$, by the convolution integral. Assuming n is white noise, the filter which maximizes the output signal to noise ratio is the matched filter, the impulse response of which is defined by $h(t) = s(t)/c$, where c is an arbitrary constant.²⁰

The maximum output signal component is then

$$b(0) = \int_{-\infty}^{\infty} h(\tau)s(\tau)d\tau .$$

^{*}The normal form is $h(t)$. This form has been used to correspond more closely to the definition which will be used in the two dimensional case.

A. PRELIMINARY DEFINITIONS AND ASSUMPTIONS

Before deriving the matched filter for a spherical surface, some preliminary definitions and assumptions will be necessary. Let vector $\vec{r}$ represent a point in a three dimensional coordinate system. A sphere of radius r_r is then defined as the set of points for which $|\vec{r}| = r_r$. As an extension of the one dimensional case, the observed signal, v , on the spherical surface will be modeled as

$$v(\vec{r}) = s(\vec{r}) + n(\vec{r}) \quad , \quad |\vec{r}| = r_r \quad .$$

In spherical coordinates, $\vec{r}$ is specified by the set of parameters (r, θ, ϕ) , so that the observed potential may be written as

$$v(\theta, \phi) = s(\theta, \phi) + n(\theta, \phi) \quad , \quad (II-1)$$

since the coordinate r is constant over the surface.

It will be assumed throughout this thesis that all surface potentials are measured with a neutral point used as a reference. In practice, other measurement techniques are often used. These techniques and their possible effects on the accuracy of the estimated dipole parameters will be discussed in chapter V.

The function $v(\theta, \phi)$ represents the voltage observed at every point on the surface at a particular instant of time. The objective of the locating procedure is to find the current dipole which produces a surface

potential most nearly the same as $v(\theta, \phi)$. In this initial derivation it will be assumed that v is known at every point. Later, the more practical case in which the available information consists of a set of samples of v at specific points on the surface will be discussed.

The signal component of v , $s(\theta, \phi)$, is the theoretical surface potential of a current dipole with a specific location and orientation. Equations defining s for the single sphere and three spheres models are available in the literature and have been included in appendix A for reference. For a specific model, s is completely specified by six parameters defining the location, magnitude, and orientation of the dipole. One possible set of parameters consists of the three parameters (r, ψ, ω) defining the position and the parameters (α, β, d) defining the orientation and magnitude (see fig. 1). Alternatively, the orientation and magnitude may be implicitly specified by the magnitudes (d_r, d_θ, d_ϕ) of the dipole components in the directions of the $\bar{a}_r$, $\bar{a}_\theta$, and $\bar{a}_\phi$ unit vectors of the spherical coordinate system.

When it is desirable to show the dependence of the signal on the dipole parameters, s will be written as

$$s(\theta, \phi | r, \psi, \omega, \alpha, \beta, d) \quad , \text{or}$$

$$s(\theta, \phi | r, \psi, \omega, d_r, d_\theta, d_\phi) \quad ,$$

where the vertical bar is used to differentiate between the surface coordinates and the dipole parameters.

Since the volume conductor is modeled as linear,

FIGURE 1. The single sphere model of the human head. Coordinates θ and ϕ define the position of an element of area, dq , on the surface, while ψ , ω , and r define the dipole position. The dipole orientation is specified by α and β .

s is proportional to the dipole magnitude, d ; so if function s_d is the surface potential distribution which results from a dipole of unit magnitude,

$$s(\theta, \phi | r, \psi, \omega, \alpha, \beta, d) = ds_d(\theta, \phi | r, \psi, \omega, \alpha, \beta) \quad . \quad (\text{II-2})$$

In addition, using the principle of superposition, the surface potential will be the sum of contributions from the three components d_r , d_θ , and d_ϕ . Let s_r , s_θ , and s_ϕ be the surface potential functions resulting from unit magnitude dipoles oriented in the $\bar{a}_r$, $\bar{a}_\theta$, and $\bar{a}_\phi$ directions, respectively. Now, s can also be expressed as

$$\begin{aligned} s(\theta, \phi | r, \psi, \omega, d_r, d_\theta, d_\phi) &= d_r s_r(\theta, \phi | r, \psi, \omega) \\ &+ d_\theta s_\theta(\theta, \phi | r, \psi, \omega) + d_\phi s_\phi(\theta, \phi | r, \psi, \omega) \quad . \quad (\text{II-3}) \end{aligned}$$

The noise component of v , $n(\theta, \phi)$, represents all sources of error, including noise in the amplifiers, other electrical activity of the brain, and inaccuracies of the model.

Two types of delta functions will be needed at this point in order to describe the noise mathematically. The ordinary one dimensional delta will be represented by $\delta_1(t)$, while $\delta_2(\bar{r})$ will be a two dimensional delta function defined on the surface of a sphere of radius r_r .

As in the one dimensional case, the noise will be modeled as white. In one dimension, the noise is white if $n(t)$ is statistically stationary and has the properties

$$E[n(t)] = 0 \quad , \text{ and}$$

$$E[n(t_1)n(t_2)] = k^2 \delta(t_2 - t_1) \quad ,$$

where E represents the expectation operator, and k is a constant. For the spherical case, these properties become

$$E[n(\bar{r})] = 0 \quad , \text{ and}$$

$$E[n(\bar{r}_1)n(\bar{r}_2)] = k^2 \delta_s(\bar{r}_2 - \bar{r}_1) \quad . \quad (\text{II-4})$$

Care must be taken in expressing the delta function in spherical coordinates.

In any coordinate system delta must have the following properties:²¹

$$\delta_s(\bar{r} - \bar{r}_0) = 0 \text{ if } \bar{r} \neq \bar{r}_0 \quad , \text{ and}$$

$$\int \delta_s(\bar{r} - \bar{r}_0) dq = 1 \quad ,$$

where $\bar{r}_0$ is a point on the sphere, and $\bar{r}$ is the position of dq . In spherical coordinates this integral becomes

$$\int_0^{2\pi} \int_0^{\pi} \delta(\bar{r} - \bar{r}_0) r_r^2 \sin\theta d\theta d\phi \quad .$$

It is easily seen that the value of this integral is one if

$$\delta_s(\bar{r}-\bar{r}_0) = \frac{\delta_1(\theta-\theta_0)\delta_1(\phi-\phi_0)}{r_r^2 \sin\theta} \quad (\text{II-5})$$

The properties assumed for the noise can then be written as

$$E[n(\bar{r})] = 0 \quad , \text{ and} \quad (\text{II-6a})$$

$$\begin{aligned} E[n(\bar{r}_1)n(\bar{r}_2)] &= k^2 \delta_s(\bar{r}_2-\bar{r}_1) \\ &= k^2 \frac{\delta_1(\theta_2-\theta_1)\delta_1(\phi_2-\phi_1)}{r_r^2 \sin\theta} \quad . \end{aligned} \quad (\text{II-6b})$$

Using this definition the sifting property of the delta function also holds in spherical coordinates. That is, for any function $f(\bar{r})$ the following relationship holds:

$$\int f(\bar{r}) \delta_s(\bar{r}-\bar{r}_0) dq = f(\bar{r}_0) \quad . \quad (\text{II-7})$$

This relationship can easily be proven by writing the integral as

$$\int_0^{2\pi} \int_0^\pi f(\theta, \phi) \frac{\delta_1(\theta-\theta_0)\delta_1(\phi-\phi_0)}{r_r^2 \sin\theta} r_r^2 \sin\theta d\theta d\phi \quad ,$$

which reduces to

$$\int_0^{2\pi} \int_0^\pi f(\theta, \phi) \delta_1(\theta-\theta_0)\delta_1(\phi-\phi_0) d\theta d\phi \quad .$$

Two applications of the sifting property of the one dimensional delta function produce the desired result.

B. DERIVATION OF THE SPHERICAL MATCHED FILTER

The matched filter may be derived for a spherical surface using a method similar to that applied in the one dimensional case.²⁰ The filter output is

$$b = \int h(\theta, \phi) v(\theta, \phi) dq, \quad (\text{II-8})$$

which consists of a signal component

$$p = \int h(\theta, \phi) s(\theta, \phi) dq \quad (\text{II-9})$$

and a noise component

$$u = \int h(\theta, \phi) n(\theta, \phi) dq. \quad (\text{II-10})$$

The matched filter problem is to maximize the output peak signal to average noise power ratio $\eta = p^2/E(u^2)$. This problem is equivalent to maximizing the ratio $p^2/[wE(u^2)]$; where w equals $\int s^2 dq$, the total energy* in the signal, which is a fixed value for a particular instant of time.

To demonstrate that the matched filter (a filter whose impulse response has the same shape as the signal) maximizes η , it will be necessary to evaluate $E(u^2)$. This quantity may be evaluated as shown in ref. 22, p. 324. Let $u_1 = \int n(\bar{r}) h_1(\bar{r}) dq$ and $u_2 = \int n(\bar{r}) h_2(\bar{r}) dq$. Then if dq_1 is the element of area

*The term energy as used here is the integral of voltage squared (power) over the surface, which is analogous to the definition for time signals as the integral of voltage squared over time.

at point $\bar{r}_1$,

$$\begin{aligned} E(u_1 u_2) &= E \left[\int n(\bar{r}_1) h_1(\bar{r}_1) dq_1 \int n(\bar{r}_2) h_2(\bar{r}_2) dq_2 \right] \\ &= E \left\{ \left[\int n(\bar{r}_1) n(\bar{r}_2) h_1(\bar{r}_1) dq_1 \right] h_2(\bar{r}_2) dq_2 \right\} . \end{aligned}$$

Taking the expected value inside the integral, and assuming white noise, the equation becomes

$$E(u_1 u_2) = k^2 \int \left[\int \delta(\bar{r}_2 - \bar{r}_1) h_1(\bar{r}_1) dq_1 \right] h_2(\bar{r}_2) dq_2 .$$

Applying the sifting property of the delta function, this equation reduces to

$$E(u_1 u_2) = k^2 \int h_1(\bar{r}_2) h_2(\bar{r}_2) dq_2 . \quad (\text{II-11})$$

If $u_1 = u_2 = u$, this equation reduces to

$$E(u^2) = k^2 \int h^2(\bar{r}) dq . \quad (\text{II-12})$$

Now,

$$\frac{\eta}{w} = \frac{p^2}{w E(u^2)} = \frac{[\int h s dq]^2}{[\int s^2 dq] [k^2 \int h^2 dq]} .$$

Schwarz's inequality states that $[\int h s dq]^2 \leq \int s^2 dq \int h^2 dq$.

Substituting this inequality in the above numerator and cancelling terms gives $\eta/w \leq 1/k^2$, or $\eta \leq w/k^2$.

Equality holds in Schwarz's inequality, and η is maximum, only when

$$h(\theta, \phi) = s(\theta, \phi)/c . \quad (\text{II-13})$$

Therefore, as in the one dimensional case, the matched filter produces the greatest possible value of the output signal to noise ratio, η .

This maximum value of η is then

$$\begin{aligned} \eta_m &= (w/k^2) = (1/k^2) \int [c h(\theta, \phi)]^2 dq \\ &= (c^2/k^2) \int h^2(\theta, \phi) dq . \end{aligned} \quad (\text{II-14})$$

To apply the matched filter to the problem of locating a current dipole, it will be convenient to define a real vector space consisting of functions of θ and ϕ , with the inner product of two elements h and v defined as $\langle h|v \rangle = \int h v d\mathbf{q}$. The cosine of the angle between two elements, γ , is then defined as²³

$$\gamma = \left[\int h v d\mathbf{q} \right] / \left[\int h^2 d\mathbf{q} \int v^2 d\mathbf{q} \right]^{1/2}. \quad (\text{II-15a})$$

Note that Schwarz's inequality implies $|\gamma| \leq 1$. (II-15b)

Using the subscript 0 to indicate the location and orientation coordinates of the actual dipole, the surface potential is

$$v(\theta, \phi) = s_d(\theta, \phi | r_0, \psi_0, \omega_0, \alpha_0, \beta_0) + n(\theta, \phi). \quad (\text{II-16})$$

The filter function h will be chosen proportional to the theoretical surface potential of an arbitrary dipole:

$$h(\theta, \phi | r, \psi, \omega, \alpha, \beta) = s_d(\theta, \phi | r, \psi, \omega, \alpha, \beta) / c(r, \psi, \omega, \alpha, \beta) \quad (\text{II-17})$$

The relationship between h and s_d will be defined such that

$$\int h^2(\theta, \phi) d\mathbf{q} = 1, \quad (\text{II-18})$$

and therefore

$$c(r, \psi, \omega, \alpha, \beta) = \left[\int s_d^2(\theta, \phi | r, \psi, \omega, \alpha, \beta) d\mathbf{q} \right]^{1/2}. \quad (\text{II-19})$$

As a result of this definition

$$\gamma = \frac{\int h(\theta, \phi) v(\theta, \phi) d\mathbf{q}}{\left[\int v^2(\theta, \phi) d\mathbf{q} \right]^{1/2}}.$$

Since v has a random component, γ is a random process

which depends on the parameters of the arbitrary dipole:

$$\gamma(r, \psi, \omega, \alpha, \beta) = \frac{\int h(\theta, \phi | r, \psi, \omega, \alpha, \beta) v(\theta, \phi) dq}{[\int v^2(\theta, \phi) dq]^{\frac{1}{2}}} \quad (\text{II-20})$$

If the assumption is made that the noise is negligible, $v = s = ds_d$, and γ reduces to the deterministic function

$$\gamma(r, \psi, \omega, \alpha, \beta) = \frac{\int h(\theta, \phi | r, \psi, \omega, \alpha, \beta) s_d(\theta, \phi | r_0, \psi_0, \omega_0, \alpha_0, \beta_0) dq}{c(r_0, \psi_0, \omega_0, \alpha_0, \beta_0)} \quad (\text{II-21})$$

Now h and s_d will be linearly dependent, and therefore γ will equal one, when the five parameters of the assumed dipole correspond to those of the actual dipole; that is, γ is one when $r=r_0$, $\psi=\psi_0$, $\omega=\omega_0$, $\alpha=\alpha_0$, and $\beta=\beta_0$. The dipole position and orientation may therefore be found by finding the set of parameter values $(r, \psi, \omega, \alpha, \beta)$ for which γ attains its maximum value of one.

If the noise is small but not negligible, it is to be expected that a maximum value of γ will still occur when its arguments are near the actual dipole position and orientation. The set of parameters which maximizes γ then provides an estimate of the actual dipole parameters.

The cosine, γ , is proportional to the matched filter output since $\int v^2 dq$ is constant for a particular experiment and $\int h^2 dq$ is constant by definition. Combining equations II-8 and II-20,

$$\begin{aligned}
 b(r, \psi, \omega, \alpha, \beta) &= \int h(\theta, \phi | r, \psi, \omega, \alpha, \beta) v(\theta, \phi) dq \\
 &= \gamma(r, \psi, \omega, \alpha, \beta) \left[\int v^2(\theta, \phi) dq \right]^{\frac{1}{2}} \quad (\text{II-22})
 \end{aligned}$$

For purposes of determining the dipole location and orientation, maximizing b is therefore equivalent to maximizing γ .

C. SEARCH PROCEDURE DEVELOPMENT

The straightforward approach to determining the source position and orientation would be to divide the range of each dipole parameter into segments, evaluate b for all possible combinations, and select the set of values for which b is a maximum. However, since five parameters are involved, the number of calculations required becomes prohibitive if the increments are chosen small enough to obtain reasonable accuracy.

The number of calculations required may be reduced by modifying the above algorithm as follows. For a specific set of position coordinates $(r_1, \gamma_1, \omega_1)$, find the set of orientation parameters which maximizes b . Repeat the procedure for all locations in the sphere and select the location and corresponding orientation for which b is a maximum. The advantage of this algorithm is that, as will now be shown, the desired orientation for a given location may be determined with a small amount of computation, and without using an iterative procedure.

In eq. II-17, the filter response function, h , was defined to be proportional to the surface potential as expressed in eq. II-2. However, in the derivation

which follows, an equivalent definition based on eq. II-3 will be more convenient.

Repeating eq. II-3,

$$s(\theta, \phi | r, \psi, \omega, d_r, d_\theta, d_\phi) = d_r s_r(\theta, \phi | r, \psi, \omega) + d_\theta s_\theta(\theta, \phi | r, \psi, \omega) + d_\phi s_\phi(\theta, \phi | r, \psi, \omega) .$$

The matched filter will now be defined proportional to the dipole surface potential as defined in eq. II-3.

$$h(\theta, \phi | r, \psi, \omega, d_r, d_\theta, d_\phi) = [d_r s_r(\theta, \phi | r, \psi, \omega) + d_\theta s_\theta(\theta, \phi | r, \psi, \omega) + d_\phi s_\phi(\theta, \phi | r, \psi, \omega)] / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-24a})$$

Since the function s_d , which forms the numerator of eq. II-17, is defined to be the surface potential produced by a dipole with unit magnitude, the numerator of eq. II-24a is equal to s_d if

$$[d_r^2 + d_\theta^2 + d_\phi^2]^{\frac{1}{2}} = 1 . \quad (\text{II-24b})$$

To determine the value of c , square both sides of the equation and integrate over the surface, giving

$$\begin{aligned} \int h^2(\theta, \phi | r, \psi, \omega, d_r, d_\theta, d_\phi) dq = & \left[d_r^2 \int s_r^2(\theta, \phi | r, \psi, \omega) dq \right. \\ & + 2d_r d_\theta \int s_r(\theta, \phi | r, \psi, \omega) s_\theta(\theta, \phi | r, \psi, \omega) dq \\ & + 2d_r d_\phi \int s_r(\theta, \phi | r, \psi, \omega) s_\phi(\theta, \phi | r, \psi, \omega) dq \\ & + d_\theta^2 \int s_\theta^2(\theta, \phi | r, \psi, \omega) dq \\ & + 2d_\theta d_\phi \int s_\theta(\theta, \phi | r, \psi, \omega) s_\phi(\theta, \phi | r, \psi, \omega) dq \\ & \left. + d_\phi^2 \int s_\phi^2(\theta, \phi | r, \psi, \omega) dq \right] / c^2(r, \psi, \omega, d_r, d_\theta, d_\phi) . \quad (\text{II-25}) \end{aligned}$$

In appendix A, it is shown that the surface potentials produced by the three component dipoles are mutually orthogonal. As a result, the cross terms of

the above equation are equal to zero, and this equation reduces to

$$\int h^2(\theta, \phi | r, \psi, \omega, d_r, d_\theta, d_\phi) dq = \left[d_r^2 \int s_r^2(\theta, \phi | r, \psi, \omega) dq + d_\theta^2 \int s_\theta^2(\theta, \phi | r, \psi, \omega) dq + d_\phi^2 \int s_\phi^2(\theta, \phi | r, \psi, \omega) dq \right] / c^2(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (II-26)$$

The left side of this equation is equal to one, by equation II-18; so, making this substitution in eq. II-26 and rearranging terms gives

$$c(r, \psi, \omega, d_r, d_\theta, d_\phi) = \left[d_r^2 \int s_r^2(\theta, \phi | r, \psi, \omega) dq + d_\theta^2 \int s_\theta^2(\theta, \phi | r, \psi, \omega) dq + d_\phi^2 \int s_\phi^2(\theta, \phi | r, \psi, \omega) dq \right]^{\frac{1}{2}} \quad (II-27)$$

For later use, a set of three filters will be defined, one matched to each component of s . Three constants, c_r , c_θ , and c_ϕ , will first be defined as follows:

$$c_r(r, \psi, \omega) = \left[\int s_r^2(\theta, \phi | r, \psi, \omega) dq \right]^{\frac{1}{2}} \quad (II-28a)$$

$$c_\theta(r, \psi, \omega) = \left[\int s_\theta^2(\theta, \phi | r, \psi, \omega) dq \right]^{\frac{1}{2}} \quad (II-28b)$$

$$c_\phi(r, \psi, \omega) = \left[\int s_\phi^2(\theta, \phi | r, \psi, \omega) dq \right]^{\frac{1}{2}} \quad (II-28c)$$

The three filter response functions h_r , h_θ , and h_ϕ , which are matched to the $\bar{a}_r$, $\bar{a}_\theta$, and $\bar{a}_\phi$ dipole components, respectively, may now be defined as

$$h_r(\theta, \phi | r, \psi, \omega) = s_r(\theta, \phi | r, \psi, \omega) / c_r(r, \psi, \omega) \quad (II-29a)$$

$$h_\theta(\theta, \phi | r, \psi, \omega) = s_\theta(\theta, \phi | r, \psi, \omega) / c_\theta(r, \psi, \omega) \quad (II-29b)$$

$$h_\phi(\theta, \phi | r, \psi, \omega) = s_\phi(\theta, \phi | r, \psi, \omega) / c_\phi(r, \psi, \omega) \quad (II-29c)$$

With c_r , c_θ , and c_ϕ as defined in eq. II-28, it is obvious

that these response functions conform to the constraint, expressed in eq. II-18, that the integral of the square of each response function be equal to unity.

