University of Louisville

ThinkIR: The University of Louisville's Institutional Repository

Electronic Theses and Dissertations

12-2016

Knowledge of algebra for teaching quadratic equations : the impact of knowledge dimensions on planning and pedagogical strategies.

Michael Raney
University of Louisville

Follow this and additional works at: <https://ir.library.louisville.edu/etd>

Part of the [Education Commons](#)

Recommended Citation

Raney, Michael, "Knowledge of algebra for teaching quadratic equations : the impact of knowledge dimensions on planning and pedagogical strategies." (2016). *Electronic Theses and Dissertations*. Paper 2563.

<https://doi.org/10.18297/etd/2563>

This Doctoral Dissertation is brought to you for free and open access by ThinkIR: The University of Louisville's Institutional Repository. It has been accepted for inclusion in Electronic Theses and Dissertations by an authorized administrator of ThinkIR: The University of Louisville's Institutional Repository. This title appears here courtesy of the author, who has retained all other copyrights. For more information, please contact thinkir@louisville.edu.

KNOWLEDGE OF ALGEBRA FOR TEACHING QUADRATIC EQUATIONS: THE
IMPACT OF KNOWLEDGE DIMENSIONS ON PLANNING AND PEDAGOGICAL
STRATEGIES

By

Mick Raney

B.S., United States Military Academy, 1974

M.S. Rensselaer Polytechnic Institute, 1981

A Dissertation Submitted to the Faculty of the College of Education and Human
Development of the University of Louisville in Partial Fulfillment of the Requirements

for the Degree of

Doctor of Philosophy

in Curriculum and Instruction

Department of Middle and Secondary Education

University of Louisville

Louisville, Kentucky

December 2016

KNOWLEDGE OF ALGEBRA FOR TEACHING QUADRATIC EQUATIONS: THE
IMPACT OF KNOWLEDGE DIMENSIONS ON PLANNING AND PEDAGOGICAL
STRATEGIES

By
Michael L. Raney

B.S. United States Military Academy, 1974

M.S. Rensselaer Polytechnic Institute, 1983

A Dissertation Approved on

November 3, 2016

By the following Dissertation Committee

Dr. Jenny Bay-Williams

Dr. Caroline Sheffield

Dr. Maggie McGatha

Dr. Mike Mays

DEDICATION

This dissertation is dedicated to the one person who has most steadfastly encouraged, loved, and supported me throughout this arduous journey: my wonderful wife, Laura Raney. Without her constant and unwavering confidence and commitment, this work could not have been realized. She has been my constant companion, source of inspiration, and consistent strength every day of the process from matriculation to manifestation. With enduring gratefulness and unending love, Thank You.

ACKNOWLEDGEMENT

After a protracted journey across several years, the list of people deserving of my appreciation and gratefulness is both long and distinguished. Of primary importance are the dedicated professionals who have given so much of their time, energy, knowledge, and personal encouragement to assure the success of this endeavor. Those select few are the members of my dissertation committee. It is not truly possible to express the gratitude I feel toward these four incredibly talented and dedicated individuals who have left an indelible impression on this work and on me as an individual. “Thank You” seems an inadequate expression to convey my sincere appreciation for the perseverance and intellectual contribution of this group of gifted educators. Beginning with my committee chair, Dr. Jenny Bay-Williams, whose contagious enthusiasm was a motivating force when the edits seemed endless, and continuing with Dr. Caroline Sheffield whose patience and consistent encouragement were a guiding force during many hours of writing – and rewriting, the four members were constant advocates for success. The two remaining members, Dr. Maggie McGatha and Dr. Mike Mays, are especially deserving of my gratitude for enduring the dissertation journey for the entire tenure from ACCLAIM COHORT 3 to final completion. Dr. McGatha’s steadfast encouragement, and Dr. Mays wonderful instructional and mentoring contribution were both integral in this successful effort. I would be remiss if I did not express my sincere gratitude to Dr.

Bill Bush as well for his tireless effort in guiding the crafting of my proposal that led to this completed work.

The entire cohort of classmates that began this journey at WVU so many years prior will always hold a special place in my memory along the way. Of special note, I have repeatedly leaned on a select subset of this group for their support and encouragement. These few have been an important, even critical component of my successful completion. Thank you so much, Robin, for your constant, quiet confidence and persistence in believing in the reality of success. My very special friends, confidants, and mentors: Sunny and Eleanor have been a consistent factor in keeping me focused and moving forward. Finally, Wayne, for his perseverance and consistency.

Living in the reality of the working world, there are many colleagues at Montgomery Bell Academy who have given both encouragement and the gift of time to ensure success in this effort to complete the task. Chief among those is LtCol. Joe Sharbel whose selfless commitment to my dissertation process has been a key ingredient in realizing this objective. His personal commitment of time and energy, allowing me an opportunity to allocate needed time to finish the mission, was invaluable at a point when time was the most critical ingredient necessary to make the revisions needed to bring the project to fruition. Several others including Courtney, Chandler, Chris, and Emmitt were a source of continuing encouragement, helping to maintain momentum throughout. There are an additional cadre of colleagues at Boone County High School who have been my constant companions during the entire process from coursework through the final product. The entire Department of Mathematics, led by Kelly Lindsey are of special note

with particular thanks to my fellow workers; Julie Stickels, Alison Sutton, and Kelly Read who were steadfast in their continued support and confidence.

A veritable Army of prayer warriors, recruited and nurtured by my loving wife, have been an ever-present source of strength and encouragement. Through so many times of challenge during writing and rewriting, research and multiple iterations, these faithful family members and friends lifted me up in prayer for guidance and perseverance. Without their constant supplication, this effort would not have been possible. I am in debt to my extended family for their unwavering confidence and commitment over the many years needed to complete this study and dissertation journey. Of particular importance was the enduring confidence of my son, Mickey, who never failed to encourage and prod me on to the finish. It is not likely I will be ever able to repay this debt, which I am confident will be enthusiastically forgiven.

Most importantly, I am eternally grateful to my parents whose steadfast commitment to the true value of education was the bedrock of all of my academic endeavors. These two people, truly the source of the essence of my persona, left an undeniable imprint on my life by the values and moral standing they embraced and transmitted to myself and all my siblings. Their unswerving commitment to the betterment of our lives through education established the foundation for the success of all of their offspring, imbuing an enduring appreciation for a productive work ethic and the intrinsic value of knowledge through a lifetime of learning.

ABSTRACT

KNOWLEDGE OF ALGEBRA FOR TEACHING QUADRATIC EQUATIONS: THE IMPACT OF KNOWLEDGE DIMENSIONS ON PLANNING AND PEDAGOGICAL STRATEGIES

Mick Raney

15 December, 2016

Planning and presenting meaningful instruction in the mathematics classroom is a demanding task requiring mental flexibility as well as a solid foundation in relevant content knowledge and pedagogical strategies. In the last three decades, there has been a growing interest in the way teachers utilize this specialized mathematical knowledge for teaching algebra in the practice of teaching, specifically in planning and presenting classroom instruction. Even more recently, a growing area of emphasis has been the application of these domains of knowledge in a specific focal context involving a singular topic or unit of instruction in the mathematics classroom. This study is such an inquiry, with a focus on a unit of instruction on solving quadratic equations in a first course in algebra. Using a mixed-methods design, the study investigates the cognitive connections between the dimensions of teachers' knowledge for teaching algebra and the planning and implementation of instruction using the KAT Framework from Michigan State University. Findings indicate that classroom teachers access and utilize their knowledge in ways that reflect the depth of KAT they possess. Further, the degree of knowledge in the three dimensions of the KAT Framework is a strong indicator of teachers' ability to enact effective instruction, extend learning

experiences, and make connections across the mathematical domain. Both internal classroom dynamics and external factors beyond the teachers' control emerged as surprisingly strong influences in these processes. The study has implications for teacher preparation, metrics for teacher effectiveness, social justice, and professional development programs.

Keywords: KAT, Teaching Knowledge, Knowledge of School Curriculum, Advanced Mathematical Knowledge, Wait-Time, High-Stakes Testing, Bridging, Trimming, Decompressing.

TABLE OF CONTENTS

ABSTRACT.....	VII
LIST OF TABLES	XIII
LIST OF FIGURES	XIV
CHAPTER 1 - INTRODUCTION.....	1
INTRODUCTION TO THE CASE STUDY.....	1
STATEMENT OF THE PROBLEM	4
RESEARCH QUESTIONS	8
SIGNIFICANCE OF THE STUDY.....	8
Terminology	10
CHAPTER 2 – LITERATURE REVIEW	13
OVERVIEW	13
CONCEPTUAL FRAMEWORK.....	15
DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING.....	19
Pedagogical Content Knowledge	20
Mathematical Knowledge for Teaching.....	22
Knowledge of Algebra for Teaching.....	24
Research on Quadratic Equations	31
Research on Students - Beliefs	31
Research on Students - Understanding	33
Research on Teacher Knowledge related to Quadratic Equations.....	36
TASKS OF TEACHING	37
Conclusion.....	39
CHAPTER 3 – METHODOLOGY	40
INTRODUCTION	40
THE CASE STUDY AS A METHODOLOGY	42
Research Methods	45
Participants.....	45
Setting	47

Data collection	48
Content Knowledge	51
Planning Data.....	52
Teaching Data	54
Data Analysis	56
Knowledge of Algebra for Teaching	57
Semi-structured interviews	61
Questionnaires.....	63
Lesson Observation.....	64
Artifacts.....	65
TRUSTWORTHINESS OF THE STUDY	67
Credibility.....	67
Transferability	69
Dependability	70
Confirmability	71
RESEARCH ASSUMPTIONS	72
LIMITATIONS	73
CHAPTER 4	75
FINDINGS	75
Introduction	75
Participants	77
Analysis of Data from Research Instruments, Coding, and Emergent Themes	79
Coding Structure	79
Analysis of Data Using the KAT Matrix	82
First Level Coding: Coding to KAT Matrix	85
Second Level Coding: Coding to KAT Themes	90
Open Coding: Emergent Themes.....	91
Individual Case Studies	96
Case Study One: Janice.....	97
Teaching Style	98
Procedural versus Conceptual Orientation.....	101
Role as Teacher.....	102

Evidence of Bridging, Trimming, and Decompressing	108
Applications to Research Objectives: Janice	116
Research Question 1.	116
Research Question 2.	117
Research Question 3.	118
Individual Case Study Dean.....	119
Teaching Style	120
Conceptual versus Procedural Orientation.....	122
Role as Teacher	125
Evidence of Bridging, Trimming, and Decompressing	129
Applications to Research Objectives: Dean	137
Research Question 1.	137
Research Question 2.	138
Research Question 3.	139
Cross-Case Analysis.....	141
Demographics and Learning Climate	141
Teaching Style	142
Procedural versus Conceptual Orientation.....	146
Testing Focus and Lack of Wait-Time	147
Role as a Teacher	150
Evidence of Bridging, Trimming, Decompressing Across KAT Dimensions in Instruction	154
Summary of Findings: Applications to Research Objectives.....	156
CHAPTER 5	158
DISCUSSION AND CONCLUSIONS	158
Purpose of the Study	159
Methodology	160
Discussion and Implications.....	162
Knowledge of Algebra for Teaching (KAT)	162
KAT Dimensions of Knowledge	163
KAT Knowledge Levels as a Lens for Instructional Approach.....	165
KAT Accessed for Planning	166
Accessing KAT in Planning to Meet Student Needs	166

KAT Accessed in Planning to Meet Teacher and Student Expectations	169
KAT Accessed in Instruction.....	171
Accessing KAT in Instruction to Meet Student Needs	172
Evidence of Bridging, Trimming, Decompressing Across KAT Dimensions in Instruction	173
Impact of Emergent Themes on KAT Accessed During Instruction	174
Impact of Wait-Time Not Provided on Instruction.....	174
Impact of High Stakes Testing on Instruction	176
Implications and Future Research	178
Limitations	182
Summary	183
REFERENCES	186
APPENDICES	198
CURRICULUM VITA	238

LIST OF TABLES

Table 1. Research Components.....	50
Table 2. Demographic Data for Teachers and Schools.....	77
Table 3. KAT Assessment Scores for Participants.	84
Table 4. Counts of Teacher Tasks vs. Teacher Knowledge.....	89
Table 5. Use of Overarching Themes	91
Table 6. Emergent Themes	94
Table 7 . Teacher Tasks vs. Knowledge Categories First Level Coding.....	95
Table 8. Teacher Tasks Used by Janice	103
Table 9. Emergent Themes Used by Janice	107
Table 10. Teacher Tasks Used by Dean.....	123
Table 11. Emergent Themes in Dean's Classroom	128
Table 12. Comparison of Emergent Themes	148

LIST OF FIGURES

Figure 1. Framework for the Case Study on KAT	17
Figure 2. Domains of Mathematical Knowledge for Teaching	23
Figure 3 Multi-Dimensional Grid for Professional Competence in Elementary Algebra	26
Figure4 KAT Matrix is used by permission of the Michigan State Knowledge of Algebra for Teaching Project and serves as the basis for data collection and analysis.	49
Figure5. KAT Matrix with Overarching Themes	86
Figure 6. Concentration of Teacher Tasks	151

CHAPTER 1 - INTRODUCTION

INTRODUCTION TO THE CASE STUDY

A contradiction that many Algebra I students encounter is that mathematics is boring and difficult to learn but, at the same time, mathematics is important and practical for life (Breiteig, Grevholm & Kislenko, 2005). Niss (1994) referred to this situation as “a paradoxical dilemma at the personal level” because students value mathematics empirically and sociologically, but mathematics presents challenges for many of them in their first algebra course. In fact, 80% of students tested on the most recent National Assessment of Educational Progress responded they agreed or strongly agreed that taking mathematics was very important and would help them in their future education or career (http://www.nationsreportcard.gov/reading_math_g12_2013/#/, retrieved 3 August, 2014).

From a policy-making perspective, the National Council of Teachers of Mathematics (NCTM) has continuously sought to create research-based guidelines that delineate standards of effective teaching. In *Principles and Standards for School Mathematics* (NCTM, 2000), mathematics teaching is described as a “complex endeavor” (p. 17) which requires knowing and understanding content, cognitive aspects of students’ learning, and pedagogical strategies. NCTM’s latest publication, *Principles to Actions* (2014), identifies eight Mathematics Teaching Practices designed to improve student learning, promote improved reasoning and problem skills, and more adequately prepare

students for post-secondary education. To meet ever-increasing demands of college and career readiness, teachers must be able to create comprehensible representations, encourage critical thinking, contextualize mathematical teaching, form connections, adapt lessons, analyze student errors, and ask mathematically productive questions (Ball & Bass, 2000; Ball, Lubienski, & Mewborn, 2001; Ball, Thames, & Phelps, 2008; Hill, Schilling & Ball, 2004; Lampert, 1989; Ma, 1999).

As students, teachers, and policy makers grapple with an ever increasing emphasis on 21st century skills, College and Career Readiness, ubiquitous standardized testing and demands for technical proficiency in the work place, schools, administrators and stakeholders at every level struggle to find more effective and efficient ways to prepare the next generation of learners. Employers need a workforce with better technical skills and critical thinking abilities, and STEM careers place high priority on advanced mathematical proficiency. Many of those same stakeholders concurrently raise the alarm at the continued decline in international standing on mathematics tests such as the Program for International Student Assessment (PISA) where 37 countries have shown higher annualized improvement than the United States (OECD, 2014) and national assessments such as the National Assessment of Educational Progress (NAEP) on which 12th grade scores remain unchanged from the previous assessment in 2009.

In response to the myriad calls for improved learning and performance in mathematics, classroom teachers draw upon their experiences, knowledge of mathematics, and pedagogical strategies to meet the needs of an increasingly diverse population of students. Inherent in the multiple layers of socio-economic backgrounds, culturally blended populations, an explosion of technology, and a generation of learners

immersed in social media, students carry a wide range of beliefs and attitudes into the classroom setting that have an impact on the learning climate. Though, as previously noted, students recognize the importance of attaining proficiency in mathematics, these attitudes and beliefs can significantly impact their confidence, goal setting, and persistence. To counter these barriers to learning, classroom teachers must be prepared to utilize adaptive and flexible strategies that make mathematics comprehensible to every learner.

