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Chaos in semiflows.

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CHAOS IN SEMIFLOWS

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DEDICATION

For Julian, who can brighten even the darkest days

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ABSTRACT

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All the common notions about dynamics in cascades - topological transitivity, periodic points, sensitive dependence, and so forth - can be formulated in the context of a general abelian semiflow. Many intricate results, such as the redundancy of Devaney chaos, remain true (with very minor qualifications) in this wider context. However, when we examine general monoid actions on a product space, it turns out that the topological and algebraic structure of $\mathbb{N}_0$ plays a large role in the preservation of chaotic properties. In order to obtain meaningful results in that arena, new ideas such as "directional" and "synnrec" are introduced, then applied.

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CHAPTER 1
PRELIMINARIES

1.1 Overview

This document is an exploration of chaos in the context of a general semiflow. The original content builds on work which began in the summer of 2012, all under the guidance and supervision of Dr. Alica Miller at the University of Louisville. The reader is presumed to have a working knowledge of analysis, abstract algebra, and topology (one year's graduate study in these areas will be sufficient).

In this opening chapter, I introduce the fundamental topic of discussion - a semiflow - in the notation which I will be using throughout. Chapter 2 provides a look at cascades and chaos (in the sense of Devaney), while Chapter 3 adapts the principles of chaos to a general abelian semiflow.

The last three chapters are composed entiredly of original research. Chapter 4 builds to a very general (and applicable) theorem regarding conditions for chaos in a general abelian semiflow. Chapter 5 explores the conditions under which chaos is and is not preserved in product spaces, and Chapter 6 answers the question, "What next?" Thank you for reading!

1.2 Semiflows

Let us begin by defining the object of our primary interest.

Definition 1.2.1. *Let T be an abelian topological monoid and X a metric space. A* **semiflow***, denoted (T, X), is a continuous left monoid action of T on X.*

If $\phi : T \times X \to X$ is the action mapping, then for $t \in T$ and $x \in X$ we will write $t \cdot x$ or simply tx in lieu of the more cumbersome $\phi(t, x)$.

Example 1.2.2. *Let $T = [0, \infty)$ with addition and the standard topology, and let $X = \mathbb{R} \times \mathbb{R} \times \mathbb{R}$ with the standard topology and usual metric. Then the action (T, X), given by $t \cdot (x, y, z) = (x + 2t, y - t, z)$ is a semiflow.*

Example 1.2.3. *Let T be any abelian topological monoid with operation $+$. Then the action (T, T) given by $t_1 \cdot t_2 = t_1 + t_2$ is a semiflow.*

Example 1.2.4. *Let X be a topological space and let $f : X \to X$ be any continuous function. This leads to a natural semiflow where $T = \mathbb{N}_0$ (with the discrete topology) and, for $n \in \mathbb{N}$, $n \cdot x = f^n(x)$, i.e. f iterated n times at x. This type of semiflow is called a cascade and is often denoted by $\langle X, f \rangle$ instead of $(\mathbb{N}_0, X)$.*

Cascades are the most commonly studied type of semiflow, followed by those in which $T = \mathbb{R}$ or $[0, \infty)$. The coming research aims to extend our knowledge of semiflows beyond these "popular" cases.

Let's examine some specific semiflows that will be of use later, and pick up a few handy definitions along the way.

Example 1.2.5. *Let $T = \mathbb{N}_0$ with the discrete topology, and give $X = [0, 1]$ the standard topology. Then (T, X) given by $n \cdot x = x^{2n}$ is a cascade. Observe that if $x = 1$, then $n \cdot x = 1 \, \forall n$. On the other hand, if $x \neq 1$, then the sequence $\{a_n\}$ given by $a_n = n \cdot x$ is convergent to zero.*

Definition 1.2.6. *Let* (T, X) *be a semiflow. A point* $x \in X$ *is called a* **fixed point***, or a point of periodicity 1, if* $\forall t \in T$*, it is true that* $t \cdot x = x$*.*

Example 1.2.7. *In Example 1.2.5, both* 0 *and* 1 *are fixed points of* X*. Without formally defining the notions, we might well say that* 0 *is an attractor and that* 1 *is a repeller. These concepts do exist in topological dynamics, but we will have no need for them.*

Definition 1.2.8. *Let* (T, X) *be a semiflow and choose* $x \in X$*. The* **orbit** *or trajectory of* x*, denoted* Tx*, is the set* $\{tx \,|\, t \in T\}$*. The orbit of a set* $A \subseteq X$ *is just* $TA = \{Tx \,|\, x \in A\}$*.*

Example 1.2.9. *In Example 1.2.5, the orbit of* $1/5$ *is* $\{1/5, 1/25, 1/625, \dots\}$*.*

Note that, while the orbit of a point in a cascade will consist of terms of a sequence, an orbit is formally a set.

Definition 1.2.10. *Let* (T, X) *be a semiflow with* $t \in T$ *and* $A \subset X$*. Then the* **image** *of* A *under* t*, denoted* tA*, is the set* $\{tx \,|\, x \in A\}$*.*

Example 1.2.11. *Let* $X = [0, 1]$ *with the standard topology and usual metric, and let the cascade* $\langle X, f \rangle$ *be given by* $f(x) = \min\{2x, 2(1-x)\}$*. This cascade is called the tent map.*

There are several interesting things to notice about the tent map. Here let us mention only a few.

Lemma 1.2.12. *Let* $\langle X, f \rangle$ *be the tent map and choose* $x, y \in X$*. If either* $\max\{x, y\} \leq 1/2$ *or* $\min\{x, y\} \geq 1/2$*, then* $|f(x) - f(y)| = 2|x - y|$*.*

Proof. This is obvious if $\max\{x, y\} \leq 1/2$. So suppose $\min\{x, y\} \geq 1/2$; then

$$|f(x) - f(y)| = |2(1-x) - 2(1-y)| = |2 - 2x - 2 + 2y| = 2|y - x|$$

which, of course, is the same as $2|x - y|$. $\square$

Proposition 1.2.13. *Let $\langle X, f\rangle$ be the tent map and choose any nonempty open interval $J \subseteq X$. Then $\exists n \in \mathbb{N}$ s.t. $nJ = X$.*

Proof. Let $D = \{a/b \in (\mathbb{Q} \cap X) \,|\, a$ is odd and b is a power of $2\}$. Since D is dense in X and J is open, we can choose $x \in D \cap J$ and $\epsilon > 0$ s.t. $(x - \epsilon, x + \epsilon) \subseteq J$.

Suppose $x = p/q$ with p odd and $q = 2^m$; wlog suppose $m \geq 2$. Then $(m-2) \cdot x$ is either $1/4$ or $3/4$; suppose wlog that it is the former. The image of an open interval under f will again be an open interval. By the preceding lemma, an open interval containing $1/4$ will have orbit X. □

The last observation we will make about the tent map deals with finite orbits.

Example 1.2.14. *Let $\langle X, f\rangle$ be as in Example 1.2.11. Then any point of $\mathbb{Q} \cap X$ has a finite orbit, while any other point has an infinite orbit.*

Definition 1.2.15. *Let $\langle X, f\rangle$ be a cascade. A point $x \in X$ is called* **periodic** *if $\exists n \in \mathbb{N}$ s.t. $n \cdot x = x$. For the least such n we say x has periodicity n. A point $y \in X$ is called* **eventually periodic** *if y is not periodic but its orbit contains a periodic point.*

Notice that this definition only makes sense for a cascade.

Example 1.2.16. *In a cascade, a point has a finite orbit iff it is periodic or eventually periodic.*

Now let us turn our attention to other useful semiflows. We will examine them in more detail later.

Example 1.2.17. *Let $X = [0, 1]$ with the standard topology and usual metric, and let $\langle X, f\rangle$ be given by $f(x) = \mu x(1 - x)$, where $\mu \in (0, 4]$. This is called the logistic map.*

EXAMPLE 1.2.18. *Let* $\mathbb{U} = [0, 1)$ *with the usual metric and standard topology, and let* $f : \mathbb{U} \to \mathbb{U}$ *be given by* $f(x) = 2x$ *if* $2x \in \mathbb{U}$ *and* $f(x) = 2x - 1$ *otherwise. Because* $\mathbb{U}$ *is essentially a circle in terms of its topology and a point on a circle can be associated with a central angle, this cascade is sometimes called the angle-doubler.*

EXAMPLE 1.2.19. *Let* $\mathbb{U}$ *be as in the previous example and choose any* $\alpha \in \mathbb{U}$*. Now let* $g : \mathbb{U} \to \mathbb{U}$ *be given by* $g(x) = x + \alpha$ *modulo 1; i.e.* $g(x) = x + \alpha$ *or* $x + \alpha - 1$*, whichever is an element of* $\mathbb{U}$*. For the same reasons as in the previous example, this cascade is called the "rotation" map.*

In this last example, it is easy to see that if $\alpha \in \mathbb{Q}$, then every single point of $\mathbb{U}$ is periodic (with periodicity equal to, at most, the denominator of α). On the other hand, irrational rotations possess deeper, more interesting properties which we will explore in the next chapter. Let us conclude this section with one last example: a semiflow which is not a cascade.

EXAMPLE 1.2.20. *Let* $T = [0, 1)$ *and define the monoid structure of* T *by setting* $s + t = \min\{s, t\}$*. The action* (T, T) *where* $s \cdot t = s + t$ *is a semiflow.*

Note that this T is not even cancellative; that is, it cannot be embedded in any group. Later we shall see that noncancellative monoids are of special interest.

CHAPTER 2
CHAOS IN CASCADES

In this chapter we will define the properties which comprise chaos, then explore what chaos looks like in a cascade. Finally we will examine the BBCDS theorem, which tells us something about how the three chaos properties are related.

2.1 Topological transitivity

Definition 2.1.1. *A cascade $\langle X, f \rangle$ is called* **topologically transitive** *if for any open and nonempty $U, V \subset X$ there is an $n \in \mathbb{N}$ such that $f^n(U) \cap V \neq \emptyset$.*

Note that there is no need to consider the case $V = X$, although for some cascades (and choices of V) it may well be that $f^n(X) \cap V = \emptyset$ for every n.

Proposition 2.1.2. *The tent map introduced in Example 1.2.11 is topologically transitive.*

Proof. Any opene subset of $[0, 1]$ contains an open interval, and so this follows immediately from Proposition 1.2.13. $\square$

Example 2.1.3. *The logistic map on $[0, 1]$, introduced in Example 1.2.17, is not topologically transitive when $\mu < 4$, as $f(x) = \mu x(1 - x)$ achieves a maximum value of $\mu/4$ when $x = 1/2$.*

The behavior of this semiflow as μ varies between 2 and 4 is a well-studied and highly interesting problem in dynamics, one of the origins of chaos theory. We will return later to the case $\mu = 4$.

One is tempted to claim that if $\langle X, f \rangle$ is a casacde and f is not a surjection, then $\langle X, f \rangle$ is not topologically transitive – but we must be careful! Instead let us say this with confidence:

EXAMPLE 2.1.4. *If $\langle X, f \rangle$ is a cascade and $\exists V \subset X$ which is open and nonempty such that $f(X) \cap V = \emptyset$, then there is no way $\langle X, f \rangle$ can be topologically transitive.*

PROPOSITION 2.1.5. *The angle-doubler map introduced in Example 1.2.18 is topologically transitive.*

Proof. This is similar to the proof of Proposition 1.2.13. Let U, V be open nonempty subsets of $\mathbb{U}$ with $J \subseteq U$ an open interval. Then J contains a point x of the form $k_1/2^m$ for some $k_1, m \in \mathbb{N}$, as such points are dense in $\mathbb{U}$. Hence $f^m(x) = 0$ according to the definition of f.

Since f is continuous, $f^m(U)$ includes an interval of the form $[0, b)$ for some $b \in X$. Any interval of this form will have image $\mathbb{U}$ after a finite number of iterations, say k_2. The choice $n = m + k_2$ guarantees $f^n(U) = \mathbb{U}$, and so in fact $V \subseteq f^n(U)$. Certainly this cascade is topologically transitive. □

This result suggests the following definition, which - as we are about to see - is strictly stronger than topological transitivity. It is mentioned here in passing, but also as an indication that there is more than one type of transitivity. This fact will come in handy in Chapter 5.

DEFINITION 2.1.6. *$\langle X, f \rangle$ is called a* **supertransitive** *cascade if for any open nonempty $U \subset X$, $\exists n \in \mathbb{N}$ such that $f^n(U) = X$.*

From this point forth, I will use the abbreviation "toptran" for "topologically transitive."

Definition 2.1.7. *Let X be a topological space. A point $x \in X$ is called* **isolated** *if $\{x\}$ is an open set.*

Lemma 2.1.8. *Let $\langle X, f \rangle$ be a cascade where X is an infinite set with no isolated points. If $\langle X, f \rangle$ contains a dense orbit, then it is toptran.*

Proof. Let $x \in X$ be a point whose orbit is dense. Then by the restrictions on X, every point in the orbit of x also has a dense orbit. Now let U, V be given opene subsets of X. Then U contains a point y in the orbit of x, and of course the orbit of y must intersect V. This suffices to show $\langle X, f \rangle$ is toptran. $\square$

Is the converse true? Does every toptran semiflow contain a dense orbit? The answer is no, even for cascades, but the answer is yes if X has a countable dense subset and is sufficiently "large" (of second category).