By substituting equations II-28 into II-27, that equation becomes

$$c(r, \psi, \omega, d_r, d_\theta, d_\phi) = \left[d_r^2 c_r^2(r, \psi, \omega) + d_\theta^2 c_\theta^2(r, \psi, \omega) + d_\phi^2 c_\phi^2(r, \psi, \omega) \right]^{\frac{1}{2}}. \quad (\text{II-30})$$

Next, define the variables

$$d'_r = d_r c_r(r, \psi, \omega) / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-31a})$$

$$d'_\theta = d_\theta c_\theta(r, \psi, \omega) / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-31b})$$

$$d'_\phi = d_\phi c_\phi(r, \psi, \omega) / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-31c})$$

Solving these equations for d_r , d_θ , and d_ϕ , substituting into eq. II-30, and dividing by c , it is found that

$$(d'_r)^2 + (d'_\theta)^2 + (d'_\phi)^2 = 1 \quad (\text{II-32})$$

The relationship between the filter response function, h , and the three component response functions may be determined by solving equations II-29 for s_r , s_θ , and s_ϕ , solving equations II-31 for d_r , d_θ , and d_ϕ , and substituting into equation II-24. After cancelling terms, it is found that

$$h(\theta, \phi | r, \psi, \omega, d_r, d_\theta, d_\phi) = d'_r h_r(\theta, \phi | r, \psi, \omega) + d'_\theta h_\theta(\theta, \phi | r, \psi, \omega) + d'_\phi h_\phi(\theta, \phi | r, \psi, \omega) \quad (\text{II-33})$$

To determine the dipole orientation which maximizes the filter output for a given position, combine

eq. II-8 and eq. II-33, to obtain

$$\begin{aligned}
 b(r, \psi, \omega, d_r, d_\theta, d_\phi) = & \\
 & d_r' \int h_r(\theta, \phi | r, \psi, \omega) v(\theta, \phi) dq \\
 & + d_\theta' \int h_\theta(\theta, \phi | r, \psi, \omega) v(\theta, \phi) dq \\
 & + d_\phi' \int h_\phi(\theta, \phi | r, \psi, \omega) v(\theta, \phi) dq . \quad (II-34)
 \end{aligned}$$

This result may be thought of as the inner product of the two vectors

$$D' = [d_r', d_\theta', d_\phi']^t \quad (II-35)$$

and

$$B = \begin{bmatrix} b_r \\ b_\theta \\ b_\phi \end{bmatrix} = \begin{bmatrix} \int h_r(\theta, \phi) v(\theta, \phi) dq \\ \int h_\theta(\theta, \phi) v(\theta, \phi) dq \\ \int h_\phi(\theta, \phi) v(\theta, \phi) dq \end{bmatrix} . \quad (II-36)$$

For a given function $v(\theta, \phi)$ and set of coordinates (r, ψ, ω) , the values of b_r , b_θ , and b_ϕ may be calculated using eq. II-36; and the magnitude of vector D' is one, from eq. II-32. Therefore, by Schwarz's inequality, b in eq. II-34 is maximum when vectors D' and B are linearly dependent. That is, the value of D' which maximizes the filter output, b , for an assumed dipole position is

$$D'_m = gB ,$$

where g is a constant of proportionality. Since

$$|D'_m| = 1 , \quad g|B| = 1, \text{ or } g = 1/|B| . \text{ So,}$$

$$D'_m = B/|B| . \quad (II-37)$$

Substituting this result into eq. II-34, the maximum filter output, b_m , for an assumed dipole location (r, ψ, ω) is

$$b_m(r, \psi, \omega) = D_m' \cdot B = (B \cdot B) / |B| = |B| \quad (II-38)$$

Equation II-38 gives in closed form the maximum value of b over the parameters α and β with r, ψ , and ω fixed. To determine the dipole location it is necessary to maximize b over the parameters r, ψ , and ω also. While this operation could be performed using an iterative procedure like that used by Schneider, the maximum can also be approximated by evaluating b_m for discrete points in the sphere and selecting the point for which b_m is a maximum.

After the dipole locations are determined, the magnitudes of the component dipoles may be calculated using equations which will now be derived.

Substitute $v = s + n$ into eq. II-36 with the signal, s , represented as in eq. II-3. If the actual dipole location is $\bar{r}_0 = (r_0, \psi_0, \omega_0)$, the equation for b_r at that point now becomes

$$b_r(\bar{r}_0) = \int h_r(\theta, \phi | \bar{r}_0) [d_r s_r(\theta, \phi | \bar{r}_0) + d_\theta s_\theta(\theta, \phi | \bar{r}_0) + d_\phi s_\phi(\theta, \phi | \bar{r}_0) + n(\theta, \phi)] dq \quad (II-39)$$

Since, as previously stated, the surface potentials due to the three component dipoles are mutually orthogonal, this equation reduces to

$$b_r(\bar{r}_0) = d_r \int h_r(\theta, \phi | \bar{r}_0) s_r(\theta, \phi | \bar{r}_0) dq + \int h_r(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq \quad (II-40)$$

Substituting eq. II-28 for s_r and solving for d_r yields:

$$d_r = \frac{b_r(\bar{r}_0)}{c_r(\bar{r}_0)} - \frac{\int h_r(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq}{c_r(\bar{r}_0)} \quad (II-41)$$

In the absence of noise, the ratio of b_r to c_r

gives the correct value of d_r . When the noise is non-zero, this ratio provides an estimate of d_r with the error given by the last term of eq. II-41.

The estimated values of the other two components may be obtained in a similar manner, giving

$$\hat{d}_r = b_r(\bar{r}_0)/c_r(\bar{r}_0) \quad (\text{II-42a})$$

$$\hat{d}_\theta = b_\theta(\bar{r}_0)/c_\theta(\bar{r}_0) \quad (\text{II-42b})$$

$$\hat{d}_\phi = b_\phi(\bar{r}_0)/c_\phi(\bar{r}_0) \quad (\text{II-42c})$$

The error in each component is then*

$$(\hat{d}_r - d_r) = [\int h_r(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq] / c_r(\bar{r}_0) \quad (\text{II-43a})$$

$$(\hat{d}_\theta - d_\theta) = [\int h_\theta(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq] / c_\theta(\bar{r}_0) \quad (\text{II-43b})$$

$$(\hat{d}_\phi - d_\phi) = [\int h_\phi(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq] / c_\phi(\bar{r}_0) \quad (\text{II-43c})$$

In summary, the simplified locating procedure consists of the following operations:

- (1) Divide the range of each of the position coordinates r, θ, ϕ into segments.
- (2) For each set of position coordinates evaluate b_r, b_θ, b_ϕ , and b_m using equations II-36 and II-38.
- (3) Compare the resulting values of b_m and select the set of coordinates which produced the largest magnitude. This point is the estimated dipole position.
- (4) The component magnitudes may then be calculated using eq. II-42.

*These equations assume the dipole location is known exactly. In general, the noise can also cause error in the estimated dipole position which will in turn cause more error in the magnitudes calculated with eq. II-42.

D. DISCRETE CASE

In a practical application the surface potential, v , is known only at a finite set of points, corresponding to the electrode positions. The matched filter detection procedure will therefore be developed for the discrete case also.

From equation II-1, the surface potentials at m electrode locations are

$$v(\theta_1, \phi_1) = s(\theta_1, \phi_1) + n(\theta_1, \phi_1), \quad i = 1, \dots, m.$$

In addition, from eq. II-2,

$$s(\theta_1, \phi_1 | r, \psi, \omega, \alpha, \beta, d) = ds_d(\theta_1, \phi_1 | r, \psi, \omega, \alpha, \beta), \quad i=1, \dots, m.$$

Letting V , S , S_d , and N be m dimensional column vectors corresponding to functions v , s , s_d , and n , respectively, these two sets of equations may be expressed as

$$V = S + N \tag{II-44a}$$

$$S(r, \psi, \omega, \alpha, \beta, d) = dS_d(r, \psi, \omega, \alpha, \beta) \tag{II-44b}$$

For this vector space, eq. II-15 for the cosine of the angle between two vectors, H and V , is replaced by

$$\gamma = \frac{H^T V}{|H| |V|} \quad ; \quad |\gamma| \leq 1. \tag{II-45}$$

As in the continuous case, applying Schwarz's inequality, γ is equal to one when H is proportional to S , assuming that the noise is negligible. The filter response vector, H , for a dipole with a given magnitude and orientation

is therefore proportional to the surface potential vector, S , of that dipole. Corresponding to eq. II-18, the constant of proportionality will be chosen such that

$$|H| = 1 \quad . \quad (II-46)$$

Therefore, define the constant c for the discrete case as

$$c(r, \psi, \omega, \alpha, \beta) = |S_d(r, \psi, \omega, \alpha, \beta)| \quad . \quad (II-47a)$$

Eq. II-46 is then satisfied if H is defined as

$$H(r, \psi, \omega, \alpha, \beta) = S_d(r, \psi, \omega, \alpha, \beta) / c(r, \psi, \omega, \alpha, \beta) \quad . \quad (II-47b)$$

Eq. II-45 then becomes

$$\gamma = H^t v / |v| \quad . \quad (II-48)$$

As in the continuous case, the magnitude of v is constant for a given set of data, so the matched filter output, b , may be defined as the numerator of eq. II-48. Therefore, eq. II-22 for the discrete case becomes

$$\begin{aligned} b(r, \psi, \omega, \alpha, \beta) &= [H(r, \psi, \omega, \alpha, \beta)]^t v \\ &= \gamma(r, \psi, \omega, \alpha, \beta) |v| \quad . \end{aligned} \quad (II-49)$$

The simplifications developed in section C may be applied to the discrete case as well. Equations II-23 through II-25 of section C are replaced by the following:

$$\begin{aligned} S(r, \psi, \omega, d_r, d_\theta, d_\phi) &= d_r S_r(r, \psi, \omega) \\ &+ d_\theta S_\theta(r, \psi, \omega) + d_\phi S_\phi(r, \psi, \omega) \end{aligned} \quad (II-50)$$

$$\begin{aligned} H(r, \psi, \omega, d_r, d_\theta, d_\phi) &= [d_r S_r(r, \psi, \omega) + d_\theta S_\theta(r, \psi, \omega) \\ &+ d_\phi S_\phi(r, \psi, \omega)] / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \end{aligned} \quad (II-51)$$

$$H^t H = \left[d_r^2 s_r^t s_r + 2d_r d_\theta s_r^t s_\theta + 2d_r d_\phi s_r^t s_\phi + d_\theta^2 s_\theta^t s_\theta + 2d_\theta d_\phi s_\theta^t s_\phi + d_\phi^2 s_\phi^t s_\phi \right] / c^2 \quad (\text{II-52})$$

At this point a possible problem arises. Equation II-26 utilized the fact that the functions h_r , h_θ , and h_ϕ are mutually orthogonal over the surface. In the discrete case, there is no guarantee that H_r , H_θ , and H_ϕ will be exactly mutually orthogonal. It is, however, to be expected that these three vectors will be approximately orthogonal if a large number of electrodes are used and they are evenly distributed over the surface.

Based on the assumption that H_r , H_θ , and H_ϕ are orthogonal, equations II-26 through II-38 may be replaced by:

$$H^t H(r, \psi, \omega, d_r, d_\theta, d_\phi) = \left[d_r^2 s_r^t s_r(r, \psi, \omega) + d_\theta^2 s_\theta^t s_\theta(r, \psi, \omega) + d_\phi^2 s_\phi^t s_\phi(r, \psi, \omega) \right] / c^2(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-53})$$

$$c(r, \psi, \omega, d_r, d_\theta, d_\phi) = \left[d_r^2 s_r^t s_r + d_\theta^2 s_\theta^t s_\theta + d_\phi^2 s_\phi^t s_\phi \right]^{1/2} \quad (\text{II-54})$$

$$c_r(r, \psi, \omega) = \left[s_r^t(r, \psi, \omega) s_r(r, \psi, \omega) \right]^{1/2} \quad (\text{II-55a})$$

$$c_\theta(r, \psi, \omega) = \left[s_\theta^t(r, \psi, \omega) s_\theta(r, \psi, \omega) \right]^{1/2} \quad (\text{II-55b})$$

$$c_\phi(r, \psi, \omega) = \left[s_\phi^t(r, \psi, \omega) s_\phi(r, \psi, \omega) \right]^{1/2} \quad (\text{II-55c})$$

$$H_r(r, \psi, \omega) = s_r(r, \psi, \omega) / c_r(r, \psi, \omega) \quad (\text{II-56a})$$

$$H_\theta(r, \psi, \omega) = s_\theta(r, \psi, \omega) / c_\theta(r, \psi, \omega) \quad (\text{II-56b})$$

$$H_\phi(r, \psi, \omega) = s_\phi(r, \psi, \omega) / c_\phi(r, \psi, \omega) \quad (\text{II-56c})$$

$$c(r, \psi, \omega, d_r, d_\theta, d_\phi) = [d_r^2 c_r^2(r, \psi, \omega) + d_\theta^2 c_\theta^2(r, \psi, \omega) + d_\phi^2 c_\phi^2(r, \psi, \omega)]^{\frac{1}{2}} \quad (\text{II-57})$$

$$d_r' = d_r c_r(r, \psi, \omega) / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-58a})$$

$$d_\theta' = d_\theta c_\theta(r, \psi, \omega) / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-58b})$$

$$d_\phi' = d_\phi c_\phi(r, \psi, \omega) / c(r, \psi, \omega, d_r, d_\theta, d_\phi) \quad (\text{II-58c})$$

$$(d_r')^2 + (d_\theta')^2 + (d_\phi')^2 = 1 \quad (\text{II-59})$$

$$H(r, \psi, \omega, d_r, d_\theta, d_\phi) = d_r' H_r(r, \psi, \omega) + d_\theta' H_\theta(r, \psi, \omega) + d_\phi' H_\phi(r, \psi, \omega) \quad (\text{II-60})$$

$$b(r, \psi, \omega, d_r, d_\theta, d_\phi) = d_r' H_r^t(r, \psi, \omega) v + d_\theta' H_\theta^t(r, \psi, \omega) v + d_\phi' H_\phi^t(r, \psi, \omega) v \quad (\text{II-61})$$

$$D' = [d'_r, d'_\theta, d'_\phi]^t \quad (\text{II-62})$$

$$B = \begin{bmatrix} b_r \\ b_\theta \\ b_\phi \end{bmatrix} = \begin{bmatrix} H_r^t(r, \psi, \omega) V \\ H_\theta^t(r, \psi, \omega) V \\ H_\phi^t(r, \psi, \omega) V \end{bmatrix} \quad (\text{II-63})$$

$$D'_m = B/|B| \quad (\text{II-64})$$

The maximum output for an assumed location (r, ψ, ω) is again

$$b_m(r, \psi, \omega) = D'_m \cdot B = |B| \quad (\text{II-65})$$

In the continuous case the approximate dipole location was determined by finding the set of location coordinates which maximized b_m in eq. II-38. Similarly, in the discrete case the location is determined by finding the set of coordinates which maximizes b_m in eq. II-65.

To determine the magnitudes of the dipole components, equations II-39 through II-43 may be revised for the discrete case as follows:

$$b_r(\bar{r}_0) = H_r^t(\bar{r}_0) [d_r S_r(\bar{r}_0) + d_\theta S_\theta(\bar{r}_0) + d_\phi S_\phi(\bar{r}_0) + N] \quad (\text{II-66})$$

$$b_r(\bar{r}_0) = d_r H_r^t(\bar{r}_0) S_r(\bar{r}_0) + H_r^t(\bar{r}_0) N \quad (\text{II-67})$$

$$d_r(\bar{r}_0) = \frac{b_r(\bar{r}_0)}{c_r(\bar{r}_0)} - \frac{H_r^t(\bar{r}_0) N}{c_r(\bar{r}_0)} \quad (\text{II-68})$$

The estimated component values are

$$\hat{d}_r = b_r(\bar{r}_0)/c_r(\bar{r}_0) \quad (\text{II-69a})$$

$$\hat{d}_\theta = b_\theta(\bar{r}_0)/c_\theta(\bar{r}_0) \quad (\text{II-69b})$$

$$\hat{d}_\phi = b_\phi(\bar{r}_0)/c_\phi(\bar{r}_0) \quad (\text{II-69c})$$

and the error components due to noise are

$$(\hat{d}_r - d_r) = H_r^t(\bar{r}_0)N/c_r(\bar{r}_0) \quad (\text{II-70a})$$

$$(\hat{d}_\theta - d_\theta) = H_\theta^t(\bar{r}_0)N/c_\theta(\bar{r}_0) \quad (\text{II-70b})$$

$$(\hat{d}_\phi - d_\phi) = H_\phi^t(\bar{r}_0)N/c_\phi(\bar{r}_0) \quad (\text{II-70c})$$

E. RELATIVE ACCURACY

In this section, the matched filter procedure will be compared to the procedure used by Schneider, which was described in Chapter I. As stated in section II-A, it will be assumed that the electrode potentials are measured with respect to a neutral reference. With this restriction, it will be shown that the estimated dipole location and orientation determined by these two procedures will be the same.

The procedure used by Schneider is to find the set of dipole location, orientation, and magnitude parameters which minimize the quantity (eq. I-5):

$$\zeta = |w_m - w_c|^2 / |w_m|^2 \quad (\text{II-71})$$

The matched filter procedure determines the dipole location and orientation by finding the set of parameters which maximizes

$$\gamma = v^t H / |v| \quad (\text{II-72})$$

It will be important to note that minimizing ζ determines the dipole location, orientation and magnitude, while minimizing γ determines only the dipole location and orientation. A further step, defined by eq. II-69, is required to determine the dipole magnitude using the matched filter procedure.

The vector W_m in eq. II-71 is the set of measured electrode potentials; vector W_m is therefore equivalent to vector V in eq. II-72. Vector W_c is the set of theoretical surface potentials produced by a dipole with arbitrary location, orientation, and magnitude. This vector is equivalent to the vector S , as defined in eq. II-44b. Therefore, using eq. II-44b,

$$W_c = S(r, \psi, \omega, \alpha, \beta, d) , \quad (II-73)$$

or

$$W_c = dS_d(r, \psi, \omega, \alpha, \beta) , \quad (II-74)$$

or, using eq. II-47b,

$$W_c = dc(r, \psi, \omega, \alpha, \beta)H(r, \psi, \omega, \alpha, \beta) . \quad (II-75)$$

In this analysis, it will be convenient to use a geometrical interpretation of these quantities. Vectors V (or W_m) and W_c may be thought of as points in an m dimensional space, where m is the number of electrodes. The unit vector H will be considered to be the direction of a line passing through the origin. Eq. II-75 then indicates that the point W_c lies on this line at a distance dc from the origin.

By substituting V for W_m and eq. II-75 for W_c , eq. II-71 becomes

$$\zeta = \frac{|V - dcH|^2}{|V|^2} \quad (II-76)$$

Since V does not change during the locating process, ζ can be minimized by minimizing only the numerator

of eq. II-76. In addition, since the magnitude of a vector is nonnegative, the squaring operation can also be omitted from the numerator. Minimizing ζ is therefore equivalent to minimizing the quantity

$$\lambda(r, \psi, \omega, \alpha, \beta, d) = |V - dc(r, \psi, \omega, \alpha, \beta)H(r, \psi, \omega, \alpha, \beta)| \quad (\text{II-77})$$

which is the distance from point V to a point, W_c , which lies on the line defined by H.