In the field of mathematics education, a growing number of research efforts have made attempts to describe the application of the classroom teachers' experiences, knowledge, and proficiencies in the dynamic and evolving environment of the mathematics classroom through the framework of an overarching concept known as Mathematical Knowledge for Teaching (MKT). This terminology was first used by Ball & Bass (2003) to capture the essence of the complex teaching skills used by mathematics teachers as they navigate the demands of the mathematics classroom. At the core of this burgeoning field of research is a motivation to more clearly define the specialized knowledge that drives instructional strategies and pedagogical actions that occur in every facet of the classroom from planning to implementation to assessment of learning. Increased attention has been directed toward the actions of the teacher in the classroom setting, encompassing the dynamic interactions between teachers, students and the learning environment itself. There is much evidence that as teachers are adaptive, flexible, and innovative, manage student misconceptions, parse content into manageable packets, construct multiple representations, facilitate mathematical discussions, and bridge various topics and content, that they draw upon their MKT, which is

contextualized for specific levels and situations (Ball et al. 2001; Ball & Bass, 2003; Ball, Thames, & Phelps, 2008; Fennema & Franke, 2002; Ferrini-Mundy, 2009; Hill, Rowan, & Ball 2005; Lampert, 1989; Ma, L., 1999). Mathematical knowledge for teaching remains an important topic for research because, as Ball and her colleagues stated, “what teachers and students are able to do *together* (emphasis added) with mathematics in classrooms is at the heart of mathematics education...” (Ball et al., 2001, p.433).

STATEMENT OF THE PROBLEM

As efforts to delineate what mathematical knowledge for teaching consists of at its core, the research community is beginning to recognize the need to understand this specialized knowledge as it varies from one topic to another and across many different academic levels. Just as content in elementary school varies from content in secondary school, the internal mechanisms of this specialized knowledge and its application within the learning environment will need modification and adaptation. Elementary teachers need to be skilled in numeracy and representations of fractions that are comprehensible to their students, while secondary teachers need to have multiple representations of functions and an understanding of the way students perceive and learn to solve algebraic equations. To this point, most extant research has been conducted at the elementary level (Ball, Hill, & Bass, 2005), with fewer, more recent efforts to examine mathematical knowledge for teaching at the secondary level (Ferrini-Mundy et al., 2005). As continued emphasis is placed on technical and problem solving skills, teachers of mathematics at the secondary level need to have access to research that is relevant and productive both at the pre-service and in-service stage of their career as they work to convey a deeper and richer understanding of their content.

In the mathematics curriculum encompassing elementary through secondary education, the topic of algebra stands as a transitional course of study, marking the maturation from basic arithmetic manipulation, to initiation, to a sometimes abstract and symbolic lexicon that establishes a foundation for the study of mathematics at advanced levels. The first course in algebra has been called a “unique and formidable gatekeeper” (RAND Mathematics Study Panel, 2003, p.47) course as it marks the initial level of mathematical instruction that provides the basis for learning further skills that underpin those requisite abilities required in the 21st century skills and the post-secondary transition to College and Career Ready opportunities. The RAND Mathematics Study Panel emphasized the importance of algebra because it serves as a basis for both mathematics and science pursuits as well as skills needed in the workplace, and ultimately could result in limited educational and career options if not mastered (RAND Mathematics Study Panel.2003).

Against the backdrop of this emphasis on algebra as a milestone in secondary education, some recent efforts have been made to contextualize the mechanisms and principles of mathematical knowledge for teaching as it is deployed in a first course on algebra. Intuitively, it is expected that pedagogical knowledge, instructional practices, and even content knowledge will increase with experience, but the how and the why of the growth of mathematical knowledge for teaching *algebra* has been elusive from a quantitative perspective. One such effort to assess teacher knowledge related to teaching algebra is the Diagnostic Test for Assessing Middle School (DTAMS) teachers’ knowledge for teaching algebra developed at the University of Louisville (Center for Research in Mathematics and Science Teacher Education (CRiMSTeD) (n.d.).

MKT for algebra is complex and diverse. Recent research has focused on establishing a definition for MKT by examining the diverse aspects of this specialized knowledge. As an example, Michigan State observed *tasks of teaching* linear equations as a proxy for application of certain aspects of MKT (Ferrini-Mundy et al., 2005). In another study effort, Artigue, Assule, Grugeon & Lenfant (2001) characterize MKT as multidimensional and multifaceted with inherent changing nuances when different topics or subject matter in the mathematics curriculum are being addressed. These conclusions frame the challenge facing current research and serve as a springboard for my study to deepen understanding of how knowledge for teaching algebra is accessed and utilized during planning and instructing in a specific topical area.

In the larger context of teacher actions that translate into creating and nurturing a productive learning climate, educational research in several areas has examined and attempted to codify what teacher moves are most effective. These actions can be related to mathematical knowledge for teaching as loosely defined by Ball and Bass (2003). They asserted that teaching mathematics is a dynamic, complex activity requiring continuous effort to design, modify, adapt and evaluate teaching strategies, multiple representations and student responses, make appropriate corrections as needed to encourage and promote student learning, and encourage mathematical discourse to produce new insights and mathematical knowledge. Although not an exhaustive list, these teacher moves dovetail well with extant literature that encourages a shift toward more student-centered instruction, promotes clear establishment of goals, meaningful discourse leading to improved problem solving skills, connections to topics across the curriculum, encouraging persistence in the face of difficulty, and timely, relevant

feedback that promotes student growth (McKewan, 2004; NCTM, 2014; Shulman, 1987; Zemelman, Daniels, & Hyde, 2005). These teacher actions also formed the basis for categories of *teaching tasks* utilized in a 2005 study by Ferrini-Mundy and her colleagues at Michigan State University, which purported to bridge the gap between a global understanding of mathematical knowledge for teaching and the specific aspects of knowledge for teaching algebra for one specified topic of instruction on solving linear equations. This effort, the Knowledge of Algebra for Teaching (KAT) project, established the basis for the conceptual framework I utilized in my case study. In the KAT study, the researchers took the first step toward exploring the distinctive manner in which MKT might be explored while more narrowly focused on a specific domain (algebra) and a topic within that domain (linear equations). The intent of this researcher's effort was to extend the previous work of the Michigan State team but to focus on the way KAT is accessed and utilized in both the planning and instructional phases of a unit of instruction on *quadratic equations*. This specific focus has produced the following research questions for investigation in my study of MKT in the classroom.

RESEARCH QUESTIONS

In an effort to address the interaction between mathematical knowledge for teaching and the planning/preparation as well as the teacher moves during classroom instruction on a unit on solving quadratic equations, I propose the following research questions:

1. What content knowledge do teachers possess for teaching a first course in algebra?
2. What mathematical knowledge for teaching algebra do teachers draw upon when preparing for a unit of instruction on solving quadratic equations?
3. What mathematical knowledge for teaching algebra is utilized during instruction in a unit on solving quadratic equations?

SIGNIFICANCE OF THE STUDY

Further examining and searching for clear definitions of teacher knowledge and how it impacts instruction has significance across stakeholders. Policy makers and other leaders grapple with developing salient and relevant measures of effectiveness for teachers and means for measuring student growth and achievement. Mathematics teacher educators can benefit from knowing more about the dynamics of student and teacher knowledge in mathematics classrooms in order to design effective teacher preparation programs and ongoing professional development. Researchers focused on MKT will benefit from evidence in a second area of algebra to see what common elements of MKT or of KAT might be generalizable and what might be unique to a topic.

Previous research on teacher knowledge has focused broadly to refine and establish a discourse on how to define the mathematical knowledge for teaching needed to improve student achievement. Because the first course in algebra is a critical gateway to further studies in mathematics which eventually leads to “attractive education and job opportunities” (Niss, p.377), more research is needed to examine the impact of mathematical knowledge for teaching in this specific context. This case study attempted to determine, in the context of a first course in algebra, what connections exist among classroom teachers’ mathematical knowledge for teaching algebra and the cognitive processes they utilize during preparation for instruction. Additionally, I examine the interaction between the teacher’s KAT and the actions taken during instruction in a unit on solving quadratic equations.

To date, virtually no research has focused on relationships among teachers’ mathematical knowledge for teaching and teacher planning and instructional decisions at the level of specific mathematical tasks. As more detailed and specialized approaches to instruction within well-defined strands of knowledge guided by frameworks such as the Common Core Standards are considered, research has the potential to provide additional insight into curricula development, teacher preparation, and policy decisions. This proposed study attempted to uncover underlying thought processes, intentional instructional decisions, and pedagogical practices in more detail than previously explored. Specifically, this study focused on the preparation and execution of instruction on the single mathematical skill of solving quadratic equations and the interaction of these activities with a teacher’s knowledge for teaching algebra within this defined skill. This research purported to construct an understanding of the mathematical knowledge for

teaching at the basic level and build on the tenets of that structure within the content area. Research on and development of these essential understandings are necessary steps in providing a framework for further efforts to develop a more robust identification of the complex skills and knowledge needed by teachers of mathematics to enhance their teaching effectiveness and, by extension, the procedural and conceptual understanding that serve as critical teacher knowledge for improved student performance.

Terminology

Throughout the study, some particular terminology will be utilized for types of teacher knowledge or actions taken by teachers in both planning and implementation of instruction in the algebra classroom. These terms are defined below for the sake of clarity.

Pedagogical Content Knowledge (PCK) is a precursor of MKT. Shulman (1986) proposed a concept incorporating elements of content knowledge and pedagogical knowledge into Pedagogical Content Knowledge (PCK). His landmark work forged a connection between the pedagogical skills needed to be an effective teacher with the content knowledge necessary to teach effectively. Classroom teachers make numerous pedagogical and cognitive decisions about instructional strategies throughout the planning and implementation of instruction across the myriad topics in any mathematics course. These decisions rely on their experiences, content knowledge, the way they understand their students' capabilities, resources available and their own internal beliefs about teaching and learning (Anderson & Holt-Reynolds, 1995). This study uses Shulman's definition throughout as that amalgam of content and pedagogical knowledge

that permeates the classroom teacher's actions before and during mathematics instruction as the student needs evolve.

Mathematical Knowledge for Teaching, or MKT, is a term used by various mathematic educators and researchers as a reference to the myriad skills used in the increasingly sophisticated mathematics classroom. The term was initially coined by Ball & Bass, (2003) as a reference to the content knowledge used in planning representations and classroom activities as well as pedagogical actions related to teachers' understanding of student difficulties and misconceptions. Since that time it has been widely used to address a kind of specialized knowledge that incorporates subject specific content knowledge, broader understanding of the mathematics curriculum as it relates to a specific topic and other courses across the discipline, and pedagogical knowledge of student learning needs. As discussed in the following chapter, MKT has been adopted as a broad conceptual definition of what teachers must do to contextualize mathematical teaching, form connections, adapt lessons, analyze student errors, and ask mathematically productive questions (Hill et al. 2004). It is used in this final context throughout this study to indicate both mental and physical actions teachers take as they traverse the evolving learning environment in their mathematics classroom.

Knowledge of Algebra for Teaching (KAT) is a term used by Michigan State University as a subset of MKT that specifically addresses that particular knowledge within the mathematics curriculum utilized in teaching algebra. Ferrini-Mundy et al. (2005) used this term as a basis for exploring connections between teachers' pedagogical actions in the classroom with certain aspects of mathematical content knowledge. Their

work, illustrated in matrix form later in figure 4, provides a conceptual basis for sorting the actions taken in both planning and instruction against categories of content knowledge. Overlaid with further concepts of Bridging, Trimming, and Decompressing, KAT provides a structure for looking at teacher knowledge in a finer grain within specific algebra topics. This study depended heavily on the Michigan State study and uses their definitions with permission from the program director.

Tasks of Teaching is a broad term used in this study to refer to the actions teachers take when they are planning for instruction and when they present that instruction in the classroom. These tasks may be planned representations, sequencing decisions for instruction, modifying instruction as student needs or difficulties are clarified, and establishing goals for their students. These tasks of teaching are enumerated in the KAT study matrix and represent a broad categorization of specific instructional strategies and teacher actions recommended by policy-makers. The *tasks of teaching* used throughout this study also stem from the KAT study and are used by permission. The rationale for this decision also reflects an analysis of various listings of best practices and recommended principles for teaching from several policy sources as described in the following chapter. These Tasks of Teaching can also be found in figure 4, which contains the KAT Matrix.

CHAPTER 2 – LITERATURE REVIEW

OVERVIEW

This literature review describes the conceptual framework for an investigation that focuses on teachers' mathematical knowledge for teaching and the ways they access and use this knowledge when planning for and conducting instruction in their mathematics classrooms, specifically during a unit on solving quadratic equations. In the review that follows, I discuss the evolution of the concept of mathematical knowledge for teaching (MKT), and its application to the act of teaching. The review describes the evolution of teacher knowledge from pedagogical content knowledge (PCK) (Shulman, 1986) to MKT (Ball & Bass, 2000) and further narrowing focus on the aspects of this specialized knowledge that apply to teaching algebra, defined by McCrory et al. 2012, as knowledge of algebra for teaching (KAT). This body of research led to increased use of terminology that encompasses teacher actions in the classroom as measures of knowledge for teaching mathematics. Although no existing research codifies the specific actions that are described, the implication from the literature is that mathematical knowledge for teaching is dynamic, adaptive, and interactive-- not merely a collection of strategies appropriately matched with a set of specified classroom conditions.

As indicated, following the development of the current definition of MKT, I will discuss research that has narrowed the focus and applied this knowledge for teaching to the domain of algebra. These research efforts have formed a foundation both for contextualizing the investigation of MKT in a more specific domain and relating teacher actions in those domains to the knowledge for teaching mathematics in a high school algebra course. The findings in these algebra-focused studies (Artigue, et al., 2001; Li, 2011; McCrory et al., 2012) have informed the initial stages of exploration into a refining of MKT to allow discussion of a more narrowed band for the specific area of algebra instruction, the Knowledge of Algebra for Teaching (KAT).

This section will also provide a review of extant literature regarding the scope of students' perceptions and difficulties with the mathematical task of solving quadratic equations as they transition into their secondary education. Though there is a paucity of research involving interaction of MKT and teacher actions in this specific unit of instruction on solving quadratic equations in a first course in algebra, the literature regarding teacher efforts to address student conceptions and challenges is helpful in guiding my research by highlighting potential areas of cognitive challenge for both teacher and student. The section concludes with a discussion of recent efforts to examine the connections between teacher knowledge of algebra for teaching and pedagogical decisions while teaching certain aspects of solving quadratic equations from an instructional and a coaching perspective. Together, these sections establish the basis for my conceptual framework that will guide my investigation.

The third stage of the review examines the literature on teacher actions in the classroom from a descriptive and an investigational perspective. Various reform efforts

and research (Ferrini-Mundy et al. 2003; Li, 2011; NCTM, 2000, 2014; Shulman, 1987; Zemelman, Daniels, & Hyde, 2005) have promulgated numerous representations of practices and processes that are intended to serve as a guide for teacher actions and instructional techniques in the classroom. This section of the review serves to establish the rationale for the specific *tasks of teaching* categories that were used in this study to analyze actions the subject teachers took during their planning and instruction. The review discusses efforts to delineate teacher actions that occur in the classroom, both from the perspective of the application of MKT and through reform efforts by educators and policy makers. Through a comparative analysis of the instructional strategies, best practices and teacher actions discussed in the literature, the tasks of teaching enumerated in the KAT Matrix emerged as the most appropriate analysis format to complete the conceptual framework. I also include research that focuses on classroom beliefs and behaviors that are malleable (Tschannen-Moran, Woolfolk & Hoy, 1998), that are based on teachers' actions (Ames & Archer, 1998), and that impact achievement. (Anderman, Eccles, Yoon, Roeser, Wigfeld, & Blumenfeld 2001; Chouinard, Karsenti, & Roy, 2007; Phan, 2009).

CONCEPTUAL FRAMEWORK

The purpose of this study was to examine the use of mathematical knowledge of algebra for teaching (KAT) by teachers in a first course in algebra. Specifically, what aspects of mathematical knowledge for teaching algebra are used when planning and preparing to instruct a unit of instruction covering solutions to quadratic equations and what mathematical knowledge for teaching algebra is utilized during the actual

instruction on solving quadratic equations? The conceptual framework used in this research is based on one developed over the past decade by a research team at Michigan State University related to teachers' Knowledge of Algebra for Teaching (KAT) (McCrory et al., 2012, Ferrini-Mundy et al., 2005). Using a matrix to relate *tasks of teaching* to *categories of knowledge*, researchers investigated how tasks relevant to solving linear equations might be connected to different knowledge dimensions of KAT. The KAT constructs will be discussed in more detail later in this chapter.