Proposition 2.1.9. *Let $\langle \mathbb{U}, g \rangle$ be the rotation map from Example 1.2.19. Then $\langle \mathbb{U}, g \rangle$ is toptran if and only if $\alpha \notin \mathbb{Q}$.*

Proof. Suppose $\alpha \in \mathbb{Q}$. Then if $x \in \mathbb{U}$, its orbit is finite (as previously discussed). Let k be the (rational) distance between x and the nearest point in its orbit, then choose $U = B(x, k/4)$ and $V = B(x + k/2, k/4)$. For every $n \in \mathbb{N}$, then, $g^n(U) \cap V = \emptyset$ and so $\langle \mathbb{U}, g \rangle$ is not toptran.

Now suppose $\alpha \notin \mathbb{Q}$ and let $M \in \mathbb{N}$ be given. The orbit $\{0, g(0), g^2(0), \dots\}$ is infinite (otherwise α is rational). There are a finite number of intervals of width $1/M$ whose union is $\mathbb{U}$, and so one of those intervals contains two points $g^m(0), g^n(0)$. Wlog $m > n$. Then the iterate g^{m-n} sends any point an arbitrarily small distance away, so the orbit of zero is dense.

$\mathbb{U}$ is an infinite set with no isolated points. By the preceding lemma, $\langle \mathbb{U}, g \rangle$ is toptran. $\square$

Corollary 2.1.10. *Every orbit of the irrational rotation map is dense.*

Proof. Zero has a dense orbit, and $g^n(x) = x + g^n(0)$ for any $x \in X, n \in \mathbb{N}$. $\square$

Note that again we showed more than we needed to. The above property is a key idea in dynamics and deserves a formal definition:

Definition 2.1.11. *A semiflow (T, X) is called* **minimal** *if every orbit is dense in X.*

To conclude this section, here is one more definition, followed by a useful interpretation of what it means to be toptran.

Definition 2.1.12. *Let (T, X) be any semiflow. A subset $Y \subset X$ is called an* **invariant** *set of (T, X) if $TY = \{ty \,|\, t \in T, y \in Y\} \subseteq Y$; that is, every orbit starting in Y stays there.*

A semiflow, then, is toptran if its phase space X cannot be decomposed into two disjoint open nonempty invariant sets.

2.2 Sensitivity

Definition 2.2.1. *Let X be a metric space with metric d. A cascade $\langle X, f\rangle$ is called* **pointwise sensitive** *if $\exists c > 0$ such that $\forall x \in X$ and $\forall U \in \mathcal{N}(x)$, $\exists n \in \mathbb{N}, y \in U$ satisfying $d(f^n(x), f^n(y)) \geq c$.*

It is noteworthy that we do not merely require orbits to become widely separated – the same iterate of f must separate x and y.

The primary significance of sensitivity is that it renders even the most precise numerical approximations worthless. This, then, is the infamous "Butterfly Effect" that laymen associate with chaos. Here is an alternate formulation:

Definition 2.2.2. *Let X be a metric space with metric d. A cascade $\langle X, f\rangle$ is called* **setwise sensitive** *if $\exists c > 0$ such that for any open and nonempty $U \subseteq X$ there exist $x, y \in U$ and $n \in \mathbb{N}$ satisfying $d(f^n(x), f^n(y)) \geq c$.*

In some cases it may be easier to use one definition or the other, but let us convince ourselves that it doesn't really matter.

Proposition 2.2.3. *Pointwise and setwise sensitivity are equivalent.*

Proof. Suppose $\langle X, f\rangle$ is pointwise c-sensitive and choose an open $U \subseteq X$ with at least two elements. Choose any $x \in U$ and the requisite y and n must exist, so $\langle X, f\rangle$ is setwise c-sensitive.

Now suppose $\langle X, f\rangle$ is setwise c-sensitive and choose any $x \in X$ and $U \in \mathcal{N}(x)$. Then $\exists y, z \in U$ and $n \in \mathbb{N}$ so that $d(f^n(y), f^n(z)) \geq c$, but

$$d(f^n(y), f^n(z)) \leq d(f^n(y), f^n(x)) + d(f^n(x), f^n(z))$$

and so one of the latter two terms is at least $c/2$, meaning $\langle X, f\rangle$ is pointwise $c/2$-sensitive. $\square$

The difference in sensitivity constant doesn't matter in practice; all we will ever care about is whether a semiflow is sensitive, or not. From now on I will drop the adjectives "pointwise" and "setwise" and use whichever formulation is more convenient.

Example 2.2.4. *The tent map is sensitive, as is the logistic map when $\mu = 4$.*

Before proceeding, note that the obvious metric on the circle $\mathbb{U}$ is given by $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$.

Proposition 2.2.5. *The angle-doubler is sensitive with $c = 1/4$.*

Proof. Let $x \in \mathbb{U}$ and $U \in \mathcal{N}(x)$. Pick any $y \in U \setminus \{x\}$; then $\exists m \in \mathbb{N}$ so that $1/2^m \leq d(x, y) < 1/2^{m-1}$. But f doubles the distance between any two points less than $1/4$ apart. Thus if $n = m - 2$, we have $1/4 \leq d(f^n(x), f^n(y)) < 1/2$. $\square$

Example 2.2.6. *Any rotation map of $[0, 1)$, be it rational or irrational, is not sensitive; in fact, it is an isometry, as $d(x, y) = d(f^n(x), f^n(y))$ for any choice of x, y, n.*

Example 2.2.7. *If X has the discrete metric and at least two points, then $\langle X, f\rangle$ is 1-sensitive for any continuous function f.*

In contrast to toptran, sensitivity is a property whose definition requires that X be metrizable and which seems to depend heavily on the particular metric. Remember this, because we will soon see that, in certain cases which are of great interest, a semiflow can be shown to be sensitive without recourse to the phase space metric at all!

Example 2.2.8. *Let $X = \mathbb{R}$ with the usual metric and $f(x) = x^3$. Then $\langle X, f\rangle$ is not sensitive because $(-1, 1)$ is an invariant set and $\forall x, y \in (-1, 1)$, we have $d(x, y) > d(f(x), f(y))$.*

2.3 Devaney chaos

The mathematical study of chaos dates to the late 1960's. It was not until 1989, however, that Devaney gave his characterization of chaos in an arbitrary cascade. In the ensuing definition we will use Per(X) to denote the set of all periodic points of X.

Definition 2.3.1 (Devaney). *A cascade $\langle X, f\rangle$ is called* **chaotic** *if it satisfies the following three conditions:*

1. $\langle X, f\rangle$ is topologically transitive;

2. $\langle X, f\rangle$ is sensitive;

3. Per(X) is dense in X.

Observe that, like toptran, density of periodic points (hereafter abbreviated DPP) is a purely topological property which has nothing to do with the metric on X.

It is worth taking a moment to examine why this particular combination of properties merits our attention. A chaotic cascade is not only sensitive, as one might expect, but is also in a sense irreducible (thanks to toptran) and, in at least one important regard, quite well-behaved (thanks to DPP). Perhaps it is this peculiar combination of innocuous simplicity and erratic dynamics which initially attracted Devaney's interest.

Put another way, if one cares to use numerical methods to try and analyze the behavior of a chaotic cascade, c-sensitivity will guarantee that any initial error in approximation will eventually inflate to an error of at least size c; however, DPP implies that a nearby point is easily understood. Meanwhile, toptran guarantees that the task cannot be made simpler.

We have already seen a few chaotic cascades; here's a demonstration of that fact. (I am not going to give a thorough proof of the following lemma, because it will be unnecessary after the next section.)

Lemma 2.3.2. *The tent map has DPP.*

Proof. Let $Y = \{a/b \in (\mathbb{Q} \cap X) \,|\, a \text{ is even, } b \text{ is odd, and } \gcd(a,b) = 1\}$. Choose any odd b and consider the set Y_b of all elements from Y with denominator b. It is easy to see that Y_b is itself a periodic orbit of the tent map; hence every point in Y is periodic. It is also a simple matter to prove Y is dense in $[0,1]$. □

Proposition 2.3.3. *Assuming the statement in Example 2.2.4 is true, then the tent map is chaotic.*

Proof. This follows from the preceding lemma and Proposition 1.2.13. □

In the next section we will prove the tent map is sensitive; we will also see that the logistic map with $\mu = 4$ is chaotic.

Lemma 2.3.4. *The angle-doubler has DPP.*

Proof. The proof is very, very similar to that of Lemma 2.3.2. The chief difference is that, since (for example) $2 \cdot 8/15 = 1/15$, now we consider a larger set, namely $Z = \{a/b \in (\mathbb{Q} \cap X) \,|\, b \text{ is odd and } \gcd(a,b) = 1\}$. Also, while each point of Z is periodic, there may be more than one orbit for each denominator b. □

Proposition 2.3.5. *The angle-doubler is chaotic.*

Proof. Combine the results of the preceding lemma with Propositions 2.1.5 and 2.2.5. □

On the other hand, even a very simple cascade can exhibit chaos.

Example 2.3.6. *Let $X = \{x, y\}$ with $f(x) = y$ and $f(y) = x$. Then if we give X the indiscrete topology and any metric, $\langle X, f \rangle$ is chaotic.*

2.4 The BBCDS theorem

Let's see if we can find examples of cascades which meet each of the other seven possible combinations of chaos's three criteria.

Proposition 2.4.1. *Let $X = \mathbb{R}$ with the usual metric and $f(x) = x + 1$. Then $\langle X, f\rangle$ satisfies none of the three chaos criteria.*

Proof. Clearly $\mathrm{Per}(X) = \emptyset$, and since f is an isometry it is impossible that this cascade is sensitive. Finally, the choice $U = (1, 2)$ and $V = (0, 1)$ shows that $\langle X, f\rangle$ is not toptran. □

Proposition 2.4.2. *A rational rotation map $\langle \mathbb{U}, g\rangle$ has DPP, but is not sensitive or toptran.*

Proof. As we've seen, every point in such a cascade is periodic. The other two claims follow from Examples 2.1.9 and 2.2.6. □

Proposition 2.4.3. *An irrational rotation map is, of the three chaos criteria, only toptran.*

Proof. Again we refer to Examples 2.1.9 and 2.2.6; as for DPP, the proof of the former example contains a statement that every point has an infinite orbit, and so $\mathrm{Per}(\mathbb{U}) = \emptyset$. □

Proposition 2.4.4. *Let $X = \mathbb{R}$ with the usual metric and $f(x) = 2x$. Then $\langle X, f\rangle$ is, of the three chaos criteria, sensitive only.*

Proof. Clearly only zero is periodic, and $|2x - 2y| = 2|x - y|$ for any $x, y \in X$, meaning this cascade is c-sensitive $\forall c$. But it isn't toptran; choose $U = (0, 1)$ and $V = (-\infty, 0)$. □

This last example suggests a straightforward way to demonstrate two of the three remaining cases.

Proposition 2.4.5. *Let $\langle X, f\rangle$ be as in Proposition 2.4.4, but give $\mathbb{R}$ the indiscrete topology. Then $\langle X, f\rangle$ is toptran and sensitive, but it lacks DPP.*

Proof. For the same reason as above, $\langle X, f\rangle$ is sensitive with only one periodic point. But now the only opene set is X, so of course the cascade must be toptran. $\square$

Proposition 2.4.6. *Let $k \in \mathbb{N} \setminus \{1\}$, let X be a discrete metric space with $n(X) = 2k$, and let $f(x) = x$ for each $x \in X$. Give X a topology generated by the sets $\{x_1, x_2\}, \{x_3, x_4\}, \dots \{x_{2k-1}, x_{2k}\}$. Then $\langle X, f\rangle$ is 1-sensitive and $Per(X) = X$, but this cascade is not toptran.*

Proof. Every point is periodic, and $\forall x$ any neighborhood of x contains exactly one point which starts (and stays) a distance of 1 away. However, X consists of k disjoint invariant open sets. $\square$

What about the eighth case? Happily, it is far more fruitful. Three years after Devaney defined chaos, a group of five mathematicians (Banks et al., whom we will refer to as BBCDS) examined the eight cases as we just have and discovered a surprising fact:

Theorem 2.4.7 (BBCDS). *If X is an infinite metric space and $\langle X, f\rangle$ is toptran with DPP, then $\langle X, f\rangle$ is sensitive and hence chaotic.*

Corollary 2.4.8. *The tent map is chaotic.*

As mentioned previously, the amazing thing about this statement is that sensitivity, which seems to depend so heavily on the particular metric chosen for X, can be forced without any information about that metric. We will omit the proof of the BBCDS theorem, because the main result of Chapter 4 has it as a corollary.

What about the requirement that X be an infinite space? Quite obviously this detail was necessary for BBCDS to prove their claim, or they wouldn't have included it as a hypothesis.

Proposition 2.4.9. *A cascade with DPP in which X is an infinite metric space must possess two orbits whose closures are disjoint.*

Proof. Since periodic orbits in a cascade are necessarily finite, there must be two (indeed, infinitely many) distinct periodic orbits in order for $\langle X, f\rangle$ to have DPP. By their nature, distinct periodic orbits must be disjoint. But metric spaces are Hausdorff, meaning finite subsets of X are closed. $\square$

This discovery of redunancy in Devaney's definition is the starting point for my original research. In the years that followed, other similar results were obtained. Here is one more, which I will state without proof:

Theorem 2.4.10 (Vellekoop, Berglund)**.** *If X is an interval of real numbers and $\langle X, f\rangle$ is topran, then $\langle X, f\rangle$ is chaotic.*

To conclude this section, let us return to the logistic map and show that it is indeed chaotic.