Let V be expressed as the sum of two components: V_h , which lies in the direction of H; and V_p , which is orthogonal to H. Thus,

$$V = V_h + V_p \quad . \quad (\text{II-78})$$

Since H is a unit vector, the component V_h is

$$V_h = (V^t H)H \quad (\text{II-79})$$

and

$$V_p = V - (V^t H)H \quad (\text{II-80})$$

Substituting these equations into eq. II-77,

$$\lambda = |V_p + (V^t H - dc)H| \quad (\text{II-81})$$

which, using the generalized Pythagorean theorem, is equal to

$$\lambda = \left[|V_p|^2 + |(V^t H - dc)H|^2 \right]^{1/2} , \quad (\text{II-82})$$

or

$$\lambda = \left[|V_p|^2 + (V^t H - dc)^2 \right]^{1/2} . \quad (\text{II-83})$$

It is easily seen that this quantity is minimized with respect to d if

$$dc = V^t H , \text{ or} \quad (\text{II-84})$$

$$d = V^t H / c \quad . \quad (\text{II-85})$$

Substituting this result back into equation II-77, the minimum value of λ with respect to d is

$$\lambda_d = |v - (v^t_H)H| \quad . \quad (II-86)$$

This equation may be expanded to give

$$\lambda_d = \left[|v|^2 - 2(v^t_H)(v^t_H) + (v^t_H)^2 \right]^{\frac{1}{2}} \quad (II-87)$$

or

$$\lambda_d = \left[|v|^2 - (v^t_H)^2 \right]^{\frac{1}{2}} \quad . \quad (II-88)$$

Substituting eq. II-72,

$$\lambda_d = \left[|v|^2 - \gamma^2 |v|^2 \right]^{\frac{1}{2}} \quad . \quad (II-89)$$

Finally, since eq. II-77 indicates that

$$\zeta = \lambda^2 / |v|^2 \quad , \quad (II-90)$$

the minimum of ζ with respect to d is

$$\zeta_d = \lambda_d^2 / |v|^2 \quad , \quad (II-91)$$

or, by eq. II-89,

$$\zeta_d = 1 - \gamma^2 \quad . \quad (II-92)$$

Since the magnitude of γ cannot be greater than one, it is obvious from eq. II-92 that ζ will reach its minimum value when γ is a maximum. The dipole location and orientation determined using the matched filter will therefore be the same as the location and orientation determined using Schneider's procedure.

III. SIMULATIONS

Tests have been performed to verify that a two dimensional matched filter can be used to indicate the location of a current dipole, and to investigate the performance of this locating procedure under various conditions. For these tests, computer programs were written which generate simulated surface potential data and then perform the filtering operation on this data. Filter outputs are evaluated for different dipole orientations as a function of the position of the filter dipole (that is, a dipole to which the filter response function is matched).

One purpose of these tests is to determine the resolution of the matched filter with respect to position. It is desirable that the matched filter output decrease very rapidly as the filter dipole moves away from the actual dipole. Such performance would be advantageous for two reasons. First, if more than one source were present, a plot of the filter output would then show a distinct spike indicating the position of each source. Second, a narrow peak should make it possible to determine the source position with greater precision when noise is present.

Another purpose of these tests is to investigate

the effects of some practical problems related to the implementation of the matched filter. Matters which have been simulated include the effect on filter resolution of the use of discrete surface potential samples rather than continuous data, changing of the number of electrodes, and the omission of electrodes on a portion of the surface. These problems will be described more fully in the discussions of the individual tests.

In all of the tests to be described, the theoretical surface potentials were calculated using the single sphere model of the human head. The signal source is a single current dipole oriented in either a radial or tangential direction. The simulation programs calculate the resulting response of one or more of the component filters whose impulse responses are defined in eq. II-29. Multiple sources, noise sources, and possible distortions due to inaccuracies of the model are not considered in these simulations.

It has been shown by Schneider¹⁹ that the surface potentials predicted by the single sphere model for dipoles near the surface are substantially inaccurate. Therefore, primary emphasis in these simulations has been placed on dipoles located between the center and 70% of the radius of the sphere.

To calculate the surface potentials, the dipole magnitude and certain physical parameters of the model

must be specified. For the single sphere model, the two required parameters are the radius of the sphere and its conductivity (see appendix A). However, from equations A19 through A21 of appendix A, it can be seen that the only result of a change in the values of these parameters, or of the dipole magnitude, is to scale the potential at all points on the surface by a constant factor. As a result, the shape of the response curves of the component filters may be determined using arbitrary values for these parameters; the actual values of the component filter outputs may then be determined, if desired, by applying a scale factor derived from the actual values of these parameters.

For purposes of these simulations, the factor $(P_1/4\pi\sigma R)$ of equations A19 through A21 is adjusted so that the integral of the square of the potential over the surface is unity. This selection produces two advantages. First, it causes the filter output to equal unity when the actual dipole and filter dipole are identical, which facilitates comparison of different curves. Second, as can be seen from equations II-28 and II-29, the constant of proportionality, c , becomes unity, so that the surface potential due to an actual dipole is now identical with the impulse response of a filter dipole having the same location and orientation. As a result, the actual dipole and filter dipole may be

interchanged without affecting the filter output, thus halving the amount of computation required in many of the simulations.

The simulation results are presented in this chapter in the form of graphs. The programs used and the actual data generated are listed and described in appendix B.

A. SURFACE POTENTIALS

Before simulating the matched filter, the surface potential functions were calculated for radial and tangential dipoles using the single sphere model.

As shown in appendix A, if the coordinate system is chosen such that the dipole lies on the z axis, the potential due to the radial component is independent of ϕ , while the potential due to a tangential component is a sinusoidal function of ϕ . Therefore, the surface potentials were calculated and plotted as a function of θ only.

Plots of surface potential as a function of dipole depth are shown in figures 2 and 3. An observation which may be made from these figures is that the shape of the curves changes only slightly with changes in dipole depth. Effects of this fact will be discussed more fully in the next section. It will also be noted that the signal tends to become concentrated in a smaller region around the z axis as the dipole moves closer to the surface.

Total signal energy* was also calculated for

*As was stated in section II-B, the energy is defined here as the integral of the square of the voltage over the surface.

FIGURE 2. Surface potential of a radial dipole on the z axis as a function of θ for different values of r .

FIGURE 3. Surface potential of a tangential dipole on the z axis as a function of θ for different values of r .

each curve and is listed in table I. It will be noted that the signal energy is much higher for dipoles near the surface and is slightly higher for radial than for tangential dipoles.

TABLE I
RELATIVE SIGNAL ENERGIES PRODUCED
BY CURRENT DIPOLES

POSITION	RADIAL DIPOLE	TANGENTIAL DIPOLE
0.1	38.3	38.2
0.2	40.4	39.7
0.3	44.2	42.5
0.4	50.6	47.1
0.5	61.4	54.7
0.6	81.0	67.9
0.7	121.3	93.9
0.8	228.8	159.4
0.9	762.2	460.2

B. RESOLUTION TESTS

To test the sensitivity of the matched filter to changes in the location of the dipole, the filter output was calculated and plotted for several types of mismatch between the filter and the dipole coordinates.

Calculations were made of the filter response as a function of the dipole location parameters r and ψ . These calculations were made for both radial and tangential dipoles. In addition, the output of a tangential filter detecting a radial dipole was calculated as a function of ψ .

Figure 4 shows the filter response for variation of ψ . Both radial and tangential dipoles lie on the z axis at $r = 0.6$ with the tangential dipole oriented in the $\bar{a}_\phi$ direction. The curves for a radial filter detecting a radial dipole and for a tangential filter and dipole are drawn from data points spaced $.02\pi$ radians apart. It will be noted that the filter is more sensitive to ψ variation for a tangential dipole than for a radial one.

The third curve of fig. 4 is the response of a radial filter to a tangential dipole (or equivalently, as explained in the introduction, the response of a tangential filter to a radial dipole).

FIGURE 4. Filter response for position change with depth constant. Outputs are shown for (1) a radial filter detecting a radial dipole; (2) a tangential filter detecting a tangential dipole; and (3) a radial filter detecting a tangential dipole. The dipole radial position is $r=0.6$ in all cases.

Top portion of above graph, with expanded scale.

This curve is drawn from data points spaced .05 radians apart. The maximum value is approximately $\gamma = 0.67$ at $\psi = 0.25\pi$. When the filter and dipole positions are the same ($\psi = 0$), the dipole potential is orthogonal to the filter response function and the output is zero, as expected.

Figure 5 shows the response of the filter to changes in r for actual dipoles located at $r = .1, .3, .5, .7, .8$, and $.9$. Comparison with fig. 4 shows that the filter is much less sensitive to variation in r than to variation in ψ . It can also be seen that in this case the filter is more selective for radial than for tangential dipoles.

This lack of sensitivity to r is not surprising since, from figures 2 and 3, the surface potential itself changes only slowly with changes in r .

FIGURE 5. Filter response for depth mismatch between filter and dipole. The filter outputs are shown for dipoles at $r = .1, .3, .5, .7, .8$, and $.9$.

C. EFFECTS OF NUMBER OF ELECTRODES

In practice, the surface potential is known only at the electrode locations and not at every point on the surface. It is therefore of interest to determine how the selectivity of the locating procedure is affected by variations in the number of electrodes.

To simulate operation with a finite number of electrodes, a computer program was written to evaluate the filter outputs given by eq. II-53 (See appendix B for the program and printouts). A series of runs were made for depth mismatch using from 8 to 99 electrodes on each hemisphere with the electrode positions determined by an algorithm described in appendix B. The actual dipole used in each case is a radial or tangential dipole at $r = 0.5$. Results are plotted in figures 6 and 7 for 8 to 66 electrodes per hemisphere.

Comparing figures 6 and 7 to fig. 5, it can be seen that the filter response shapes approach those of the continuous case as the number of electrodes increases. Using 8 electrodes per hemisphere, the filter shows much less sensitivity to changes in r near the peak of the curve. Increasing the number of electrodes to 21 per hemisphere causes the filter response to approach the continuous case over the range from $r = 0.4$ to $r = 0.6$. Further increases in the number of electrodes widen the

FIGURE 6. Response of a radial filter for depth mismatch with varying numbers of electrodes. The simulated source is a radial dipole at $r=0.5$ in all cases.

FIGURE 7. Response of a tangential filter for depth mismatch with varying numbers of electrodes. The simulated source is a tangential dipole at $r=0.5$ in all cases.

range over which the filter output for the discrete case agrees with the continuous case.

Based on these results, some observations may be made regarding the number of electrodes per hemisphere to be used.

(1) Due to the flatness of the curve, a substantial loss of accuracy may be expected in the case of 8 electrodes when the effect of noise is considered.

(2) Because the filter output with 21 electrodes nearly matches the continuous case in the region near the peak of the curve, it is not clear whether the use of more than 21 electrodes per hemisphere will produce a significant increase in accuracy.

A detailed analysis of the effects of noise on the results will be required before the latter question can be resolved.

Both observations must be qualified in that they are based on changes in filter sensitivity to variation in r only. Changes in sensitivity to variations in γ and ω should be investigated also.

D. EFFECT OF SURFACE TRUNCATION

The simulations described in the previous section assumed that the electrode locations were spread evenly over the entire spherical surface. In reality, the possible locations are limited by the shape of the head to approximately one hemisphere. To investigate the effect of this limitation, the filter outputs described in section C were recalculated using only the electrode locations on the hemisphere for which z is greater than zero. These results are shown in figures 8 and 9.

The curves of figures 8 and 9 show very little change from the curves of figures 6 and 7. One reason for the lack of change is the fact that the dipole is near the z axis. From figures 2 and 3, it can be seen that the surface potential is small on the hemisphere for which z is negative (i. e., $\theta > \pi/2$). Therefore, very little of the signal energy is lost by neglecting the contribution of this hemisphere. While simulations were not made for other dipole locations, it is expected that filter selectivity using one hemisphere will decrease as the dipole moves closer to the unused hemisphere and more signal energy is lost due to surface truncation.

FIGURE 8. Radial filter response for depth mismatch with electrodes on one hemisphere only. The simulated source is a radial dipole at $r=0.5$.

FIGURE 9. Tangential filter response for depth mismatch with electrodes on one hemisphere only. The simulated source is a tangential dipole at $r=0.5$.

E. CONCLUSIONS

In summary, the following conclusions may be reached regarding the matched filter locating procedure as a result of the computer simulations which have been performed:

- (1) The matched filter detection process exhibits more sensitivity to changes in ψ and ω than to changes in r . Therefore, it is expected that the depth of a dipole source cannot be determined to the same degree of precision as its lateral position.
- (2) Filter selectivity tends to increase as the number of electrodes increases. In particular, for the cases simulated, it was found that use of 8 or fewer electrodes per hemisphere produces a substantial loss of selectivity; while the use of more than 21 electrodes per hemisphere produces only slight increases in selectivity.
- (3) Inability to measure potentials over a portion of the surface does not necessarily result in a large decrease in selectivity. For the cases simulated, the potentials omitted were of low magnitude; and under these conditions the effect of their omission

on the filter selectivity was slight.

In the course of these simulations, some areas which merit further investigation were identified.

These areas are as follows:

(1) In the simulations for a finite number of electrodes, the effects of modifications on filter sensitivity to changes in r were observed. Further tests should be made to determine the effects of these modifications on the sensitivity to ψ and ω variation.

(2) The electrode positions used in these simulations were chosen to provide an even distribution over the surface area. However, improvements in selectivity for a given number of electrodes may be possible by modifying the electrode positions. Investigations to determine an optimum electrode pattern should be conducted.

(3) Since the single sphere model is known to be inaccurate for dipoles near the surface, the results of these simulations should be verified using the three spheres model.

(4) While the obtainable accuracy of the locating procedure is obviously related to the selectivity of the filter, a quantitative statement of their relation-

ship will require an analysis of the effects of noise on the filter output. Such an analysis has been started; the incomplete results are shown in appendix C.

IV. SUMMARY

A new technique has been presented for determining the location, orientation, and magnitude of a dipole current source in a volume conductor by measuring the surface potential at a finite number of points. This technique is based on the application of the matched filter concept of communications theory to a spherical surface. In addition, it has been shown that the amount of computation required can be greatly reduced by resolving an arbitrary dipole into three components with fixed orientation and considering each component separately. Computer simulations have also been performed to demonstrate the operation of this technique.

It was shown in section II-E that the matched filter should produce approximately the same result as other approaches which have been used previously; improved accuracy is therefore not an advantage of this technique. The matched filter method does, however, offer some advantages which will now be discussed.

The previous locating method, as used by Schneider, was based on an iterative search for the set of parameters providing a best fit to the observed data. An iterative procedure requires the selection of an initial set of values

for the unknown quantities and the specification of some criterion to determine when the search is to be terminated. As a result, failure to obtain convergence may occur either because the source being sought does not exist or because the search was started or terminated improperly.

Since the matched filter method does not require the use of an iterative procedure, this limitation does not apply to it. If no peak is observed in the filter output, it can then be stated conclusively that no source exists which can be modeled as a single current dipole. The only requirement is that the filter output be calculated for points close enough together that the peak cannot fall between output samples and be missed. This spacing can be determined from curves, such as those presented in chapter III, which show the width of the filter output peak.

The matched filter procedure also has other potential advantages. One of these advantages is the possibility of performing a theoretical analysis of the available accuracy, given the signal to noise ratio at the electrodes. Another potential advantage is the possibility of decreasing the processing time by increasing the amount of memory used. The beginning of an error analysis is presented in appendix C; the second advantage is discussed more fully in the next section.

V. RECOMMENDATIONS FOR FUTURE WORK

The matched filter algorithm has been presented, and its operation has been demonstrated using simulated data. The next step appears to be a series of tests to determine how well this algorithm will perform using real input data.

Such tests should be used not only to verify that this algorithm works correctly, but also to characterize the noise that is present in actual data. Characteristics of importance include the statistical parameters of the noise as a function of surface position and the correlation of noise samples at different points on the surface. This information will be useful in the theoretical analysis of available accuracy and will also permit simulations to be performed which more closely resemble the real world.

Since, as stated previously, it has been reported in the literature that the single sphere model introduces substantial inaccuracy, it is suggested that these and all subsequent studies be performed using the three spheres model.

Assuming the initial tests produce satisfactory results, three areas related to the matched filter algorithm have been identified which deserve further study.

These areas will now be discussed.

One problem with the matched filter locating procedure, as well as Schneider's procedure, is that the amount of computation required to calculate the theoretical potentials increases substantially as the complexity of the model increases. Therefore, for the three spheres model in particular, it may be advantageous to decrease the processing time by using a lookup table to provide the theoretical surface potentials for a large number of dipole locations within the volume conductor. For the matched filter procedure, such a table would contain for each location the filter response vectors for the three component dipoles.

By way of example, if the coordinates r , ψ , and ω are divided into ten segments each (producing 1000 dipole locations), and if 21 electrodes are used, a total of 63 000 words of memory would be required to store the three response vectors for each location. This amount of storage is within the capacity of many mini-computers, so such an approach is definitely feasible.

The primary questions to be answered in future investigations are the number of points to be used and their locations. For example, the simulations presented in chapter III showed that the filter procedure has low sensitivity to change in dipole depth. Therefore, the r coordinate would probably be divided into fewer segments

than were assumed in the above example. In addition, it is expected that the number of segments for the ψ and ω coordinates would be decreased for the locations near the center. The number and locations of points will be a compromise between the requirements, first, that the points be sufficiently close together that at least one calculated output will be near the peak of the filter output, and second, that they be chosen as far apart as possible so that the amount of memory and computation required will be minimized.

A second area to be investigated is the possibility that the matched filter can be used even if more than one dipole source is present. Because the three component filters perform linear operations on the data, the filter output for two dipoles will be the sum of the outputs which would be produced by the dipoles individually. Two peaks should therefore be observed in the filter output, one corresponding to the location of each dipole. The primary question to be answered in this regard is whether the two peaks at the dipole locations can be distinguished from other peaks such as those observed in section III-B caused by the interaction of a radial filter and a tangential dipole.

The third question to be studied is the effect that different sensing techniques have on the obtainable accuracy. Three types of connections are commonly used

in studies of the EEG.²⁴ First, the potentials may be measured differentially between pairs of electrodes. Second, the potentials may be measured with the average of all electrode potentials used as a reference. Third, a neutral point, such as an ear clip, may be used as a reference.

The matched filter procedure implicitly assumes the use of the neutral reference, since the surface potentials due to the three component dipoles will not necessarily be orthogonal if either of the first two types of connections is used. A question which therefore arises is whether this restriction has any effect on the obtainable accuracy. An advantage of the third type is that the number of independent data samples is equal to the number of electrodes, whereas the maximum number of independent samples is one less than the number of electrodes for the other two schemes.

Preliminary tests using the simulation programs were made to compare the second and third types of connections (see appendix B, figures B14 and B15, for the applicable printouts). These results indicate that the use of a neutral reference may provide an improvement in performance. However, because these simulations do not include noise effects, it cannot yet be stated with certainty which of the three connection methods is best.

APPENDICES

APPENDIX A. SURFACE POTENTIAL EQUATIONS

In this appendix the equations for the surface potential produced by a single current dipole in a spherical volume conductor will be presented for both the single sphere and the three concentric models. It will then be shown that the surface potentials produced by the three components of the dipole are mutually orthogonal.

The single sphere potential equations are available in the literature; the following set of equations* for a rectangular coordinate system are taken from ref. 6.

With the origin at the center of the sphere,
let

X, Y, Z	be the coordinates of point A on the spherical surface;
R	be the radius of the sphere;
σ	be the conductivity of the medium;
P_1, P_2, P_3	be the coordinates of the dipole center;
P_4, P_5, P_6	be the X, Y, and Z components, respectively, of the dipole moment.

*The notation used here follows that of reference 6, rather than that of the main body of this thesis.