Classroom teachers make numerous pedagogical and cognitive decisions about instructional strategies throughout the planning and implementation of instruction across the myriad topics in any mathematics course. These decisions rely on their experiences, content knowledge, the way they understand their students' capabilities, resources available and their own internal beliefs about teaching and learning (Anderson & Holt-Reynolds, 1995). Ultimately the specific actions and teacher moves used in the teaching and learning process can be viewed through the filter of the mathematical knowledge for teaching they bring to the classroom and their personal beliefs about teaching. The specific actions and teacher moves are also situated in a broader collection of teacher tasks. These teacher tasks are informed by the classroom teacher's understanding of standards, expectations, and best practices accumulated through academic exposure, personal experience, or professional development. This notion will be further developed in the following paragraphs. My conceptual framework connected a form of MKT that relates specifically to teaching a first course in algebra, or (KAT), to actions in the planning and implementation of instruction in the mathematics classroom. This framework is represented in the flow diagram below.

Figure 1. Framework for the Case Study on KAT

This diagram depicts the framework used throughout the study to explore connectivity between the mathematical content knowledge teachers possess and the actions they take while planning and implementing instruction. As teachers manage the classroom instruction, they are guided by overarching teaching objectives. These objectives stem from curriculum guidelines, local policies, or departmental directives. Each classroom teacher interprets these objectives through the filter of their own pedagogical and content knowledge of algebra for teaching. In the diagram above, the KAT represents a subset of mathematical knowledge for teaching (MKT) and is categorized by elements of specific content knowledge, knowledge of the curriculum, and pedagogical knowledge that facilitates student learning of algebra. These specific categories are found as “categories of knowledge” in the KAT Matrix in figure 4. As teachers construct a teaching plan for a specific topic, they consider how specific *tasks of teaching* will create and maintain the learning environment during the unit of instruction. These “tasks of teaching” are the second dimension of the KAT Matrix that describes how classroom teachers convey the content to their students. They include actions such as

establishing goals, analyzing student work, and designing specific tasks for students. The schematic above reflects a dynamic interaction between these tasks of teaching and the knowledge for teaching in a first course in algebra. The manifestation of internal processing between knowledge and actions is the primary focus of the study. Further, the tasks of teaching inform both the instructional plan and the implementation in the evolving classroom. The connection between planning and implementation of instruction indicates the researcher's view that classroom decisions are flexible and adaptable based on the individual teacher's perceptions of student abilities or changing conditions within the classroom.

This framework assumes a dynamic interaction between the broader categories of knowledge and tasks for teaching with the evolving and flexible categories of planning and implementation of instruction. Previous work has made progress in eliciting some connective activity between these categories and other dimensions of interaction between a teacher's MKT and actions taken in the classroom (Artigue, et al., 2001; Li, 2011; McCrory et al., 2012) and formed a basis for further investigation along these pathways. Through semi-structured interviews, analysis of documents such as lesson plans, transcribed audio observations, and teacher reflections, and then performing qualitative analysis of gathered data for emerging themes, I develop a robust representation of teacher decisions made during both planning and instruction in a unit on solving quadratic equations. These data were further analyzed to investigate connections that exist between categories of KAT and pedagogical decisions using assessments of teacher knowledge for teaching algebra gathered through administration of the KAT assessment instrument. The research further explored commonality or distinctions in teacher moves

that relate to specific categories of KAT themselves or which manifest differently based on different levels of knowledge in those dimensions. These ideas, which I have utilized and built on in constructing this framework, will be developed and discussed in the sections that follow.

DEVELOPMENT OF MATHEMATICAL KNOWLEDGE FOR TEACHING

The fact that knowledge necessary for teaching is different from the knowledge a professional or accomplished student possesses is not a new claim. At a time when formal schooling emerged as an expectation for all youth, Dewey (1920) offered the following observation in his *Reconstruction in Philosophy*:

“Every study or subject thus has two aspects: one for the scientist as a scientist; the other for the teacher as a teacher. These two aspects are in no sense opposed or conflicting. But neither are they immediately identical. For the scientist, the subject matter represents simply a given body of truth to be employed in locating new problems, instituting new researches, and carrying them through to a verified outcome. . . . The problem of the teacher is a different one What concerns him as teacher is the ways in which that subject may become part of experience, what there is in the child’s present that is usable with reference to it; how such elements are to be used; how his own knowledge of the subject-matter may assist in interpreting the child’s needs and doings, and determine the medium in which the child should be placed in order that his growth may be properly directed. He is concerned, not with the subject-matter as such, but with the subject-matter as a related factor in a total and growing experience.” (Pp. 19-20).

Pedagogical Content Knowledge

The effective classroom teacher must be able to utilize their own content knowledge to create pedagogically useful representations that form a basis for comprehension by the student. Dewey's notion that teachers hold a specialized content knowledge gained further interest after a seminal article by Lee Shulman (1986). He and his colleagues noted a lack of content knowledge in teacher preparation programs in favor of heavier emphases on teaching and management strategies. In "Those Who Understand: Knowledge Growth in Teaching," he used the term "the missing paradigm" (p. 7) to describe a lack of focus on the breadth of knowledge necessary to teach effectively. Shulman (1986) proposed a concept incorporating elements of content knowledge and pedagogical knowledge into Pedagogical Content Knowledge (PCK). His landmark work forged a connection between the pedagogical skills needed to be an effective teacher with the content knowledge necessary to teach effectively. Shulman's PCK construct added clarity regarding what knowledge teachers needed to teach content. Deborah Ball and her colleagues, as well as many other researchers, applied PCK to the context of teaching of mathematics. Shulman (1987) characterizes pedagogical content knowledge as the actions a teacher takes to comprehend, decompose, and restructure content knowledge to a form that makes sense to the student as transformation. Transformation involves several phases in the instruction process from decompressing content knowledge to preparation of appropriate lesson plans, representations and strategies, and adapting instruction as needed. He refers to these teacher actions as "pedagogical reasoning" (p.17), actions that permeate the teaching process from initiation through the action of reflection after instruction is complete.

In early work on this subject, Ball (1991) argued that conceptions of subject matter knowledge were inadequate in defining the knowledge teachers needed to be effective in teaching mathematics. She argued that the most effective teachers intuitively combined subject matter knowledge with an understanding of student learning, a deep knowledge of methods for appropriate representation of ideas in the classroom, and their own beliefs about teaching. Lampert (1990) proposed that teachers must be able to reassess and draw upon their understanding of their own mathematical content knowledge and their teaching practices to establish an environment that results in student learning of mathematics. The knowledge needed for teaching mathematics has become defined more in terms of teacher actions in the classroom rather than lists of strategies or topics. Ma (1999) described this understanding in terms of *knowledge packets* situated in what she termed a “profound understanding of fundamental mathematics” (p. 122). Ma’s contention was that good teachers have an adaptive and flexible knowledge base from which they deploy context dependent approaches to teaching. Ma compared this “flexible and confident” knowledge to that of a taxi driver who knows “several alternative routes” (p.123) between destinations. The concepts of flexibility and adaptation were reflected in further research that emphasized continuous interactions among students, the content, and the pedagogical skills of the teacher (Ball et al., 2001; Hill et al. 2009; Schoenfeld, 2002). Ball, et al. (2001) suggest that “what teachers and students are able to *do* (emphasis added) together with mathematics in classrooms is at the heart of mathematics education” (p. 433).

Mathematical Knowledge for Teaching

Ball and Bass (2003) described the complex interactions among content knowledge, pedagogical knowledge, and a knowledge of students as the “Mathematical Knowledge for Teaching” (MKT) to describe the understanding and knowledge teachers of mathematics needed to respond, react, and mobilize appropriate strategies in classroom settings. Over the next decade, a number of researchers were involved in efforts to further the overarching goal of refining a definition for this specialized knowledge and how it appears in the classroom (Ball, et al., 2001; Ball, Hill & Bass 2005; Hill et al. 2004; Stylianides & Ball, 2004). The results of these efforts reinforced that teaching mathematics is a complex activity that requires knowledge that intertwines content, beliefs, and pedagogy and is dynamic, interactive and adaptive (Butterfield & Chinnapan, 2010; Calderhead, 2002; Sowder, 2007). Further, Hill & Ball (2009) address this specialized knowledge at both pre-service and in-service levels saying that MKT is about “Teaching – helping others learn to know and do” (p.71) and is an integral component of “skillful teaching” (p.71). In discussing MKT in the classroom, Ball, et al., (2001) described their focus on mathematical knowledge for teaching as one intended to ascertain “what it takes mathematically to manage these both routine and non-routine activities of practice...to unearth this kind of *pedagogically useful mathematical understanding*” (p.453). Two other studies (Iszak, 2008; Rhoads, 2011) developed a more robust definition of the connectivity essential in mathematical knowledge for teaching and validated Ma’s findings that evidence exists of certain knowledge elements that are utilized in a contextual, conceptually founded, and longitudinal manner.

What emerged is a practice-based theory of MKT that manifests itself in teacher actions within the domains of teaching mathematics (Ball & Bass, 2003, Ball, et al., 2008, Schoenfeld, 2002; Stylianides & Ball, 2004). Ball, et al. (2008) proposed a framework, shown in Figure 1 below, that is “an amalgam of knowledge of content and pedagogy that is central to the knowledge needed for teaching” (p.392).

Figure 2. Domains of Mathematical Knowledge for Teaching

These domains describe what teachers must know in order to carry out the actual work of teaching, not merely in the classroom interactions, but also in planning, evaluating, and assessing the learning process in a dynamic and reflective manner.

The MKT framework for defining the knowledge needed by teachers for effective mathematics instruction, however, leaves many questions unanswered. For example, what specific knowledge is needed and most consistently used, and what types of teacher knowledge result in student learning? Do the types of specialized teacher knowledge differ from one course or grade level to another? In the next section, two research efforts focused specifically on the knowledge needed to teach Algebra will be discussed.

Knowledge of Algebra for Teaching

Several studies describing the mathematical knowledge for teaching needed in a *specific* course or task setting have been conducted. Intuitively, teachers of higher-level mathematics courses require a deeper content knowledge to address more complex mathematical concepts and processes (Ferrini-Mundy et al., 2005). Additionally, student attitudes and beliefs toward mathematics likely will change as they mature emotionally and intellectually (Ding, 2007; Iszak et al., 2012; Kleve, 2009). Thus the knowledge for teaching mathematics could vary greatly among different mathematics courses and academic levels (Ball, et al., 2005).

Most efforts to delineate and measure particular sets of skills, strategies and mathematical knowledge for teaching have been undertaken in the elementary classroom (Ball, et al., 2005; Hill, et al., 2004; Hill, Rowan & Ball, 2005; Iszak, 2008). In separate research efforts, Ball, et al. (2005) and Hill et al. (2004), for example, found that beyond ascribing content knowledge for teaching fractions to elementary students, it is important for the mathematical community to address the nature of that knowledge; how teachers conceive of their own mathematical knowledge used in teaching and how they routinely use it to “scrutinize, interpret, correct, and extend” (Ball, et al, 2005, p.17) student knowledge. They contend teachers’ mathematical knowledge for teaching is multidimensional and includes content knowledge, but also domain specific knowledge that differs based on the subject matter and level of mathematics. Additionally, these studies emphasize the strong connection between good teachers’ professional knowledge about their subject area and learning theories about various kinds of student learning methods. In another research effort at the elementary level, Hill, Rowan & Ball looked at

two cohorts of elementary students and their teachers to explore what subject matter skills led to increased student achievement with the underlying premise that these skills reflected the MKT teachers brought into the classroom and developed across the three-year study. Iszak (2008) analyzed the abilities of sixth grade teachers to use multiple representations when teaching fraction multiplication. He concluded that the teachers' skills reflected domain-specific content knowledge that was often insufficient for making connections across the curriculum.

Effort to extend the process of defining mathematical knowledge for teaching to include mathematics instruction at the secondary level has occurred at a lesser degree. The RAND Mathematics Study Panel (2003) recommended the teaching and learning of algebra as an initial research area in mathematics education. The panel, noting the growing importance of algebra because of its fundamental importance in other areas of mathematical inquiry as well as the workplace, identified the need to identify how teachers use their knowledge to bridge the gaps between previous mathematical experiences and the abstract nature of algebra. In addition, college and career readiness at the secondary level has a strong emphasis on algebra (CCSSO, 2010). In light of this new and increased emphasis on research in the study of algebra, and as a natural extension of earlier work cited above, some attempts to address the specific area of knowledge for teaching algebra have been undertaken. Two of these efforts serve as a basis for a framework for investigating the mathematical knowledge for teaching a first course in algebra.

Artigue, et al. (2001) studied the specific content and pedagogical knowledge teachers needed to teach algebra through the lens of the development of teachers'

professional competence through the “Multidimensional Grid for Professional Competence in Elementary Algebra (MGPCA).” Recognizing that improvement in student learning was intricately related to improved teacher knowledge, they studied “teachers’ culture in algebra, expectations and practices and their potential effects on students’ learning” (p.2). The researchers believed that knowledge for teaching algebra was an evolutionary process and dependent on personal coursework, internal belief systems, and experiences in actual practice. Therefore, they used a multidimensional analysis. The grid included three dimensions classified as “non-independent” (p.6): epistemological, cognitive, and didactic. These categories, or dimensions were similar to those used by Shulman in his 1986 discussion of pedagogical content knowledge (PCK), but were tailored toward the teaching and learning of algebra specifically. A visual depiction of the dimensions and their components is included here in figure 3.

Figure 3 Multi-Dimensional Grid for Professional Competence in Elementary Algebra

As illustrated in this graphic, the MGPCA addresses the content knowledge, pedagogical knowledge, and curricular knowledge in the same fashion as Shulman, but with specific attention to the teaching knowledge and skills needed for teaching algebra. As with many other dimensional analyses, there is always overlap between the categories, and knowledge or experience in one dimension impact knowledge in the other categories.

In the study, the epistemological dimension focused on: the structure of algebra; its abstract symbolic system and its historical development; the diversity of semantics and syntax; the complexities the use of algebraic objects present to students; and the breadth of connectivity of the algebraic structure to numerous other mathematical disciplines. They hypothesized that, in response to an unclear epistemology, teachers with less experience tended to reduce algebra to a procedural discipline reflecting their experiential knowledge. The cognitive dimension focused on the knowledge of learning processes and “resistant learning difficulties” (p.6). They found that these cognitive challenges both students and teachers encountered were related to the transition from the straightforward nature of arithmetic to the symbolic manipulation necessary to use the tools of algebra as catalysts to construct knowledge of the representations used in algebra and to formulate strategies that explain algebraic rationality. Initial observations indicated that this cognitive dimension was highly reactive to experiential development but required extensive, and sometimes external, effort to fully utilize in a productive manner. As an additional component, the grid proposed the didactic dimension which included influence from the previous dimensions but also referring to knowledge of the curriculum, divisions of topical units, activities and resources available for the teaching of algebra.

Though the study identified modifications in the external actions of the teachers, no specific connections of observed knowledge in these dimensions to the teaching of algebra was evident. However, teachers acquired a delineation of attributes over time, as well as the processes facilitating this knowledge acquisition. The MPCGA multidimensional grid analysis revealed that the knowledge needed to teach algebra was multifaceted and multidimensional and consists of multiple factors derived from both internal and external contributions. These dimensions converge with other dimensions of mathematical knowledge for teaching as the epistemological knowledge addresses core proficiency in the content, the cognitive dimension relates to knowledge of content and students, and the didactic dimension indicates knowledge of content and teaching.

Ferrini-Mundy and her colleagues (2005) at Michigan State University proposed a Knowledge of Mathematics for Teaching Algebra framework focused on how knowledge can be characterized and how assessment tools for measuring this conceptualized knowledge might be designed within a specific unit of instruction on Linear Equations. Their objective was to link categories of knowledge of algebra for teaching with teaching tasks in which the knowledge might be applied to develop a conceptual framework for teaching algebra. As shown in Figure 4 below, the horizontal and vertical axes represent different domains of mathematical knowledge for teaching, ranging from the core content knowledge of mathematics to sensitivity to various means by which students learn and knowledge of how algebra connects outside the course with other courses within the broader mathematics curriculum.

In a continuation of the initial effort by Ferrini-Mundy et al., the KAT project (McCrory, et al., 2012) team purports to explore further the connections between

knowledge for teaching algebra and the tasks of teaching as indicated in the matrix utilized by the initial project. The objectives, framework, and results of these investigations are described in more detail in the following paragraphs.