Definition 2.4.11. *Two functions $f : A \to B$ and $g : C \to D$ are said to be* **topologically conjugate** *if there is a homeomorphism $h : B \to C$ such that $h[f(a)] = g[h(b)]$ for every $a \in A, b \in B$.*

Recall that homeomorphisms preserve all topological properties, though not necessarily metric ones.

Example 2.4.12. *The logistic map with $\mu = 4$ is topologically conjugate to the tent map, with the homeomorphism being $h(x) = [\sin(\pi x/2)]^2$.*

Proposition 2.4.13. *The logistic map with $\mu = 4$ is chaotic.*

Proof. This follows from the previous example and the BBCDS theorem. $\square$

CHAPTER 3
CHAOS IN GENERAL SEMIFLOWS

This chapter examines the importance (and prevalance) of semiflows which are not cascades, adapts the notions explored in the previous chapter to the context of a general semiflow, and completes our exposition of prior knowledge. The way is then clear to obtain original results.

3.1 Suspended semiflows

Much past work on semiflows deals exclusively with cascades. In this section we will see that any cascade can be naturally extended to a semiflow in which the acting monoid is $[0, \infty)$.

Definition 3.1.1 (Li, Zhou)**.** *Let $\langle X, f \rangle$ be a cascade with X a compact metric space. The* **suspension** *of $\langle X, f \rangle$ is an equivalence relation $\mathcal{R}$ on the product space $Y = [0,1] \times X$, defined $\forall (w_1, x_1), (w_2, x_2) \in Y$ by $(w_1, x_1)\mathcal{R}(w_2, x_2)$ iff either*

1. the pairs are identical, or

2. $w_1 = 1, w_2 = 0$, and $x_2 = f(x_1)$.

This construction has the effect of tying the function f to a circle, and equating points from Y which feature compatible first and second coordinates. It has been shown that Y is also a compact metric space.

Example 3.1.2. *If $X = \mathbb{Z}$ and $f(x) = x + 5$, then in the suspension of $\langle X, f \rangle$ it is true that $(1, 6)\mathcal{R}(0, 11)$.*

EXAMPLE 3.1.3. *If $X = \mathbb{R}$ and $f(x) = x^3 - x$, then in their suspension $(1,0)\mathcal{R}(0,0)$ and $(1,1)\mathcal{R}(0,0)$. Note that this forces $(1,1)\mathcal{R}(1,0)$, which makes sense because in X, the points 0 and 1 have the same image.*

In the next definition we will use $[(w,x)]$ as notation for "the equivalence class containing (w,x)".

DEFINITION 3.1.4 (Li, Zhou)**.** *Let $\langle X, f\rangle$ be a cascade with suspension $\mathcal{R}$ as defined above. The* **suspended semiflow** *of $\langle X, f\rangle$ is the semiflow with acting monoid $[0,\infty)$ and phase space Y defined as follows: $\forall \alpha \in [0,\infty)$ and $\forall [(w,x)] \in Y$, we have*

$$\alpha \cdot [(w,x)]) = [w + \alpha - n, f^n(x)]$$

where $n \in \mathbb{N} \cup \{0\}$ satisfies $n \leq w + \alpha < n + 1$.

The obvious question now is how much of the theory developed in Chapter 2 remains true in the context of an arbitrary (abelian) semiflow.

3.2 Toptran and sensitivity revisited

The first question I tackled in my original research is whether Theorem 2.4.7 is true for any semiflow, or only for a cascade. Because $\mathbb{N}_0$ is discrete, the transition is far from trivial. In order to formally pose the question, we must first define Devaney's three chaos criteria in this general setting.

DEFINITION 3.2.1. *A semiflow (T, X) is called* **topologically transitive** *if for any open and nonempty $U, V \subset X$ the dwelling set $D(U, V) = \{t \in T \mid tU \cap V \neq \emptyset\}$ is nonempty.*

I will continue to use the abbreviation "toptran." The above formulation includes Definition 2.1.1 as a special case, just as the next definition supersedes Definition 2.2.1.

DEFINITION 3.2.2. *Let X be a metric space with metric d. A semiflow (T, X) is called* **sensitive** *if $\exists c > 0$ such that for any open and nonempty $U \subseteq X$ the set $R(U, c) = \{t \in T \mid \exists x, y \in U \text{ s.t. } d(tx, ty) \geq c\}$ is nonempty.*

As before, there is no need to distinguish between pointwise and setwise sensitivity. I will also take this opportunity to mention that we lose nothing of interest by considering only phase spaces which lack isolated points. Now for examples where $T \neq \mathbb{N}_0$:

EXAMPLE 3.2.3. *Let $X = \mathbb{U}$ with the usual metric and topology, and let $T = [0, \infty)$ with the usual topology and operation (addition). Define $t \cdot x$ to be the decimal portion of $t + x$.*

We may as well refer to the above as the "general rotation semiflow" – think of this construction as a smooth wheel, versus the gears we have worked with so far.

Proposition 3.2.4. *The general rotation semiflow is toptran but not sensitive.*

Proof. As with the rotation cascades, this action preserves distance; to be precise, $d(x, y) = d(tx, ty)$ for every $x, y \in X$ and $t \in T$. Hence (T, X) is not sensitive.

Now choose any open nonempty $U, V \in X$ and any irrational $t \in T$. We already know the irrational rotation map is toptran, and so there is some (positive integer) multiple of t - let's call it nt - which satisfies $(nt)U \cap V \neq \emptyset$, meaning (T, X) is toptran. □

In passing, note that the general rotation semiflow is also minimal (every orbit is dense).

Proposition 3.2.5. *Define (T, X) as in Example 1.2.20, with $X = T$ and the usual metric and topology. This semiflow is neither sensitive nor toptran.*

Proof. Choose any $x, y \in X$ and note that $\forall t \in T$, we have $d(tx, ty) \leq d(x, y)$; thus (T, X) is not sensitive. It is also not toptran; choose $U = X$, $t = 1/2$, and $V = [0, 1/2)$. □

Proposition 3.2.6. *Let $T = \{x \in \mathbb{Q} \,|\, x \geq 1\}$ with the discrete topology and $X = [1, \infty)$ with the usual topology and metric. Define (T, X) by $tx = x^t$. This semiflow is sensitive, but it is not toptran.*

Proof. Since $tx \geq x$ for every $t \in T, x \in X$, there is no way (T, X) could be toptran – simply choose $U, V \subset X$ so that the infimum of U is greater than the supremum of V.

Now choose any distinct $x, y \in X$ and let $c \geq 0$ be given. As T is unbounded above, $\exists t \in T$ so that $d(tx, ty) \geq c$, so we might call (T, X) supersensitive. □

EXAMPLE 3.2.7. *If we revisit Example 3.2.3 and give X the discrete metric, then (T, X) is both toptran and sensitive.*

The next statement was published in 2013; the proof follows largely from the pertinent definitions.

THEOREM 3.2.8 (Li, Zhou)**.** *A cascade $\langle X, f \rangle$ is toptran iff its suspended semiflow is toptran; the same is true for sensitivity.*

Using this, we can get toptran $[0, \infty)$ semiflows from toptran cascades, or indeed generally complicated examples from simple ones.

3.3 WLF periodicity

The idea of a fixed point translates immediately to the environment of an arbitrary semiflow:

Definition 3.3.1. *Let (T, X) be a semiflow. A point $x \in X$ is called* **fixed** *if its orbit $Tx = \{x\}$.*

However, the notion of periodicity does not generalize quite as obviously as the other two of Devaney's criteria. (For example, it is gibberish to refer to a fixed point as "a point with periodicity 1.") In this and the next section we will examine some of the possibilities.

Definition 3.3.2 (Wang, Long, Fu)**.** *Let (T, X) be any (abelian) semiflow. A* **WLF periodic orbit** *is a finite subset $W = \{x_1, x_2, \dots x_n\} \subseteq X$ satisfying the following two conditions:*

1. Each $t \in T$ acts on W either as a cyclic permutation or the identity map.

2. $\exists t_0 \in T$ so that t_0 acts on W as a cyclic permutation.

Note that the first condition in the above definition is strictly stronger than asking that W be invariant. Another way to think about this notion is that if W is a WLF periodic orbit, then every element of T permutes the elements of W, with at least one of those permutations being nontrivial.

Definition 3.3.3. *Any point contained in a WLF periodic orbit is called a* **WLF periodic point** *of (T, X).*

As with periodic orbits in cascades, WLF periodic orbits are equal or disjoint.

Lemma 3.3.4. *Let x be a WLF periodic point of (T, X). Then x is contained in exactly one WLF periodic orbit.*

Proof. To suppose the contrary is to immediately violate condition 1 of Definition 3.3.2. □

Now we come to the reason Wang et al. defined periodicity in such a fashion:

Theorem 3.3.5 (Wang, Long, Fu). *Let (T, X) be a toptran semiflow with dense WLF periodic points. Then (T, X) is sensitive.*

Unfortunately, while they are valid constructions, WLF periodic orbits do not generalize the notion of periodic orbits in a cascade. To see why, let us ask how might we define periodicity under this new definition.

Definition 3.3.6. *The* **WLF periodicity** *of a WLF periodic point x is the cardinality of the (unique) WLF periodic orbit which contains x. We say x is WLF n-periodic.*

Now the problem arises: it is quite easy to construct a cascade containing a point with periodicity 4.

Example 3.3.7. *In the rational rotation map with $\alpha = 1/4$, every point has periodicity 4.*

Note in particular that $2 \cdot x \neq x$ for every $x \in \mathbb{U}$ in this case.

Proposition 3.3.8. *No WLF periodic orbit can have composite cardinality.*

Proof. Suppose that W is a WLF periodic orbit of (T, X) with cardinality pq, where $p, q \in \mathbb{N} \setminus \{1\}$, and that t_0 is as given in Definition 3.3.2. Let pt_0 denote the element of T which equals $t_0 + t_0 + \cdots + t_0$, a total of p times. (This is a convenient and harmless abuse of notation.) Then $(pt_0)x_1 = x_{p+1}$, $(pt_0)x_{p+1} = x_{2p+1}$, and so forth until finally $(pt_0)x_{(q-1)p+1} = x_1$.

Hence W contains a WLF periodic orbit W_q of cardinality q (necessarily as a proper subset), and x is an element of both W and W_q. This violates Lemma 3.3.4, and we have our contradiction. □

Corollary 3.3.9. *In any cascade, a point with composite periodicity n is not WLF n-periodic.*

Example 3.3.10. *In the rational rotation map with $\alpha = a/b$, where b is composite and $gcd(a, b) = 1$, every point is periodic but no point is WLF periodic.*

For some choices - including very natural ones - of the acting monoid T, the situation is even worse.

Proposition 3.3.11. *Let (T, X) be any semiflow where $T = [0, \infty)$. Then a point $x \in X$ is WLF periodic iff it is fixed.*

Proof. Obviously a fixed point is WLF periodic. Now suppose x is WLF periodic with orbit W and let $t \in T$ be any element which induces a cyclic permutation on W.

The fatal question: does $t/[n(W)]$ induce a cyclic permutation on the orbit of x? Whether the answer to this question is yes or no, we have a contradiction because $t \cdot x = x$. □

3.4 Syndetic subsets

In the last section we saw that there are at least two approaches to defining periodicity in a semiflow. Both share one characteristic – they require periodic orbits to be either equal or disjoint. But consider the following:

EXAMPLE 3.4.1. *In the general rotation semiflow,* $\forall x \in \mathbb{U}, n \in \mathbb{N}$ *there is an orbit containing* x *which has cardinality* n*. Some of these orbits intersect, and some (except for* x*) do not.*

It seems to make sense to call such orbits "periodic," despite their highly interactive nature (some are even subsets of others). For that matter, is it at all sensible to require *a priori* that, when T is not discrete, a periodic orbit must be finite?

There is an idea which truly incorporates the fundamental aspect of periodic orbits in a cascade. To express it, we first need to explore the monoid structure of T in conjunction with a key topological idea which we haven't really taken advantage of yet: compactness.

DEFINITION 3.4.2. *Let* T *be an abelian monoid.* $S \subseteq T$ *is called* **syndetic** *in* T *if there is a compact* $K \subseteq T$ *so that* $(t + K) \cap S \neq \emptyset$ *for every* $t \in T$*.*

We sometimes call sets of the form $t + K$ the "t-translate of K" or simply "a translate of K."

EXAMPLE 3.4.3. T *is always syndetic in itself; choose* $K = \{0\}$*.*

In practice, we say "S is syndetic" instead of "S is syndetic in T" when the ambient monoid T is understood. K is also often referred to as "the compact corresponding to S."

PROPOSITION 3.4.4. *If $T = \mathbb{N}_0$, then $S \subset T$ is syndetic iff $\exists n \in \mathbb{N}$ such that no sequence of n consecutive integers belongs to $T \setminus S$.*

Proof. Suppose such an n exists. Choose $K = \{0, 1, \dots, n\}$. Now let $t \in T$; then $t + K$ contains a string of $n + 1$ consecutive integers, one of which (by assumption) belongs to S.