Now, define

$$D = (P_1^2 + P_2^2 + P_3^2)^{1/2} \quad A1$$

$$d = D/R \quad A2$$

$$H_1 = (P_1 P_4 + P_2 P_5 + P_3 P_6)/D \quad A3$$

$$H_2 = (P_4 X + P_5 Y + P_6 Z)/R \quad A4$$

$$H = (P_1 X + P_2 Y + P_3 Z)/(DR) \quad A5$$

$$L = (1 + d^2 - 2Hd)^{1/2} \quad A6$$

$$S = \frac{2}{L^3} + \frac{d - H}{d(1-H^2)L} + \frac{H}{d(1 - H^2)} \quad A7$$

$$T = \frac{2d}{L^3} + \frac{dH - 1}{d(1-H^2)L} + \frac{1}{d(1 - H^2)} \quad A8$$

The potential at point A due to the dipole is then

$$V(A) = (SH_2 - TH_1)/(4\pi\epsilon R) \quad A9$$

For later use it will be necessary to convert to a spherical coordinate system. First, the rectangular coordinate system may be defined without loss of generality such that the dipole is located on the positive z axis. This definition implies that P_1 and P_2 become zero. Now, equations A1 through A9 may be modified to become:

$$D = P_3 \quad A10$$

$$d = P_3/R \quad A11$$

$$H_1 = P_6 \quad A12$$

$$\begin{aligned} H_2 &= (P_4 X + P_5 Y + P_6 Z)/R \\ &= P_4 \sin\theta \cos\phi + P_5 \sin\theta \sin\phi + P_6 \cos\theta \end{aligned} \quad A13$$

$$H = Z/R = \cos\theta \quad A14$$

$$L = (1 + d^2 - 2d\cos\theta)^{1/2} \quad A15$$

$$S = \frac{2}{L^3} + \frac{d - \cos\theta}{dL\sin^2\theta} + \frac{\cos\theta}{d\sin^2\theta} \quad A16$$

$$T = \frac{2d}{L^3} + \frac{d\cos\theta - 1}{dL\sin^2\theta} + \frac{1}{d\sin^2\theta} \quad A17$$

$$V(A) = (SP_4 \sin\theta \cos\phi + SP_5 \sin\theta \sin\phi + SP_6 \cos\theta - TP_6)/(4\pi\sigma R) \quad A18$$

The $\bar{a}_x$, $\bar{a}_y$, and $\bar{a}_z$ unit vectors are equivalent to the $\bar{a}$, $\bar{a}$, and $\bar{a}_r$ unit vectors, respectively, for a point on the z axis. Therefore, since P_4 , P_5 , and P_6 are the dipole components in the $\bar{a}_x$, $\bar{a}_y$, and $\bar{a}_z$ directions, respectively, the surface potential contributions from the three components of the dipole are:

$$V_r(A) = \frac{S\cos\theta - T}{4\pi\sigma R} P_6 \quad A19$$

$$V_\theta(A) = \frac{S\sin\theta}{4\pi\sigma R} (\cos\phi) P_4 \quad A20$$

$$V_\phi(A) = \frac{S\sin\theta}{4\pi\sigma R} (\sin\phi) P_5 \quad A21$$

The equations defining the surface potential for the three spheres model are also available in the literature. The set of equations presented here is from ref. 19.

The coordinate system for the three spheres model is again defined such that the center of the coordinate system is at the center of the three spheres and the dipole lies on the positive z axis and is in the inner sphere. The

brain is modeled as the inner sphere of conductivity g , the skull is represented by the intermediate shell of conductivity kg , and the scalp is the outer shell of conductivity jg . The current dipole has magnitude p with components p_r , p_θ , and p_ϕ . The dimensions of the model are shown in fig. A1, while typical values of the constants are listed in table A1.

For point A on the surface, with coordinates (R, θ, ϕ) , the surface potential contribution from each of the dipole components is:

$$V_r(A) = \frac{1}{4\pi g j R^2} \left[\sum_{n=1}^{\infty} \frac{(2n+1)3kf^{n-1}}{(n+1)D_n} P_n^0(\cos\theta) \right] p_r \quad A22$$

$$V_\theta(A) = \frac{1}{4\pi g j R^2} \left[\sum_{n=1}^{\infty} \frac{(2n+1)3kf^{n-1}}{n(n+1)D_n} P_n^1(\cos\theta) \right] (\cos\phi)p_r \quad A23$$

$$V_\phi(A) = \frac{1}{4\pi g j R^2} \left[\sum_{n=1}^{\infty} \frac{(2n+1)3kf^{n-1}}{n(n+1)D_n} P_n^1(\cos\theta) \right] (\sin\phi)p_r \quad A24$$

where P_n^0 and P_n^1 are the Legendre polynomials, and

$$\begin{aligned} D_n &= (kn + k + n)(nk/j + n + 1)/(n + 1) \\ &\quad - (kn + k + n)(1 - k/j)c^{2n+1} \\ &\quad + (1 - k)(n + k(n + 1)/j)b^{2n+1} \\ &\quad - n(1 - k)(1 - k/j)(b/c)^{2n+1} \end{aligned} \quad A25$$

FIGURE A1. The three spheres model of the human head.

TABLE A1

PARAMETERS OF THE THREE SPHERES MODEL

(From Reference 4)

PARAMETER	VALUE
R	9.2 cm
b	0.87
c	0.92
g	0.0045 mho/cm
j	1.0
k	0.0125

To prove that the surface potential functions of the three component dipoles are mutually orthogonal, note that for both the single sphere and three spheres models the surface potentials due to the three components are of the forms

$$V_r(A) = F_1(\theta)p_r \quad A26$$

$$V_\theta(A) = F_2(\theta)(\cos\phi)p_\theta \quad A27$$

$$V_\phi(A) = F_2(\theta)(\sin\phi)p_\phi \quad A28$$

where F_1 and F_2 are functions which depend on the dimensional and electrical constants of the model and on θ , but not ϕ . Therefore, defining the inner product of two functions of θ and ϕ as the integral over the entire surface of their product, the inner product of the functions V_r and V_θ is

$$\begin{aligned} \langle V_r | V_\theta \rangle &= \int_0^{2\pi} \int_0^\pi V_r(R, \theta, \phi) V_\theta(R, \theta, \phi) R^2 \sin\theta d\theta d\phi \\ &= p_r p_\theta R^2 \int_0^\pi F_1(\theta) F_2(\theta) \sin\theta d\theta \int_0^{2\pi} \cos\phi d\phi \quad A29 \end{aligned}$$

The integral of $\cos\phi$ over 2π is zero; therefore, the inner product is zero and the two functions are orthogonal.

Similarly, the inner product of V_r and V_ϕ is

$$\langle V_r | V_\phi \rangle = p_r p_\phi R^2 \int_0^\pi F_1(\theta) F_2(\theta) \sin\theta d\theta \int_0^{2\pi} \sin\phi d\phi \quad A30$$

which is also zero since the integral of $\sin\phi$ over 2π is zero.

Finally, the inner product of V_θ and V_ϕ is

$$V_\theta V_\phi = p_\theta p_\phi R^2 \int_0^\pi F_2(\theta) \sin\theta d\theta \int_0^{2\pi} \sin\phi \cos\phi d\phi \quad A31$$

Again, the integral of the product of $\sin\phi$ and $\cos\phi$ over 2π is zero, so these functions are also orthogonal.

Therefore, the three functions V_r , V_θ , and V_ϕ are mutually orthogonal.

APPENDIX B. COMPUTER PROGRAMS AND OUTPUTS

In this appendix the programs used to simulate the matched filter are presented along with the output data. This data is also presented in the form of graphs in the main body of this thesis.

The first program was used to calculate the relative surface potentials for radial and tangential dipoles. The method used is a straightforward evaluation of equations A15 through A21 of appendix A. The magnitude of the dipole component is assumed to be unity, and the constant $1/(4\pi\epsilon R)$ is also set equal to unity. Since the variation with ϕ is sinusoidal, this variable is ignored and the voltages are calculated as a function of radial position, d , and θ only.

This program was written in Hewlett-Packard Basic. Referring to the listing in figure B-1, variable L9 determines whether a radial or tangential dipole is assumed. If L9 is false (zero), potentials are calculated for a radial dipole; if it is true (nonzero), a tangential dipole is used. Variables θ , d , S , and T of the equations are represented in the program by Q , D , Sl , and Tl , respectively.

Outputs for radial and tangential dipoles are

```

1000 REM PROGRAM TO CALCULATE SURFACE POTENTIALS.
1005 INPUT L9,N0
1006 FOR D9=1 TO 3
1010 PRINT " THETA"," ", " DEPTH"
1015 PRINT " DEG.",(3*D9-2)/10,(3*D9-1)/10,(3*D9)/10
1020 PRINT
1030 FOR Q1=1 TO N0-1
1040 LET Q=(Q1/N0)*3.14159
1045 PRINT Q*180/3.14159,
1050 FOR D1=3*D9-2 TO 3*D9
1055 LET D=D1/10
1060 LET C=COS(Q)
1070 LET S=SIN(Q)
1080 LET L=SQRT(1+D*(D-2*C))
1090 LET S1=2/(L+3)+(D-C)/(D*L*S+2)+C/(D*S+2)
1100 LET T1=2*D/(L+3)+(D*C-1)/(D*L*S+2)+1/(D*S+2)
1110 IF L9 THEN 1140
1120 LET V=S1*C-T1
1130 GOTO 1150
1140 LET V=S1*S
1150 PRINT V,
1160 NEXT D1
1165 PRINT
1170 NEXT Q1
1180 PRINT
1190 PRINT
1200 NEXT D9
1210 STOP
1220 END

```

FIGURE B1. Program to calculate theoretical surface potentials. Potentials produced by radial and tangential dipoles are calculated using the single sphere model.

THETA DEG.	.1	DEPTH .2	.3
9	3.51856	4.26785	5.31394
18	3.33773	3.96071	4.77085
27	3.04993	3.4926	3.99518
36	2.67332	2.91615	3.11918
45	2.22986	2.28555	2.25095
54	1.74261	1.64722	1.45769
63	1.2336	1.03537	.769696
72	.722295	.471657	.192348
81	.224738	-3.27552E-02	-.282346
90	-.246661	-.474241	-.667821
99	-.68292	-.853994	-.978449
108	-1.07799	-1.17508	-1.22739
117	-1.42816	-1.44491	-1.42583
126	-1.73148	-1.66637	-1.5828
135	-1.9872	-1.84515	-1.70547
144	-2.19535	-1.98555	-1.79932
153	-2.35635	-2.09112	-1.86847
162	-2.47078	-2.1646	-1.91588
171	-2.5391	-2.20788	-1.94357

THETA DEG.	.4	DEPTH .5	.6
9	6.83645	9.16516	12.932
18	5.82736	7.17715	9.76035
27	4.52222	4.97367	5.13562
36	3.21893	3.12223	2.72677
45	2.08348	1.7499	1.2494
54	1.16574	.783514	.347094
63	.452497	.112099	-.217481
72	-9.19461E-02	-.356691	-.582448
81	-.505187	-.68033	-.826273
90	-.819135	-.926686	-.994122
99	-1.05858	-1.10068	-1.11269
108	-1.242	-1.22942	-1.19823
117	-1.38291	-1.32564	-1.26092
126	-1.4911	-1.39793	-1.3073
135	-1.57365	-1.45214	-1.34167
144	-1.63562	-1.49231	-1.36691
153	-1.68063	-1.52119	-1.38494
162	-1.71118	-1.54064	-1.39703
171	-1.72889	-1.55189	-1.40397

FIGURE B2. Theoretical surface potentials of a radial dipole. (Page 1 of 2)

THETA DEG.	.7	DEPTH .8	.9
9	19.3186	29.6008	35.4922
18	10.1151	9.80815	5.75987
27	4.66906	3.28323	1.14373
36	1.98167	.965157	-.107807
45	.63183	-9.45759E-03	-.574946
54	-9.14044E-02	-.480885	-.787371
63	-.506062	-.73469	-.897704
72	-.758488	-.882989	-.960732
81	-.920109	-.975318	-.999386
90	-1.02799	-1.03574	-1.02442
99	-1.10247	-1.07685	-1.04132
108	-1.15528	-1.10569	-1.05311
117	-1.19347	-1.12638	-1.06153
126	-1.22144	-1.14144	-1.06765
135	-1.24202	-1.15246	-1.07211
144	-1.25703	-1.16048	-1.07535
153	-1.26771	-1.16616	-1.07764
162	-1.27485	-1.16995	-1.07917
171	-1.27893	-1.17212	-1.08004

FIGURE B2. Theoretical surface potentials of a radial dipole. (Page 2 of 2)

THETA DEG.	.1	DEPTH .2	.3
9	.610298	.822846	1.16037
18	1.19187	1.57887	2.16523
27	1.71899	2.21493	2.91303
36	2.17117	2.6999	3.37615
45	2.53458	3.02563	3.58478
54	2.80226	3.20204	3.59654
63	2.97345	3.2499	3.47173
72	3.05233	3.19416	3.26069
81	3.04646	3.0592	3.00028
90	2.96541	2.86631	2.71531
99	2.81954	2.63272	2.42136
108	2.61912	2.37161	2.12757
117	2.37377	2.09266	1.83884
126	2.09216	1.8027	1.55736
135	1.78187	1.50637	1.28367
144	1.44945	1.20667	1.01731
153	1.1005	.905429	.757217
162	.739856	.603663	.502024
171	.371728	.301812	.250166

THETA DEG.	.4	DEPTH .5	.6
9	1.73199	2.78238	4.92484
18	3.08857	4.59726	7.12252
27	3.89947	5.27162	7.08021
36	4.21301	5.18375	6.18593
45	4.17824	4.74041	5.17986
54	3.94208	4.18895	4.29669
63	3.60953	3.644	3.57276
72	3.24376	3.14771	2.98766
81	2.87843	2.7094	2.51162
90	2.5293	2.32551	2.11851
99	2.20231	1.98839	1.78796
108	1.89828	1.68987	1.50464
117	1.61561	1.42253	1.25716
126	1.35167	1.18003	1.03691
135	1.10348	.957055	.837334
144	.86805	.74916	.653319
153	.642627	.552585	.480758
162	.42446	.364082	.316254
171	.211054	.180772	.156881

FIGURE B3. Theoretical surface potentials of a tangential dipole. (Page 1 of 2)

THETA DEG.	.7	DEPTH .8	.9
9	9.90637	23.2036	57.7889
18	11.2739	17.258	22.9485
27	9.18258	11.0589	11.9271
36	7.02555	7.47468	7.40535
45	5.40767	5.3789	5.11637
54	4.25014	4.06566	3.78301
63	3.41097	3.1854	2.92572
72	2.78472	2.56039	2.33255
81	2.30223	2.09467	1.898
90	1.91887	1.73315	1.56462
99	1.60546	1.4425	1.29881
108	1.3424	1.20153	1.07979
117	1.11621	.996246	.894053
126	.917313	.816994	.732425
135	.738737	.656906	.588451
144	.575222	.510905	.457402
153	.42267	.375095	.335676
162	.277768	.246362	.220412
171	.137709	.122098	.109221

FIGURE B3. Theoretical surface potentials of a tangential dipole. (Page 2 of 2)

shown in figures B-2 and B-3. Plots of these data are shown in chapter III, figures III-1 and III-2.

DEPTH MISMATCH SIMULATIONS

Two programs were written to calculate the theoretical effect of depth mismatch on the filter output (see figures B-4 and B-7 for program listings). These programs evaluate the functions b_r , b_θ , or b_ϕ as defined in eq. II-36. The function $v(\theta, \phi)$ in these equations is the surface potential of a current dipole on the z axis having the same orientation as the dipole to which the filter is matched.

The equations for the single sphere model from appendix A are again used with the component dipole magnitude, the radius R , and the constant $1/(4\pi\epsilon R)$ all set equal to one. For a radial dipole, the integral to be evaluated is

$$b_r = \int b_r(\theta, \phi) v(\theta, \phi) dq \quad B1$$

Using equation A26, this equation becomes

$$b_r = \frac{1}{c_r} \int_0^{2\pi} \int_0^\pi F_1(\theta|r_1) F_1(\theta|r_0) R^2 \sin\theta d\theta d\phi \quad B2$$

where r_0 and r_1 are the radial position coordinates of the actual dipole and the filter dipole, respectively. (dependence of F_1 on the other dipole parameters is not

```

1000 REM MAX. DEPTH ACCURACY HERMBERG)
1010 DIM T(10)
1020 FOR K=1 TO 9
1030 D=K*10
1040 FOR K1=1 TO K
1050 D1=K1*10
1060 PRINT LIN(2) "DEPTH SETTINGS: "; D; D1
1070 PRINT " CE, B INTEGRAL VALUES
1080 A=0
1090 FOR I=1 TO 9
1100 Q=1.314159
1120 GOSUB 2000
1130 NEXT I
1140 T(1)=936961*A
1145 N=10
1150 PRINT USING "3EXX, 3D5D; N, T(1, 1)
1160 FOR N=2 TO 5
1170 N9=5.2*N
1174 N8=N9*K1
1180 FOR I=1 TO N9 STEP 2
1190 Q=3.14159/J1/HN9K11
1200 GOSUB 2000
1210 NEXT I
1220 T(N)=936961JA/N8
1225 PRINT USING "3EXX, 3D, 5D; N3L T(N, 1)
1230 FOR J=2 TO N
1240 T(NJ)=H4*(J-1)*T(NJ-1)/H4*(J-1)*T(NJ-1)
1260 PRINT USING "B, 3D, 5D; T(N, J)
1270 NEXT J
1275 PRINT
1280 NEXT N
1290 NEXT K1
1300 NEXT K
1310 FOR I=1 TO 20
1320 PRINT BB;
1330 NEXT I
1999 STOP
2000 C=COS(Q)
2010 S=SIN(Q)
2020 L=SQRT(1+D*(HD-2JCI))
2030 L1=SQRT(1+D1*(HD1-2JCI))
2040 V1=2*S*L*(3*(D-C)*LN(C)/HD*S)
2050 H1=2*S*L1*(3*(HD1-C1)/L1+C1*(D1*S1)
2060 A=AKV1H1*S
2070 RETURN
9999 END

```

FIGURE B4. Program to test numerical integration for depth mismatch.

DEPTH SETTINGS: .1 .1
 # INTEGRAL VALUES
 10 37.32111
 20 38.08510 38.33976
 40 38.27370 38.33656 38.33635
 80 38.32069 38.33636 38.33634 38.33634
 160 38.33242 38.33633 38.33633 38.33633 38.33633

DEPTH SETTINGS: .2 .1
 # INTEGRAL VALUES
 10 37.87026
 20 38.71487 38.99641
 40 38.92282 38.99214 38.99185
 80 38.97462 38.99189 38.99187 38.99187
 160 38.98756 38.99187 38.99186 38.99186 38.99186

DEPTH SETTINGS: .2 .2
 # INTEGRAL VALUES
 10 39.09229
 20 40.04820 40.36684
 40 40.28287 40.36109 40.36071
 80 40.34128 40.36075 40.36073 40.36073
 160 40.35589 40.36076 40.36076 40.36076 40.36076

DEPTH SETTINGS: .3 .1
 # INTEGRAL VALUES
 10 38.36150
 20 39.34529 39.67323
 40 39.58646 39.66685 39.66642
 80 39.64646 39.66647 39.66644 39.66644
 160 39.66142 39.66641 39.66640 39.66640 39.66640

DEPTH SETTINGS: .3 .2
 # INTEGRAL VALUES
 10 40.30577
 20 41.44241 41.82128
 40 41.72005 41.81261 41.81203
 80 41.78908 41.81209 41.81205 41.81205
 160 41.80630 41.81205 41.81204 41.81204 41.81204

DEPTH SETTINGS: .3 .3
 # INTEGRAL VALUES
 10 42.33932
 20 43.71443 44.17280
 40 44.04852 44.15989 44.15903
 80 44.13147 44.15912 44.15907 44.15907
 160 44.15218 44.15909 44.15909 44.15909 44.15909

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 1 of 8)

DEPTH SETTINGS: .4 .1
INTEGRAL VALUES

10	38.74809				
20	39.96657	40.37273			
40	40.26271	40.36143	40.36068		
80	40.33627	40.36079	40.36074	40.36074	
160	40.35459	40.36070	40.36069	40.36069	40.36069

DEPTH SETTINGS: .4 .2
INTEGRAL VALUES

10	41.45846				
20	42.89120	43.36877			
40	43.23808	43.35371	43.35270		
80	43.32417	43.35287	43.35281	43.35282	
160	43.34570	43.35287	43.35287	43.35287	43.35287

DEPTH SETTINGS: .4 .3
INTEGRAL VALUES

10	44.41293				
20	46.17145	46.75762			
40	46.59467	46.73575	46.73429		
80	46.69957	46.73454	46.73446	46.73446	
160	46.72572	46.73444	46.73443	46.73443	46.73443

DEPTH SETTINGS: .4 .4
INTEGRAL VALUES

10	47.58245				
20	49.85817	50.61674			
40	50.40029	50.58100	50.57862		
80	50.53440	50.57910	50.57897	50.57898	
160	50.56786	50.57901	50.57900	50.57900	50.57900

DEPTH SETTINGS: .5 .1
INTEGRAL VALUES

10	38.92082				
20	40.55629	41.10145			
40	40.94686	41.07705	41.07542		
80	41.04353	41.07576	41.07568	41.07568	
160	41.06767	41.07572	41.07571	41.07571	41.07571

DEPTH SETTINGS: .5 .2
INTEGRAL VALUES

10	42.42207				
20	44.37388	45.02448			
40	44.83782	44.99247	44.99033		
80	44.95258	44.99084	44.99073	44.99074	
160	44.98124	44.99079	44.99079	44.99079	44.99079

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 2 of 8)