The framework utilizes a data collection device that consists of a 6 X 6 matrix matching Tasks of Teaching with Categories of Knowledge developed over an extended period of research using an iterative process of observations and interviews with one thousand in-service teachers of varying experience levels and college majors. During classroom observations, review of taped lessons, and interviews, the project team assigned a dual coding to teaching actions reflecting a category of the teaching tasks coupled with knowledge accessed by subject teachers that corresponded with a single cell of the matrix. In Chapter 3, a more detailed explanation of the categories of tasks and knowledge and a visual representation of the KAT Matrix is included in the methodology discussion.

Three categories overlay the two-dimensional array and serve as connections between the other two categories. The first of these categories, decompressing, indicates the need for teachers with a deep knowledge of mathematics to unpack that knowledge in ways that makes sense to students. Algebra often provides substantial difficulty for new learners because of its symbolic representations and syntax and the abstractness of using algebraic objects in constructing models. The act of decompressing addresses student misconceptions and obstacles to learning centered on the unique challenges of developing facility with the language of algebra. In the second overarching category *trimming*, teachers decide where and how to omit certain facets or details of problems in order to reduce the complexity of the problems to a manageable level. Trimming involves a

dynamic decision process in which teachers employ multiple representations, reduce the rigor of a problem without losing its essence, and differentiate instruction to meet the needs of various students in a non-homogeneous classroom. Bridging, the third category, requires teachers to connect ideas, concepts, and applications and to provide paths toward more complete understandings of algebra as the basis for developing deeper understandings of core mathematics principles. In bridging, teachers take student conceptions and connect them to instructional goals or transform student-generated definitions to more conventional ones in the discipline. It involves making explanatory comments that demonstrate the veracity of the mathematical or algebraic procedure in the context of established norms and definitions of algebra.

Li (2011) observed the actions a teacher took in the classroom while instructing students on use of the quadratic formula as a means of solving quadratic equations. Li identified three areas of MKT for the study: knowledge of the mathematical subject matter, knowledge of pedagogical representations, and knowledge of learners' conceptions. The findings in this study of MKT in the single domain of solving equations using the quadratic formula in a high school algebra class indicate a heavy reliance on subject matter knowledge with other areas of knowledge for teaching algebra occupying a secondary, but important role in guiding representations and goal modification. Further discussion of the teacher actions observed in Li's research are included in the following section.

All of these research efforts convey an increased interest by policy makers and stakeholders to clarify the most productive actions and teacher moves for improved learning in the mathematics classroom. A review of multiple sources on teaching tasks

and best practices produced a condensed listing of those tasks which most appropriately categorize the actions that teachers utilize routinely in the mathematics classroom. Because no single source or instrument contained this focused list of teacher actions, or tasks, the researcher examined the available instruments for the one that most completely reflected the combined listing of routine tasks. Though all of the sources have merit, this research has concluded the matrix used by the Michigan State team provides the most compatible and useful composite listing of tasks of teaching specifically related to Knowledge of Algebra for Teaching. Used by permission of the Project Leader, their categorization of tasks of teaching related to associated categories of knowledge for teaching algebra comprises the most complete and encompassing list of teaching practices and activities and most appropriately addresses the research questions posed in a specific focus on a unit of instruction on solving quadratic equations.

Research on Quadratic Equations

For a topic as specific as quadratic equations, there is not a significant research base. However, there are several studies on students and on teachers that can provide insights into the KAT that may be needed for quadratic equations.

Research on Students - Beliefs

Classroom teachers in a first course on algebra encounter a variety of misconceptions and difficulties as their students are confronted with the newly abstract nature of mathematics in algebra compared to the more concrete nature of arithmetic operations utilized in earlier courses. The primary research in the area of the challenges students face in understanding and solving quadratic equations has focused on student

beliefs and attributes they bring to the mathematics classroom. These studies concur that one of the most challenging tasks for students in an initial algebra course is finding solutions to quadratic equations. The extant research explores these issues from pedagogical content and mathematical beliefs perspectives. Methods for finding solutions to a quadratic equation include: a graphical method, factorization, the geometric approach of completing the square, and the quadratic formula. Each method has its own challenges from both procedural and conceptual perspectives. Identifying the obstacles to learning in the context of instruction on quadratic equations situates the effort by this researcher in the domain of MKT addressing knowledge of the way students learn, and teacher decisions that address these student conceptions.

Student beliefs about mathematics have powerful influences on their actions and perspectives. In a study on student beliefs about mathematics, Spangler (1992) cited previous research that described students' mathematical beliefs about mathematics as a computational, procedural, intelligence-based subject in which getting the right answer is the primary objective. Her research found that most students preferred to know only one method of solution for solving quadratic equations, and that the majority believed that they could get the correct solution without understanding what they were actually doing.

Rauff (1994) provided further insight into the role of beliefs as they relate to methods to solve polynomials, including quadratic equations, using factoring. He relied on the work of Gärdenfors (1988) to define a belief set and a means of modifying student beliefs through "expansion, contraction, and revision" (p.421). Rauff described student's beliefs about factoring as a process of "Reversal, Deconstructive, Evaluative, Formal, and

Numeric” (p.423) and indicated that each of these beliefs resulted in a different solution process based on the “collection of beliefs” in each category. Reversal belief implies “reversing a multiplication.” Deconstructive belief breaks down the expression. The Evaluative belief involves solution by substituting values. A Formal belief recognizes the relationship between the factored expression and a product. Numeric belief decomposes numbers to their prime factors. According to Rauff, students hold these beliefs in what he calls “epistemic entrenchment” (p. 422) and will often continue using their preferred process until their belief system is modified through one of the three methods described by Gärdenfors (1988). His study indicated that teachers should understand student beliefs about these solution techniques and use their understanding to adapt or modify student beliefs to move them toward the “Formal” belief set. In this belief set, students have a more flexible approach and are capable of adopting new techniques for solving quadratic equations, particularly when the structure is different than anticipated.

Research on Students - Understanding

Solving quadratic equations is one of the most conceptually difficult tasks for secondary students (Kotsopoulos, 2007). Lima (2008), for example, found that students demonstrated a complete lack of understanding of the connection between the procedure of solving quadratic equations with the quadratic formula and algebraic principles for solution of equations. Lima reported a very small proportion (7 of 77 students) achieved success in solving equations with non-standard structures. Students were dependent on the procedure represented by the quadratic formula to get the correct answer, simply followed a set of rules regardless of the problem structure, and demonstrated no comprehension of underlying concepts.

A particular challenge with quadratic functions is interpreting the structure of the situation. Several studies have indicated that students lack an understanding of such structures. Didiş, Baş, and Erbaş (2011) examined the methods used by students to solve quadratic equations by factoring. They noted that students found the work challenging for a variety of reasons, one of which was that students primarily used instrumental, or procedural, approaches without considering underlying concepts or structures in which “they tried to apply these rules thinking neither about why they did so, nor whether if what they were doing was mathematically correct” (p. 7). In a related study, students’ failures to understand the internal structure of quadratic equations presented difficulty when expressions were presented in nonstandard form (Vaiyavutjamai, Ellerton, and Clements, 2005). Similarly, Hoch and Dreyfus (2005) found that students were generally not familiar with the concept of structure and, as such, could not address more complex factoring problems.

Student understanding of quadratics is a reflection of the instruction they receive. Vaiyavutjamai and Clements (2006) examined the way particular pedagogical decisions affected students’ understanding of the solution process for equations with different structures. In six algebra classes, teachers used a traditional approach of instruction. The definition of traditional instruction was where the teacher presented an example or model, and then students practiced the technique with example problems. Using Skemp’s (1976) definition of “instrumental” and “relational” (p.53) which are interpreted as procedural and conceptual understandings, the authors administered a pre-test and post-test accompanied by interviews. In the overwhelming majority of cases, students pursued a procedural approach that demonstrated little understanding of the mathematics involved

in solving quadratic equations. Not only did students struggle when equations were structured differently than the model used by the teacher, they also had difficulty understanding the concept of variable in this context. Furthermore, they were unable to explain how the variable(s) in the solution related to the variable in the initial problem. Lima and Tall (2010) conducted a follow-up study with a sample of three secondary school classes after concluding in an earlier study that these students demonstrated a “procedural embodiment” (p. 1) involving following a set of rules, or a procedure, to solve linear equations. Although teachers in the study were encouraged to follow their own ideas and plans, they anticipated difficulty with factorization and completing-the-square solution strategies. The teachers in the study concentrated more heavily on the use of the quadratic formula as a means of providing students a strategy, and to meet the demands of the syllabus. The study found that students failed to develop an understanding of formal principles, but continued to persist in a procedural approach to solving equations. As expected, the increasing complexity inherent with quadratic equations presented difficulties for the majority of students. Citing previous work by Gray and Tall (1994), their research reinforced the notion of a “proceptual divide” (p. 17) where students either develop the flexibility to adapt knowledge and develop new strategies or pursue more complicated sequences of rules and procedures. In this case, most of their subjects failed to move to a more flexible, conceptual embodiment. A recommendation of this study was that teachers must avoid a path that leads to quick success and encourage students to develop conceptual, flexible strategies that enhance understanding of algebraic ideas.

Research on Teacher Knowledge related to Quadratic Equations

A paucity of research relating mathematical knowledge for teaching to a specific mathematical task currently exists. In fact, the only efforts to connect any of these teacher beliefs or knowledge levels in a specific context have come from the KAT studies conducted at Michigan State University (McCrory et al., 2009), and from two recent studies by Nebesniak (2012) and Li (2011) in which the actions by a teacher planning for instruction in a particular aspect of solving quadratic equations was examined. The work by McCrory and her colleagues extended the study of mathematical knowledge for teaching that was conducted previously in elementary classrooms and investigated the specific mathematical knowledge that teachers use when teaching first-year algebra students to solve linear equations.

In 2011, Li studied an individual teacher during a three-lesson sequence on solving a quadratic equation using the method of applying the quadratic formula. For the study, Li focused on the teacher's knowledge of subject matter, pedagogical representations and learner's conceptions. The research suggested that the teacher utilized knowledge of the subject matter more heavily in making pedagogical decisions with less dependence on representations. The teacher also drew on her knowledge of her own students' level of mathematical proficiency in making instructional decisions on representations and decisions on trimming learning objectives during classroom instruction. Additionally, Li concluded it was difficult to gauge the level of conceptual understanding developed because there was an emphasis initially on procedure, and many students likely remained at this level of understanding. In another recent study, Nebesniak (2012) addressed the instruction of solving quadratic equations from the

perspective of an instructional coach assisting a young teacher in planning for this unit of instruction. Her effort focused primarily in the domains of teacher content knowledge, reasoning and sense-making skills, and pedagogical decisions prior to instruction. Quoting Lampert (1985) Nebesniak concludes that improved knowledge for teaching comes through the daily process of dealing with “pedagogical teaching dilemmas” (p.289) that require a deep content knowledge as well as knowledge of student conceptions and learning. Further, the study’s emphasis on reasoning and sense making accessed the domain of knowledge of the curriculum by making appropriate connections across the topics within the unit of instruction using various representations and discourse.

TASKS OF TEACHING

The *Tasks of Teaching* defined in the opening chapter have been operationalized in this study following a comprehensive review of a number of studies and theoretical papers that purported to enumerate teaching practices or strategies attributed to mathematical content knowledge. A substantial number of these studies were undertaken with the specific purpose of codifying the actions, decisions, strategies and practices that were evident during planning and instruction and matching them to categories of mathematical knowledge for teaching (MKT) (Ball et al., 2001; Ball et al. 2008; Hill & Ball, 2004; Lampert, 1989; Li, 2011; Nebesniak, 2012). Others like Ma (1999) and Shulman (1987) contend that effective teachers must possess a knowledge base of content and pedagogy from which a set of identifiable actions naturally proceeds.

From the perspective of policy-makers, tasks of teaching are expressed as standards or *best practices* that stem from educational research that explores how

teachers transform what they know into instructional strategies and practices in the classroom. Examples of these include the efforts in 2000 and 2014 by NCTM to establish standards for classroom instruction in publications such as the *Principles to Standards*, and the book *Best Practices* that relies on both educators and cognitive research to recommend a new paradigm for teacher practices in the classroom (Zemelman, Daniels, & Hyde, 2005).

The primary source for the *tasks of teaching* used in the current research is the KAT project (McCrory et al., 2012). This project developed the tasks of teaching list found in the KAT Matrix, figure 4, and correlated with the categories of knowledge in that same matrix. These tasks include:

- Analyzing students' mathematical work and thinking.
- Designing, modifying, and selecting mathematical tasks.
- Establishing and revising mathematical goals for students.
- Accessing and using tools and resources for teaching.
- Explaining mathematical ideas and solving mathematical problems.
- Building and supporting mathematical community and discourse.

The rationale for the adoption of these tasks is based in a cross-analysis of all referenced lists of teacher actions, strategies, and practices. This list most concisely and efficiently captures the essence of all other source lists. This list of tasks of teaching is used by permission of the KAT program director and is discussed in greater detail in the following chapter.

Conclusion

The current literature demonstrates a burgeoning interest in forming new connections between knowledge for teaching and classroom practices in ways that inform teachers' decisions throughout the planning and implementation of instruction in the mathematics classroom. A growing amount of research continues to more clearly define mathematical knowledge for teaching and more clearly define the categories of knowledge within this domain. Policy makers and stakeholders continue to examine best practices for improving student learning in mathematics. This current research purported to continue emerging attempts to merge these two efforts into a coherent understanding of connections that exist in specific contexts both for use in the classroom and in mathematics education to raise awareness of these cognitive pathways.

CHAPTER 3 – METHODOLOGY

INTRODUCTION

Classroom teachers constantly make pedagogical decisions as they prepare for and present instruction. Many researchers have recognized the complex and dynamic nature of mathematics teaching (Ball et al., 2001; Hill et al. 2009; Schoenfeld, 2002) and the necessary skill set requiring continuous flexibility and adaptation to meet the needs of a diverse student population. Past research efforts have identified categories of tasks of teaching in the mathematics classroom and the multi-dimensional knowledge of mathematics for teaching needed to enact these tasks in a manner that is comprehensible to students (Ball, et al., 2001; Ball et al. 2005; Hill, et al., 2004; Stylianides & Ball, 2004). As research in this vein continues, there is a recent emergence of research that is more contextualized in a specific domain of mathematics. Beyond continuing attempts to elucidate a global understanding of what has now become known as Mathematical Knowledge for Teaching (MKT), some research undertakes to focus more narrowly within specific mathematical contexts and courses to explain the impact of knowledge for teaching mathematics within a bounded set of topics. This research builds on the recent trend to identify specific pedagogical decisions associated with particular types of mathematical knowledge for teaching mathematics. Specifically, this study asks, what aspects of knowledge for teaching mathematics apply to planning and instructing within the single unit of instruction on solving quadratic equations in a first course in algebra? It

explores the cognitive and pedagogical processes innately utilized by classroom teachers during the planning phase and subsequent instruction to investigate any trends or commonalities that may exist within the context of a first course in algebra.

Specific research questions addressed in this research are:

1. What knowledge of algebra for teaching do teachers possess for teaching a first course in algebra?
2. What knowledge of algebra for teaching do teachers draw upon when preparing for a unit of instruction on solving quadratic equations?
3. What knowledge of algebra for teaching is utilized during instruction in a unit on solving quadratic equations?

Planning and preparation for instructing a unit on quadratic equations is highly contextualized and dependent on a myriad of context rich factors including established curriculum, internal school directives or policies, materials and technology available and the individual teacher's knowledge of content. Additionally, the pedagogical knowledge and knowledge of students' learning strategies, conceptions and proficiencies the individual teachers possess are an important component of the preparation and implementation of instruction. This research attempted to answer the how and why questions about those teacher decisions within this bounded context of a first course in algebra when planning for and instructing students on solution processes for quadratic equations.

THE MIXED METHODS METHODOLOGY

The objective of this research is to explore the thinking behind decisions that are made when planning for and implementing a unit of instruction. The focus of this exploration is not on the decisions themselves, but the process by which the decisions are made; the “how and why” that underpin the final plan for instruction and its application in the classroom. The methodology employed in this research effort was a mixed method design. A mixed method approach to research provides opportunity to address situations that may be too complicated for a single methodology such as a case study using qualitative data collections alone. Yin, (2014), suggests that a mixed methods study allows the researcher to “collect a richer and stronger array of evidence” (p.66). Reaching similar conclusions, other investigations on the effectiveness of various research methodologies conclude that it is both beneficial and necessary to employ a mixed-method design to address the breadth and diversity of data to adequately develop an accurate picture of the evidence within a case study (Mestre, J. 2000; Cline, H. & Mandinach, E., 2000). Using qualitative tools such as semi-structured interviews combined with quantitative methods of data collection for comparative purposes, this study established a means of relating qualitatively categorized teacher responses and classroom actions recorded during observations to numerical measures of content knowledge and observable patterns of activity that emerged from tabular data.