Now suppose S is syndetic with corresponding compact K. Since $K \subset \mathbb{N}$, K has a largest element m. Choose $n = m + 2$. □

EXAMPLE 3.4.5. *If T is a compact group, then $\{0\}$ is syndetic (one can simply choose $K = T$).*

LEMMA 3.4.6. *If A is syndetic in T and $A \subseteq S$, then S is syndetic in T.*

Proof. This follows at once from the definition of a syndetic set. □

Intuitively, then, a syndetic set has "bounded gaps." Often this is a useful mental shortcut for appreciating what a syndetic set is – but be warned that the two notions are not quite the same.

PROPOSITION 3.4.7. *Let T be as in Example 1.2.20. Then $S \subset T$ is syndetic iff $\exists a \in T$ such that $[a, 1) \subseteq S$.*

Proof. Suppose such an a exists. Choose $K = [0, a]$ and any $t \in T$. Then if $t \leq a$, we have $t + a = a$, and so $a \in t + K$. But $a \in S$ too. If $t > a$, then $t + K = \{t\}$ and of course $t \in S$.

Now suppose S is syndetic with corresponding compact K. Then $\exists b \in T$ such that $K \subseteq [0, b]$; otherwise K could not be compact. Choose $a = b$ and note that, $\forall t > b, t + K = \{t\}$ and so, if $t \notin S$, then S cannot be syndetic. Hence $[a, 1) \subseteq S$. □

Proposition 3.4.8. *Let $T = [0, 1]$ with the same operation from Example 1.2.20. Then a subset of T is syndetic iff it includes 1.*

Proof. Suppose $1 \notin S$. Then no matter what compact set K is chosen, $1+K = \{1\}$ and S is not syndetic.

Now suppose $1 \in S$. Choose $K = \{1\}$, and $t + K = K$ for each $t \in T$, so S is syndetic with this corresponding K. □

It's time to see how the notion of syndetic sets helps us along.

Definition 3.4.9 (Gottschalk, Hedlund)**.** *Let (T, X) be any semiflow. A point $x \in X$ is called* **GH periodic** *if Fix$(x) = \{t \in T \mid tx = x\}$ is syndetic in T.*

The above definition has one immediate appeal: it takes care of the issue raised by Example 3.4.1.

Example 3.4.10. *In the general rotation semiflow, every point $x \in \mathbb{U}$ is periodic, because Fix$(x) = \mathbb{Q} \cap [0, \infty)$ and this set is syndetic in T.*

Let's see if we run into any problems with cascades.

Proposition 3.4.11. *In a cascade, a point is periodic iff it is GH periodic.*

Proof. Let $\langle X, f\rangle$ be a cascade and $x \in X$ a periodic point. Then $\exists n \in \mathbb{N}$ such that $f^n(x) = x$, or $n \in \mathrm{Fix}(x)$. Thus $\mathrm{Fix}(x) \supseteq \{0, n, 2n, 3n, \dots\}$, and by Proposition 3.4.4, $\mathrm{Fix}(x)$ is syndetic in $\mathbb{N}$ and so x is GH periodic.

Now suppose $x \in X$ is GH periodic. Then $\mathrm{Fix}(x)$ is syndetic in $\mathbb{N}$; in particular, $\exists n \in \mathbb{N}$ so that $n \cdot x = x$. Hence x is (cascade) periodic. □

This is what we want. But what about WLF periodicity? How compatible is it with the GH idea?

PROPOSITION 3.4.12. *Let (T, X) be any semiflow. Then any WLF periodic point is GH periodic.*

Proof. Let $x_1 \in X$ be WLF n-periodic with orbit $W = \{x_1, x_2, \ldots, x_n\}$. Then we can assume $\exists k \in T$ so that $kx_i = x_{i+1}$ for each $i \in \{1, 2, \ldots, n-1\}$ and $kx_n = x_1$. Let $K = \{0, k, 2k, 3k, \ldots, (n-1)k\}$. Obviously K is compact. Choose any $t \in T$. If $tx_1 = x_1$, we are done. If not, certainly $tx_1 \in W$, hence $\exists k_i \in K$ such that $(t + k_i)x_1 = x_1$, meaning $(t + K)\cap$ Fix$(x_1) \neq \emptyset$, so x_1 is GH periodic. $\square$

Of course, the converse to the above proposition doesn't hold.

In light of these results, from this point forth I will use the word "periodic" and all its variations exclusively to mean "GH periodic." If necessary, we can refer to cascade periodicity and WLF periodicity explicitly as such.

We are now in a position to complete the original statement of Theorem 3.2.8, as the reader has likely anticipated.

THEOREM 3.4.13 (Li, Zhou)**.** *Using the notation from Definition 3.1.4, $x \in X$ is periodic iff $[(w, x)]$ is periodic $\forall w \in [0, 1)$.*

COROLLARY 3.4.14. *A cascade and its suspended semiflow are either both chaotic or both not.*

CHAPTER 4
A NEW THEOREM

In this chapter we will first introduce stronger versions of two of Devaney's chaos criteria, then use them to obtain a very general and elegant statement about chaos in a general semiflow. This statement, the Redundancy Theorem, includes the BBCDS result as one of its many corollaries.

4.1 Syndetic transitivity and sensitivity

The next (original) construction is an excellent source of counterexamples. I call it the Chickenfoot.

Example 4.1.1. *Let $T \subset \mathbb{R} \times \mathbb{R}$ be the union of all line segments connecting points of the form $(0, n)$ and $(1, 0)$, where $n \in \mathbb{N} \cup \{0\}$. In other words, let any point $(x, y) \in T$ satisfy the equation $y = n(1 - x)$ for some such n.*

Now to give the Chickenfoot a semigroup structure.

Example 4.1.2. *Let T be as above and $(x, y) \in T \backslash \{(1, 0)\}$. Since there is a unique n satisfying $y = n(1-x)$, we may as well denote the point by (x, n). Using the latter notation, define $(t, n) + (s, m) = (\max\{t, s\}, \max\{n, m\})$ and $(t, n) + (1, 0) = (1, 0)$.*

Note that the resulting Chickenfoot semigroup is neither cancellative nor contained in a compact subset of $\mathbb{R} \times \mathbb{R}$. However, it is complete (with the usual metric). Should we so desire, we can easily do away with that:

Example 4.1.3. *Let T be as in Example 4.1.2 but with the point $(1, 0)$ deleted. The resulting semigroup is called the punctured Chickenfoot.*

Now let's introduce two definitions which will allow us to state and prove a much stronger version of the BBCDS theorem, thereby completing the agenda set forth by Wang et al. in 2012. The definitions themselves are identical to some we have already seen, save the last word:

Definition 4.1.4 (Moothathu)**.** *(T, X) is called* **syndetically transitive** *if for any open and nonempty $U, V \subset X$ the dwelling set $D(U, V)$ from Definition 3.2.1 is syndetic.*

Definition 4.1.5 (Moothathu)**.** *Let X be a metric space with metric d. (T, X) is called* **syndetically sensitive** *if $\exists c > 0$ such that for any open and nonempty $U \subseteq X$ the set $R(U, c)$ from Definition 3.2.2 is syndetic.*

Syndetic sets must be nonempty, so obviously these ideas are at least as strong as toptran and sensitivity. It remains to show that, in fact, they are strictly stronger.

Lemma 4.1.6. *A subset of the Chickenfoot semigroup is syndetic iff it includes the point $(1, 0)$.*

Proof. Adapt the proof of Proposition 3.4.8. □

If we let the Chickenfoot semigroup act on itself via the operation, we have a Chickenfoot semiflow.

Proposition 4.1.7. *The Chickenfoot semiflow is not syndetically sensitive.*

Proof. The set $R(U, c)$ from Definition 4.1.5 cannot include $(1, 0)$, as the distance between any two translates by this point is zero. By the preceding lemma, the Chickenfoot semigroup cannot be syndetically sensitive. □

PROPOSITION 4.1.8. *The Chickenfoot semiflow is sensitive for any $c > 0$, but it is not syndetically sensitive.*

Proof. Let $c > 0$ be given, and let U be an open and nonempty subset of the Chickenfoot. Then wlog $\exists n$ so that we can find $(x_1, n), (x_2, n) \in U$ with $x_1 < x_2 < 1$. Let M be an integer greater than $c/(x_2 - x_1)$ and choose $t = (x_1, M)$. Then

$$\begin{aligned} d[t(x_1, n), t(x_2, n)] &= d[(x_1, M), (x_2, M)] \\ &= \sqrt{(x_2 - x_1)^2 + [M(1 - x_2) - M(1 - x_1)]^2} \\ &= (M + 1)(x_2 - x_1) > c \end{aligned}$$

and so this semiflow is sensitive. $\square$

Now let's see a similar result for transitivity.

PROPOSITION 4.1.9. *Let (T, X) be any toptran semiflow and consider the abelian semigroup $G = \{0, 1\}$, where $0 + 0 = 0$ and $1 + 0 = 1 + 1 = 1$. If we give G the discrete topology and let it act on itself via the operation, then $(T \times G, X \times G)$ is toptran but not syndetically transitive.*

Proof. Let $U \times \{0\}, V \times \{0\}$ be open nonempty subsets of $X \times G$. Since (T, G) is toptran, $\exists t \in T$ such that $(t, 0) \cdot (U \times \{0\})$ intersects $V \times \{0\}$ and so $(T \times G, X \times G)$ is still toptran.

Because of the way our action is defined, there is no way $D(U \times \{0\}, V \times \{0\})$ can contain any element of $T \times G$ whose second coordinate is 1. But any syndetic subset of $T \times G$ must contain such elements (as in the proof of Proposition 3.4.8), and so $(T \times G, X \times G)$ is not syndetically transitive. $\square$

4.2 The Redundancy Theorem

In this section we will develop a few necessary preliminaries, then state and prove my first major result.

Lemma 4.2.1. *Any translate of a syndetic set is also syndetic.*

Proof. Suppose $S \subset T$ is syndetic and let $t_1 \in T$. Let K be any compact set corresponding to S; we will show $t_1 + S$ is syndetic with corresponding compact $t_1 + K$.

First of all, $t_1 + K$ is also compact (from the definition of compactness and the continuity of the operation $+$). Now let $t_2 \in T$. Since $\exists x \in (t_2 + K) \cap S$, we must have $(t_1 + x) \in (t_2 + t_1 + K) \cap (t_1 + S)$, i.e. the intersection is nonempty as desired. $\square$

Proposition 4.2.2. *Let (T, X) be toptran with dense periodic points. Then (T, X) is syndetically transitive.*

Proof. Choose open and nonempty $U, V \subset X$; since (T, X) is toptran, $\exists t \in T$ such that $tU \cap V \neq \emptyset$. But V contains a periodic point - let's call it v - and therefore $(t + \mathrm{Fix}(v)) \subseteq D(U, V)$. Since $\mathrm{Fix}(v)$ is syndetic, so is $t + \mathrm{Fix}(v)$ by the preceding lemma; hence so is $D(U, V)$. $\square$

Corollary 4.2.3. *Any chaotic semiflow is syndetically transitive.*

The natural question is this: do there exist syndetically transitive semiflows which do not have dense periodic points?

Proposition 4.2.4. *Any toptran cascade is syndetically transitive.*

Proof. Let $\langle X, f \rangle$ be toptran and $U, V \subset X$ be open and nonempty. By assumption, $\exists n_1 \in D(U, V)$. For that matter, $\exists n_2 \in D(V, V)$. But this means $\forall k \in \mathbb{N}$, we have $(n_1 + kn_2) \in D(U, V)$. By Proposition 3.4.4, $D(U, V)$ is syndetic. $\square$

We know (see Proposition 2.4.3) that there exist toptran cascades with no periodic points, and so "(T, X) is syndetically transitive" is strictly intermediate in strength between "(T, X) is toptran" and "(T, X) is toptran with DPP."

Definition 4.2.5. *Let (T, X) be a semiflow where X has metric d, with $A \subseteq T$ and $x \in X$. We say A acts* **equicontinuously** *on x if $\forall \epsilon > 0$ there is a neighbourhood U of x such that $\forall y \in U$, $d(ax, ay) \leq \epsilon$ for each $a \in A$.*

The following two lemmas are common knowledge in topological dynamics but, as they figure so prominently in the coming proof, will also be proven here.

Lemma 4.2.6. *Let (T, X) be a semiflow and let $K \subset T$ be compact. Then $\forall x \in X$, K acts equicontinuously on x.*

Proof. Let d be the metric on X. Choose $x \in X$ and $\epsilon > 0$. Because the action of T on X is continuous, $\forall k \in K$ we can find a neighborhood U_k of k and $\delta_k > 0$ satisfying

$$\forall k' \in U_k \cap K, d(k'y, kx) < \epsilon/2 \ \forall y \in B_{\delta_k}(x).$$

In words, points in K near k map points near x to points near kx. Note in particular that we may always choose $y = x$.

Now select any open cover of K; this cover has a finite subcover. Wlog each open set in this subcover contains at least one element of K, so $\exists k_1, \ldots, k_n \in K$ enabling us to express the subcover as $U_{k_1}, \ldots, U_{k_n}$. In turn, each of these k_i has a corresponding δ_{k_i} by the above. Set $\delta = \min\{\delta_{k_i} \mid 1 \leq i \leq n\}$.