DEPTH SETTINGS: .5 .3
 # INTEGRAL VALUES
 10 46.38442
 20 48.80902 49.61722
 40 49.38142 49.57223 49.56923
 80 49.52281 49.56995 49.56979 49.56980
 160 49.55807 49.56982 49.56982 49.56982 49.56982

DEPTH SETTINGS: .5 .4
 # INTEGRAL VALUES
 10 50.82604
 20 53.99547 55.05194
 40 54.73505 54.98158 54.97689
 80 54.91736 54.97813 54.97790 54.97791
 160 54.96277 54.97791 54.97789 54.97789 54.97789

DEPTH SETTINGS: .5 .5
 # INTEGRAL VALUES
 10 55.63528
 20 60.08755 61.57163
 40 61.10333 61.44193 61.43327
 80 61.35284 61.43601 61.43562 61.43566
 160 61.41493 61.43563 61.43560 61.43560 61.43560

DEPTH SETTINGS: .6 .1
 # INTEGRAL VALUES
 10 38.58018
 20 41.05865 41.88480
 40 41.62622 41.81541 41.81079
 80 41.76578 41.81229 41.81209 41.81211
 160 41.80050 41.81207 41.81206 41.81206 41.81206

DEPTH SETTINGS: .6 .2
 # INTEGRAL VALUES
 10 42.83566
 20 45.82964 46.82763
 40 46.51146 46.73874 46.73281
 80 46.67893 46.73476 46.73450 46.73452
 160 46.72057 46.73446 46.73444 46.73444 46.73444

DEPTH SETTINGS: .6 .3
 # INTEGRAL VALUES
 10 47.82143
 20 51.57694 52.82879
 40 52.42540 52.70822 52.70019
 80 52.63352 52.70290 52.70255 52.70258
 160 52.68532 52.70258 52.70256 52.70256 52.70256

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 3 of 8)

DEPTH SETTINGS: .6 .4
 # INTEGRAL VALUES
 10 53.63584
 20 58.58380 60.23312
 40 59.68745 60.05534 60.04348
 80 59.95760 60.04765 60.04713 60.04720
 160 60.02473 60.04712 60.04709 60.04708 60.04708

DEPTH SETTINGS: .6 .5
 # INTEGRAL VALUES
 10 60.23586
 20 67.22510 69.55484
 40 68.74875 69.25664 69.23676
 80 69.12033 69.24419 69.24336 69.24347
 160 69.21262 69.24338 69.24333 69.24333 69.24333

DEPTH SETTINGS: .6 .6
 # INTEGRAL VALUES
 10 67.00139
 20 77.96053 81.61357
 40 80.25545 81.02043 80.98090
 80 80.81076 80.99586 80.99422 80.99443
 160 80.94852 80.99443 80.99434 80.99434 80.99434

DEPTH SETTINGS: .7 .1
 # INTEGRAL VALUES
 10 36.79640
 20 41.30150 42.80320
 40 42.26224 42.58248 42.56776
 80 42.49416 42.57146 42.57072 42.57077
 160 42.55169 42.57087 42.57083 42.57083 42.57083

DEPTH SETTINGS: .7 .2
 # INTEGRAL VALUES
 10 41.57220
 20 47.05629 48.88432
 40 48.22025 48.60824 48.58984
 80 48.50091 48.59446 48.59354 48.59360
 160 48.57049 48.59368 48.59363 48.59363 48.59363

DEPTH SETTINGS: .7 .3
 # INTEGRAL VALUES
 10 47.33946
 20 54.25237 56.55667
 40 55.71030 56.19629 56.17226
 80 56.06130 56.17831 56.17711 56.17719
 160 56.14824 56.17722 56.17714 56.17714 56.17714

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 4 of 8)

DEPTH SETTINGS: .7 .4
 # INTEGRAL VALUES
 10 54.29671
 20 63.41976 66.46078
 40 65.32584 65.96120 65.92789
 80 65.78358 65.93617 65.93451 65.93462
 160 65.89694 65.93474 65.93465 65.93465 65.93465

DEPTH SETTINGS: .7 .5
 # INTEGRAL VALUES
 10 62.50282
 20 75.33609 79.61385
 40 77.97847 78.85927 78.80898
 80 78.61035 78.82098 78.81842 78.81857
 160 78.76669 78.81880 78.81865 78.81865 78.81865

DEPTH SETTINGS: .7 .6
 # INTEGRAL VALUES
 10 71.34486
 20 91.12369 97.71663
 40 95.11847 96.45007 96.36565
 80 96.06551 96.38118 96.37660 96.37679
 160 96.29955 96.37756 96.37733 96.37733 96.37735

DEPTH SETTINGS: .7 .7
 # INTEGRAL VALUES
 10 77.85780
 20 112.13425 123.55974
 40 119.11887 121.44708 121.30623
 80 120.73689 121.27625 121.26486 121.26421
 160 121.13521 121.26799 121.26744 121.26749 121.26749

DEPTH SETTINGS: .8 .1
 # INTEGRAL VALUES
 10 31.04420
 20 40.48463 43.63145
 40 42.69365 43.42999 43.41656
 80 43.19066 43.35633 43.35143 43.35040
 160 43.31245 43.35306 43.35284 43.35286 43.35287

DEPTH SETTINGS: .8 .2
 # INTEGRAL VALUES
 10 35.58067
 20 47.08428 50.91882
 40 49.77675 50.67423 50.65793
 80 50.38166 50.58331 50.57725 50.57597
 160 50.52980 50.57919 50.57891 50.57894 50.57895

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 5 of 8)

DEPTH SETTINGS: .8 .3
 # INTEGRAL VALUES
 10 41.18556
 20 55.65137 60.47330
 40 59.03963 60.16904 60.14877
 80 59.79940 60.05267 60.04491 60.04326
 160 59.98545 60.04746 60.04712 60.04716 60.04717

DEPTH SETTINGS: .8 .4
 # INTEGRAL VALUES
 10 48.11955
 20 67.06851 73.38484
 40 71.51479 72.99687 72.97101
 80 72.50894 72.84033 72.82990 72.82767
 160 72.75223 72.83333 72.83286 72.83290 72.83292

DEPTH SETTINGS: .8 .5
 # INTEGRAL VALUES
 10 56.52834
 20 82.75034 91.49101
 40 88.93369 90.99481 90.96173
 80 90.30942 90.76801 90.75288 90.74957
 160 90.64572 90.75783 90.75714 90.75720 90.75723

DEPTH SETTINGS: .8 .6
 # INTEGRAL VALUES
 10 65.90335
 20 105.00052 118.03291
 40 114.37607 117.50127 117.46582
 80 116.44173 117.13029 117.10556 117.09984
 160 116.94577 117.11378 117.11269 117.11281 117.11285

DEPTH SETTINGS: .8 .7
 # INTEGRAL VALUES
 10 73.39093
 20 137.28430 158.58209
 40 153.69971 159.17151 159.21082
 80 157.23471 158.41306 158.36249 158.34903
 160 158.09372 158.38007 158.37787 158.37811 158.37820

DEPTH SETTINGS: .8 .8
 # INTEGRAL VALUES
 10 70.48421
 20 180.75055 217.50598
 40 218.62714 231.25269 232.16913
 80 226.36603 228.94568 228.79190 228.73828
 160 228.22040 228.83353 228.83139 228.83203 228.83243

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 6 of 8)

DEPTH SETTINGS: .9 .1
 # INTEGRAL VALUES
 10 17.84621
 20 32.75415 37.72346
 40 41.42445 44.31454 44.75394
 80 43.53635 44.24033 44.23537 44.22714
 160 44.00644 44.16314 44.15799 44.15676 44.15649

DEPTH SETTINGS: .9 .2
 # INTEGRAL VALUES
 10 20.65593
 20 38.77641 44.81657
 40 49.36316 52.89208 53.43045
 80 51.94231 52.80202 52.79602 52.78595
 160 52.51622 52.70753 52.70123 52.69972 52.69939

DEPTH SETTINGS: .9 .3
 # INTEGRAL VALUES
 10 24.17368
 20 46.84447 54.40140
 40 60.17184 64.61430 65.29517
 80 63.41963 64.50223 64.49477 64.48207
 160 64.14200 64.33280 64.37483 64.37292 64.37250

DEPTH SETTINGS: .9 .4
 # INTEGRAL VALUES
 10 28.59248
 20 58.01803 67.82655
 40 75.48256 81.30406 82.20256
 80 79.74065 81.16002 81.15042 81.13373
 160 80.68710 81.00259 80.99211 80.98959 80.98903

DEPTH SETTINGS: .9 .5
 # INTEGRAL VALUES
 10 34.04769
 20 74.11070 87.46504
 40 98.29401 106.35510 107.61444
 80 104.19609 106.16345 106.15068 106.12746
 160 105.50627 105.94299 105.92830 105.92477 105.92398

DEPTH SETTINGS: .9 .6
 # INTEGRAL VALUES
 10 40.28468
 20 98.32802 117.67580
 40 134.61407 146.70944 148.64502
 80 143.48981 146.44839 146.43100 146.39587
 160 145.45551 146.11075 146.08826 146.08279 146.08154

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 7 of 8)

DEPTH SETTINGS: .9 .7
 # INTEGRAL VALUES
 10 45.62609
 20 136.09888 166.25647
 40 197.89185 218.48950 221.97171
 80 213.10883 218.18115 218.16061 218.10010
 160 216.46152 217.57910 217.53897 217.52911 217.52686

DEPTH SETTINGS: .9 .8
 # INTEGRAL VALUES
 10 44.98096
 20 191.67960 240.57913
 40 322.45270 366.04370 374.40802
 80 355.86694 367.00507 367.06915 366.95270
 160 363.11066 365.52521 365.42657 365.40051 365.39441

DEPTH SETTINGS: .9 .9
 # INTEGRAL VALUES
 10 30.99894
 20 216.04062 277.72119
 40 585.06165 708.06873 736.75854
 80 724.99939 771.64539 775.88391 776.50500
 160 753.37219 762.82971 762.24207 762.02551 761.96875

FIGURE B5. Integral values for depth mismatch with a radial dipole. (Page 8 of 8)

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DEPTH SETTINGS: .1 .1
C INTEGRAL VALUES

10	38.18136				
20	38.17668	38N17496			
40	38.17636	38.17625	38N17635		
80	38N17635	38.17634	38.17635	38N17635	
160	38.17634	38.17634	38.17634	38.17634	38N17634

DEPTH SETTINGS: .2 .1
INTEGRAL VALUES

10	38.67245				
20	38.66609	38.66398			
40	38.66570	38.66557	38.66567		
80	38.66566	38.66566	38N66566	38N66566	
160	38.66566	38.66566	38.66566	38N66566	38N66566

DEPTH SETTINGS: .2 N2
C INTEGRAL VALUES

10	39.69132				
20	39.68321	39.68051			
40	39.68272	39.68256	39.68270		
80	39.68269	39.68269	39.68269	39N68269	
160	39.68269	39.68269	39N68269	39.68269	39.68269

DEPTH SETTINGS: .3 N1
C INTEGRAL VALUES

10	39N17677				
20	39.16817	39.16531			
40	39.16766	39.16749	39N16763		
80	39.16763	39.16763	39.16763	39N16763	
160	39N16762	39.16761	39.16761	39N16761	39.16761

DEPTH SETTINGS: .3 .2
INTEGRAL VALUES

10	40.76611				
20	40.75484	40N75103			
40	40.75413	40.75396	40.75415		
80	40.75414	40N75413	40N75414	40N75414	
160	40N75413	40.75413	40N75413	40N75413	40.75413

DEPTH SETTINGS: .3 N3
C INTEGRAL VALUES

10	42.46990				
20	42.47391	42.46859			
40	42N47293	42.47266	42N47293		
80	42.47293	42.47292	42.47293	42N47293	
160	42N47291	42.47290	42N47290	42N47293	42.47289

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 1 of 8)

DEPTH SETTINGS: .4 N1
INTEGRAL VALUES

10	39N69643				
20	39.68349	39.67918			
40	39.68275	39N68251	39.68273		
80	39N68269	39.68267	39N68269	39N68269	
160	39.68269	39.68269	39N68270	39N68270	39N68270

DEPTH SETTINGS: .4 .2
INTEGRAL VALUES

10	41.90273				
20	41.88536	41.87957			
40	41.88433	41.88406	41N88436		
80	41.88433	41.88432	41.88433	41.88433	
160	41.88434	41.88435	41N88435	41N88435	41N88435

DEPTH SETTINGS: .4 .3
INTEGRAL VALUES

10	44N36707				
20	44N34204	44N33370			
40	44.34065	44.34019	44.34061		
80	44N34059	44N34058	44.34060	44.34060	
160	44.34057	44.34057	44.34057	44.34056	44N34056

DEPTH SETTINGS: .4 .4
INTEGRAL VALUES

10	47N13818				
20	47N09856	47.08535			
40	47N09648	47.09579	47.09649		
80	47N09635	47.09631	47.09634	47N09634	
160	47N09630	47.09628	47.09628	47.09628	47.09628

DEPTH SETTINGS: .5 N1
C INTEGRAL VALUES

10	40.23554				
20	40.21263	40.20499			
40	40.21144	40N21104	40.21145		
80	40.21137	40.21135	40.21137	40N21137	
160	40N21131	40.21130	40.21130	40N21130	40.21130

DEPTH SETTINGS: .5 N2
C INTEGRAL VALUES

10	43.11098				
20	43.07975	43.06934			
40	43.07814	43.07761	43.07816		
80	43.07804	43.07801	43.07803	43.07803	
160	43.07807	43.07809	43.07809	43N07809	43.07809

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 2 of 8)

DEPTH	SETTINGS:	.5	.3		
C		INTEGRAL VALUES			
10	46.42437				
20	46N37393	46.36373			
40	46N37665	46.37533	46.37669		
80	46N37650	46.37644	46.37643	46.37648	
160	46N37648	46.37647	46.37647	46.37647	46N37647

DEPTH	SETTINGS:	.5	.4		
#		INTEGRAL VALUES			
10	50.23342				
20	50.21092	50.18676			
40	50.20743	50.20627	50N20757		
80	50.20722	50.20715	50.20721	50N20720	
160	50.20720	50.20719	50.20719	50N20719	50N20719

DEPTH	SETTINGS:	N5	N5		
#		INTEGRAL VALUES			
10	54.34145				
20	54N70949	54.66550			
40	54.70364	54.70169	54N70410		
80	54N70329	54N70316	54N70326	54N70325	
160	54.70326	54.70325	54N70326	54.70326	54N70326

DEPTH	SETTINGS:	N6	N1		
#		INTEGRAL VALUES			
10	40.30661				
20	40.75653	40.73990			
40	40N75423	40.75352	40.75442		
80	40.75414	40.75410	40N75413	40N75413	
160	40.75413	40.75413	40N75413	40N75413	40N75413

DEPTH	SETTINGS:	.6	.2		
C		INTEGRAL VALUES			
10	44.41244				
20	44.34337	44.32101			
40	44.34076	44.33972	44N34097		
80	44.34056	44N34049	44.34055	44.34054	
160	44.34059	44N34061	44N34061	44.34062	44.34062

DEPTH	SETTINGS:	.6	N3		
C		INTEGRAL VALUES			
10	43.70733				
20	43N60311	43.57435			
40	43.60363	43.60220	43N60403		
80	43.60342	43.60333	43.60341	43N60340	
160	43.60342	43.60342	43N60342	43N60342	43N60342

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 3 of 8)

DEPTH SETTINGS: .6 .4
 C INTEGRAL VALUES
 10 53.90766
 20 53.75023 53N69775
 40 53.74345 53.74119 53.74409
 80 53.74306 53.74293 53.74304 53N74303
 160 53.74300 53N74299 53.74300 53.74300 53.74300

DEPTH SETTINGS: .6 .5
 # INTEGRAL VALUES
 10 60.33267
 20 60.05597 59.96374
 40 60.04453 60.04072 60N04535
 80 60N04388 60.04367 60N04336 60N04383
 160 60.04336 60.04385 60.04336 60N04336 60N04386

DEPTH SETTINGS: .6 N6
 # INTEGRAL VALUES
 10 63.48164
 20 67.94493 67.76610
 40 67N92265 67.91521 67N92514
 80 67.92142 67.92100 67N92140 67.92134
 160 67N92136 67.92134 67N92136 67.92134 67N92134

DEPTH SETTINGS: .7 .1
 C INTEGRAL VALUES
 10 41.43032
 20 41.31772 41.23019
 40 41.31193 41.31000 41.31199
 80 41.31161 41.31152 41.31162 41.31161
 160 41.31162 41.31162 41.31163 41N31163 41.31163

DEPTH SETTINGS: .7 N2
 # INTEGRAL VALUES
 10 45.83925
 20 45.63613 45.63510
 40 45.67827 45.67565 45.67835
 80 45.67786 45.67773 45N67737 45.67786
 160 45.67781 45N67780 45N67780 45.67780 45.67780

DEPTH SETTINGS: .7 N3
 # INTEGRAL VALUES
 10 51.27926
 20 51.06013 50.93709
 40 51.04839 51.04514 51.04902
 80 51.04827 51.04806 51.04826 51.04825
 160 51.04824 51.04823 51.04823 51N04823 51N04823

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 4 of 8)

DEPTH SETTINGS: N7 .4

95

#	INTEGRAL VALUES					
10	58N14497					
20	57N81055	57.69907				
40	57N79329	57N78754	57N79343			
80	57N79236	57.79205	57.79236	57.79234		
160	57N79228	57.79226	57.79227	57N79227	57N79227	

DEPTH SETTINGS: .7 N5

#	INTEGRAL VALUES					
10	67.05649					
20	66N50650	66N32317				
40	66.47731	66.46757	66.47720			
80	66N47580	66.47530	66N47581	66.47578		
160	66.47569	66.47566	66.47568	66.47568	66.47568	

DEPTH SETTINGS: .7 .6

#	INTEGRAL VALUES					
10	79N00438					
20	78.06433	77N75081				
40	78.00691	77.98773	78N00359			
80	78.00404	78N00310	78N00412	78.00412		
160	78N00389	78N00384	78N00389	78N00389	78N00389	

DEPTH SETTINGS: N7 .7

#	INTEGRAL VALUES					
10	95.31302					
20	94.06223	93N64531				
40	93.91492	93N86583	93.88054			
80	93.90831	93N90611	93.90880	93.90924		
160	93.90790	93.90776	93.90788	93.90787	93.90785	

DEPTH SETTINGS: .8 N1

#	INTEGRAL VALUES					
10	41.99715					
20	41.90944	41.88020				
40	41.88551	41N87753	41N87734			
80	41.88441	41.88406	41.88449	41.88461		
160	41.88435	41.88432	41.88435	41.88435	41.88435	

DEPTH SETTINGS: .8 N2

#	INTEGRAL VALUES					
10	47.24583					
20	47N13045	47N09199				
40	47.09788	47.08704	47.08670			
80	47N09643	47N09595	47.09654	47.09669		
160	47.09632	47.09626	47.09631	47N09630	47.09629	

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 5 of 8)

DEPTH SETTINGS:		.3			.3
C	INTEGRAL VALUES				
10	53.94770				
20	53.79185	53.73990			
40	53.74525	53N72973	53N72905		
80	53.74315	53N74245	53N74330	53.74353	
160	53.74303	53N74299	53N74303	53.74303	53.74302

DEPTH SETTINGS:		.8			N4
#	INTEGRAL VALUES				
10	62.75647				
20	62N54353	62.47256			
40	62.47225	62N44849	62N44689		
80	62.46908	62.46802	62N46932	62N46968	
160	62.46892	62.46886	62.46892	62N46891	62N46891

DEPTH SETTINGS:		.8			.5
#	INTEGRAL VALUES				
10	74N74194				
20	74.47186	74.38184			
40	74.35223	74N31236	74N30772		
80	74.34692	74.34515	74.34735	74.34798	
160	74.34663	74.34654	74.34663	74.34662	74.34662

DEPTH SETTINGS:		N8			.6
#	INTEGRAL VALUES				
10	91.65601				
20	91N52360	91.47948			
40	91.29304	91.21619	91N19864		
80	91.28297	91.27962	91.28336	91N28522	
160	91.28241	91.28223	91.28241	91N28233	91N28236

DEPTH SETTINGS:		N8			.7
C	INTEGRAL VALUES				
10	115.87997				
20	117N53708	118.08946			
40	116.98511	116N80113	116.71525		
80	116.96175	116.95396	116N96416	116N96311	
160	116.96049	116N96008	116.96049	116N96043	116.96040