The mixed method employing multiple case studies is chosen as a method of investigation precisely because the objective is to answer these kinds of questions, to develop insight, make discovery, to discover the underlying motivations or influences classroom teachers experience that lead to specific decisions in a situation where context

is important, and to provide thick descriptions of a phenomenon which occurs in a bounded context (Merriam, 2009; Stake, 2006; Yin, 2003; Yin, 2014). The case study niche is in recording the experience of the phenomenon without attempting to make evaluations or judgments of the contextual factors. As Romberg (1992) indicates, the purpose of the case study is illumination and explanation, intentionally avoiding “making judgments about a program or testing a theoretical assumption” (p.57).

The mixed method design can be a valuable form of investigation that can reveal underlying connections between actions and internal decision-making processes across a number of disconnected cases. The quantitative data collected provided a foundation from which to observe patterns and establish a basis for comparison of pedagogical strategies enacted in the classroom as they relate to measured levels of knowledge in the KAT Dimensions, while the case study is useful when investigating underlying motivations or explanations to phenomena through observation within a given context. Multiple case studies are appropriate when the purpose of investigation is to answer questions such as “how” and “why”, when you intentionally refrain from manipulating the behavior of those being studied, and you are researching phenomena that occur within and perhaps due to a certain context. They also support the objective to gain insight into a particular phenomenon when looking at several cases connected only by common access or participation in the phenomenon being studied (Stake, 1995; Yin, 2003).

In this research, quantitative data collection along with a multiple case study format was used as a means of replicating a single case study across cases involving two different classroom teachers. The rationale for this decision is a motivation to investigate the similarities and differences that may emerge in the planning and implementation of

instruction in a common unit on solving quadratic equations in diverse demographic or administrative circumstances. This process of investigation across several cases lends itself to a cross-case analysis of emerging themes and also provides a means of strengthening the internal validity and the reliability of the study (Yin, 2014). In this research, the multiple case study format is intended as a literal replication, examining two cases in an overarching bounded context of the first course in algebra, but recognizing the potential differences in the local context of the individual schools or districts.

The challenge to the researcher is to recognize a tacit obligation to analyze commonalities that may exist across the cases, but also to be cognizant that differences may also emerge that are equally important to document and investigate. This conflict between objectives of the multiple case study and an obligation to respect the integrity of each individual case demands a determination to maintain an attitude of flexibility and adaptability which Stake categorizes as “not just a procedural dilemma, but an epistemological one as well” (p. 7). Using the mixed methods design embeds the teacher planning and pedagogical decisions within a framework that strengthens and supports connections to numerical measures of KAT Dimensions of knowledge and uncovers patterns of behavior that elucidate teacher dispositions toward certain instructional strategies. When combined in the analyses, these parallel data collection methods create a richer and more elaborate explanation of the phenomena under investigation. Taken in concert these dual pathways of analysis give a stronger basis for providing the reader a vicarious experience of the case in a manner that offers explanation of the phenomenon and clarification of the findings.

Research Methods

Participants

This research is focused on the complex process of teacher-decision making as it relates to content knowledge situated in planning and teaching a specific unit of instruction. In order to assure the maximum possible sample variation it was important to seek participants in different settings using purposeful sampling as recommended by Patton (2002). Specifically, the objective in choosing participants was to find subject teachers that represented diverse teaching environments, experience in education, and diversity in their personal and educational demographics. Within the wider geographic proximity of Nashville, there are 75 public and private high schools varying from urban to rural settings and with populations from approximately 500 to 2100 students. Based on the proposed case study methodology, the guiding objective was to find two schools with as much difference in culture, climate and demographic as possible. After consideration of many alternatives, and based on accessibility offered by administrators, one suburban public and one private school in an urban setting were chosen. After interviewing administrators and faculty leaders at each school, then filtering teachers at both institutions based on their current teaching responsibilities, academic and experience backgrounds, and availability, several teachers were considered as the final participants. Further interviews were utilized to identify two teachers who were teaching a first course in algebra for a multiple case study analysis. Following my investigation and interview of all teachers in a first course of algebra at the two final schools, I selected one male and one female teacher with varying degrees of experience and different educational backgrounds to reflect gender differences and potential differences in Knowledge of

Algebra for Teaching based on past experiences. The rationale for these particular subject teachers was that the classroom teachers in these settings likely have different levels of autonomy in choosing how and when to instruct the topic of solving quadratic equations, and both schools are in Tennessee, a Common Core State Standards (CCSS) state. Selection of teachers from these two schools also provided opportunity to observe different school climates and cultures inherent in the student demographics and the presence or absence of external factors involving standardized testing requirements.

Using criteria described above, the researcher identified teachers in two distinct categories; those with between five and ten years of experience that have spent the majority of their career teaching curriculum recommended in the Standards-Based, High Stakes Testing environment as defined by criteria such as that found in the Common Core Standards, and teachers with longer tenure of more than ten years who have taught prior to widespread implementation of these Standards-Based Tests with minimal experience with Common Core Standards. In addition, case study teachers had more than two years' experience teaching algebra I so that they were aware of the scope and sequence from previous years. The final participants were selected based, in order of priority, on 1.) Specific experience levels in Standards-Based Testing curriculum, 2.) Experience teaching algebra I, 3.) Gender considerations, and 4.) Educational background.

My first subject teacher, Janice, had been teaching in the same public school for a total of ten years with a two year break in service early in her career. She had a Bachelor's Degree with a major in math and specialization in secondary education. Janice has teaching experience only in Pre-Algebra and Algebra 1 courses. The school where Janice teaches uses a form of the Common Core Standards tailored for Tennessee schools

by the state policy makers. An annual accountability index for each public school is based in part on the results of an end of course test called the TNReady Test.

Dean, the second subject teacher, has taught at three schools over thirteen years and in his fourth year in his current position at a private school in Nashville. He also has a Bachelor's Degree with a major in mathematics, but also has a Master's Degree in Education. Dean has taught a variety of courses including Algebra 1, Geometry, Calculus and AP Calculus. Like Janice, Dean has taught Algebra 1 for the last several years. There is an end of course exam for his Algebra 1 class that serves as a final course exam, but does not contribute to any accountability index.

Setting

The Nashville area includes many private, rural, and urban schools. Tennessee schools began phasing in the CCSS-Mathematics curriculum in the 2011-2012 school year with full implementation in the 2013-2014 academic year. The content and sequencing guides in the CCSS have established the prevailing curricular approach across school districts nationally and therefore provide the most widely utilized document for planning and implementation of instruction in all core disciplines. The private school is not governed by the implementation of the CCSS in the state of Tennessee, and establishes curriculum based on a college-preparation program of studies with autonomy in sequencing and content. Both schools are accredited by the Southern Association of Colleges and Schools (SACS). In each case, the schools reportedly subscribe to the standards and principles to teach mathematics with understanding, address sound Mathematical Practices, and pursue conceptual understanding and procedural proficiency.

Data collection

Data collection focused on gathering information about the teacher's content knowledge, their planning decisions, and their actions taken during classroom instruction on solving quadratic equations. Data were collected in these three categories through use of demographic surveys, use of the KAT Instrument to measure mathematical content and teaching knowledge, direct observation of classroom lessons, semi-structured interviews, and analysis of artifacts including classroom handouts and assessment tools as described in the paragraphs below.

The strategy and mechanics for collecting the data described above depended entirely on the Knowledge of Algebra for Teaching (KAT) Matrix framework and its components. This framework, discussed in detail in the previous chapter, describes the interaction between tasks of teaching used in the classroom and categories of teacher knowledge used in selecting these tasks. Each decision point or classroom action utilized by the teacher during this unit of instruction on solving quadratic equations was assigned a two dimensional coding that situated the observed instructional task in one of thirty-six cells of the matrix. The tasks and cognitive actions by the teachers were coded using the tasks of teaching versus categories of knowledge as a priori coding guides. A summary of these tasks and categories as described in the previous chapter is presented below as a point of reference.

The KAT matrix has *Tasks of Teaching* as the rows in the matrix, and *Categories of Knowledge* as the columns in the matrix. The array in the figure below serves as both a representation of the a priori codes used in initial data coding and as a data collection instrument for this coding effort.

Figure 4 KAT Matrix is used by permission of the Michigan State Knowledge of Algebra for Teaching Project and serves as the basis for data collection and analysis.

As each task in the planning and instructional phases was observed, it was assigned one of the above codes that most closely represented the intent or motivation for that action. Further, the task was analyzed in context to determine the category of (teacher) knowledge that was accessed in utilizing the specific task for a specific objective during planning or instruction, resulting in its inclusion in one cell of the two dimensional KAT Matrix representation.

Collecting qualitative data is about “asking, watching, and reviewing” (Merriam, 2009, p.85) the subject teachers as they manage the teaching process and access various pedagogical tools or knowledge for teaching algebra. These collection techniques and instruments reflect the objectives and purposes of the study and provided access and

insights to various aspects of the thought processes subject teachers used during their planning and implementation of instruction. Table 1 provides an advanced organizer of the instruments that will be used to access information in each of the three components of the study: teacher content knowledge, planning, and teaching. The decision to collect data in these three areas was informed by the KAT Framework above and provided opportunities to investigate the actions along the pathways indicated in the framework discussed in the previous chapter relating content knowledge, planning activities, and tasks of teaching. Each instrument is further discussed related to each of these three components in the sections that follow.

Table 1. Research Components

Research Components	Teacher Content Knowledge	Planning	Teaching
KAT	Yes	-	-
Pre-Unit Interview	Yes	Yes	-
Unit Artifacts	Yes	Yes	Yes
Pre-Lesson Questionnaire	-	Yes	Yes
Lesson Observation	-	-	Yes
Post-Lesson Interview	-	-	Yes
Post-Unit Interview	Yes	Yes	Yes

Content Knowledge

The initial phase of data collection of content knowledge was accomplished using the Knowledge of Algebra for Teaching (KAT) Assessment Instrument, which is included at Appendix 1. This instrument was designed by researchers at Michigan State and has been utilized in various study efforts conducted by their research team (Ferrini-Mundy, Floden, McCrory, Burrill, & Sandow, 2009). The instrument provides a measure of the content knowledge teachers have for teaching algebra and provides a metric for their familiarity with algebraic topics such as quadratic functions and equations. The results of the assessment informed the semi-structured interviews and served as a basis for further development of the questionnaires as described below.

Second, teachers were asked to provide all artifacts used during their course of instruction for the unit. These artifacts included lesson plans, student handouts, power point presentations, and assessment instruments. All of these documents provide insight into the teacher content knowledge as demonstrated by decisions regarding sequencing, representations for instruction, emphasis on procedural versus conceptual mastery, and the trimming or decompressing actions taken based on conceptions about student understanding.

The post-unit interview itself was used to elucidate the data gathered from the artifacts and the questionnaire referenced above as it pertained to content knowledge. During this interview, the connections to content knowledge were explored with the teachers as they were asked to explain their rationale for choosing the instructional

approaches or representations used during the unit. This interview was also used to gather the teacher's reflections on the unit of instruction and provoke thoughts on modifications needed in future instruction on this topic. As part of the interview, teachers were asked to explain their reasoning for concluding that any proposed alternate instructional strategies or pedagogical approaches should be used in future planning and implementation of their unit on quadratic equations. The post-unit interview included specific questions about the teacher's breadth of knowledge and reasoning for preference of those specific instructional decisions. This line of questioning provided a means of investigating whether there were any limitations in the teacher's content knowledge that would impact planning or teaching decisions.

Planning Data

Planning data was gathered at both the unit and lesson level. At the unit level, a semi-structured interview was administered prior to and following the completion of the unit. The guiding questions for the semi-structured pre- and post-interviews are included at Appendix 3. The objective of the interview was to investigate the decisions made prior to instruction that reflect the teacher's content knowledge, pedagogical knowledge, and understanding of the students' learning needs. Answers to the questions during the interview were categorized using the KAT Matrix as described below in the Data Analysis section.

Second, classroom artifacts are a data source for planning. Artifacts are defined in this study as all written or e-materials used by the teacher or the student for a lesson. Artifacts include lesson plans, class notes, handouts, PowerPoint Presentations, and assessment tools the teacher used during instruction. This data was used to analyze the

connections that may exist between the teacher's planning decisions and content knowledge. These artifacts were also analyzed for evidence of tasks of teaching the teacher planned that reflect the teacher's knowledge of students' learning conceptions and appropriate representations that result in productive mathematical discussions.

Third, data was gathered across three lessons using an electronic pre-observation questionnaire completed prior to each observed lesson. The pre-observation questionnaire, included at Appendix 2, was designed to gather data on specific instructional strategies and planning decisions the teacher made for each lesson. As a part of this questionnaire, teachers were asked to address what they saw as the important mathematics concepts and prerequisite content. They were then asked to indicate any modifications they made to their original instructional plans prior to the lesson and to discuss any student misconceptions or difficulties they anticipated during the course of the lesson. In the questionnaire teachers were asked whether they modified representations they intended to use during this individual instructional period and to explain their basis for these changes.

Finally, the post-unit questionnaire and interview described above was utilized as a source for investigating planning decisions made throughout the unit of instruction. Teachers were asked to reflect on their planning and explain the evolution of those plans throughout the unit of instruction along with an explanation of their reasons for altering plans as the instruction progressed. This questioning strategy drew the teachers into a deeper process of analysis of their own cognitive decisions and the basis for utilizing certain strategies in lieu of others. All of these sources of planning data were also incorporated into the KAT Matrix to further clarify the teacher's content and pedagogical

knowledge, and the ways they accessed that knowledge during the planning phase for the unit.

Teaching Data

Several sources of data are used to gather data on teaching and how content knowledge is reflected in teaching. The pre-lesson questionnaire described in the data collection sections above was also utilized to examine the tasks of teaching for the specific lesson from a practical and implementation perspective. By looking at specific activities planned for the lesson, the researcher was able to see which tasks of teaching were accessed, preferences of representations teachers used, and emphases placed on addressing student conceptions or difficulties. The decisions teachers make as they adapt, modify, or extend learning goals for students are indicators of the complex management of an evolving set of circumstances within the classroom environment that most aptly define the act of teaching mathematics and “the kinds of knowledge of mathematics and sensitivity to nuances of mathematical differences that perhaps are uniquely the province of teachers” (Marcus & Chazan, 2010, p.186). The pre-lesson questionnaire allowed the researcher to examine teaching in light of the dynamic interaction between student and teacher as the unit proceeded and was impacted by both internal and external forces such as unexpected student difficulties or circumstances beyond the control of the classroom teacher.

Second, three audio-recorded classroom observations were completed for each teacher, along with field notes. These observations gave the researcher opportunity to see the actions the teachers took in a dynamic, interactive and unscripted environment in which the plan for instruction and student needs merge to create a somewhat

unpredictable amalgam. In these situations, the researcher could most clearly observe the teachers as they constantly accessed mathematical knowledge for teaching as classroom instruction unfolded. The researcher maintained a record of perceptions and thoughts during these observations by immediately journaling and reviewing field notes for each lesson observed, then analyzed transcriptions of the observed classes upon their completion to validate and supplement these observations.

Following each lesson, a post-lesson interview was conducted to discuss actions the teacher took in response to the situation inside the classroom. The teachers were asked to explain their use of particular tasks of teaching and reflect on the effectiveness of these actions. Further, they were asked to explain what alternate pedagogical decisions they would deem appropriate based on the flow of instruction and their consideration of the current level of understanding exhibited by their students.

Fourth, using lesson plans and other artifacts, the researcher was able to determine where adaptations such as trimming, decompressing or extending knowledge were made in response to the needs of the students. This environment provided rich opportunities to observe the teacher tasks described in the KAT Matrix and investigate further connections between classroom actions and knowledge of algebra for teaching.

Finally, a post-unit interview as described in the first section above was conducted in which the teachers were asked to provide student artifacts from students they perceive as being both strong and weak in quadratic equations. During the interview, the researcher queried the teacher on the teacher tasks that they believe impacted the student work and how they might modify their approach to address any misconceptions discovered. Use the student artifacts was directed at drawing the teacher into a more

robust discussion on the effect certain teacher tasks have on student learning and outcomes. As the teachers reflected on their teaching strategies throughout the unit, the researcher continued to explore any trends or emerging themes with the participating teachers. As with the previous data collected, teacher actions and decisions proceeding from the classroom observations and interviews were added to the KAT Matrix to build an overall representation of the teachers' knowledge of algebra for teaching.