Next fix any $k' \in K$ and $y \in B_\delta(x)$; then $\exists i \in \{1, \ldots, n\}$ so that $k' \in U_{k_i}$. Hence

$$d(k'x, k'y) \leq d(k'y, k_i x) + d(k_i x, k'x) < \epsilon/2 + \epsilon/2$$

and $B_\delta(x)$ is the desired neighborhood of x. □

Lemma 4.2.7. *Let T be a topological monoid and $K_1, K_2 \subseteq T$ be compact sets. Then $K_1 + K_2 = \{k_1 + k_2 \mid k_1 \in K_1, k_2 \in K_2\}$ is compact.*

Proof. Tychonoff's Lemma tells us that $K_1 \times K_2$ is compact, and the function $\phi : K_1 \times K_2 \to K_1 + K_2$, defined by $\phi(k_1, k_2) = k_1 + k_2$, is continuous. $\square$

Theorem 4.2.8. *Suppose (T, X) is syndetically transitive and not minimal. Then (T, X) is syndetically sensitive.*

Proof. Since (T, X) is not minimal, there is a point $p \in X$ whose orbit is not dense; that is, $\exists q \in X \setminus \overline{Tp}$. This means the infimum of the set $\{d(q, z) \mid z \in \overline{Tp}\}$ is not zero. Call this infimum $3c$. We will show that (T, X) is syndetically c-sensitive.

Choose an open nonempty $U \subset X$. Define $V = \{v \in X \mid d(q, v) < c\}$. By assumption $D(U, V)$ is syndetic with corresponding compact K_1. By Lemma 4.2.6, K_1 acts equicontinuously on p, and so some neighborhood W of p so that $\forall w \in W$ and $\forall k_1 \in K_1$, we have $d(k_1 w, k_1 p) < c$.

Again by assumption, $D(U, W)$ is syndetic in T with corresponding compact K_2. We will now show that $K_1 + K_2$ is a compact corresponding to $R(U, c)$. Fix any $t \in T$ and observe that $\exists u_1 \in U, k_2 \in K_2$ such that $z = (t + k_2)u_1 \in W$.

Any translate of K_1 must intersect $D(U, V)$, and so $\exists u_2 \in U, k_1 \in K_1$ such that $(t + k_2 + k_1)u_2 \in V$, meaning $d(q, [t + k_2 + k_1]u_2) < c$. Finally,

$$\begin{aligned}3c &\leq d(q, k_1 p)\\ &\leq d(q, [t + k_1 + k_2]u_2) + d([t + k_1 + k_2]u_2, [t + k_1 + k_2]u_1) + d(k_1 z, k_1 p).\end{aligned}$$

The first of these three distances is less than c, as is the third (by the definition of W). Hence $t + k_1 + k_2 \in R(U, c)$. Since t is arbitrary, this completes the proof. $\square$

4.3 Corollaries of the Redundancy Theorem

In this section we examine the power of Theorem 4.2.8. Let's commence with a characterization of toptran cascades.

Corollary 4.3.1. *A topologically transitive cascade is syndetically sensitive or minimal.*

Proof. Let $\langle X, f\rangle$ be a toptran cascade. By Proposition 4.2.4, $\langle X, f\rangle$ is syndetically transitive. Now apply Theorem 4.2.8. $\square$

Of course, we have also generalized Theorem 2.4.7 (the BBCDS redundancy theorem), which was our original intent.

Corollary 4.3.2. *Let $\langle X, f\rangle$ be toptran with DPP and let X be an infinite set. Then $\langle X, f\rangle$ is sensitive.*

Proof. By Proposition 4.2.2, $\langle X, f\rangle$ is syndetically transitive. Since X is infinite and periodic orbits in a cascade are finite, $\langle X, f\rangle$ cannot be minimal (no periodic orbit is dense). Apply Theorem 4.2.8. $\square$

It is worth pointing out exactly how we have improved upon the BBCDS result:

Firstly, the sole reason for the requirement "X is infinite" is to guarantee non-minimality. Secondly, the requirement that $\langle X, f\rangle$ be syndetically transitive is strictly weaker than the requirement that $\langle X, f\rangle$ be toptran with DPP. Thirdly, we have seen that syndetic sensitivity is strictly stronger than sensitivity. Fourthly, of course, we are no longer limited to cascades.

Now let's examine our new theorem in the context of WLF periodicity. It is worthwhile to remember that Theorem 3.3.5 (Wang, Long, and Fu's version of BBCDS) is actually independent of Theorem 2.4.7.

Corollary 4.3.3. *Let (T, X) be a toptran semiflow with dense WLF periodic points. Then (T, X) is (syndetically) sensitive.*

Proof. Proposition 3.4.12 tells us that WLF periodic points are a special subset of periodic points. Hence we may again combine Proposition 4.2.2 and Theorem 4.2.8. □

One naturally wonders what happens if a toptran semiflow with DPP is, in fact, minimal. After introducing a few new ideas, we will be in a position to find out.

Definition 4.3.4. *A semiflow (T, X) is called* **uniformly equicontinuous** *if $\forall \epsilon > 0$ we can find $\delta > 0$ such that T acts equicontinuously on each $x \in X$ with the choice of neighborhood $B_\delta(x)$.*

Lemma 4.3.5. *Let $x \in X$ be periodic and $y \in Tx$. Then $x \in Ty$.*

Proof. Let K be the compact corresponding to $\mathrm{Fix}(x)$. Now $\exists t \in T$ such that $y = tx$, so apply the fact that x is periodic to obtain $x \in (t + K)x$. Hence $\exists k \in K$ such that $k(tx) = ky = x$. □

Lemma 4.3.6. *Suppose $x \in X$ is periodic with $\mathrm{Fix}(x)$ having corresponding compact $K \subset T$. Then $Tx = K^{-1}x = \{y \in X \mid Ky \cap \{x\} \neq \emptyset\}$.*

Proof. Choose any $t \in T$. If $tx \notin K^{-1}x$, then $t + K$ fails to intersect $\mathrm{Fix}(x)$, a contradiction to the definition of K. □

PROPOSITION 4.3.7. *Suppose that $x \in X$ is a periodic point with Fix(x) having corresponding compact $K \subset T$. Then $Tx = Kx$.*

Proof. Combine the two preceding lemmas. $\square$

LEMMA 4.3.8. *Let (T, X) be a semiflow. If X consists of a single periodic orbit, then X is compact and (T, X) is uniformly equicontinuous.*

Proof. Choose any $x \in X$ with K being the compact corresponding to $\mathrm{Fix}(x)$. Then $X = Kx$. Since K is compact (in T) and Kx is the continuous image of a compact set (the mapping in question is $\phi(k) = kx$), X is compact.

$K + K$ is compact by Lemma 4.2.7. $K + K$ acts uniformly equicontinuously on X by Lemma 4.2.6. So let $\epsilon > 0$ be given and choose $\delta > 0$ so that if $r \in K + K$ and $x_1, x_2 \in X$ with $d(x_1, x_2) < \delta$, then $d(rx_1, rx_2) < \epsilon$.

Choose $y \in X$ and $t \in T$. Then $\exists k \in K$ so that $(t + k)x = x$. Moreover, since $X = K(kx)$, $\exists k' \in K$ so that $(k' + k)x = tx$. Thus $(2k + k')x = x$. By symmetry $(2k + k')ty = ty$, forcing $(k' + k)y = ty$.

Finally, whenever $d(x, y) < \delta$ we can say $d(tx, ty) = d([k' + k]x, [k' + k]y)$, so the choice $r = k' + k$ gives $d(tx, ty) < \epsilon$. $\square$

Now we are in a position to answer our earlier question. What happens when a syndetically transitive semiflow is also minimal?

COROLLARY 4.3.9. *Suppose (T, X) is toptran with DPP. Then exactly one of the following statements is true:*

1. *(T, X) is syndetically sensitive; or*

2. *(T, X) is a minimal, uniformly equicontinuous semiflow in which every point is periodic with orbit X. Moreover, X is compact.*

Proof. First suppose that (T, X) is syndetically sensitive. Then it cannot be uniformly equicontinuous – the definitions are in direct conflict.

Now suppose (T, X) is not syndetically sensitive. By Proposition 4.2.2 (T, X) is syndetically transitive. Thus by Theorem 4.2.8, it must be that (T, X) is minimal. But the orbit of any periodic point is both dense and compact. Hence X is compact, all orbits coincide, and (T, X) is indeed uniformly equicontinuous. $\square$

More corollaries to Theorem 4.2.8 follow.

COROLLARY 4.3.10. *Let (T, X) be toptran with DPP and suppose X has an isolated point, i.e. $\exists y \in X$ such that $\{y\}$ is open. Then X is finite.*

Proof. Because $\{y\}$ is open, (T, X) is not sensitive. By the preceding corollary, (T, X) is minimal, etc. If $X = \{y\}$, there is nothing further to do. So assume $\exists x \in X \setminus \{y\}$; then $\exists t, t' \in T$ such that $tx = y$ and $t'y = x$.

There is a neighborhood U of x satisfying $tU = \{y\}$, or $(t' + t)U = \{x\}$. If x is also isolated, this is okay – but if not, choose any $z \in U \setminus \{x\}$ and note that $t' + t$ is an element of $\mathrm{Fix}(x)$ but not $\mathrm{Fix}(z)$, meaning x and z cannot lie in the same periodic orbit. But this semiflow has only one orbit, which is periodic, and so every point of X must be isolated. Because X is compact, X must be finite. $\square$

The next two statements provide a generalization of a 1997 result by Touhey . The first is true of any toptran semiflow (and originally required Y to be invariant), while the second follows from Theorem 4.2.8.

PROPOSITION 4.3.11. *Let (T, X) be toptran with $Y \subset X$ dense. Let $U_1, \ldots, U_n$ be any open nonempty subsets of X. Then $\exists x \in Y$ such that $(Tx \cap U_i) \neq \emptyset$ $\forall i \in \{1, 2, \ldots, n\}$.*

Proof. Since (T, X) is toptran $\exists t_1 \in D(U_{n-1}, U_n)$, i.e. $\exists z \in U_{n-1}$ such that $t_1 z \in U_n$. Since the action of T on X is continuous, we can find a neighborhood V_{n-1} of z satisfying $t_1 V_{n-1} \subseteq U_n$. Similarly, $\forall k \in \{1, 2, \ldots, n-1\}$ we can find t_k and an open nonempty $V_{n-k} \subseteq U_{n-k}$ such that $t_k V_{n-k} \subseteq U_{n-k+1}$. Since Y is dense, we may choose any $x \in (Y \cap V_1)$. □

COROLLARY 4.3.12. *Let (T, X) be chaotic with X infinite and let $U_1, \ldots, U_n$ be any open nonempty subsets of X. Then X contains infinitely many points x such that $(Tx \cap U_i) \neq \emptyset$ $\forall i \in \{1, 2, \ldots, n\}$*

Proof. Because (T, X) is chaotic, $\mathrm{Per}(X)$ is dense (and, of course, invariant). The preceding proposition guarantees one such x, while Cororally 4.3.10 guarantees x is not isolated, i.e. $\mathrm{Per}(X) \cap V_1$ is infinite. □

In fact it is possible to generalize Theorem 4.2.8.

PROPOSITION 4.3.13. *Let (T, X) be syndetically transitive and let $S \subset T$ be a closed syndetic subsemigroup with corresponding compact K. If $\exists p, q \in X$ such that $Kq \cap \overline{Tp} = \emptyset$, then the subsemiflow (S, X) is syndetically sensitive.*

Note that, if $S = T$ and $K = \{0\}$, this is identical to Theorem 4.2.8. The requirements on S might seem peculiar until we realize that if q is periodic, then $\mathrm{Fix}(q)$ is a closed syndetic subsemigroup of T, and $\overline{Tq} = Kq$.

The basic idea behind the above statement is that we will be able to find a sensitivity constant $c > 0$ s.t. for any opene $U \in X$, the set $R(U, c)$ is not only syndetic in T, but $R(U, c) \cap S$ is syndetic in S. Perhaps surprisingly, the difficult part of proving that statement is this:

PROPOSITION 4.3.14. *Let $S \subset T$ be a closed syndetic subsemigroup with $A \subset S$. If A is syndetic in T, then A is syndetic in S.*

The proof of this seemingly innocuous statement will take us too far afield; the entire affair is discussed in a recent paper (jointly authored with Dr. Alica Miller) which I include in this dissertation's bibliography.

CHAPTER 5
CHAOS IN PRODUCT SEMIFLOWS

This chapter explores how chaotic properties propagate - or fail to - among semiflows of the form (T, X), (T, Y) and $(T, X \times Y)$. The structure of T turns out to play a major role, which we will explore in depth.

5.1 Motivation

We have spent a great amount of time thinking about when and why toptran and DPP imply sensitivity. The question has been fully answered. Where do we go next?

Recall that we have already seen (in Section 2.4) that no analogue of the BBCDS theorem is possible in the other two configurations; that is, no other pair of chaos conditions can force the third. Let's look at some examples which involve nontrivial topologies.

Example 5.1.1. *Let $X = \{0, 1\} \times \mathbb{U}$ with $f(x, y) = (x, 2y)$. Then $\langle X, f \rangle$ is sensitive with DPP but is not toptran.*

This is a much simpler way to ruin toptran while maintaining the other chaotic properties of the angle-doubler.