DEPTH SETTINGS:		.8			.8
C	INTEGRAL VALUES				
10	147.62323				
20	161N10431	165.59799			
40	159N44022	158.88553	158N43302		
80	159.35745	159N32986	159.35947	159N37411	
160	159.35321	159.35181	159.35324	159N35315	159.35306

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 6 of 8)

DEPTH	SETTINGS:	.9	.1		
#		INTEGRAL VALUES			
10	41.70350				
20	42.50838	42.77502			
40	42.43525	42.47755	42N45772		
80	42.47356	42N46966	42N46913	42N46931	
160	42N47304	42N47286	42N47303	42N47314	42N47315

DEPTH	SETTINGS:	N9	.2		
#		INTEGRAL VALUES			
10	47N55993				
20	48N65105	49N01475			
40	48.62004	48.60971	48.58271		
80	48.60417	48.59888	48.59316	48N59341	
160	48.60345	48.60322	48.60351	48N60359	48N60361

DEPTH	SETTINGS:	.9	.3		
#		INTEGRAL VALUES			
10	56N22525				
20	56N79311	57.31573			
40	56.74988	56.73547	56.69678		
80	56N72726	56N71972	56.71867	56.71902	
160	56N72622	56.72587	56.72628	56.72640	56.72643

DEPTH	SETTINGS:	N9	.4		
#		INTEGRAL VALUES			
10	65N60475				
20	68.02049	68.82574			
40	67.95737	67.93632	67.87703		
80	67N92297	67N91151	67N90987	67.91033	
160	67.92145	67.92094	67.92157	67.92175	67N92180

DEPTH	SETTINGS:	.9	.5		
#		INTEGRAL VALUES			
10	80.20787				
20	84.31776	85.68773			
40	84.22104	84N18330	84.08387		
80	84.16391	84N14437	84N14194	84.14278	
160	84.16138	84.16054	84N16159	84.16191	84N16197

DEPTH	SETTINGS:	.9	.6		
#		INTEGRAL VALUES			
10	101.54285				
20	109.66370	112.37065			
40	109.51311	109N46291	109.26906		
80	109N40536	109.36945	109N36322	109N36470	
160	109N40060	109.39903	109.40102	109N40161	109.40176

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 7 of 8)

DEPTH SETTINGS: .9		N7			
#	INTEGRAL VALUES				
10	133.05438				
20	153.06970	159.74148			
40	152.91453	152.86233	152N40430		
80	152.66522	152N53212	152N56341	152N56595	
160	152.65420	152N65054	152N65512	152.65658	152.65695

DEPTH SETTINGS: N9		.8			
#	INTEGRAL VALUES				
10	175.31088				
20	237N82312	258N66717			
40	239.59750	240.13728	238N95529		
80	238N75351	238.47217	238N35732	238N34333	
160	238.71673	238.70447	238.71994	238N72571	238.72717

DEPTH SETTINGS: N9		N9			
#	INTEGRAL VALUES				
10	213.03949				
20	412.16974	473.54645			
40	466N13220	484N11969	484N49121		
80	460.53174	458.66498	456.96795	456.53106	
160	460.20934	460.10254	460.19336	460.24963	460N26422

FIGURE B6. Integral values for depth mismatch with a tangential dipole. (Page 8 of 8)

shown explicitly since these parameters are the same for both dipoles and are held constant) and c_r is the normalizing constant defined in equation II-28a. Since R has been defined to be unity and the integrand is independent of ϕ , equation B2 reduces to

$$b_r = \frac{2\pi}{c_r} \int_0^\pi F_1(\theta|r_1) F_1(\theta|r_0) \sin\theta d\theta \quad B3$$

Similarly, for a dipole oriented in the a_0 direction, the integral is

$$b_\theta = \int h_\theta(\theta, \phi) v(\theta, \phi) dq \quad B4$$

or,

$$b_\theta = \frac{1}{c_\theta} \int_0^{2\pi} \int_0^\pi F_2(\theta|r_1) (\cos\phi) F_2(\theta|r_0) (\cos\phi) R^2 \sin\theta d\theta d\phi \quad B5$$

Since the integral of $\cos^2\phi$ over 2π radians equals π , this equation reduces to

$$b_\theta = \frac{\pi}{c_\theta} \int_0^\pi F_2(\theta|r_1) F_2(\theta|r_0) \sin\theta d\theta \quad B6$$

Finally, for a dipole oriented in the a_ϕ direction, the integral is

$$b_\phi = \int h_\phi(\theta, \phi) v(\theta, \phi) dq \quad B7$$

or,

$$b_\phi = \frac{1}{c_\phi} \int_0^{2\pi} \int_0^\pi F_2(\theta|r_1) (\sin\phi) F_2(\theta|r_0) (\sin\phi) R^2 \sin\theta d\theta d\phi \quad B8$$

which similarly reduces to

$$b_{\phi} = \frac{\pi}{c_{\phi}} \int_0^{\pi} F_2(\theta|r_1) F_2(\theta|r_0) \sin\theta d\theta \quad B9$$

The program listed in fig. B4 was written to determine as acceptable way of performing the numerical integrations necessary to evaluate the integrals in equations B3, B6, and B9. This program uses the Romberg integration method.²⁵ Program results are shown in figure B5 for a radial dipole and figure B6 for a tangential dipole (note that the integrals for $\bar{a}_{\theta}$ and $\bar{a}_{\phi}$ directed dipoles are identical). These outputs give the values of the quantities $b_{r,r} c_r$ and $b_{\theta,\theta} c_{\theta}$ (or $b_{\phi,\phi} c_{\phi}$) as functions of r_0 and r_1 .

From the program outputs, it can be seen that the integral converges rapidly as the number of points used increases for dipole locations near the center. However, for values of r_0 and r_1 greater than approximately 0.7, convergence is much slower. This result is to be expected since the surface potential curves change more rapidly for dipoles near the surface (see figures III-1 and III-2). Based on these results, it was decided that 160 points and a third order approximation should be used in evaluating the integrals (i. e., fifth row, third column, of integral values on the printouts).

Another program was written to present the same

data in more readable form (see figure B7 for listing). The basic changes from the previous program are that the constant c_r , c_θ , or c_ϕ is included in the computation and that the total signal energy is normalized to unity. This program therefore calculates the parameter γ as defined in equation II-16.

The first step of this program is to calculate the integrals of equation B3, B6, or B9 as follows:

$$I_1(r_1, r_0) = \int_0^\pi F_1(\theta|r_1)F_1(\theta|r_0)\sin\theta d\theta \quad B10$$

$$I_2(r_1, r_0) = \int_0^\pi F_2(\theta|r_1)F_2(\theta|r_0)\sin\theta d\theta \quad B11$$

It can be seen that the values of r_1 and r_0 may be interchanged without affecting the value of the integral, so these integrals are calculated only for values such that r_1 is greater than or equal to r_0 .

The program output is as follows. First, for a radial dipole, the integral of the square of the surface potential (the power) is printed for each depth used. This value is equal to the integral I_1 when $r_1 = r_0$. In addition, the average value of the surface potential is printed. It will be noted that the average is zero to within the accuracy of the computer used (approximately six decimal digits). Next, the values of the parameter γ are printed for values of r_0 and r_1 from 0.1 to 0.9 in 0.1 steps. The value of γ is calculated as

```

0000 REM  FILTER RESPONSE FOR DEPTH MISMATCH.
0010 L9=0
0020 GOTO 1170
0030 C=COS(Q)
0040 S=SIN(Q)
0050 L=SQR(1+D*(D-2*C))
0060 L1=SQR(1+D1*(D1-2*C))
0070 IF L9 THEN 1130
0080 V1=(2*(C-D)/(L*L)+(1-L)/D)/L
0090 H1=(2*(C-D1)/(L1*L1)+(1-L1)/D1)/L1
0100 S1=S1+V1*H1*S
0110 S2=S2+V1*S
0120 RETURN
0130 V1=2*S/L+3+((D-C)/L+C)/(D*S)
0140 H1=2*S/L1+3+((D1-C)/L1+C)/(D1*S)
0150 GOTO 1100
0160 DIM B[9],C[9],R[9,9]
0170 PRINT "R","MAG+2","AVERAGE","NORM. MAG+2"
0180 FOR K9=1 TO 9
0190 D=D1=K9/10
0200 GOSUB 1680
0210 IF NOT L9 THEN 1230
0220 B6=0
0230 B[K9]=B6
0240 C0=A6-B6+2/(4*3.14159)
0250 C[K9]=SQR(C0)
0260 PRINT D,A6,B6,C0
0270 NEXT K9
0280 IF NOT L9 THEN 1300
0290 MAT B=ZER
0300 PRINT
0310 FOR K1=1 TO 9
0320 FOR K2=1 TO K1
0330 D=K1/10
0340 D1=K2/10
0350 GOSUB 1680
0360 R[K1,K2]=(A6-B[K1]*B[K2]/(4*3.14159))/(C[K1]*C[K2])
0370 NEXT K2
0380 PRINT "";
0390 NEXT K1
0400 PRINT
0410 PRINT "FILTER RESPONSE FOR DEPTH MISMATCH";
0420 IF NOT L9 THEN 1450
0430 PRINT "    TANGENTIAL DIPOLE"
0440 GOTO 1460
0450 PRINT "    RADIAL DIPOLE"
0460 PRINT "    R          .1          .2          .3          .4          .5"
0470 FOR K1=1 TO 9
0480 R=K1/10
0490 PRINT USING "#,D.DX";R

```

FIGURE B7. Final program to evaluate filter response for depth mismatch. (Page 1 of 2)

```

1500 FOR K2=1 TO 5
1510 R0=R(K1 MAX K2,K1 MIN K2)
1520 PRINT USING "#,3D.6D";R0
1530 NEXT K2
1540 PRINT
1550 NEXT K1
1560 PRINT LIN(1)
1570 PRINT " R .6 .7 .8 .9"
1580 FOR K1=1 TO 9
1590 R=K1/10
1600 PRINT USING "#,D.DX";R
1610 FOR K2=6 TO 9
1620 R0=R(K1 MAX K2,K1 MIN K2)
1630 PRINT USING "#,3D.6D";R0
1640 NEXT K2
1650 PRINT
1660 NEXT K1
1670 STOP
1680 REM ROMBERG INTEGRATION
1690 S1=S2=0
1700 FOR I=39 TO 1 STEP -1
1710 Q=I*3.14159/40
1720 GOSUB 1030
1730 NEXT I
1740 A1=3.14159+Z/20*S1
1750 B1=3.14159+Z/20*S2
1760 FOR I=79 TO 1 STEP -2
1770 Q=I*3.14159/80
1780 GOSUB 1030
1790 NEXT I
1800 A2=3.14159+Z/40*S1
1810 B2=3.14159+Z/40*S2
1820 FOR I=159 TO 1 STEP -2
1830 Q=I*3.14159/160
1840 GOSUB 1030
1850 NEXT I
1860 A3=3.14159+Z/80*S1
1870 B3=3.14159+Z/80*S2
1880 A4=(4*A2-A1)/3
1890 B4=(4*B2-B1)/3
1900 A5=(4*A3-A2)/3
1910 B5=(4*B3-B2)/3
1920 A6=(16*A5-A4)/15
1930 B6=(16*B5-B4)/15
1940 IF NOT L9 THEN 1970
1950 A6=A6/Z
1960 B6=B6/Z
1970 RETURN
1980 END

```

FIGURE B7. Final program to evaluate filter response for depth mismatch. (Page 2 of 2)

R	MAG: 2	AVERAGE	NORM. MAG: 2
.1	38.3363	-1.23766E+05	38.3363
.2	40.3607	-2.56994E-06	40.3607
.3	44.1591	-2.30582E-06	44.1591
.4	50.5739	-1.68921E+06	50.5739
.5	61.4355	-5.54538E+06	61.4355
.6	80.9942	-1.49553E-06	80.9942
.7	121.267	-1.03248E-06	121.267
.8	228.331	-5.34455E-06	228.331
.9	762.222	-3.64765E+04	762.222

FILTER RESPONSE FOR DEPTH MISMATCH			RADIAL DIPOLE		
R	.1	.2	.3	.4	.5
0.1	1.000000	0.991264	0.964067	0.916577	0.846333
0.2	0.991264	1.000000	0.990402	0.959517	0.903511
0.3	0.964067	0.990402	1.000000	0.988376	0.951693
0.4	0.916577	0.959517	0.988376	1.000000	0.986264
0.5	0.846333	0.903511	0.951693	0.986264	1.000000
0.6	0.750359	0.817392	0.881241	0.938166	0.981614
0.7	0.624359	0.694588	0.767676	0.841894	0.913161
0.8	0.462865	0.526299	0.597344	0.676994	0.765444
0.9	0.253322	0.300463	0.350333	0.412490	0.489506
R	.6	.7	.8	.9	
0.1	0.750359	0.624359	0.462865	0.253322	
0.2	0.817392	0.694588	0.526299	0.300463	
0.3	0.881241	0.767676	0.597344	0.350333	
0.4	0.938166	0.841894	0.676994	0.412490	
0.5	0.981614	0.913161	0.765444	0.489506	
0.6	1.000000	0.972467	0.860240	0.537955	
0.7	0.972467	1.000000	0.950746	0.715519	
0.8	0.860240	0.950746	1.000000	0.874978	
0.9	0.537955	0.715519	0.874978	1.000000	

FIGURE B8. Computed radial filter response for depth mismatch. The simulated source is a radial dipole.

R	MAG+2	AVERAGE	NORM. MAG+2
.1	33.1763	0	33.1763
.2	39.6326	0	39.6326
.3	42.4723	0	42.4723
.4	47.0962	0	47.0962
N5	54.7031	0	54.7031
.6	67.9211	0	67.9211
.7	93.9077	0	93.9077
.8	159.353	0	159.353
.9	460.197	0	460.197

FILTER RESPONSE FOR DEPTH MISMATCH			TANGENTIAL DIPOLE		
R	.1	.2	.3	.4	.5
ON1	1.000000	0.993409	0.972639	0.935359	0.879924
0.2	0.993409	1.000000	0.992692	0.968352	0.924533
ON3	0.972639	0.992692	1.000000	0.991407	0.962133
0.4	0.935359	0.968352	0.991407	1.000000	0.959159
ON5	0.879924	0.924533	0.962133	0.989159	1.000000
0.6	0.800334	0.854073	0.904914	ON950223	0.985051
0.7	0.689960	0.743262	0.803302	0.869011	0.927431
0.8	0.537000	0.592251	0.653259	0.721090	0.796295
0.9	0.320433	0.359662	0.405747	0.461362	0.530437

R	.6	.7	.8	.9
0.1	0.800334	0.689960	0.537000	0.320433
0.2	0.854073	0.743262	0.592251	0.359662
ON3	0.904914	0.803302	0.653259	0.405747
0.4	0.950223	0.869011	0.721090	0.461362
ON5	0.985051	0.927431	0.796295	0.530437
0.6	1.000000	0.976702	0.877412	0.618793
0.7	0.976702	1.000000	0.956109	0.734324
0.8	0.877412	0.956109	1.000000	ON381523
0.9	0.618793	0.734324	0.881523	1.000000

FIGURE B9. Computed tangential filter response for depth mismatch. The simulated source is a tangential dipole.

$$\gamma_1 = I_1(r_1, r_0) / [I_1(r_1, r_1) I_1(r_0, r_0)]^{1/2} \quad B12$$

Output for a tangential dipole is similar. In this case, the average value of the surface potential is set to zero by the program. The average value is zero in this case because of the sinusoidal variation of the potential with θ . The parameter γ is here calculated as

$$\gamma_2 = I_2(r_1, r_0) / [I_2(r_1, r_1) I_2(r_0, r_0)]^{1/2} \quad B13$$

Outputs for radial and tangential dipoles are listed in figures B8 and B9. These data are also plotted in figure III-4.

POSITION VARIATION SIMULATION

A fourth program was written to evaluate the filter output for the case in which the actual dipole and the filter dipole locations are independent. This program is required to perform a two dimensional numerical integration. The listing is shown in fig. B10 and the results are shown in figure B11.

For this program, the coordinate system is defined such that the actual dipole lies on the positive z axis. The filter dipole may be located anywhere within the sphere. Parameters which must be specified include the actual dipole orientation and depth and the filter dipole orien-

```

0001 C    SPHERICAL INTEGRATION PROGRAM.
0002 C    JOSEPH ROTHWEILER    JULY 20, 1976
0003      DIMENSION ICHAR(3)
0004      COMMON ERROR, IV, IH, DV, DH, GAM1, GAM2
0005      COMMON STH, CTH, SG1, CG1, SG2, CG2
0006      COMMON PI, PI50, PI51, PID3, P21, PI13
0007      PI = 3.1415926535
0008      PI25 = .25*PI
0009      PI50 = .50*PI
0010      PI75 = .75*PI
0011      PD16 = PI/16.
0012      PI156 = 15.*PI/16.
0013      PI51 = 1.5*PI
0014      PID3 = PI/3.
0015      P21 = 2.*PI
0016      PI13 = 1.3333333*PI
0017      SPI25 = SIN(PI25)
0018      SPI5 = .5*SIN(PD16)
0019      ERROR = 1.E-5
0020      ICHAR(1) = 130B
0021      ICHAR(2) = 131B
0022      ICHAR(3) = 132B
0023 C    READ REQUIRED PARAMETERS
0024      WRITE(6,2000)
0025 2000  FORMAT(15H ENTER IV,DV,IH)
0026      READ(1,*) IV,DV,IH
0027      WRITE (6,2010)
0028 2010  FORMAT(25H ENTER DH LIMITS AND SPACING)
0029      READ(1,*) IDHS,IDHE,IDHP
0030      WRITE(6,2020)
0031 2020  FORMAT(30H ENTER GAM1 LIMITS AND SPACING)
0032      READ(1,*) IG1S,IG1E,IG1P
0033      IG1S = IG1S+1
0034      IG1E = IG1E +1
0035      WRITE (6,2030)
0036 2030  FORMAT(30H ENTER GAM2 LIMITS AND SPACING)
0037      READ(1,*) IG2S,IG2E,IG2P
0038      IG2S = IG2S+1
0039      IG2E = IG2E+1
0040 C    WRITE HEADING
0041 1500  WRITE (6,1300)
0042 1300  FORMAT(1H0,2X,6HACTUAL,4X,6HFILTER,5X,13HX ROTATE  Y ROTATE,
0043 1 3X,4HLAST,7X,6HOUTPUT)
0044 C    START LOOPS
0045      DO 1600 IDH = IDHS,IDHE,IDHP
0046      DH = FLOAT(IDH)/100.
0047      DO 1600 IG1 = IG1S,IG1E,IG1P
0048      GAM1 = PI*FLOAT(IG1-1)/100.
0049      SG1 = SIN(GAM1)
0050      CG1 = COS(GAM1)

```

FIGURE B10. Program to calculate filter response for general position mismatch. (Page 1 of 4)

```

0051      DO 1600 IG2 = IG2S,IG2E,IG2P
0052      GAM2 = PI*FLOAT(IG2-1)/100.
0053      SG2 = SIN(GAM2)
0054      CG2 = COS(GAM2)
0055  C**** BEGIN INTEGRATION OVER SURFACE OF SPHERE *****
0056      AREA4 = (PHINT(PI25)+PHINT(PI75))*SPI25+PHINT(PI50)
0057      AREA6 = ((PHINT(PI16)+PHINT(PI156))*SPI5*.5+AREA4)*PI25
0058      M = 4
0059  100    M = 2*M
0060  C      THE NUMBER OF DIVISIONS DOUBLES FOR EACH LOOP.
0061      DVM = 1./FLOAT(M)
0062      IF(64-M)200,105
0063  105    AREA3 = AREA6
0064      AREA5=(PHINT(PI25*DVM)+PHINT(PI-PI25*DVM))*SIN(PI25*DVM)*.5
0065      DO 110 J=2,M,2
0066      THETA = FLOAT(J-1)*PI*DVM
0067  110    AREA4 = AREA4 +PHINT(THETA) *SIN(THETA)
0068      AREA6 = (AREA4+AREA5*.5)*PI*DVM
0069      IF(ABS(AREA6)-1.E-8)150,140
0070  140    IF(ERROR-ABS((AREA6-AEA3)/AREA6))100,200
0071  150    IF(1.E-8-ABS(AREA3))100,200
0072  200    CONTINUE
0073  C**** END OF INTEGRATION OVER SPHERE *****
0074      WRITE(6,1400)ICHAR(IV),DV,ICHAR(IH),DH,GAM2,GAM1,AREA3,AREA6
0075  1400    FORMAT(2(1X,A1,2X,F4.2,2X),2F10.5,2F13.5)
0076  1600    CONTINUE
0077  1      STOP
0078      END