Data Analysis

The data collected was analyzed continually throughout the study. Each data source underwent multiple analysis efforts to examine content knowledge for teaching algebra, connections between planning decisions and tasks of teaching, implementation of classroom instruction using the tasks of teaching, and modifications in instruction each classroom teacher made based on conditions in the learning environment or student needs. The overarching guide for these analyses was the KAT Matrix described previously and included here as a reference for the data analysis discussion. Each data source underwent coding to match planning and teaching decisions to the tasks of teaching in the matrix, which correlate with knowledge for teaching algebra. The coding utilized the tasks of teaching and categories of knowledge as *a priori* codes to place each significant action in the appropriate cell of the matrix. These *a priori* codes form the rows and columns of the KAT Matrix as discussed above. The rows are comprised of the various tasks of teaching that were deemed most common and inclusive in a first course in algebra, and the columns consist of categories of knowledge that span knowledge of the content, pedagogical knowledge, and advanced mathematical knowledge that contemplates future mathematical work.

As indicated in the previous section, data sources included an assessment of the teacher's content knowledge, semi-structured interviews, classroom artifacts, and field notes from classroom observations. Each instrument listed in Table 1 above underwent analysis to develop explanations and connections between the research components of content knowledge, planning, and teaching. As suggested by Merriam (2009), the data analysis began at the onset of data collection and was both emergent and recursive. The researcher continuously utilized the KAT Matrix, referred to in this section and the previous review of literature, combined with an open coding to suggest and incorporate emerging categories for further development or investigation. Because, as Merriam (2009) suggests, qualitative data analysis is "primarily inductive and comparative" (p.175), all further data collected was subjected to additional coding and refinement of categories when compared to existing categories. The analysis performed on each research component is discussed in greater detail in the following paragraphs.

Analysis of each of the research instruments; KAT Assessment, Interviews, Questionnaires, Observations, and Artifacts provided valuable input in developing a rich explanation of the means by which teachers understand and access their knowledge for teaching algebra during a unit on solving quadratic equations. The analysis is discussed in more detail in the following paragraphs.

Knowledge of Algebra for Teaching

This instrument, provided by the KAT Research Team at Michigan State University, provided an objective measure of content knowledge with a specific focus on algebra. Content knowledge was assessed prior to initial interviews using the rubric provided with the KAT Assessment Tool. The results of this assessment served to

establish some connection between the knowledge each teacher had for teaching algebra, specifically quadratic equations, and the trends observed in both planning and teaching decisions. These data addressed the initial research question and provided a basis for discussing the potential connections between actions in the classroom and differing levels of content knowledge as well as a comparison between the subjects in the case study. The KAT Instrument gathers demographic data regarding the personal data and experience levels teaching algebra as well as data regarding teacher preparation, specifically curriculum and major selection in post-secondary training. The assessment also produced a scaled score on the teacher's Knowledge of School Algebra, Advanced Mathematical Knowledge, and Teaching Knowledge described in more detail below.

These three dimensions of knowledge defined by the, KAT Project team (McCrory et al, 2012), formed the basis for situating the subject teachers' knowledge and actions in the context of teachers in a first course in algebra. Teaching Knowledge indicates knowledge of student learning. It addresses teachers' awareness of what difficulties and common misperceptions students have concerning different mathematical topics. This is knowledge that often comes from experience rather than coursework and includes the ability to establish appropriate goals, choose between tasks, and consider trajectories in the curriculum that need emphasis or clarification. Knowledge of School Algebra reflects knowing the intended curriculum for high school algebra. This dimension of knowledge is an indicator of the expectations the teacher has for the students during the course of instruction. Advanced Knowledge of Mathematics measures a breadth and depth of knowledge of mathematical work that may be encountered in the future by algebra students. It provides a foundation for bridging the topics in a first

course in algebra with other topics the students will likely see in later high school or undergraduate math courses. These knowledge categories work in concert to guide the teacher's planning and instruction in an effective and productive manner most beneficial to their students. They address the teacher's knowledge in the specific topics covered in a typical algebra course, as well as their understanding of connections and the "trajectory and growth of mathematical ideas beyond school algebra" (Knowledge of Algebra for Teaching Project, Michigan State University, 2004-2008, IRT Scale Scores, used by permission of Floden, R., 2009). These knowledge dimensions also provided indicators of pedagogical knowledge that enable teachers to make appropriate modifications that address their specific students' needs.

Beyond this initial analysis effort, all data sources were further analyzed for evidence of the three overarching themes indicated in the KAT Matrix; *Bridging*, *Trimming*, and *Decompressing*. These three actions indicate how the teachers actually use their knowledge of algebra for teaching to further the learning experience for the students. They show the ways teachers enact the dimensions of knowledge described above to provide depth and breadth of learning of both procedure and concept, and how they build connections between and across topics. Analysis of the application of these overarching themes provided the most in-depth perspective of the way the subject teachers held and utilized their mathematical knowledge for teaching a first course in algebra.

Bridging in a first course in algebra involves making or clarifying connections with previous learning and with future work. Students in algebra frequently encounter levels of abstraction that obscure how ideas and processes are connected. Their initial

confusion when exposed to terminology such as variable and function are potential obstacles to learning that teachers need to anticipate. Bridging means providing a link between algebra and other mathematical work. McCrory et al (2012) say “Bridging, in practice, means providing students with the big picture of mathematics, making explicit connections across topics, keeping a range of ideas in play in the classroom, and presenting mathematics as a coherent, connected endeavor” (p. 608). Establishing a sense of continuity in their instruction demonstrates the teacher’s ability to utilize bridging both with previous learning objectives and the uses of current knowledge in future learning opportunities.

Trimming describes the way good teachers demonstrate their understanding of mathematically complex ideas by their ability to present difficult ideas in a way that students can intellectually process. They are able to do this by scaffolding ideas and processes without loss of integrity within the context of the unit of instruction. This skill reflects the teacher’s understanding of the proficiency of their students and a willingness to present concepts in a manner consistent with that level of capability without sacrificing the mathematical requirements of the concept. Ma (1999) discusses having a “clearer idea of what is the simplest form of a certain mathematical idea” (p. 46), so that difficult ideas become accessible to struggling students. This is an intentional act of modifying the rigor of a topic to meet the specific need of the students, either by omitting or supplementing instruction with detail needed to clarify what may be initially unclear.

Decompressing is an intentional effort by classroom teachers to open up or make clear the underlying logic or concept behind material that may be opaque to students. Teachers often encounter confusion or lack of understanding when introducing new ideas

or concepts. This could be a consequence of the inherent capabilities of the students, a lack of familiarity by students with ideas that are essential to algebraic understanding, or misinterpretation by the teacher of the level of preparedness of the class. Teachers of mathematics know more mathematics than is needed for instruction at the secondary level, but sometimes overlook the complications that result from terminology or expectations they brought to the classroom nestled within a more advanced mathematical knowledge. Procedures teachers may have learned by rote, or have become accustomed to using over a period of years, may seem obvious to them but opaque to their students. One of the marks of robust mathematical knowledge is the ability to decompress, or break into more simple components, the parts of that knowledge in a way that is comprehensible to their students. Teachers may lose sight of the difficulty students have when encountering the abstractions of algebra, and they become incapable of seeing the mathematics through the eyes of their students. Because teachers have a natural grasp of processes and definitions used over years of learning and experience, it is sometimes a challenge to understand why students struggle with the notion of variable or the use of a letter as a quantitative surrogate. The ability to simplify while maintaining mathematical integrity demonstrates a deep level of understanding of algebraic content that translates into effective and productive instruction.

Semi-structured interviews

Interviews were used in three contexts for this case study. As indicated by Dewalt and Dewalt (2002), interviews have been shown to be a valuable way to gather data regarding decisions as they relate to knowledge for teaching because they situate the teacher as participant in the data collection process. All interviews were audio recorded

and transcribed immediately following completion, then coded using the *a priori* codes from the KAT Matrix above. An additional open coding was conducted on the interviews to explore other themes or patterns that might exist within or between the cases in the study. These coding efforts, conducted simultaneously with data collection also informed future interviews and other data collection. In this way, all data contributed to the primary research objective of data analysis as a “process of making sense of the data” (Merriam, 2009, p.175). All of the interviews were also independently coded by a colleague holding a PhD from Ohio University to ensure consistency of interpretation and provide a measure of reliability for the results. An initial pre-unit interview was conducted following administration of the KAT Assessment and before instruction of the unit on quadratic equations. This interview was informed by the results of that assessment and served as a means of exploring the planning decisions made by the subject teachers. As teachers prepared for the instruction, the initial interview was directed at establishing an initial explanation for pedagogical strategies and connecting their plans with the tasks of teaching included in the matrix. This interview also established a basis for observing modifications and evolving teaching strategies as classroom conditions changed or student misconceptions arose requiring a shift in instructional approach. As Merriam (2009) suggests, each phase of the data collection and analysis process was utilized to inform further steps in the analysis including construction of other probing questions that arose as a result of uncovering the rationale that motivated the teachers to make certain pedagogical decisions or pursue courses of action in the planning and teaching process that the researcher had not anticipated. During the unit of instruction, teachers were observed on three instructional days. Following each of these observations, a post-lesson

interview was conducted to focus on the teaching decisions made during the classroom instruction. The focus of these interviews was the actions and decisions teachers made that digressed from original plans, or addressed student needs regarding difficulties or knowledge gaps. These interviews were also coded against the KAT Matrix and results compared to the initial coding of pre-unit interviews to discover any emerging themes. At the conclusion of the unit of instruction, a post-unit interview was conducted and analyzed against earlier results to explore the consistency of planning and teaching decisions. This interview was utilized as a data source for all three research components and serves as a capstone analysis instrument for the themes developed throughout the case study. After coding this interview independently, all interview results were compared in developing a more complete explanation of connections observed between the content knowledge, planning decisions, and teaching decisions made during the unit of instruction. Because the case study is not substantially large, the data was collected and categorized using the capabilities of Microsoft Word and Excel to accumulate coding results and analyze density of teacher tasks and access to related knowledge categories.

Questionnaires

Two questionnaires, the KAT Assessment Instrument Open-Ended Questions, and a pre-lesion interview were used during the case study underwent analysis and coding similar to that conducted on the interviews. The coding for the KAT Instrument included examining the responses on the open-ended questions for evidence of the tasks of teaching in the KAT Matrix and comparison with categories resulting from open-coding of the interviews to discover consistencies or new themes emerging creating additional categories of data. A pre-lesson questionnaire, administered electronically, addressed

both planning and teaching decisions. The questionnaire gave teachers opportunity to indicate any student conceptions or difficulties observed that might require a modification in planned instruction. The rationale for these changes provided insight into the content knowledge accessed as classroom conditions intersected with teacher intentions resulting in modified tasks of teaching per the KAT Matrix. Additionally, the intended teaching strategies listed in the pre-lesson questionnaire demonstrate the teacher's understanding of student learning and appropriate tools to address those needs as reflected in the tasks of teaching from the KAT Matrix and evidence of the overarching themes of bridging, trimming, and decompressing as defined in the review of literature. For example, use of particular representations or extensions of conceptual presentations were indications of the teacher's efforts to either remediate deficiencies, or change the mathematical discourse to appropriately challenge the students. These documents also informed the post-lesson interview and provided a rich source for observing the teaching process as the subject teachers wrestled with the dynamics of classroom instruction. As these documents were analyzed, the researcher looked for any emerging themes or practices that might assist in developing a more complete picture of the connections between the flow of instruction across a specific topic and the content knowledge for teaching algebra. The questionnaires also informed the post-unit interviews in which the researcher attempted to clarify and further develop an explanation for the teacher's actions and connections to the tasks of teaching.

Lesson Observation.

Audio recordings of lesson observations were transcribed immediately following the classes observed. In conjunction with field notes, the transcriptions were read in depth

and coded using the matrix as described above. Along with other data sources, these notes were also independently coded by a colleague to assist in assuring clear and rich explanations of teaching decisions and their connections to content knowledge. Additionally, field notes and perceptions of the researcher during observation periods were recorded to provide a robust picture of the teacher's response to the dynamic situations in the classroom when unexpected events or student difficulties occurred. The analysis of the transcribed observations and field notes was focused on teaching decisions made during instruction. Classroom observations gave the researcher valuable insight into the pedagogical flexibility of the teachers as they encountered unexpected questions or student difficulties. In this context the researcher was able to observe instantaneous modification of scheduled or planned teaching strategies in favor of more appropriate tasks of teaching to address evolving student needs. The field notes from observations along with transcriptions were concurrently analyzed for evidence of the overarching concepts of decompressing, trimming, or bridging to further explain the teacher's decisions to employ particular tasks of teaching, or lack of facility to adapt flexibly to student difficulties.

Artifacts

The artifacts provided by teachers were used as data sources for all three research components. Lesson plans, student handouts, and presentation documents were all analyzed for evidence of connections between the various components and tasks of teaching. The coding of these artifacts overlaid the interview results and indicated the degree of integrity and consistency between written documents and spontaneous responses given during interviews. Using the research components as a guide, the

researcher went through each artifact three times coding once for content knowledge, then planning and finally teaching decisions. Using the lesson plans, student handouts, and assessment tools prepared by the teachers, the researcher examined the tasks of teaching demonstrated in these documents. Each artifact provided a means of observing the underlying content knowledge and pedagogical decisions teachers demonstrated as they addressed the dynamic nature of classroom instruction and changing student needs. These data sources were coded against the KAT Matrix by the researcher and underwent an independent analysis and coding by a colleague to ensure consistent results and interpretation.

Analysis by the teachers' on their students' work during the post-unit interview provided an opportunity to develop a robust picture of the teacher's thought processes as they reflected on possible modifications for future instruction that could address observed gaps in student knowledge or conceptions. Using this artifact gave the researcher opportunity to engage in an interactive investigation with the subject teachers into the entire spectrum of content knowledge accessed during planning and teaching and the connections that exist with the tasks of teaching utilized. As the student work was discussed, the teacher's explanations for mistakes or solution strategies provided insight into their content knowledge for teaching and their understanding of the student's approach to learning. The teachers were also able to provide ideas on how they would modify future instruction to improve student learning. This discussion gave opportunity to perform some member checking with the participants and use their input to clarify and refine explanations of phenomena previously considered. As the researcher looked back over the post-unit interview and notes recorded by the researcher, the strongest evidence

for use of particular tasks of teaching, pedagogical strategies, and depth of content knowledge proceeded from the discussion of actual student work and responses during classroom instruction. In conjunction with other data, this discussion enabled the researcher to solidify previous categories and clarify emergent themes.

TRUSTWORTHINESS OF THE STUDY

Qualitative Research Studies have been more widely used during the past decade, but are sometimes viewed with skepticism because they use a naturalistic approach rather than a quantitative approach to investigate phenomena and establish some explanation for study results. A growing number of authors have demonstrated ways qualitative research can utilize measures and strategies that address concerns of reliability and validity (Guba, 1981; Pitts, 1994; Silverman, 2001). Shenton (2004), relying on previous work by Guba and others, addresses four criteria that serve to establish trustworthiness of qualitative research by utilizing measures that parallel the quantitative results favored by positivists. These measures: credibility, transferability, dependability, and confirmability (sic), are addressed in the following paragraphs as they pertain to the processes used in this case study to validate the reliability of the results discussed in the following chapters.

Credibility

At the outset, establishing credibility stands as a major consideration for the qualitative researcher. Lincoln and Guba (1985) contend that ensuring credibility is a critical part of ensuring the trustworthiness of the research. Merriam (2009) adds that credibility deals with the question “How congruent are the findings with reality?” (p.213). The first means of establishing credibility in this study is the researcher’s reliance on previous methods used by the KAT Project at Michigan State University

under the supervision of Dr. Robert Floden (2009). These methods and framework formed the basis for data collection and analysis of the research components: content knowledge, planning decisions, and teaching decisions.

Secondly, the researcher has used the Common Core School Standards (CCSS) since their initial implementation and is familiar with the curricular components of these math standards. Additionally, the researcher is, by virtue of his position as a classroom instructor in secondary school, familiar with the school systems in the local area and the specific Algebra I curriculum used in the institutions where the subject teachers are employed. Because of this knowledge of the culture, the researcher was able to more clearly understand the climate in which subject teachers worked. A side effect of this familiarity was that the researcher was able to anticipate biases that may exist and address these issues in the context of the study. For example, classroom teachers in the state of Tennessee are required to administer a standardized End of Course (EOC) test that weighs heavily in their professional evaluation. This foreknowledge informed the questioning strategies during interviews and provided opportunity to address rival explanations for certain pedagogical decisions and curricular choices.