Example 5.1.2. *Select any $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ and choose $X = \mathbb{U} \times \mathbb{U}$. Now define $f(x, y) = (2x, y + \alpha)$. The cascade $\langle X, f \rangle$ is toptran and sensitive, but no point is periodic.*

Here we took two well-known maps, one chaotic and one which was only toptran. In the product semiflow, toptran was preserved, sensitivity survived, and DPP did not. It would seem that constructing products in this fashion provides a straightforward and productive way to examine the preservation (or lack thereof) of chaotic properties. So we shall formalize this approach.

Definition 5.1.3. *Let (T, X) and (T, Y) be semiflows. We may construct the* **product semiflow** *$(T, X \times Y)$ by defining $t(x, y) = (tx, ty)$. We call (T, X) and (T, Y) the* **factors** *of the product semiflow.*

Note that, while we could, we are not examining product semiflows of the form $(T_1 \times T_2, X \times Y)$ or even $(T \times T, X \times Y)$. To my knowledge, only one heretofore published paper considers such semiflows. For now we are only considering products where T acts on both phase spaces in lockstep.

Now let's see exactly how this type of construction can affect each of the three Devaney properties. The easiest case is sensitivity. First we'll need to define a metric on the product $X \times Y$. The most obvious option is to simply add distances, but we can be more general:

Definition 5.1.4. *Let (X, d_X) and (Y, d_Y) be metric spaces and let $p \in [1, \infty)$. The* p-**product metric** *on $X \times Y$ is defined as follows:*

$$d_p[(x_1, y_1), (x_2, y_2)] = ([d_X(x_1, x_2)]^p + [d_Y(y_1, y_2)]^p)^{1/p}.$$

The ∞-product metric is defined by

$$d_\infty[(x_1, y_1), (x_2, y_2)] = \max\{d_X(x_1, x_2), +d_Y(y_1, y_2)\}.$$

All of these are valid metrics. The 2-product metric is probably the most common, but in regards to our sensitivity results it makes no difference.

PROPOSITION 5.1.5. *$(T, X \times Y)$ is sensitive with respect to any product metric if and only if at least one factor is sensitive.*

Proof. First assume one factor is c-sensitive, say (T, X). Choose any open nonempty $U \subset X$ and $V \subset Y$ and note that $R(U, c) \subseteq R(U \times V, c)$ in any product metric, and of course we're assuming $R(U, c)$ is nonempty. Hence $(T, X \times Y)$ is (at least) c-sensitive.

Now suppose that neither (T, X) nor (T, Y) is sensitive and let $c > 0$ be given. This means $\exists x \in X$ s.t. T acts equicontinuously on x for any choice of $\epsilon \in [c/2, c]$. Similarly, T acts equicontinuously on Y $\forall \epsilon \in [c/2, c]$. Hence T acts equicontinuously on (x, y) for $\epsilon = c$ regardless of the product metric, and $(T, X \times Y)$ is not c-sensitive. □

It is not surprising that sensitivity is handled so simply. First of all, as we've seen, it is in some sense the least fundamental of the three chaos sisters. Second of all, its definition has something in common with that of a product semiflow: the idea of lockstep.

The reader may wonder whether Proposition 5.1.5 can be adapted to apply to syndetic sensitivity. Interestingly, the answer is a resounding no! The reason is actually the same reason that is going to give us trouble with DPP, so let us move on to that arena.

EXAMPLE 5.1.6. *Let $T = [0, \infty)$ with $S_1 = \mathbb{N}_0$ and $S_2 = \sqrt{2}\mathbb{N}_0$. Both S_1 and S_2 are syndetic in T, but $S_1 \cap S_2 = \{0\}$ which is decidedly not syndetic.*

So the intersection of two syndetic sets need not be syndetic (note here that S_1 and S_2 are even closed submonoids). Now let's construct X and define an action so that the product of two DPP semiflows lacks DPP.

EXAMPLE 5.1.7. *Choose $X = Y = \mathbb{U}$, and let T, S_1, and S_2 be as in the previous example. Define $tx = t + x$, and define $ty = t + y/\sqrt{2}$. Then $\forall x \in X, y \in Y$, we have $Fix(x) = S_1$ and $Fix(y) = S_2$. But no point of $(T, X \times Y)$ is periodic.*

Let's find some characteristic of certain monoids which prevents this from happening.

EXAMPLE 5.1.8. *Let $T = [0, \infty)$ with $S_1 = 3\mathbb{N}_0$ and $S_2 = 5\mathbb{N}_0$. Then $S_1 \cap S_2 = 15\mathbb{N}_0$ is syndetic.*

EXAMPLE 5.1.9. *If $T = \mathbb{Z}$ or any finitely generated submonoid of $\mathbb{N}_0$, then any intersection of syndetic submonoids is syndetic.*

One may wonder if the only "well-behaved" monoids with regard to DPP are discrete. Before answering that, let's give this idea a name.

DEFINITION 5.1.10. *A monoid T is called* **ISS** *if the intersection of any two syndetic submonoids of T is also syndetic.*

Happily, some non-discrete monoids - indeed, rather complicated ones! - can also be cooperative in this respect.

PROPOSITION 5.1.11. *The Chickenfoot is ISS.*

Proof. This is immediate from Lemma 4.1.6. $\square$

5.2 Toptran in products

The third chaos property offers the richest results with regard to product semiflows.

Example 5.2.1. *Let $X = [0, 2]$ and define*

$$f(x) = \begin{cases} 2x + 1, & 0 \leq x \leq 1/2 \\ 3 - 2x, & 1/2 < x \leq 1 \\ 2 - x, & 1 < x \leq 2. \end{cases}$$

Then $\langle X, f\rangle$ is toptran.

The proof that this cascade is toptran is very similar to those we saw in Chapter 2 and will be omitted. The point is the following:

Proposition 5.2.2. *If X and f are as in the previous example, then the product semiflow $\langle X \times X, f\rangle$ is not toptran.*

Proof. Choose $U = (0, 1) \times (0, 1)$ and $V = (0, 1) \times (1, 2)$. These are open and nonempty sets in $X \times X$. But because f will always send $(0, 1)$ to $(1, 2)$ and vice versa, we have $f^n U \cap V = \emptyset\ \forall n$. □

So the product of two toptran semiflows - indeed, toptran cascades - need not be toptran. In fact, we can go further:

Corollary 5.2.3. *The product of two chaotic semiflows need not be chaotic.*

Proof. Combine the previous proposition with Theorem 2.4.10. □

In what follows our goal is to answer this question: is there a condition stronger than toptran so that if one of the factor semiflows has this condition and the other is toptran, the product must be toptran? There is an obvious place to start.

Definition 5.2.4. *A semiflow* (T, X) *is called* **strongly mixing** *if for any opene* $U, V \subset X$*, there is a compact* $K \subset T$ *such that* $\forall t \in T \setminus K, tU \cap V \neq \emptyset$*.*

Quite clearly, this is stronger than either toptran or syndetic transitivity. Note in passing that if T itself is compact, (T, X) will be vacuously strongly mixing regardless of X or the action.

The reason we might be interested in strong mixing is this result, well-known in dynamics but here acting only as an impetus:

Theorem 5.2.5. *If* (T, X) *and* (T, Y) *are both strongly mixing semiflows, then so is* $(T, X \times Y)$*.*

And there is hope that we can weaken that assumption, because of this recent finding:

Theorem 5.2.6 (Li, Zhou)**.** *Let* $T = \mathbb{N}_0$ *or* $[0, \infty)$ *with* (T, Y) *strongly mixing. If* (T, X) *is toptran, then so is* $(T, X \times Y)$*.*

As it happens, the method used to prove the preceding statement does not work for general T. The question now becomes this: what specific property do $\mathbb{N}_0$ and $[0, \infty)$ share (hopefully with many other monoids) that enables one to prove such a result?

It is time to introduce two new notions which will answer both of these questions.

Definition 5.2.7. T *is called a* **directional** *monoid if for any compact* $K \subset T$*,* $\exists t_0 \in T$ *such that* $(t_0 + T) \subseteq T \setminus K$*.*

In words, a directional monoid "points" away from a part of itself we might call the "beginning"; there is (at least) one direction in which the elements move upon translation.

EXAMPLE 5.2.8. *If T is a group (compact or not), then T is not directional because $t + T = T$ for every $t \in T$.*

EXAMPLE 5.2.9. *$\mathbb{N}_0$ is directional, because any compact subset of $\mathbb{N}_0$ is contained in a set of the form $K' = \{1, 2, \ldots, n\}$, and $(n+1) + \mathbb{N}_0$ will not intersect K'.*

EXAMPLE 5.2.10. *$[0, \infty)$ is directional, because any compact subset is contained in a set of the form $K' = [0, b]$, so we can choose any $t_0 > b$.*

It is this property that allows statements such as Theorem 5.2.6. Now let's examine how rich the family of directional monoids can be.

PROPOSITION 5.2.11. *$T = \mathbb{N}_0 \times \mathbb{Z}$ is directional.*

Proof. Let K be a compact subset of $\mathbb{N}_0 \times \mathbb{Z}$. Then we can find $n \in \mathbb{N}_0$ such that $K \subseteq \{1, 2, \ldots, n\} \times \mathbb{Z}$. Choose $t_0 = (n+1, 0)$; then every element of $t_0 + T$ is of the form (m, z) where $z \in \mathbb{Z}$ and $m > n$. Thus $t_0 + T$ does not intersect $T \setminus K$. □

PROPOSITION 5.2.12. *$[0, \infty) \times \mathbb{R}$ is directional.*

Proof. Adapt the proof of the previous proposition. □

We can easily generalize this to obtain a welcome surprise.

THEOREM 5.2.13. *Let T_1 be a directional monoid and let T_2 be any monoid. Then $T_1 \times T_2$ is directional.*

Proof. Let $K \subset T_1 \times T_2$ be compact. Then $K \subseteq K' \times T_2$, where K' is a compact subset of T_1 (if not, we have an immediate contradiction). By assumption $\exists t_1 \in T_1$ satisfying $(t_1 + T_1) \cap K' = \emptyset$. Let 0_2 be the identity of T_2. Then $(t_1, 0_2) + (T_1 \times T_2)$ does not intersect $K' \times T_2$, so it certainly cannot intersect K. □

In short, any monoid with a directional factor will be directional (we could even adapt the preceding proof to cover uncountable products). And that's not all!

Proposition 5.2.14. *The Chickenfoot is not a directional monoid; however, the punctured Chickenfoot is.*

Proof. The choice $K = \{(1,0)\}$ suffices to show the Chickenfoot is not directional, because for any t_0 we have $t_0 + (1,0) = K$.

If T is the punctured Chickenfoot, any compact K will necessarily fail to intersect some neighborhood of $(1,0)$; call this neighborhood A. Then choosing any $t_0 \in A$ will guarantee $t_0 + T \subset A$. □

The other new notion applies not only to the monoid T, but to the whole semiflow.

Definition 5.2.15. *(T, X) is called a* **strongly transitive** *semiflow if for any open and nonempty $U, V \subset X$ and any compact $K \subset T$, there is a $t \in T \setminus K$ satisfying $tU \cap V \neq \emptyset$.*

Proposition 5.2.16. *If (T, X) is strongly transitive, it is toptran.*

Proof. The empty set is compact. □

5.3 Exploring strong transitivity

In the next section, we find a relation between directional monoids and strongly transitive semiflows, then use it to obtain a stronger version of Theorem 5.2.6.

Henceforth we will need to assume that the semiflow action is surjective; that is, $tX = X$ for each $t \in T$. This is a common assumption in dynamics, but I will point out why (and where) we need it.

Proposition 5.3.1. *Let T be a directional monoid. Then (T, X) is strongly transitive iff it is toptran.*

Proof. In light of Proposition 5.2.16, it suffices to prove that if T is directional and (T, X) is toptran, then (T, X) is strongly transitive.

Choose any open and nonempty $U, V \subset X$ and compact $K \subset T$. Since T is directional, choose $t_0 \in T$ so that $(t_0 + T) \cap K = \emptyset$. Now the pullback set $t_0^{-1}V = \{x \in X \mid t_0 x \in V\}$ is open because V is open and the action of T on X is continuous. We must now ask, is $t_0^{-1}V$ empty?

Because of our assumption that T acts surjectively, the answer is yes. Note that we actually need much less than this; for example, we could have stipulated that tX is dense in X for each t, or that for any compact $K \subset T$, at least one element $\tau \in T \setminus K$ exists such that τX is dense.

At any rate, $t_0^{-1}V$ is open and nonempty, so $\exists t_1 \in D(U, t_0^{-1}V)$, which is the same as saying $(t_0 + t_1) \in D(U, V)$. Since $(t_0 + t_1) \notin K$ by construction, (T, X) is indeed strongly transitive. $\square$

Recall that if T is compact, any semiflow (T, X) is (vacuously) strongly mixing. On the other hand, if T is compact, no semiflow (T, X) can be strongly transitive. But as we saw in the last chapter, compact monoids act equicontinuously, so they're not interesting to our study of chaos. Let's see now that the rest of the time, strong transitivity is intermediate in strength between the two properties we've been discussing.