```

FIGURE B10. Program to calculate filter response for general position mismatch. (Page 2 of 4)


```

FUNCTION PHINT(THETA)
COMMON ERROR, IV, IH, DV, DH, GAM1, GAM2
COMMON STH, CTH, SG1, CG1, SG2, CG2
COMMON PI, PI50, PI51, PID3, P21, PI13
STH = SIN(THETA)
CTH = COS(THETA)
ERR2 = ERROR/(2.*STH+1.)
SUM1 = FIL(0.) + FIL(PI)
SUM2 = FIL(PI50)+FIL(PI51)
AREA2 = PID3*(SUM1+2.*SUM2)
I = 4
100 I=2*I
DVI = 1./FLOAT(I)
IF(64-I)200,105
105 AREA1 = AREA2
SUM1 = SUM1+SUM2
SUM2 = 0.
DO 110 J=2, I, 2
PHI = P21*DVI*FLOAT(J-1)
110 SUM2 = SUM2+FIL(PHI)
AREA2 = PI13*(SUM1+2.*SUM2)*DVI
IF(AREA2)120,150,120
120 IF(ERR2-ABS((AREA2-AREA1)/AREA2))100,200
150 IF(AREA1)100,200,100
200 PHINT = AREA2
RETURN
END

```

FIGURE B10. Program to calculate filter response for general position mismatch. (Page 3 of 4)

```

0001 FUNCTION FIL(PHIP)
0002 COMMON ERROR, IV, IH, DV, DH, GAM1, GAM2
0003 COMMON STH, CTH, SG1, CG1, SG2, CG2
0004 COMMON PI, PI50, PI51, PID3, P21, P113
0005 CPH = COS(PHIP)
0006 SPH = SIN(PHIP)
0007 X = STH*CPH
0008 Y = STH*SPH
0009 YSZC = Y*SG2-CTH*CG2
0010 CTH1 = X*SG1-YSZC*CG1
0011 X1 = X*CG1+YSZC*SG1
0012 Y1 = Y*CG2+CTH*SG2
0013 STH1 = SQRT(X1*X1+Y1*Y1)
0014 IF(STH1)100,110,100
0015 110 CPH1 = 1.
0016     SPH1 = 0.
0017     GO TO 120
0018 100 CPH1 = X1/STH1
0019     SPH1 = Y1/STH1
0020 120 CONTINUE
0021 C THE FOLLOWING SECTION REPLACES FUNCTION POT.
0022 RL = SQRT(1.+DV*(DV-2.*CTH))
0023 GO TO (1000,1010,1020,1030), IV
0024 1020 POT = (2.*(CTH-DV)/(RL*RL)+(1.-RL)/DV)/RL
0025     GO TO 11
0026 1030 POT = 1.
0027     GO TO 11
0028 1000 IF(STH)1005,1001,1005
0029 1001 POT = 0.
0030     GO TO 11
0031 1005 POT = (2.*STH/(RL*RL*RL)+((DV-CTH)/RL+CTH)/(DV*STH))*CPH
0032     GO TO 11
0033 1010 IF(STH)1015,1011,1015
0034 1011 POT = 0.
0035     GO TO 11
0036 1015 POT = (2.*STH/(RL*RL*RL)+((DV-CTH)/RL+CTH)/(DV*STH))*SPH
0037 C
0038 11 RL = SQRT(1.+DH*(DH-2.*CTH1))
0039     GO TO (1100,1110,1120,1130), IH
0040 1120 FIL = POT*((2.*(CTH1-DH)/(RL*RL)+(1.-RL)/DH)/RL)
0041     RETURN
0042 1130 FIL = POT
0043     RETURN
0044 1100 IF(STH1)1105,1101,1105
0045 1101 FIL = 0.
0046     RETURN
0047 1105 FIL = POT*((2.*STH1/(RL*RL*RL)+((DH-CTH1)/RL+CTH1)
0048     1 / (DH*STH1))*CPH1)
0049     RETURN
0050 1110 IF(STH1)1115,1111,1115
0051 1111 FIL = 0.
0052     RETURN
0053 1115 FIL = POT*((2.*STH1/(RL*RL*RL)+((DH-CTH1)/RL+CTH1)
0054     1 / (DH*STH1))*SPH1)
0055     RETURN
0056     END

```

FIGURE B10. Program to calculate filter response for general position mismatch. (Page 4 of 4)

ACTUAL	FILTER	X ROTATE	Y ROTATE	LAST	OUTPUT
Y .60	Z .60	.00000	.00000	.00000	-.00000
Y .60	Z .60	.31416	.00000	-33.50371	-33.50447
Y .60	Z .60	.62332	.00000	-43.50033	-43.49399
Y .60	Z .60	.94248	.00000	-47.91561	-47.91491
Y .60	Z .60	1.25664	.00000	-40.97373	-40.97340
Y .60	Z .60	1.57030	.00000	-32.75732	-32.75768

ACTUAL	FILTER	X ROTATE	Y ROTATE	LAST	OUTPUT
Y .60	Z .60	.00000	.30000	.00000	-.00000
Y .60	Z .60	.00000	.31416	-.00000	-.00000
Y .60	Z .60	.00000	.62332	-.00001	-.00001
Y .60	Z .60	.00000	.94248	-.00001	-.00000
Y .60	Z .60	.00000	1.25664	-.00000	-.00000
Y .60	Z .60	.00000	1.57030	-.00000	-.00000

ACTUAL	FILTER	X ROTATE	Y ROTATE	LAST	OUTPUT
Z .60	Z .30	.00000	.00000	52.91133	52.75607
Z .60	Z .30	.31416	.00000	43.22216	43.00351
Z .60	Z .30	.62332	.00000	36.17762	36.03723
Z .60	Z .30	.94248	.00000	21.08678	21.04243
Z .60	Z .30	1.25664	.00000	6.60571	6.59706
Z .60	Z .30	1.57030	.00000	-5.43933	-5.42252
Z .60	Z .40	.00000	.00000	60.31775	60.11652
Z .60	Z .40	.31416	.00000	53.74206	53.57832
Z .60	Z .40	.62332	.00000	37.38654	37.79374
Z .60	Z .40	.94248	.00000	19.94769	19.91149
Z .60	Z .40	1.25664	.00000	4.55190	4.55099
Z .60	Z .40	1.57030	.00000	-7.31892	-6.99X65
Z .60	Z .50	.00000	.00000	69.61311	69.33864
Z .60	Z .50	.31416	.00000	60.22122	60.02132
Z .60	Z .50	.62332	.00000	39.31763	39.22755
Z .60	Z .50	.94248	.00000	18.35203	18.32653
Z .60	Z .50	1.25664	.00000	2.40842	2.41482
Z .60	Z .50	1.57030	.00000	-8.42299	-8.40003
Z .60	Z .60	.00000	.00000	31.53996	31.13612
Z .60	Z .60	.31416	.00000	67.32545	67.53229
Z .60	Z .60	.62332	.00000	40.27853	40.19913
Z .60	Z .60	.94248	.00000	16.23593	16.27257
Z .60	Z .60	1.25664	.00000	.24350	.25621
Z .60	Z .60	1.57030	.00000	-9.63235	-9.60819
Z .60	Z .70	.00000	.00000	97.27993	96.61661
Z .60	Z .70	.31416	.00000	76.69778	76.41437
Z .60	Z .70	.62332	.00000	40.52637	40.46732
Z .60	Z .70	.94248	.00000	13.77016	13.76912
Z .60	Z .70	1.25664	.00000	-1.87343	-1.85573
Z .60	Z .70	1.57030	.00000	-10.64139	-10.61587

FIGURE B11. Simulation results for general position variation.

tation. In addition the range of values for the filter dipole location parameters (variables DH, GAM1, and GAM2 in the program) are specified.

Referring to fig. B10, lines 55 through 72 perform the integration over θ . This integration is performed using a rectangular integration method, as diagrammed in fig. B12. Initially, lines 56 and 57 calculate the integral using four points. Lines 59 through 71 form a loop which evaluates the integral for successively larger numbers of points, M. Variable AREA6 is the new value of the integral, AREA3 is the value from the previous iteration, AREA4 is the sum of the values of all points calculated (so that points from previous iterations need not be recalculated), and AREA5 is an additional factor due to the two end points shown in figure B12. The loop terminates when any one of the following conditions is met:

- (1) The integral is less than 10^{-8} on two successive passes.
- (2) The fractional change in the value of the integral between two passes is less than the variable named ERROR.
- (3) The integral has been evaluated for 64 points.

The function PHINT is called by the main program to perform the integration over φ . This integral is evaluated using Simpson's method. Variables SUM1 and

FIGURE B12. Rectangular integration method used for integration over theta. The diagram is drawn for $M=4$.

SUM2 hold the totals of the calculated values for even and odd points, respectively. Variable AREA2 is the latest value of the integral, while AREAL is the previous value. Criteria for terminating this integration are the same as for the θ integration, except that ERR2 rather than ERROR is the variable used in the second test.

Function FIL is called by PHINT to evaluate the product of the surface potential and the filter response function at a given point. The surface potential and filter response are both calculated using equations A16 through A18. Since these equations assume that the dipole lies on the z axis, a coordinate transformation must be performed to evaluate the filter response. This transformation may be developed in two steps. First, rotate the coordinate system about the x axis through an angle* γ_2 , producing the coordinate (x'', y'', z'') . Second, rotate the new coordinate system about the y'' axis through an angle γ_1 , producing the coordinate system (x', y', z') . Given a point with coordinates (x, y, z) , the coordinates after the first transformation are²⁶

$$x'' = x \quad \text{B14a}$$

$$y'' = y \cos \gamma_2 + z \sin \gamma_2 \quad \text{B14b}$$

$$z'' = -y \sin \gamma_2 + z \cos \gamma_2 \quad \text{B14c}$$

*Note that γ_1 and γ_2 are not related to the function γ defined in chapter II.

The second transformation produces

$$x' = -z''\sin\gamma_1 + x''\cos\gamma_1 \quad B15a$$

$$y' = y'' \quad B15b$$

$$z' = z''\cos\gamma_2 + x''\sin\gamma_2 \quad B15c$$

or, combining the two sets of equations,

$$x' = x\cos\gamma_1 + y\sin\gamma_1\sin\gamma_2 - z\sin\gamma_1\cos\gamma_2 \quad B16a$$

$$y' = y\cos\gamma_2 + z\sin\gamma_2 \quad B16b$$

$$z' = x\sin\gamma_1 - y\cos\gamma_1\sin\gamma_2 + z\cos\gamma_1\cos\gamma_2 \quad B16c$$

In function FIL, the coordinates of the surface point at which the potentials are to be evaluated are given by the variables CTH and STH, the cosine and sine of θ , and PHIP, the ϕ coordinate. These are then changed to the rectangular coordinates x, y, z represented by variables X, Y, and CTH. New coordinates x', y', z' are then calculated (variables X1, Y1, and CTH1). Finally, the position variables STH1, CPH1, and SPH1, the sine of θ' and the cosine and sine of ϕ' , respectively, for the new coordinate system are calculated.

The surface potential is calculated assuming the dipole is on the positive z axis. The filter response is calculated for a dipole on the z' axis.

Outputs of this program are shown in figure B11. The printout format is as follows. First, the direction (x, y , or z) and radial position of the actual dipole are printed. Second, the direction (x', y' , or z') and radial

position of the dipole to which the filter is matched are printed. Third, the value of γ_2 in radians is printed (labeled X ROTATE). Fourth, γ_1 is printed (labeled Y ROTATE). Fifth, the integral approximation obtained in the next to last iteration of the integration over θ is printed. Sixth, the final integral approximation is printed (labeled OUTPUT).

The value from the next to last integration is printed as a check on the convergence of the integration algorithm. Comparison of this value with the final value, and comparison with the results of the previous program, for depth mismatch, indicate the accuracy of this program to be approximately three significant figures. Execution time was found to be 1 to 2 minutes per integration on a Hewlett-Packard 21MX computer.

SIMULATION FOR DIFFERENT NUMBERS OF ELECTRODES

A program was written in Basic to simulate the filter for a finite number of electrodes. This program consists of a main program and six subroutines (see fig. B13). The starting line numbers and functions of the subroutines are as follows:

- (1) Line 5100. This subroutine calculates the


```

3533 FOR N1=1 TO 5
3518 LET L1=1
3525 REM: TEST FOR NUMBER OF ELECTRODES
3533 DIM L[2,3]
3548 DIM A[18],B[18],C[18]
3553 GOSUB 5100
3563 LET Q5=1
3573 LET Q4=Q6=3
3583 LET L[1,1]=L[2,1]=3
3593 LET L[1,2]=L[1,3]=T[11]/2
3103 LET L[2,2]=ATN(E[2,2]/E[2,1])
3113 LET L[2,3]=(L[2,2]+ATN(E[3,2]/E[3,1]))/2
3123 PRINT
3121 PRINT
3122 PRINT "ROWS:"N1;"ELECTRODES:"J1
3133 PRINT
3131 PRINT "    DIPOLE POSITIONS"," V MAGNITUDE"
3143 PRINT "THETA","PHI","SPHERE","HEMISPHERE","HEMI, MEAN=0"
3153 MAT A=ZER
3163 MAT B=ZER
3173 MAT C=ZER
3183 FOR S1=1 TO 3
3193 LET Q1=.5
3203 LET Q2=L[1,S1]
3213 LET Q3=L[2,S1]
3223 GOSUB 6333
3233 GOSUB 6533
3243 PRINT Q2,Q3,K2,K9,SQR(K9+2-K5+2/J1)
3253 FOR S2=1 TO 9
3263 LET Q1=S2/10
3273 GOSUB 6333
3283 GOSUB 7333
3293 GOSUB 7533
3303 LET A[S2]=A[S2]+K5/(3*K3*K2)
3313 LET B[S2]=B[S2]+K4/(3*K3*K9)
3323 LET C[S2]=C[S2]+(K4-K5*K17/J1)/SQR((K9+2-K5+2/J1)*(K9+2-K7+2/J1))/3
3333 NEXT S2
3343 NEXT S1
3353 PRINT
3363 PRINT " ", "NORMALIZED AVERAGE FILTER OUTPUTS"
3361 PRINT
3373 PRINT " R","SPHERE","HEMISPHERE","HEMISPHERE (MEAN= 0)"
3383 FOR S1=1 TO 9
3393 PRINT S1/10,A[S1],B[S1],C[S1]
3403 NEXT S1
3413 NEXT N1
4503 WAIT (500)
4513 PRINT ""
4523 GOTO 4533

```

FIGURE B13. Simulation program for the discrete case.
(Page 1 of 5)

```

5000 STOP
5010 REM: SUBROUTINES ***
5020 DIM T(10),N(10),E(200,3),V(200),H(200)
5100 REM PROGRAM TO CALCULATE ELECTRODE POSITIONS
5110 LET N2=2*N1+1
5120 LET T0=1.5708/N2
5130 LET N3=INT(.999999+1/(1-COS(T0)))
5140 LET T1=2*T0
5150 LET T2=3*T0
5160 LET LC=10=0
5170 LET I1=1
5180 FOR C0=2 TO N3
5190 LET I0=I0+1
5200 LET A0=1-C0/N3
5210 IF A0#0 THEN 5240
5220 LET T9=1.5708
5230 GOTO 5270
5240 LET T9=ATN(SQR(1-A0+2)/A0)
5250 IF A0>0 THEN 5270
5260 LET T9=T9+3.14159
5270 IF T9<T1 THEN 5330
5280 IF LC THEN 5310
5290 LET T(I1)=T9
5300 LET LC=1
5310 IF T9<T2 THEN 5380
5320 LET N(I1)=I0-1
5330 LET I1=I1+1
5340 LET I0=1
5350 LET T1=T1+2*T0
5360 LET T2=T1+T0
5370 LET LC=0
5380 NEXT C0
5390 IF I1>N1 THEN 5410
5400 LET N(I1)=I0-1
5410 LET N(I1)=N(I1)+1
5420 GOTO 5300
5430 FOR J0=1 TO N1
5440 NEXT J0
5500 REM ELECTRODE POSITIONS IN SPHERICAL COORDS. (Z>0 ONLY)
5510 LET E(1,1)=1
5520 LET E(1,2)=E(1,3)=2
5530 LET J1=1
5540 FOR I2=1 TO N1
5550 FOR J2=0 TO N(I2)-1
5560 LET J1=J1+1
5570 LET E(J1,1)=1
5580 LET E(J1,2)=T(I2)
5590 LET E(J1,3)=(.95*ADS(2*I2/N1-1)+J2)*6.28318/N(I2)
5600 NEXT J2
5610 NEXT I2

```

FIGURE B13. Simulation program for the discrete case.
(Page 2 of 5)

```

6100 REM CONVERSION TO RECTANGULAR COORDS.
6110 FOR J3=1 TO J1
6120 LET J4=E[J3,1]
6130 LET J5=E[J3,2]
6140 LET J6=E[J3,3]
6150 LET E[J3,1]=J4*SIN(J5)*COS(J6)
6160 LET E[J3,2]=J4*SIN(J5)*SIN(J6)
6170 LET E[J3,3]=J4*COS(J5)
6180 NEXT J3
6190 RETURN
6500 REM DEFINE V VECTOR
6510 FOR K1=1 TO J1
6520 LET P7=E[K1,1]
6530 LET P8=E[K1,2]
6540 LET P9=E[K1,3]
6550 GOSUB 6500
6560 LET V[K1]=V0
6570 NEXT K1
6580 IF NOT L1 THEN 6660
6590 FOR K1=J1+1 TO 2*J1
6600 LET P7=E[K1-J1,1]
6610 LET P8=E[K1-J1,2]
6620 LET P9=-E[K1-J1,3]
6630 GOSUB 6500
6640 LET V[K1]=V0
6650 NEXT K1
6660 REM CALCULATE V MAGNITUDE
6670 LET K2=M4=0
6680 FOR K1=1 TO J1
6690 LET K2=K2+V[K1]^2
6695 LET M4=M4+V[K1]
6700 NEXT K1
6705 LET M5=M4
6710 LET K9=SQR(K2)
6720 IF NOT L1 THEN 6760
6730 FOR K1=J1+1 TO 2*J1
6740 LET K2=K2+V[K1]^2
6745 LET M4=M4+V[K1]
6750 NEXT K1
6760 LET K2=SQR(K2)
6770 RETURN

```

FIGURE B13. Simulation program for the discrete case.
(Page 3 of 5)

```

7320 REM DEFINE H VECTOR
7310 LET M3=M6=0
7320 FOR K1=1 TO J1
7330 LET P7=E(K1,1)
7340 LET P8=E(K1,2)
7350 LET P9=E(K1,3)
7360 GOSUB 8530
7370 LET H(K1)=V3
7380 LET K3=K3+V3+2
7390 LET M6=M6+V3
7400 NEXT K1
7410 LET M7=M6
7420 LET K3=SQR(K3)
7430 IF NOT L1 THEN 7200
7440 FOR K1=J1+1 TO 2*J1
7450 LET P7=E(K1-J1,1)
7460 LET P8=E(K1-J1,2)
7470 LET P9=-E(K1-J1,3)
7480 GOSUB 8530
7490 LET H(K1)=V0
7500 LET K3=K3+V0+2
7510 LET M6=M6+V0
7520 NEXT K1
7530 LET K3=SQR(K3)
7540 RETURN
7550 REM SUBROUTINE TO MULTIPLY V*H.
7560 LET K4=0
7570 FOR K1=1 TO J1
7580 LET K4=K4+V(K1)*H(K1)
7590 NEXT K1
7600 LET K5=K4
7610 IF NOT L1 THEN 7630
7620 FOR K1=J1+1 TO 2*J1
7630 LET K5=K5+V(K1)*H(K1)
7640 NEXT K1
7650 RETURN