One of the major contributors to credibility is triangulation of data strategies. In this study, the researcher used several triangulation methods to ensure consistency of results, and build credibility into the resulting explanation of the phenomena studied. Data triangulation was achieved through the use of the KAT assessment, a questionnaire, multiple interviews and classroom observations, as well as the collection of classroom artifacts (including lesson plans, student handouts, PowerPoint presentations, pre-lesson questionnaire, assessment tools and student work.). Using these various artifacts, the

researcher was able to observe the consistency, or its absence, in all planning and teaching decisions. Using multiple data sources, the researcher was able to refine interview questions and pursue lines of questioning to clarify any apparent disagreement among the collected data. In addition to the artifacts, a colleague in the mathematics education field independently reviewed and coded all interviews against the KAT Matrix to validate or refine the interpretations of the researcher. As an internal means of triangulation, multiple interviews over the entire period of the research effort were compared for consistency of responses by the subject teachers. The repetitive nature of the interview process described above allowed the researcher to clarify and refine explanations of conflicts that were observed during the study. The entire project was frequently open to peer scrutiny through the interim meetings with my advisory committee and input from colleagues during the length of the study.

As a final source of credibility, the researcher actively sought member checking by the subject classroom teachers during the final phase of the study. At the post-unit interview, and during the weeks following, participants were asked to read the explanatory notes of the researcher to ensure the integrity of these descriptions and explanations of the phenomena under study. In this way, the researcher was able to verify the accuracy of emerging theories as an explanation of the actual situations in the context of their occurrence (Shenton, 2004).

Transferability

Because of the prevailing emphasis on the implementation and assessment of the CCSS, a substantial number of readers will be intimately familiar the application of these standards in their own circumstances and the importance of test preparation for the EOC

tests conveyed by both administrators and education policy makers. For this reason, they will be able to readily make comparison to their own situation and reflect on the cognitive processes they undergo in planning for and implementing instruction as a result of their own knowledge for teaching algebra. This familiarity of the reader to relate the parameters of the study as well as the results to their own position provides a ready basis for transferability of the case study and an inherent strengthening of its application (Bassey, 1981). Educators in the field of mathematics, and specifically involved in the teaching of algebra, will recognize the vocabulary and the curriculum requirements in the descriptions of the specific communities in which the participants operate because of the overarching commonality of the Common Core Standards themselves. The researcher has carefully included the methodology for data collection in an instructional time frame familiar to teachers of algebra and curriculum coordinators alike. As such, there is an inherent degree of transferability of these results within the limitations of a small sample size and a small educational community. In essence, the transferability of these results will vary from one audience to another and belies the primary objective of the study which is to establish a foundation for further research in the connections between mathematical knowledge for teaching and the practice of teaching at the secondary level.

Dependability

As Shenton (1998) points out, positivists and quantitative researchers rely on statistical analyses and other techniques to demonstrate that if their work was repeated in the same context, the same or similar results should be achievable in confirmation of the study. The naturalistic approach of a qualitative case study is inevitably influenced by the context in which the study is conducted, and the unique characteristics of the participants

or subjects of the case. The objective of this research is in part to validate the prototype model utilized by researchers at Michigan State University in the KAT Project, and to extend the precepts of that research by presenting a design for investigating at a deep level the cognitive processes employed by classroom teachers in their planning and implementation of instruction for a specific unit of instruction. This research leans on the positive gains in understanding the intersection between a teacher's knowledge for teaching algebra and tasks of teaching that reflect an amalgam of best practices over the past decade as described in the review of literature included earlier in this study. As detailed in the methodology above, the researcher has presented a comprehensive outline of the data collection and analysis plan that enables future research to follow a similar line of inquiry in the well-defined context of this case study. Further details of the effectiveness, strengths, and weaknesses of the project addressed in the results and discussion of the research help to guide any effort to replicate the study in a similar or related unit of instruction to investigate the cognitive processes and connections classroom teachers exhibit during the planning and implementation of mathematics instruction at the secondary level.

Confirmability

At best, it is difficult to attain objectivity in qualitative research. As Patton (2002) reminds us, the investigator's bias is nearly inevitable even when designing objective instruments for collecting survey data, and the qualitative researcher plan carefully to ensure the results reported are a product of interaction with the situated experiences of the subject and not the preferences of the researcher. The primary tool utilized in this study to address this challenge was the independent review and analysis of all interview

transcriptions and field notes by a colleague in the field of mathematics education. Additionally, the use of multiple sources: artifacts, questionnaires and classroom observations, served to collaborate or clarify emerging themes observed in the interview process. These triangulation strategies serve as a means to reduce the bias or preliminary belief system brought to the study by the researcher. Additionally, addressing rival explanations was a major part of the planning process for the study. These rivals, discussed in the following paragraph, necessitated continuous, disciplined monitoring by the researcher to avoid interjecting personal biases or beliefs into the interpretation of the case. Including this *a priori* strategy served as a strong inhibitor for a *prima facie* explanation of the phenomena being studied and helped curb the influx of researcher bias into the naturalistic inquiry.

RESEARCH ASSUMPTIONS

When conducting case study research in which discovery, explanation, or elucidation of some behavior or phenomenon is the objective, it is important to consider alternative rival influences that may play a part in predicting or even precipitating the phenomenon. (Yin, 2014) Among the chief rivals that may influence a teacher's pedagogical decisions concerning classroom instruction, the local policies must be addressed as an important factor. Site-based councils, district curriculum directors, or school administrators are often involved to various degrees in making decisions regarding the content included in course instruction as well as sequencing of varied topics. The researcher is aware of the potential impact of this rival influence, and will address the influence of pacing or curriculum guides as necessary within each case. Additionally, the use of End of Course (EOC) exams can often exert a major influence over instructional

decisions as teachers attempt to address as many topics as practicable in an effort to prepare the students for the EOC exam. These decisions may result in actions that change the emphasis on certain topics in any course to allow sufficient time and energy to address those topics that are likely to appear on these high stakes exams. Because the use of such testing and accountability strategies is ubiquitous in the current climate, the researcher recognizes the potential effect these EOC exams might have yet is of the opinion the effect is mitigated by the widespread usage of such tests. Another potential rival is the influence of textbooks chosen for the first course in algebra. Classroom teachers, especially those with little experience or a lower level of content knowledge may rely heavily on the course materials to guide planning and instruction. The researcher intends to address this issue during interviews to explore the potential of this experience level as a factor in the strategies used by teachers as they plan their instruction sequencing and representations.

LIMITATIONS

A primary limitation of the study stemmed from the use of a small, purposeful sample. Using experience with the Common Core Standards as a guiding factor, the subject teachers were selected to observe the potential impact experience with these standards might have on the planning and teaching processes. Coupled with the filter of experience level, the level of autonomy afforded the participants in planning the curriculum utilized in the classroom provided an additional factor that to a degree mitigated the limited sample size. Although the narrowed focus of the study provided opportunity for a robust explanation of instructional practices in a very specific period of planning and instruction, this same focus may limit the transferability of these results to

any other topic in a first course of algebra or other secondary math courses. Because the study observed a single unit of instruction, the brevity of the research may be viewed as a limitation. It is possible that as the academic year progresses, the marginal experiential growth of the teacher, combined with an increasing understanding of the student needs and conceptions may effect change in the planning and instructional process. The reflection in the final interview was intended to gain insight into the potential growth experienced during the study and shed light on the strategies teachers might use to increase professional or content knowledge that influences the learning environment and pedagogical decisions. Finally, the limited exposure to a diverse student demographic may have decreased the breadth of pedagogical moves needed to adequately address student needs on the extreme boundaries of the academic ability spectrum. Hopefully, the insights provided in this research will encourage further investigation that broadens the scope and application of this research.

CHAPTER 4

FINDINGS

Introduction

Over the past three decades, interest in the skillful application of Mathematical Knowledge for Teaching (MKT) has fueled research efforts to define and codify how this special content and pedagogical knowledge translates into teacher actions in the classroom. Prior to a continuing study through a Michigan State project directed at first year algebra teachers (Ferrini-Mundy et al, 2005; Floden, 2009), this research was almost exclusively focused on teachers at the elementary school level (Ball, Hill, & Bass, 2005; Ball, Lubienski, Mewborn, 2001; Hill, Schilling, Ball, 2004). The research by the Knowledge of Algebra for Teaching Project focused on teacher tasks and elements of teacher knowledge during instruction on a unit in linear equations in a first course in algebra. The purpose of this research was to extend the research to the secondary level to examine the intersection between a subset of Mathematical Knowledge for Teaching and the Tasks of Teaching utilized by classroom teachers in a first course in algebra. This current research was designed to continue that investigation by looking at teacher actions and access to knowledge for teaching algebra during a unit of instruction in solving quadratic equations. Specifically, the research objective was to examine the Knowledge of Algebra for Teaching (KAT) as it impacted the planning and execution of a unit of instruction on solving quadratic equations.

As a point of reference, my research questions were:

1. What knowledge of algebra for teaching do teachers possess for teaching a first course in algebra?
2. What knowledge of algebra for teaching do teachers draw upon when preparing for a unit of instruction on solving quadratic equations?
3. What knowledge of algebra for teaching is utilized during instruction in a unit on solving quadratic equations?

To explore these questions, I studied two algebra teachers using a mixed methods case study design. In the sections that follow, I describe the subjects of this case study including their academic backgrounds, the climate in which they teach, objective measures of their actual KAT, and findings based on classroom observations, field notes, and semi-structured interviews. Emergent themes which provided a surprisingly complex picture of classroom dynamics and external factors that impacted planning and instruction are also presented.

Initially, the selection of appropriate subject teachers is described, followed by a detailed description of the data that was collected, coding procedures, and emergent. In the individual case study analyses, and later in the cross-case analysis, the data is situated in the context of the research questions to address how the subject teachers demonstrated their knowledge for teaching algebra, how that knowledge influenced their planning activities, and the ways their knowledge translated to classroom instruction. The data is then further discussed relative to the ongoing interactions between the unit plan and the actual classroom instruction to develop a robust explanation of the impact of internal and external factors on tasks the teachers utilized. Adjustments to the unit plan and

instructional strategies based on evolving classroom dynamics and student needs provided a rich source of data for observing the flexibility of the teachers' knowledge and their depth of KAT.

Participants

My primary objective in selecting case study teachers was finding diversity both in the school environment and their level of experience, including diverse teaching environments, experience in education, and diversity in their personal and educational demographics. Following my investigation and interview of all teachers in a first course of algebra at the two final schools, I selected one male and one female teacher with varying degrees of experience and different educational backgrounds to reflect gender differences and potential differences in KAT based on past experiences. School demographics and teacher characteristics for the two subject teachers, Janice, who teaches at a suburban public school, and Dean, a teacher at an urban private school are shown in tabular form in Table 2 below, and described in more detail in the individual case studies.

Table 2. Demographic Data for Teachers and Schools

Characteristic	Janice	Dean
Gender	Female	Male
Degree(s) Earned	B. S. Math	B. S. Math M. S. Education

Post-Secondary Math and Education Courses		Calculus Linear Algebra Advanced Geometry Number Theory Differential Equations Methods of Teaching	Calculus Linear Algebra Real / Complex Analysis Number Theory Differential Equations Methods of Teaching Psychology of Learning Assessment in Math Instruction
Years Teaching		10	13
Courses Taught		Pre-Algebra Remedial Algebra Algebra 1	Algebra 1 Geometry Calculus AP Calculus
Student/Teacher Ratio		33:1	14:1
Student Ethnicity	Caucasian	87.1%	89.75%
	Hispanic	6.7%	0
	Asian	1.7%	5.34%
	African American	7.8	4.92%
Student Gender	M/F %	53/47	100/0
Graduation Rate		98%	100%
Student Population		2050	720
N = 2			

In this summary table form, the student population and educational opportunity appeared surprisingly similar. Further examination showed this to be generally true except for some factors not readily apparent. The median family income for the private school students was higher and the students there were afforded additional opportunities through summer enrichment programs, subsidized trips to international educational institutions for immersion in language, arts, or STEM programs, and on campus enrichment classes at no expense to the families or students. Janice and Dean both majored in mathematics in their undergraduate curriculum, but Dean had some additional course work in the area of mathematics education while obtaining a Master's Degree. One stark quantifiable difference in these cases is the student to teacher ratio as seen in the above table. While many educators believe class size is a critical factor in student achievement, only the Tennessee STAR project analyzing elementary school classes (Mosteller, 1995) indicated any measurable change in achievement levels with smaller classes. The other major observable distinction between these two cases, the prevailing educational climate within their respective schools, became obvious during field observation and subsequent interviews. As will be discussed in the following sections, the impact of testing requirements and constraints on autonomy created very different priorities and venues for the classroom teachers in these two distinct environments.

Analysis of Data from Research Instruments, Coding, and Emergent Themes

Coding Structure

Teaching mathematics is often a complex task requiring integration of various techniques and representations that must be presented in a way that is accessible to students. That task calls on aspects of all three dimensions of knowledge in the KAT

Framework; Knowledge of School Mathematics, Teaching Knowledge, and Advanced Knowledge of Mathematics as described in the previous chapter. Because the application of this knowledge is seen primarily when teachers are actually *practicing* their craft in either planning or instruction, that complexity manifested in trying to authentically categorize the “often invisible, mathematical work that teachers may do” (McCrary, et al, 2012, p.594).

The intersection of the above knowledge dimensions with actual use of mathematical knowledge in the classroom can be observed in three different areas described in the KAT Project as Decompressing, Trimming, and Bridging. These overarching themes are previously described in more detail in Chapters 2 and 3. These three ideas, described as overarching concepts in the KAT Project, provided the conduit for observing the teacher in the actual practice of classroom instruction as they utilized their available Teaching, School, or Advanced Knowledge in their work to construct problems, representations, and context. The findings provided insight into the deployment of their respective teacher knowledge within these dimensions, and exposed the capabilities and limitations in planning and instruction associated with varying degrees of that knowledge.

In the evolving classroom environment, external demands such as policy decisions or curricular constraints that are real or perceived may have a dramatic impact on teacher decisions and actions. In this study, those external demands coupled with teaching practices reflecting poor pedagogical strategies presented additional factors that significantly influenced the planning and instruction that occurred during this unit. These three facets which resulted in a multi-level coding structure coupled with the results of

the assessment instrument and other data collection will be further elucidated and discussed in detail in the case studies below.

Interviews, observations, questionnaires and artifacts were coded and analyzed in 3 levels. Initially, the interviews and observational data contained in field notes were transcribed, reviewed, and subjected to first level coding to the KAT Matrix using a priori codes described in the previous chapter. Each observed teacher action and associated knowledge accessed generated an entry to a cell of the KAT Matrix. This first level coding provided a concise and visual representation of the data that could be manipulated and analyzed for patterns of behavior.

Second level coding involved revisiting the transcripts and determining the application of the overarching themes of Bridging, Trimming, and Decompressing to enrich or enhance the learning experience for the students. This coding scheme provided a deeper insight into the impact of teacher knowledge levels as they influenced either planning or instructional activities individually or in concert.

Finally, an open coding scheme was utilized to discover emergent themes that appeared in the interviews and field data. These additional factors, which added an unexpected dimension of complexity, were also analyzed to determine their interaction with the planning phase or the classroom instruction as teachers accessed and utilized their knowledge of algebra for teaching. All three coding levels are explained in further detail in the following paragraphs.

Analysis of Data Using the KAT Matrix

As indicated in the previous chapter, the methodology for this study was a qualitative multiple case study. Data was collected through use of demographic surveys, use of the KAT Instrument to measure mathematical content and teaching knowledge, direct observation of classroom lessons, semi-structured interviews, and analysis of artifacts including classroom handouts and assessment tools. The actions the teachers took in class, and their connection to content and pedagogical knowledge as well as knowledge of student learning, were coded using a priori codes generated from the previously described KAT Matrix and codes developed by the researcher based on the emergent themes that arose during interviews and observation. Using the codes defined for these categories, the researcher assigned teacher actions and interview responses during field observation and subsequent readings of transcripts for audio recordings of classroom instruction and interviews a coding as prescribed through a priori methodology and emerging, ongoing analysis. As a means of triangulation, both field notes and classroom artifacts were also coded using the same criteria. These aspects of the data analysis are described in more detail below.