PROPOSITION 5.3.2. *Let T be a noncompact monoid. Then a strongly mixing semiflow (T, X) is strongly transitive.*

Proof. Let K be the compact set from Definition 5.2.4, and let K' be any compact subset of T. Then $K \cup K'$ is also compact, hence not equal to T, and so we can satisfy the requirement of Definition 5.2.15. $\square$

Now let's see two examples (where T is not compact) which demonstrate the reverse implications are untrue.

EXAMPLE 5.3.3. *Let $T = \mathbb{Z}$ with addition. The semiflow (T, T) is toptran by inspection - T is a discrete group! - but it is not strongly transitive, beucase if we choose $U = \{2\}$ and $V = \{5\}$, then $D(U, V)$ is the compact set $\{3\}$.*

Note in passing that, if we try this same situation with $T = \mathbb{N}_0$, it does not serve as a counterexample to the above proposition, because (T, X) is no longer toptran (in the above example, $D(V, U)$ would be quite empty).

EXAMPLE 5.3.4. *Let T be as in the preceding example and choose $X = \{0, 1\}$ with the discrete topology. Define $t \cdot x = (t + x) \mod 2$. Now if $U = \{0\}$ and $V = \{1\}$, the set $D(U, V)$ is precisely the set of odd integers. Since its complement is not compact, (T, X) is not strongly mixing. However, $D(U, V)$ is not compact either. Examining the other possible choices for U and V (there are only four total), we see that $D(U, V)$ is never finite, hence (in this topology) never compact, meaning (T, X) is strongly transitive.*

With these tools in hand, we can craft the sort of statement we're after. First, a simple preliminary.

Lemma 5.3.5. *If $W \subset X \times Y$ is an opene set, then it is possible to find opene $U \subseteq X, V \subseteq Y$ so that $U \times V \subseteq W$.*

Proof. In the product topology, W is a union of sets of precisely the form $U \times V$. □

Theorem 5.3.6. *If (T, X) is strongly transitive and (T, Y) is strongly mixing, then $(T, X \times Y)$ is toptran.*

Proof. Let $W_1, W_2 \subset X \times Y$ be open and nonempty. By the preceding lemma, we can find open nonempty $U_1, U_2 \subseteq X$ and $V_1, V_2 \subseteq Y$ so that $U_1 \times V_1 \subseteq W_1$ and $U_2 \times V_2 \subseteq W_2$. Because (T, Y) is strongly mixing, there is a compact $K \subset T$ such that $[T \setminus K] \subseteq D(V_1, V_2)$.

Because (T, X) is strongly transitive, $D(U_1, U_2)$ must intersect $T \setminus K$. Choose $t \in D(U_1, U_2) \cap [T \setminus K]$. Then $t \in D(U_1, U_2) \cap D(V_1, V_2)$, but this is a subset of $D(W_1, W_2)$ and so $(T, X \times Y)$ is toptran. □

Corollary 5.3.7. *Let T be a directional monoid with (T, Y) strongly mixing. If (T, X) is toptran, then so is $(T, X \times Y)$.*

Proof. Combine the preceding theorem with Proposition 5.3.1. □

Note that while Li and Zhou did not explicitly address the issue of T acting surjectively, the assumption is necessary to their proofs, so this is indeed a version of Theorem 5.2.6 which works for any directional T.

The reader has perhaps wondered when (or if) weak mixing was going to make an appearance.

DEFINITION 5.3.8. *(T, X) is called* **weakly mixing** *if $(T, X \times X)$ is toptran.*

Before proceeding, let's verify that the name is appropriate.

PROPOSITION 5.3.9. *If T is not compact and (T, X) is strongly mixing, then (T, X) is weakly mixing.*

Proof. Let (T, X) be strongly mixing and wlog (because of Lemma 5.3.5) choose open and nonempty $U_1 \times U_2, V_1 \times V_2 \subset X \times X$. Let K_i be compact such that $\forall t \in (T \setminus K_i), tU_i \cap V_i \neq \emptyset$. Any element of $T \setminus (K_1 \cup K_2)$ (happily T is not compact!) will also be an element of $D(U_1 \times U_2, V_1 \times V_2)$. □

Now let's see if we can craft a statement for weak mixing analogous to that of Corollary 5.3.7.

THEOREM 5.3.10. *Let T be a noncompact directional monoid with (T, Y) strongly mixing. If (T, X) is weakly mixing, so is $(T, X \times Y)$.*

Proof. This is very similar to the proof of Theorem 5.3.6. Wlog choose open and nonempty $U_1, U_2, U_3, U_4 \subset X$ and $V_1, V_2, V_3, V_4 \subset Y$. Our goal is to show that $D(U_1 \times V_1 \times U_2 \times V_2, U_3 \times V_3 \times U_4 \times V_4) \neq \emptyset$, or equivalently that the four sets $D(U_1, U_3), \ldots, D(V_2, V_4)$ have a common intersection.

Since (T, Y) is strongly mixing, we can find compact $K_1, K_2 \subset T$ satisfying $[T \setminus (K_1 \cup K_2)] \subseteq D(V_1 \times V_3, V_2 \times V_4)$. But $(T, X \times X)$ is strongly transitive, so $D(U_1 \times U_3, U_2 \times U_4)$ must intersect $T \setminus (K_1 \cup K_2)$.

We have shown that if T is noncompact, $(T, X \times X)$ is strongly transitive, and (T, Y) is strongly mixing, then $(T, X \times Y)$ is weakly mixing. To finish the proof, combine this statement with Proposition 5.3.1. □

5.4 Synnrec monoids

We continue our exploration of transitivity in product spaces, motivated by the following chaos-themed definition.

Definition 5.4.1 (Touhey). (T, X) *has the* **Touhey** *property if for any open nonempty $U, V \subset X$, we can find $t \in T$ and $x \in$ Per$(X) \cap U$ such that $tx \in V$.*

Proposition 5.4.2. *A Touhey semiflow is toptran with DPP, and hence is either minimal or chaotic.*

Proof. This follows from the definition and Theorem 4.2.8. □

Example 5.4.3. *The general rotation semiflow from Example 3.2.3 has the Touhey property (and is minimal).*

We would like to say that if (T, Y) is strongly mixing and (T, X) is Touhey, then $(T, X \times Y)$ is toptran. If T is directional, this would be redundant – but no proof is apparent in the general case. So let's take another approach.

Definition 5.4.4. *A monoid T is called* **synnrec** *if no syndetic subset of T is relatively compact.*

"Synnrec" is an abbreviation for "syndetic not relatively compact."

Example 5.4.5. *The Chickenfoot is not synnrec, since $\{(1, 0)\}$ is syndetic.*

Proposition 5.4.6. *If T is compact, it is not synnrec.*

Proof. Immediate from the definition. □

This eliminates the chore of avoiding vacuously strongly mixing semiflows. Even better, synnrec is more general than the property we've been using.

Proposition 5.4.7. *A noncompact directional monoid is synnrec.*

Proof. Assume S is a subset of the directional monoid T and that $S \subset K$ for some compact K. Let K' be any compact subset of T; then of course $K \cup K'$ is also compact. Since T is directional, $\exists t \in T$ satisfying $(t+T) \cap (K \cup K') = \emptyset$. Thus by assumption $(t+T) \cap S = \emptyset$, meaning $(t+K') \cap S = \emptyset$. But K' is arbitrary, meaning S cannot be syndetic. □

We also have a nice trait of synnrec monoids, just as we did for directional ones:

Proposition 5.4.8. *If T_1 is synnrec and T_2 is any monoid, then $T_1 \times T_2$ is synnrec.*

Proof. Trivial, because of the definition of compactness in the product topology. □

I will hypothesize that any cancellative monoid is also synnrec. The proof of this for discrete monoids involves the same methods as Proposition 4.3.14 and will not be discussed here (see Section 6.2). Certainly nondirectional synnrec monoids exist:

Example 5.4.9. $\mathbb{R}$ *with addition is synnrec.*

After a bit of searching, the reader may wonder whether synnrec monoids exist which are neither directional nor cancellative. The answer is yes!

Example 5.4.10. *Let T be any nondirectional, noncancellative monoid. Then $\mathbb{R} \times T$ is synnrec.*

There is a special relation between our various types of transitivity which applies to synnrec monoids.

Proposition 5.4.11. *If T is synnrec, then for any semiflow (T, X) strong mixing implies syndetic transitivity, which implies strong transitivity.*

Proof. Let (T, X) be strongly mixing. Choose open nonempty $U, V \subset X$ and let $K \subset T$ be compact satisfying $(T \setminus K) \subseteq D(U, V)$. Since T is synnrec, $T \setminus K$ must be syndetic (one choice for its corresponding compact is $K \cup \{t\}$, where $t \notin K$), and so (T, X) is syndetically transitive.

Now let (T, X) be syndetically transitive, choose open nonempty $U, V \subset X$, and let $K \subset T$ be any compact set. Because $D(U, V)$ is syndetic, it cannot be a subset of K, meaning (T, X) is indeed strongly transitive. $\square$

5.5 Creating a chaotic product semiflow

In this section we will put everything together to obtain a working theory of chaos in products, which includes an application of synnrec monoids to Touhey semiflows. We begin with a couple of easily answered, but as yet unasked, questions.

PROPOSITION 5.5.1. *If $(T, X \times Y)$ is toptran, so are both (T, X) and (T, Y). The same is true if we replace "toptran" by "syndetically transitive" or "weakly mixing."*

Proof. Select any open nonempty $U_1, U_2 \subset X$ and $V_1, V_2 \subset Y$. It is apparent that $A = D(U_1 \times V_1, U_2 \times V_2) \subseteq [D(U_1, U_2) \cap D(V_1, V_2)]$. Of course if $(T, X \times Y)$ is toptran, any superset of A is nonempty, and if $(T, X \times Y)$ is syndetically transitive, any superset of A is syndetic. For weak mixing, mimic this proof beginning with eight open nonempty subsets as in the proof of Theorem 5.3.10. □

PROPOSITION 5.5.2. *If $(T, X \times Y)$ has DPP, so do both (T, X) and (T, Y).*

Proof. Suppose (x, y) is periodic, i.e. $\mathrm{Fix}((x, y))$ is syndetic. But $\mathrm{Fix}((x, y))$ is a subset of both $\mathrm{Fix}(x)$ and $\mathrm{Fix}(y)$. □

Now let's make sure we cannot relax the assumptions on some earlier findings.

EXAMPLE 5.5.3. *Let $(\mathbb{Z}, X)$ be any strongly mixing semiflow with open nonempty $U_1, U_2 \subset X$ and $k \in \mathbb{N}_0$ satisfying $kU_1 \cap U_2 = \emptyset$. Now in the cascade $(\mathbb{Z}, \mathbb{Z})$ where $m \cdot n = m + n$, choose $V_1 = \{0\}$ and $V_2 = \{k\}$. Then the product of these two semiflows is not toptran.*

So the product of a strong mixing semiflow and a toptran semiflow need not be toptran. What about the product of two strongly transitive semiflows?

EXAMPLE 5.5.4. *Let T and $X = Y$ be as in Example 5.3.4. Two open nonempty subsets of $X \times Y$ are $U = \{(0, 0)\}$ and $V = \{(0, 1)\}$. Now $D(U, V)$ consists of all integers which are both even and odd, i.e. $(T, X \times Y)$ is not toptran.*

This means the following statement is sharp.

Theorem 5.5.5. *Let (T, X) be strongly transitive and (T, Y) strongly mixing. If $(T, X \times Y)$ has DPP and either (T, X) or (T, Y) is sensitive, then $(T, X \times Y)$ is chaotic.*

Proof. Theorem 5.3.6 guarantees $(T, X \times Y)$ is toptran, whereas Proposition 5.1.5 guarantees it is sensitive. □

There are many ways to create a chaotic product. Let's examine some others.

Corollary 5.5.6. *Let T be directional with (T, X) toptran and (T, Y) strongly mixing, with one of these semiflows also sensitive. If $(T, X \times Y)$ has DPP, it is chaotic.*

Proof. Combine the previous theorem with Proposition 5.3.1. □

Corollary 5.5.7. *Let T be directional with (T, X) toptran and (T, Y) strongly mixing. If $(T, X \times Y)$ is nominimal with DPP, it is chaotic.*

Proof. Combine the previous corollary with Theorem 4.2.8. □

The possible combinations are numerous and easily discovered. Let's now answer a question posed earlier in the chapter, and then give as general a statement as we can about how to create a chaotic product.

Lemma 5.5.8. *If A is dense in X and B is dense in Y, then $A \times B$ is dense in $X \times Y$.*

Proof. Let $W \subset X \times Y$ be open. Lemma 5.3.5 permits us to choose opene $U \subseteq X$, $V \subseteq Y$ so that $(U \times V) \subseteq W$. Thus $\exists a \in A \cap U$ and $b \in B \cap V$, meaning $(a, b) \in (A \times B) \cap W$. □

Note that this lemma is still true for arbitrary products.