```

FIGURE B13. Simulation program for the discrete case.
(Page 4 of 5)

```

3300 REM COORDINATE CONVERSION SUBROUTINE
3310 LET Q7=SIN(Q2)
3320 LET Q8=COS(Q2)
3330 LET Q9=SIN(Q3)
3340 LET Q0=COS(Q3)
3350 LET P4=Q4*Q7*Q0+Q5*Q8*Q0-Q6*Q9
3360 LET P5=Q4*Q7*Q9+Q5*Q8*Q9+Q6*Q0
3370 LET P6=Q4*Q0+Q5*Q7
3380 LET P1=Q1*Q7*Q0
3390 LET P2=Q1*Q7*Q9
3400 LET P3=Q1*Q0
3410 RETURN
3500 REM POTENTIAL EVALUATION SUBROUTINE
3510 LET H4=SQR(P1*2+P2*2+P3*2)
3520 LET H1=(P1*P4+P2*P5+P3*P6)/H4
3530 LET H2=P4*P7+P5*P8+P6*P9
3540 LET H3=(P1*P7+P2*P8+P3*P9)/H4
3550 LET H3=SQR(1+H4*H4-2*H2*H4)
3560 IF ABS(H3)=1 THEN 3590
3570 LET V5=((H4-H3)*H2-(H4*H3-1)*H1)/H3+(H3*H2-H1)
3571 LET V3=V5/(H4*(1-H3*2))+2*(H2-H4*H1)/H3+3
3580 RETURN
3590 LET V3=(2*(H2-H4*H1)/H3+2+.5*(H2+SGN(H3)*H3))/H3
3600 RETURN
9999 END

```

FIGURE B13. Simulation program for the discrete case.
(Page 5 of 5)

ROWS: 1 ELECTRODES: 3

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	10.6776	10.0000	9.5107
.5932	.352718	6.43319	5.47935	5.38673
.5932	-.26928	6.43333	5.43335	5.34214

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.861613	.87369	.969196
.2	.831755	.904537	.98174
.3	.963422	.9757	.991633
.4	.992172	.994869	.99795
.5	1	1.	1
.6	.993439	.994664	.998311
.7	.963395	.968031	.994446
.8	.871819	.853487	.993337
.9	.651921	.453311	.987624

ROWS: 2 ELECTRODES: 21

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	13.3386	11.9335	13.9367
.352274	0	12.8781	11.4342	13.454
.352274	.443799	12.8604	11.4142	13.4331

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.80279	.802033	.899263
.2	.876297	.862243	.863263
.3	.948992	.946779	.96339
.4	.983541	.985165	.989325
.5	1	1.	1
.6	.984255	.98476	.988419
.7	.938526	.943393	.956514
.8	.884269	.892735	.92327
.9	.782773	.76977	.903473

FIGURE B14. Simulation results for the discrete case using a radial dipole. (Page 1 of 3)

ROWS: 3 ELECTRODES: 40

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	19.4739	17.5934	15.8316
.225513	.264239	19.3393	17.5353	15.3695
.225513	.733038	19.3339	17.5347	15.3684

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.341139	.343891	.98491
.2	.390797	.390221	.917335
.3	.951167	.952937	.963746
.4	.933413	.937045	.993157
.5	1.	1	1
.6	.933439	.934354	.93664
.7	.919376	.92333	.931331
.8	.308635	.313131	.332043
.9	.716855	.719424	.754341

ROWS: 4 ELECTRODES: 66

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	24.9118	22.4644	20.119
.174969	.426359	24.3747	22.4503	20.1375
.174969	.675158	24.3746	22.4504	20.1374

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.342355	.342463	.932794
.2	.394513	.393645	.922464
.3	.950206	.950349	.935103
.4	.935376	.936233	.939533
.5	1	1.	1.
.6	.932179	.932923	.936613
.7	.915481	.912356	.927773
.8	.777905	.783399	.831436
.9	.632774	.640372	.666453

FIGURE B14. Simulation results for the discrete case using a radial dipole. (Page 2 of 3)

ROWS: 5 ELECTRODES: 99

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	29.8257	26.7332	23.7128
.159587	.51163	33.0362	26.9627	23.9774
.159587	.962429	33.0361	26.9627	23.9774

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.6493	.648294	.956415
.2	.901556	.89995	.92352
.3	.952376	.952446	.966464
.4	.986622	.98632	.989349
.5	1	1.	1.
.6	.982159	.982142	.985371
.7	.911936	.911934	.922439
.8	.756379	.757727	.777729
.9	.573365	.574123	.606596

FIGURE B14. Simulation results for the discrete case using a radial dipole. (Page 3 of 3)

ROWS: 1 ELECTRODES: 3

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	6.94821	6.29876	6.29876
.5932	.352718	7.15447	6.44495	6.26233
.5932	-.25923	7.37663	6.35392	6.17333

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.926944	.957502	.952636
.2	.959354	.973035	.976237
.3	.961836	.986106	.987936
.4	.995241	.995324	.996539
.5	1	1	1
.6	.993392	.992883	.994972
.7	.964393	.959333	.974823
.8	.893725	.873957	.923374
.9	.762632	.715624	.829565

ROWS: 2 ELECTRODES: 21

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	13.3706	12.1974	12.1974
.352274	0	13.6039	12.7221	12.5497
.352274	.443799	13.5555	12.6703	12.4978

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.832325	.915995	.91539
.2	.926395	.944397	.943915
.3	.934023	.973039	.970763
.4	.995376	.991564	.991563
.5	1.	1	1
.6	.983357	.989372	.989473
.7	.953293	.953438	.95459
.8	.832934	.837064	.892464
.9	.791069	.790273	.806233

FIGURE B15. Simulation results for the discrete case using a tangential dipole. (Page 1 of 3)

ROWS: 3 ELECTRODES: 40

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	18.3532	17.1455	17.1455
.225513	.284239	13.5249	17.3029	17.1974
.225513	.733333	18.5217	17.2994	17.1939

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.379112	.910074	.929955
.2	.922333	.933737	.933539
.3	.963352	.93372	.966553
.4	.93332	.939741	.939723
.5	1	1	1
.6	.933201	.934366	.934351
.7	.921439	.925754	.925761
.8	.839096	.817527	.816149
.9	.379645	.690263	.693477

ROWS: 4 ELECTRODES: 66

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	23.7667	22.255	22.255
.174969	.426359	23.9161	22.3976	22.3309
.174969	.875153	23.9157	22.3973	22.3335

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.833956	.914557	.914463
.2	.92512	.942467	.942396
.3	.962294	.939264	.939224
.4	.939135	.990637	.990677
.5	1	1	1
.6	.934445	.935596	.935539
.7	.92244	.926283	.926264
.8	.73744	.794619	.794725
.9	.610216	.61976	.621016

FIGURE B15. Simulation results for the discrete case using a tangential dipole. (Page 2 of 3)

ROWS: 5 ELECTRODES: 99

DIPOLE POSITIONS		V MAGNITUDE		
THETA	PHI	SPHERE	HEMISPHERE	HEMI, MEAN=0
0	0	29.4451	27.6514	27.6514
.159587	.51163	29.6329	27.6259	27.7276
.159587	.963429	29.6323	27.6257	27.7273

NORMALIZED AVERAGE FILTER OUTPUTS

R	SPHERE	HEMISPHERE	HEMISPHERE (MEAN= 0)
.1	.873276	.913633	.913523
.2	.923692	.942059	.941962
.3	.961318	.969265	.969266
.4	.989146	.990322	.990305
.5	1.	1.	1
.6	.93552	.936806	.936784
.7	.931095	.935342	.935243
.8	.808936	.816229	.816066
.9	.617549	.626311	.626355

FIGURE B15. Simulation results for the discrete case using a tangential dipole. (Page 3 of 3)

position coordinates for the surface electrodes.

(2) Line 6500. This subroutine calculates the vector V containing the simulated surface potentials and its magnitude.

(3) Line 7000. This subroutine calculates the vector H containing the filter response function and its magnitude.

(4) Line 7500. This subroutine calculates the inner product of V and H .

(5) Line 8000. This subroutine performs a coordinate conversion from spherical to rectangular coordinates.

(6) Line 8500. This subroutine calculates the theoretical surface potential of a current dipole.

The main program and the subroutines will be described in order.

The main program performs a test for depth mismatch between the actual dipole and the filter dipole with varying numbers of electrodes. The actual dipole is located at a radial position of $r = 0.5$. Three positions are used, corresponding to locations directly beneath one electrode, midway between two electrodes, and approximately equidistant from three electrodes. The output printed is the average of the filter responses to the three dipoles.

The filter outputs are calculated five times, with increasing numbers of electrodes. Each time, these outputs

are calculated for three cases. In the first case, the electrode positions are distributed over the entire sphere. In the second case, the electrode positions are on only one half of the sphere, namely the hemisphere for which the z coordinate is greater than zero. The third case is similar to the second, with the additional constraint that the average of the surface potentials of all electrodes is zero. This third case was included to investigate the effects of alternate electrode connection schemes as discussed in chapter IV.

Referring to the results shown in figures B14 and B15, the program first prints the number of rows of electrodes and the total number of electrodes in each hemisphere. Then, the three dipole positions used are printed, along with the calculated magnitude of the surface potential vector for each of the three cases which have been described. Then, the average filter response is printed for each case as a function of the depth of the filter dipole.

The first subroutine, which begins at line 5100, was written to generate sets of electrode coordinates for varying numbers of electrodes. To achieve an even distribution of these electrodes over the surface, the sphere was partitioned into segments of equal area, with one electrode in each segment. A further constraint was that the electrodes should be approximately equidistant

from one another. The algorithm used for electrode positioning will now be described.

The first step of the algorithm is to divide the top ($z > 0$) hemisphere into sections of equal width, which will be called bands. One band will be centered around the z axis, and n_1 additional bands will be defined in this hemisphere (See fig. B16 for the case $n_1 = 2$). The total number of bands in the hemisphere is therefore $n_1 + 1$. The width of each band is then

$$\Delta\theta = \pi/(2n_1 + 1) \quad \text{B17}$$

It can be seen from fig. B16 that the θ coordinate of the edge of the band centered around the z axis is

$$\theta_0 = \Delta\theta/2 \quad \text{B18}$$

Note that θ_0 is represented by variable T0 in the program.

The hemisphere is next divided into n_3 sections of equal area, which will be called rings (The way in which n_3 is calculated will be described later). This operation may be performed by dividing the z axis from 0 to 1 into n_3 segments, each of length $\Delta z = 1/n_3$, as shown in fig. B17. The proof of this statement follows.

Given the sphere defined by the equation

$$x^2 + y^2 + z^2 = 1 \quad \text{B19}$$

and the segment of this sphere defined by the inequality

$$x^2 + y^2 \leq a^2 \leq 1 \quad ; \quad \text{B20}$$

the surface area of this segment is²⁷

$$A = 2\pi(1 - (1 - a^2)^{1/2}) \quad . \quad \text{B21}$$

But from fig. B17, using the Pythagorean theorem,

FIGURE B16. Division of a hemisphere into bands of equal width. The dots indicate the θ coordinates of the electrode positions.

FIGURE B17. Division of a hemisphere into rings of equal area.

$$1 - a^2 = z_1^2, \quad \text{B22}$$

or

$$A = 2\pi(1 - z_1) \quad \text{B23}$$

since only the upper hemisphere is being considered. Therefore, the area of the portion of the sphere between two planes parallel to the xy plane and passing through points z_1 and z_2 on the z axis is:

$$\Delta A = A(z_1) - A(z_2) = 2\pi(z_2 - z_1) \quad \text{B24}$$

From this equation it can be seen that dividing the z axis into equal segments of length Δz will divide the hemisphere into rings of equal area $2\pi\Delta z$.

In order to make the area of each ring approximately equal to the area of the top band, the number of rings, n_3 , is chosen so that Δz will be approximately equal to the distance $\Delta z'$ shown in fig. B16. Since θ_0 is the coordinate of the edge of the top band, this distance is

$$\Delta z' = 1 - \cos(\theta_0) \quad \text{B25}$$

To meet the above constraint and also produce an integral number of divisions on the z axis, n_3 is defined as

$$n_3 = \text{INT} \left[0.999999 + \frac{1}{\Delta z'} \right] \quad \text{B26}$$

where function $\text{INT}(x)$ is the greatest integer less than or equal to x . This value is calculated in line 5130 of the program.

After these rings and bands have been defined, the electrode positions are defined as follows. One electrode is placed at the top of the sphere; that is, at the position $r = 1, \theta = 0, \phi = 0$. Next, electrodes

are placed in each of the bands such that the i 'th band contains j_i electrodes, where j_i is the number of rings in that band. A ring which is split by the boundary between two bands is considered to be in the lower of the two bands. For the electrodes in one band, the θ position coordinate is set equal to the coordinate of the lower edge of the ring which is in the center of the band. The ϕ spacing between electrodes in a given band is then $2\pi/j_i$.

The loop consisting of lines 5180 through 5380 counts the number of rings in each band and stores the results in array N. The θ coordinates for the electrodes are stored in array T.

After the electrode positions are calculated in spherical coordinates, they must be converted to rectangular coordinates for use by the other subroutines. This transformation is accomplished by the portion of the subroutine which begins at line 6100.

The next subroutine, which begins at line 6500, calculates the vector V containing the simulated surface potentials. The magnitude of this vector is also calculated. The dipole location, orientation, and magnitude

are specified in the calling program. The electrode positions for the upper hemisphere are the coordinates calculated by the first subroutine. If variable L1 is zero, the vector contains values only for points on the upper hemisphere. If L1 is one, potentials are calculated for the entire sphere. In this case, the electrode coordinates for the lower hemisphere are the same as for the upper hemisphere, except that z is negated. After the vector is calculated, its magnitude is evaluated by the portion of the subroutine beginning at line 6660. This section calculates the magnitude as the square root of the sum of the squares of the vector elements.

The third subroutine, which begins at line 7000, calculates the vector H containing the filter weighting function. The operation of this subroutine is the same as that of the previous subroutine for calculating V.

The fourth subroutine, beginning at line 7500, calculates the inner product of vectors V and H.

The fifth subroutine, beginning at line 8000, performs coordinate conversions. This operation is included so that the main program can use spherical coordinates, while the potential evaluation subroutine uses rectangular coordinates. Two conversions must be performed by this subroutine. First, the position coordinates of the dipole must be converted from spherical to rectangular coordinates. This operation is accom-

plished by lines 8050 through 8070. Second, the values of the $\bar{a}_r$, $\bar{a}_\theta$, and $\bar{a}_\phi$ component dipoles are specified in the main program. The $\bar{a}_x$, $\bar{a}_y$, and $\bar{a}_z$ component dipole magnitudes are computed from these values by lines 8080 through 8100 of this subroutine.

The sixth subroutine, beginning at line 8500, calculates the potential at any point on the surface given the location, orientation, and magnitude of the current dipole. This program evaluates equations A1 through A9 of appendix A with the radius of the sphere set equal to one and the constant $1/(4\pi\sigma R)$ of equation A9 also equal to one.

APPENDIX C. EFFECTS OF NOISE

This appendix presents a preliminary investigation of the effects of noise on the apparent magnitude of the dipole. As stated in chapter II, it will be assumed that the noise component of the surface potential has zero mean and is white. That is, if the surface potential at point $\bar{r}$ is

$$v(\bar{r}) = s(\bar{r}) + n(\bar{r}) \quad C1$$

where $s(\bar{r})$ is the signal component and $n(\bar{r})$ is the noise component, then n has the following properties (repeating equations II-6a and II-6b):

$$E[n(\bar{r})] = 0 \quad C2$$

$$E[n(\bar{r}_1)n(\bar{r}_2)] = k^2 \delta_s(\bar{r}_2 - \bar{r}_1) \quad C3$$

For the discrete case, these assumptions of zero mean and whiteness imply that the mean of the noise potential at any electrode is zero, and that the cross correlation between noise components at any two different electrodes is zero. That is, the surface potential vector V is given by the equation

$$V = S + N \quad C4$$

where S is a vector of the signal potentials and N is a vector containing the noise components. Now, the assumption of zero mean implies that

$$E(N) = 0$$

C5

where 0 in this case implies the zero vector. The assumption of whiteness implies that the covariance matrix, Q , of N is proportional to the identity matrix, or

$$Q = E[NN^t] = k^2 I \quad C6$$

Equation C6 is equivalent to the statement that for any two elements n_i and n_j of N ,

$$E[n_i n_j] = \begin{cases} k^2, & i=j \\ 0, & i \neq j \end{cases} \quad C7$$

The question to be answered is how much inaccuracy the noise adds to the observed magnitudes of the component dipoles. The errors in the component dipole magnitudes for the continuous case will be presented first; then the errors for the discrete case will be derived.

EFFECTS ON MAGNITUDE FOR THE CONTINUOUS CASE

From equation II-40a, the error in the $\bar{s}_r$ directed component of the dipole will be

$$u_r = (\hat{d}_r - d_r) = \left[\int h_r(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq \right] / c_r(\bar{r}_0) \quad C8$$

The mean value of the error is then

$$E[u_r] = E \left[\int h_r(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq \right] / c_r(\bar{r}_0) \quad C9$$

or,

$$E[u_r] = \int h_r(\theta, \phi | \bar{r}_0) E[n(\theta, \phi)] dq / c_r(\bar{r}_0) \quad C10$$

which is equal to zero since the expected value of n is zero.

Since the mean of u_r is zero, the variance is simply the expected value of u_r^2 . This parameter may be determined as follows.

First,

$$E[u_r^2] = E\left[\int h_r(\theta, \phi | \bar{r}_0) n(\theta, \phi) dq\right]^2 / c_r^2(\bar{r}_0) \quad C11$$

Applying the result of eq. II-12, eq. C11 reduces to

$$E[u_r^2] = k^2 \int h_r^2(\theta, \phi | \bar{r}_0) dq / c_r^2(\bar{r}_0) \quad C12$$

However, the integral of h_r^2 over the surface was defined to be unity in eq. II-18, so that the variance of u_r is then

$$E[u_r^2] = k^2 / c_r^2(\bar{r}_0) \quad C13$$

A similar development may be applied to the other two component dipoles, so that the mean values of the noise induced error in the component magnitudes are

$$E(u_r) = 0 \quad C14a$$

$$E(u_\theta) = 0 \quad C14b$$

$$E(u_\phi) = 0 \quad C14c$$

The variances of these errors are

$$E(u_r^2) = k^2/c_r^2(\bar{r}_0) \quad C15a$$

$$E(u_\theta^2) = k^2/c_\theta^2(\bar{r}_0) \quad C15b$$

$$E(u_\phi^2) = k^2/c_\phi^2(\bar{r}_0) \quad C15c$$

In summary, it can be seen from eq. C14 that the mean values of the component dipole magnitudes as calculated using the matched filter technique will be equal to the actual values. In addition, the variances of the errors in the calculated magnitudes will be a function of the input noise variance as given in eq. C15.

EFFECTS ON MAGNITUDE FOR THE DISCRETE CASE

From eq. II-60a, the error in the magnitude of the $\bar{a}_r$ directed component of the dipole will be

$$u_r = (\hat{a}_r - d_r) = H_r^t(\bar{r}_0)N/c_r(\bar{r}_0) \quad C16$$

The mean value of u_r is then

$$E(u_r) = E[H_r^t(\bar{r}_0)N]/c_r(\bar{r}_0) \quad C17$$

or

$$E(u_r) = H_r^t(\bar{r}_0)E[N]/c_r(\bar{r}_0) \quad C18$$

which is zero since the expected value of N is the

zero vector, from eq. C5.

Since the mean of u_r is zero, the variance is simply the expected value of u_r^2 :

$$E(u_r^2) = E[(H_r^t(\bar{r}_0)N)(H_r^t(\bar{r}_0)N)]/c_r^2(\bar{r}_0) \quad C19$$

or,

$$E(u_r^2) = E[H_r^t(\bar{r}_0)NN^tH_r(\bar{r}_0)]/c_r^2(\bar{r}_0) \quad C20$$

or,

$$E(u_r^2) = H_r^t(\bar{r}_0)E[NN^t]H_r(\bar{r}_0)/c_r^2(\bar{r}_0) \quad C21$$

which, by eq. C6, reduces to

$$E(u_r^2) = k^2 H_r^t(\bar{r}_0)H_r(\bar{r}_0)/c_r^2(\bar{r}_0) \quad C22$$

Now, by eq. II-39, this equation becomes

$$E(u_r^2) = k^2/c_r^2(\bar{r}_0) \quad C23$$

By repeating this derivation for the other two components, it is found that the average error for each component is

$$E(u_r) = 0 \quad C24a$$

$$E(u_\theta) = 0 \quad C24b$$

$$E(u_\phi) = 0 \quad C24c$$

The variances of these noise components are

$$E(u_r^2) = k^2/c_r^2(\bar{r}_0) \quad C25a$$

$$E(u_\theta^2) = k^2/c_\theta^2(\bar{r}_0) \quad C25b$$

$$E(u_\phi^2) = k^2/c_\phi^2(\bar{r}_0) \quad C25c$$

Therefore, for the discrete case as well as the con-

tinuous case, the average value of the error in the dipole component magnitudes is zero. In addition, the output noise variance is again proportional to the input variance, as shown in eq. C25.

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