PROPOSITION 5.5.9. *If T is an ISS monoid and (T, X) and (T, Y) both have DPP, then $(T, X \times Y)$ has DPP.*

Proof. Let $x \in X$ and $y \in Y$ be periodic. Then because T is an ISS monoid, $\mathrm{Fix}((x, y)) \subseteq \mathrm{Fix}(x) \cap \mathrm{Fix}(y)$ is syndetic and (x, y) is periodic. By the preceding lemma, $\mathrm{Per}(X \times Y)$ is dense in $X \times Y$. $\square$

THEOREM 5.5.10. *Let T be directional and ISS, with (T, X) and (T, Y) each chaotic. If one of these semiflows is strongly mixing, then $(T, X \times Y)$ is chaotic.*

Proof. Wlog assume (T, Y) is strongly mixing. Then (T, X) is strongly transitive by Proposition 5.3.1, so $(T, X \times Y)$ is toptran by Theorem 5.3.6. Since (T, X) is sensitive, so is $(T, X \times Y)$ by Proposition 5.1.5. And because T is ISS, $(T, X \times Y)$ also has DPP by the preceding proposition. $\square$

To conclude this chapter, let us return to our investigation of the Touhey property to obtain an alternate approach.

PROPOSITION 5.5.11. *Let T be a synnrec monoid. If (T, X) is Touhey and (T, Y) is strongly mixing, then $(T, X \times Y)$ is toptran.*

Proof. Choose open nonempty $U_1 \times V_1, U_2 \times V_2 \subset X \times Y$. Since (T, Y) is strongly mixing, let K be a compact subset of T satisfying $(T \setminus K) \subseteq D(V_1, V_2)$. As (T, X) is Touhey, choose $x \in \mathrm{Per}(X) \cap U_1$ and $t \in T$ so that $tx \in U_2$.

By definition $\mathrm{Fix}(x)$ is syndetic. Lemma 4.2.1 assures us that $t + \mathrm{Fix}(x)$ is also syndetic, hence - because T is synnrec - not contained in K. Choose $s \in \mathrm{Fix}(x)$ so that $(t + s) \notin K$.

Now $x \in U_1$ and $(t+s) \cdot x = t \cdot (sx) = tx \in U_2$, so $(t+s) \in D(U_1, U_2)$. Because $(t + s) \notin K$, we must have $(t + s) \in D(V_1, V_2)$. Hence $(t + s) \in D(U_1 \times V_1, U_2 \times V_2)$ and $(T, X \times Y)$ is toptran. $\square$

Now we can obtain a second string of results.

THEOREM 5.5.12. *Suppose T is synnrec with (T, X) Touhey and (T, Y) strongly mixing, and that one of these semiflows is sensitive. If $(T, X \times Y)$ has DPP, it is chaotic.*

Proof. The preceding proposition guarantees $(T, X \times Y)$ is toptran, and Proposition 5.1.5 guarantees it is sensitive. $\square$

PROPOSITION 5.5.13. *If T is a synnrec monoid, (T, X) is Touhey, (T, Y) is strongly mixing, and $(T, X \times Y)$ is nonminimal with DPP, then $(T, X \times Y)$ is chaotic.*

Proof. By Proposition 5.5.11 $(T, X \times Y)$ is toptran, and hence syndetically transitive by the assumption of DPP and Proposition 4.2.2. Theorem 4.2.8 then guarantees syndetic sensitivity. $\square$

THEOREM 5.5.14. *Let T be synnrec and ISS with (T, X) Touhey and (T, Y) both strongly mixing and chaotic. Then $(T, X \times Y)$ is chaotic.*

Proof. $(T, X \times Y)$ is toptran because of Proposition 5.5.11. It is sensitive because (T, Y) is sensitive (and thus we can apply Proposition 5.1.5). And it has DPP because of Proposition 5.5.9. $\square$

CHAPTER 6
CONCLUSION

As we approach the end of this dissertation, I will briefly discuss my current research, ask some unanswered questions, and discuss how my work is relevant.

6.1 Current work

Wu and Zhu have recently obtained some results for product cascades which, considering our interests in the previous chapter, we would like to obtain for a general semiflow. Let's see if this is possible.

Theorem 6.1.1 (Wu, Zhu)**.** *Let $\{X_i\}$ be a sequence of metric spaces without isolated points and let $\{f_i : X_i \to X_i\}$ be a sequence of continuous functions. Use the notation $Per(\prod X_i)$ to denote the set of all periodic points in the product cascade $\langle X_1 \times X_2 \times \ldots, f_1 \times f_2 \times \ldots \rangle$. Then $\overline{Per(\prod X_i)} = \overline{\prod(PerX_i)}$ iff $\sup\{Q_i\} \in \mathbb{N}_0$, where Q_i is the minimal period of a periodic point in $\langle X_i, f_i \rangle$.*

In words, as long as the set of minimal periods is bounded above, the infinite product of periodic points has the same closure as the periodic points of the infinite product.

We already saw (in the proof of Proposition 5.5.9) that for any finite product, DPP is preserved in product semiflows in which T is ISS. (If T is not ISS, we can say nothing.) In order to obtain a statement similar to the above theorem, we first need to contrive an analogue to the notion of a minimal period Q_i.

One way to do this, of course, is to impose a countably-ISS condition on T. Let's verify such a condition is equivalent to Wu and Zhu's requirement that $\sup\{Q_i\}$ be finite. Remember that any fixer is always a (closed) submonoid.

PROPOSITION 6.1.2. *Let Q_i be as in Theorem 6.1.1. If $T = \mathbb{N}_0$, the following are equivalent:*

1. In the countable collection $\{\langle X_1, f_1\rangle, \langle X_2, f_2\rangle, \langle X_3, f_3\rangle, \dots\}$ of cascades, the set $\{Q_1, Q_2, Q_3, \dots\}$ is bounded above.

2. In the countable collection $\{(T, X_1), (T, X_2), (T, X_3), \dots\}$ of semiflows, there is a syndetic submonoid S of T such that $\forall X_i$, there exists some $x_i \in X_i$ with $S \subseteq Fix(x_i)$.

Proof. Any set of natural numbers which has an upper bound is finite. Hence $\{Q_1, Q_2, Q_3, \dots\}$ has a least common multiple k. Since $T = \mathbb{N}_0$, the syndetic set $\{k, 2k, 3k, \dots\}$ fixes at least one element of X_i (namely, the one with period Q_i) and so the first statement implies the second.

Now, any nontrivial submonoid of $\mathbb{N}_0$ must be syndetic; in fact, it must contain a subset of the form $\{n, 2n, 3n, \dots\}$. Hence n fixes an element of each X_i and is an upper bound on each Q_i, so the second statement implies the first. □

From here we can (and intend to) proceed to reformulate Theorem 6.1.1 in the context of a general semiflow when T is ISS.

As long as we're thinking about infinite products, let's prove the following statement:

Theorem 6.1.3. *Let $\{X_\alpha\}_{\alpha \in A}$ be any collection of metric spaces, with A any nonempty set. Let $(T, \prod X_\alpha)$ be a semiflow s.t. $\forall \alpha \in A$ the factor semiflow (T, X_α) is nonminimal. Then if $(T, \prod X_\alpha)$ is chaotic, every factor semiflow is chaotic.*

Proof. Fix any $\beta \in A$. Because of Theorem 4.2.8, it suffices to show that (T, X_β) is syndetically transitive. Now $(T, \prod X_\alpha)$ is toptran with dense periodic points, and so by Theorem 4.2.2 it is syndetically transitive. Choose open nonempty $U, V \in X_\beta$ and let $U' = U \times \prod_{\alpha \neq \beta} X_\alpha$. Define V' in the same fashion.

Since U' and V' are opene subsets of $\prod X_\alpha$, the set $S = \{t \in T \mid tU' \cap V' \neq \emptyset\}$ is syndetic in T. But S is a subset of $\{t \in T \mid tU \cap V \neq \emptyset\}$, and so the latter set is syndetic in T as well. Therefore (T, X_β) is syndetically transitive, as desired. $\square$

Note that there is no restriction on T; in particular, it need not be ISS. The converse to this theorem, of course, is false.

Here is a further statement which can be adapted to our general theory, using our established techniques.

Theorem 6.1.4 (Wu, Zhu [10])**.** *If X is compact and $\langle X, f \rangle$ is chaotic, then $Per(X)$ is infinite.*

Our take on this is as follows:

Theorem 6.1.5. *Let T be ISS and let (T, X) be a chaotic semiflow with a dense orbit. Further suppose that X is not compact. Then the set of all fixer submonoids $\{S \subset T \mid \exists x \in X : S = Fix(x)\}$ is infinite.*

Proof. Contrariwise, assume $S_1, \ldots, S_n$ are all such fixer submonoids and call their intersection S. By hypothesis S is syndetic in T. Now let $p \in \mathrm{Per}(X)$. It is plain to see that $Sp = \{p\}$, i.e. S fixes every periodic point. Now we claim that $\mathrm{Per}(X) = X$.

Choose any $x \in X$. Since (T, X) is DPP, we can find a sequence of periodic points $\{p_i\}$ such that $p_i \to x$. Choose $s \in S$ and let $y = sx$. Then by continuity $sp_i \to sx$, or in other words $p_i \to y$, forcing $y = x$. Therefore $Sx = \{x\}$ and x is indeed periodic.

Finally, let z be a point with dense orbit. Then $S_z = \mathrm{Fix}(z)$ is syndetic with corresponding compact K, meaning $X = \overline{Tz} = \overline{Kz} = Kz$. But we assumed X is not compact. $\square$

Corollary 6.1.6. *Let T be ISS and (T, X) a DPP semiflow with finitely many fixer submonoids of periodic points. Then (T, X) is not sensitive.*

Proof. Let S be the intersection of all fixer submonoids of periodic points; since T is ISS, S is syndetic with some corresponding compact K. As we saw in the preceding proof, in this situation S fixes any point of X.

Let $c > 0$ be given. Certainly $\overline{Tx} = Kx$ and $\overline{Ty} = Ky$, but Lemma 4.2.6 reminds us that the action of K on a point is equicontinuous. Hence we can choose $\delta > 0$ s.t. if $d(x, y) < \delta$, then $\max\{d(tx, ty) \mid t \in T\} < c$. $\square$

It is clear that there is ample room to explore further in this terrain.

6.2 Open questions

There are many, many places to go next, apart from more investigation into product semiflows. Here are just a few ideas:

1. Referring to Proposition 4.3.13, it has been shown that such p, q always exist when X is a Baire space (i.e. the intersection of countably many open dense sets remains dense) and T is both separable (T has a countable dense subset) and a C-semigroup. Are all of these conditions necessary?

2. Proposition 4.3.14 is provable, as far as I can tell, only by embedding T into its Stone-Čech compactification βT. (We must require T to be a Tychonoff space, or there may be no such embedding.) The same problem - and solution - arises when trying to show that cancellative monoids are synnrec. Something very important is lurking here, and it is standing in the way of a general theory of abelian semiflows.

3. Du in 1998 posed the following question: suppose (T, X) is sensitive and S is a syndetic subsemigroup of T. Is (S, X) necessarily sensitive? This is partially answered by Proposition 4.3.13, but there is more to do.

4. In 2007 Moothathu wondered if it were possible to characterize syndetic sensitivity for toptran cascades. We can ask a related question: given that a product semiflow is toptran (or perhaps less), what are necessary and sufficient conditions for it to be syndetically sensitive? The same applies to cocompact sensitivity, etc.

5. We can study semiflows of the form $(S \times T, X \times Y)$.

6. Li and Zhou have begun exploring cascades in which f is not necessarily continuous. Many of their results still hold. We would have to contend with a few pullback problems in some of our proofs, but this is worth exploring for general T.

7. How much of our emergent theory applies to semiflows where T is not necessarily abelian, or perhaps merely a magma, quasigroup, etc.?

6.3 Applications

Perhaps surprisingly, a great many topics in mathematics can be thought of as semiflows.

Example 6.3.1. *Let $X = \mathbb{N}$, define an open set of X to be a set that contains $\{1, 2\}$ (or is empty), and define $f : X \to X$ by*

$$f(x) = \begin{cases} x/2, & x = 0 \mod 2 \\ (3x+1)/2, & x = 1 \mod 2 \end{cases},$$

The Collatz conjecture states that $\langle X, f \rangle$ is minimal.

To be specific, Collatz hypothesized that every point is eventually periodic and that $\{1, 2\}$ is the only periodic orbit. Of course, this formulation also permits exploration of the suspended semiflow as per Definition 3.1.4.

Example 6.3.2. *Let $X_1, X_2, \ldots, X_n$ be any phase spaces corresponding to the motion of a particle or particles (quantized, linear, multidimensional space, etc.). Choose an initial state $y(0) = (x_1(0), x_2(0), \ldots, x_n(0))$ and let $T = [0, \infty)$ or any abelian monoid which you wish to represent the flow of time. The dynamical behavior of the entire system at any time can now be explored at will, and all the results in these pages apply.*

It would be nice if T could be a matrix group (quantum mechanics thrives on such operators), hence our seventh open question from the preceding section. The point is that virtually all dynamical systems currently under scrutiny (in particular, cascades and real flows) are subject to the same rules with regard to chaos, a fact which will doubtless become even more useful as mathematicians construct new and ever more innovative models.

6.4 Epitaph

"Panta rhei [everything flows]."

- Simplicius (c. 490 - c. 560), summarizing Heraclitus

"Well here's another place you can go, where everything flows."

- The Beatles, "Glass Onion" (1968)

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Analytic number theory

Pedagogy of mathematics

Algebra, especially group theory

University Services

Represented Department of Mathematics at Kentucky K-12 Common Core Standards Focus Group, University of Louisville, summer 2013