The KAT Assessment Instrument was administered prior to interviews or observations to establish a basis for comparison of the knowledge of algebra for teaching each of the subject teachers possessed. The assessment produced a scaled score based on normalized data from prior data collected by the KAT Project Team. The results of that assessment acted as a lens through which to view the decisions made by the teachers in their planning and instructional activities. These results also provided a valuable opportunity to explore the relationship between the levels of the teacher's knowledge and

the type and frequency of decisions made to enact that knowledge in the classroom. The specific scores and their interpretation is included in the paragraphs that follow in the individual case studies.

The initial data was collected using the KAT Instrument for measuring knowledge for teaching a first course in algebra. Utilizing Form 1 of the test, demographic data was collected along with results of an assessment on the teachers' mathematical content and pedagogical knowledge. As described in Appendix 3, the assessment measured knowledge in three areas; Teaching Knowledge, Knowledge of School Algebra, and Advanced Knowledge of Mathematics. Each of these knowledge categories, discussed in Chapter 2, are the underpinning to the framework for analyzing the way teachers actively utilized their knowledge of algebra for teaching. Teachers can spontaneously perform a task in different circumstances and times in their classroom while accessing different types of knowledge to achieve different goals or objectives with their students. The presence of these underlying knowledge dimensions added a depth of understanding of how and when teachers were employing their knowledge during planning and instruction while concurrently adding an unexpected complexity in making that determination.

The results of the assessments were interpreted using the tools provided by the KAT Project as indicated in Appendix 3. All scores and their relative standing to the mean are shown in Table 3 below, which compares the case study teachers' knowledge to the mean of how teachers in the KAT Database with the same experience levels, and also teachers with College Majors in Mathematics, performed as observed by the KAT Project.

Table 3. KAT Assessment Scores for Participants.

Overall Scale Scores	Mean = 50.0, Standard Dev.= 10	Janice	Dean
<i>Raw Scores</i>			
	Teaching Knowledge	47.6	50.0
	Knowledge of School Mathematics	44.4	53.8
	Advanced Knowledge of Mathematics	45.5	69.3
<i>z-Scores</i>			
VS. Teaching Service	Teaching Knowledge	- .095	0.142
	Knowledge of School Mathematics	- .426	0.520
	Advanced Knowledge of Mathematics	- .360	2.041
VS. College Major	Teaching Knowledge	- .595	- .328
	Knowledge of School Mathematics	- .893	0.107
	Advanced Knowledge of Mathematics	- .767	1.934
KAT Form 1			

The first subject teacher, Janice, scored below the mean in all three categories, with Knowledge of School Mathematics score nearly one standard deviation below the mean versus College Majors in Mathematics. Dean scored above the mean across the board except for Teaching Knowledge versus College Majors in Mathematics, in which his score was only slightly below the mean. These results establish a measure of the particular knowledge available to the teachers as addressed by the first research objective. The impact of these results on planning and instruction decisions are discussed in greater detail in the following sections as the individual and cross-case analyses are developed.

First Level Coding: Coding to KAT Matrix

Following scoring of the KAT Assessment, first level coding of observational and interview data was accomplished by reading through the transcripts of pre-instruction interviews, classroom observation, immediate post-lesson interviews, and post-unit instruction interviews. The methodology for applying coding was to examine the statements made by the teachers during interviews or active instruction and actions taken during instruction, then compare them to the tasks of teaching and knowledge categories in the KAT Matrix as explained in the previous chapter. The KAT Matrix is an array of actions and teacher knowledge that enables categorizing a specific teacher action in the context of one of the several categories of knowledge of algebra for teaching utilized in the KAT Project. That coding decision was accomplished by coupling an action with the appropriate teacher knowledge category accessed to reflect the teacher's desired objective, then making a corresponding entry in the appropriate cell of the matrix. For reference, these teacher tasks and knowledge categories are again summarized in Figure 5 below.

Tasks of Teaching	Core Content Knowledge	Representations	Content Trajectories	Applications and Contexts	Language and Conventions	Reasoning and Proof
Analyze Students Mathematical Work and Thinking						
Designing, Modifying, and Selecting Mathematical Tasks						
Establishing, Revising Mathematical Goals						
Accessing Tools, Resources for Teaching						
Explain Ideas, Solve Problems						
Building Mathematical Community and Discourse

Figure5. KAT Matrix with Overarching Themes

This organizational structure allowed deeper analysis of not only the actions teachers were employing, but what cognitive processes they accessed to make those teaching decisions. The three other overarching concepts of Bridging, Trimming, and Decompressing as defined by the KAT Project are also included for reference. These additional conceptual underpinnings were instrumental in helping clarify the relationships between teacher actions and three types of teacher knowledge, described in the previous section, at work underneath the planning and instruction for solving quadratic equations. These three overarching concepts also provided the basis for the second level coding effort as described below.

As an example, in response to my question about what students should expect to gain from this unit of instruction on solving quadratic equations, Janice replied:

“I want you to understand that the solution in solving quadratics is the place where the graph crosses the X axis. Um, I want you to make connections between the reason for factoring and solving quadratic equations.”

This response was coded as an entry to the (Establishing and Revising Mathematical Goals / Core Content) because Janice was *Establishing a Mathematical Goal* for her students and making allowance for revision of these goals as instruction proceeded. She also accessed her own *Core Content* knowledge of how to solve quadratic equations using a variety of methods. Similarly, during instruction on factoring, Dean said,

“Okay. So, I'm gonna start with this. Now, we're gonna use a technique which is really, really, really useful. It's called the zero product property. You guys recall we touched on that awhile back?”

Because Dean had purposely designed and used this particular problem approach as an introduction to the procedure for solving quadratic equations by factoring, this statement was coded in the cell (*Designing, Modifying, and Selecting Mathematical Tasks / Language and Conventions*) because he knew it would exemplify certain ideas and concepts needed in instruction, and would also establish the appropriate language used later to refer to this mathematical condition. Using the defined categories for teacher actions and algebra Knowledge for teaching in the KAT Matrix, each action and reaction during planning and instruction was then coded to a particular cell in the matrix during this initial coding effort.

As a result of this initial effort, the transcripts yielded a cumulative accounting, for both subject teachers combined, of the occurrences for each task recorded in the KAT Matrix template. In examining the density of occurrences for each cell in the matrix, it became more apparent which categories of knowledge the teachers were relying on in their decision processes. The instructional actions and decisions by the teachers, depicted in the matrix below (see Table 4), showed a heavy reliance on the Core Content knowledge category, and reflected a concentration in the teacher tasks that manifested in explaining or telling during instruction. Raw counts of each Teacher Task utilized and Teacher Knowledge accessed reflected in the data recorded is discussed as it reflected teachers' knowledge of algebra for teaching, impacted their planning, and translated to instruction in the following sections involving individual case studies and cross-case analysis.

Table 4. Counts of Teacher Tasks vs. Teacher Knowledge.

Tasks of Teaching	Categories of Knowledge					
	Core Content	Representations	Content Trajectories	Applications and Context	Language and Conventions	Reasoning and Proof
Analyze Students' Work and Thinking	22	4	6	2	3	4
Design, Modify, Select Mathematical Tasks	15	4	10	1	2	1
Establish and Revise Mathematical Goals	20	2	8	2	5	0
Access, Use Tools, Resources for Teaching	3	0	1	0	1	0
Explain Ideas, Solve Mathematical Problems	16	3	3	2	9	0
Build and Support Mathematical Community	5	1	1	3	3	3
Totals	81	14	29	10	23	8

These results initially suggested a tendency by the classroom teachers to use processes or instructional strategies that were expedient to ensure the students' familiarity with core content required by the curriculum. That tendency is further discussed in the case analyses that follow.

Second Level Coding: Coding to KAT Themes

During second level coding the transcripts of classroom observations and interviews were read again and, in this case, examined for the overarching themes described in the KAT Matrix; *Bridging, Trimming, and Compressing*. The presence of these overarching concepts in both the planning and implementation phases added depth to the understanding of the cognitive processes employed by the teachers because they provided a window of opportunity for insight into the application of the Teaching, School, and Advanced Mathematical Knowledge possessed by the two classroom teachers. These actions were sometimes based on conscious decisions, intended to clarify a concept, explain a language convention, or provide a basis for future connections both within the algebra classroom and across the mathematics curriculum. Interestingly, they also occurred spontaneously illustrating a richness of teacher knowledge that underpinned the classroom planning and instruction even though they were transparent to the students. The three overarching concepts that served as the basis for the second level coding effort required a contextual analysis of the observational and interview data to determine the intent behind the teacher actions throughout planning and instruction.

This coding effort also served as a validation of earlier coding since it required examining for a second time the teacher actions in context to determine the presence of the teacher's effort to clarify or connect the instruction to some other topic or concept across the discipline. Additionally, teachers sometimes used the opportunities to extend the concept or procedure, or to decompress the instruction, adding depth to the discussion. The raw counts of the occurrences for each of these overarching concepts is shown in Table 5 below.

Table 5. Use of Overarching Themes

Overarching Themes	Janice	Dean
Bridging	14	21
Trimming	8	9
Decompressing	6	10

Although this table provides an overview of how the subject teachers employed these connecting, explanatory, or extending actions, further analysis of when and where they manifested provided a much clearer picture of the intersection of the teachers' knowledge of algebra for teaching and their actions in the classroom to meet the needs of their students during both the planning and the instruction phases of the unit. The density of use for these elucidating behaviors within the context of the KAT Matrix gave insight into both the underlying motivation for teacher actions in the classroom and the depth of the teacher's knowledge across the topics and across the discipline. These issues and their contribution to teacher actions during either planning or instruction will be discussed in more detail below during cross-case analysis.

Open Coding: Emergent Themes

Initially, observations and interviews fit nicely into the established coding scheme. During subsequent readings of follow-up interviews, however, two additional themes emerged which were important factors in developing a more complete understanding of teacher motivations and priorities that guided their planning and instruction. First among these was the observation of very short wait-times during the course of instruction. A number of studies emphasize the critical need for productive

questioning techniques (Cashin, 1995; Mckewan, 2004; Pressley, El-Dinary, Gaskins, et al., 1992) and the importance of wait-time (Cornell Center for Teaching Excellence, retrieved January 2016; Fletcher-Wood, 2013; Fries-Gaither, 2008; Kelly, 2016) as a means of encouraging cognitive struggle and development of sound reasoning. In both cases, the subject teachers missed opportunities to clarify misunderstanding and provide a forum for increased analysis by students before moving forward in the instructional plan. Based on the guidelines in the studies mentioned above, for the purposes of this research, an intentional decision was made to classify wait-time not provided as occurring when the subject teachers allowed less than three seconds before providing the answer or moving on to the next step in a process. For example, when Dean was moving from one level of complexity to another in solving factorable quadratic equations, he stated without pause,

“Negative three and positive two. So you did all that in your head about signs and zeroes. Okay, cool. Now we have an equation that looks like this, don't we?”

and Janice, struggling with a graphing calculator projected for the class said,

“It's not an ordered pair. This is negative one, positive one up here. I'm not crossing the X axis up there. Don't give it to me in parenthesis. Just give me the two answers. If you want something around it, do your little braces that we do. Otherwise, just write your two numbers. All we're looking for is the place where the graph crosses the X axis. Now hopefully I wrote the second one right. Are we following? I have my daughter's calculator and she has it set to shut-off very quickly. So it's gonna shut off a lot.”

Then, she moved quickly to a different example. In both cases, student questions remained, but were subjugated to address the need to maintain the instructional flow so

that all of the learning objectives would be met during this period. These instances were coded as (*Wait-Time Not Provided*) for further investigation.

Secondly, there was an inordinate number of references to specific preparation for high stakes, end of course testing as a primary motivator during both active instruction and follow-up interviews. For example, during instruction on calculator use for solving quadratic equations by graphical means, Janice told her students,

“It's the first part of your TCAP, only the problems are gonna be much more difficult than what you saw on the TCAP test. So what we're doing this week is I'm making sure that I'm covering the material that I would expect, that you can easily see.”

In the previous comment, the standardized, end of course test known as TNReady was being addressed. On the other hand, Dean, whose students were not subject to standardized tests, still used language and procedures that alluded to test preparation saying about the students' mindset,

“I missed points that should count against me, small points mind you. Maybe I should know this for the test. So by giving them a two point quiz, a small quiz, requires them to know the formula and to use it in two very easy problems, and they don't, they get a zero out of three. Okay, that might, often times encourage them to learn it for the test.”

The impact of this issue became apparent during post-lesson interviews when the subject teachers were recounting their rationale for taking certain actions in class during their lecture and activities. Testing issue occurrences were coded as (*Testing Issues*) for future reference.

These two emergent themes, summarized in table form below, added an unexpected dimension to the individual case studies and cross-case analysis. They each provided a rich source for further developing the motivations for specific planning and instructional decisions and the cognitive processes teachers utilize in the evolving classroom climate influenced by factors both inside and outside their purview. The impact of these emergent themes will be discussed in more detail in the Individual Case Studies and the Cross-Case analysis.

Table 6. Emergent Themes

Emergent Themes	Janice	Dean
Wait-Time Not Provided (< 3 seconds)	21	20
Concessions to Testing Schedule	89	0

Field notes and artifacts collected during research were analyzed to extract further evidence of teacher actions and associated mathematical knowledge related to elements of the KAT Matrix. These additional data were added to previous occurrences, and are reflected cumulatively in Table 7.

Table 7 . Teacher Tasks vs. Knowledge Categories First Level Coding

Teacher Tasks and Knowledge		Janice	Dean
<i>Teacher Tasks</i>	<i>Code</i>		
Analyze Students' Work and Thinking	ASW	19	23
Design, Modify, Select Mathematical Tasks	DMT	18	15
Establish and Revise Mathematical Goals	ERG	18	20
Access, Use Tools, Resources for Teaching	ATT	3	3
Explain Ideas, Solve Mathematical Problems	ESP	14	19
Build and Support Mathematical Community	BCD	4	12
<i>Knowledge</i>			
Core Content	CC	43	40
Representations	REP	4	11
Content Trajectories	CT	13	16
Applications and Contexts	AC	4	6
Language and Conventions	LC	11	12
Reasoning and Proof	RP	1	7

In this table an initial cumulative count reflecting first level coding illustrates the teacher tasks each teacher actually utilized during planning and instructional activities coupled with the knowledge accessed by categories during these same activities. As the

table indicates, the greatest number of tasks utilized were in performing actions that generally involved direct instruction or explaining procedures and routines. That type of engagement with students in a telling or explanatory mode was indicative of the preponderance of teacher tasks utilized, either in the planning considerations or the implementation of that plan in the classroom. Specifically, teacher tasks and intentional efforts to access KAT occurred when there was a direct interaction with the student in an effort to clarify or emphasize creation of a routine or standard practice for solving quadratic equations. This concentration of tasks reinforced the observation that the focus of both subject teachers was on actions that explained the content or use of a procedure rather than seeking to deepen conceptual understanding or encourage development of reasoning skills. Teachers demonstrated adaptability and flexibility to meet the evolving needs of their students by making numerous clarifying explanations, using additional representations, and revising the goals they established for the classroom instruction. Further, knowledge accessed was heavily weighted, not surprisingly, in the area of teacher's content knowledge, followed by higher occurrences in utilization of content trajectory to make connections between previous instruction, current topics, and future application of algorithms and processes in the current unit of instruction.

Individual Case Studies

In the case study presentations below, an explanatory narrative is utilized to develop a portrayal of the fluidity that existed in the subject teachers' interactions between knowledge, planning activities, and implementation of instruction in the classroom. Because this unit of instruction occurred in the ongoing stream of topics in this first course in algebra, many of the elements of planning were already in place based

on curriculum guidelines or previous introductory lessons on related topics such as graphing linear equations or factoring expressions. This was at the same time beneficial and counter-productive. The classroom teachers had established knowledge structures on which to build, but also tended to rely too heavily on the belief that students retained that knowledge as preparation for the current unit on quadratic equations.

Following the two individual cases, the research objectives will be addressed for each teacher, and again following a cross-case analysis. In the cross-case analysis, the actions, decisions, and adjustments made by the two teachers are analyzed and discussed related to their KAT. Finally, a comparative and cumulative discussion of the research objectives is included to summarize findings related to each question, as well as address the efficacy of the research effort in elucidating the questions posed.

Case Study One: Janice

My first subject teacher, “Janice”, is employed at a public school with a student population of just over two thousand students located in a rapidly growing suburban community. She has been teaching for a total of 10 years at the same high school with a break of two years earlier in her tenure. The school district is growing rapidly, with a new high school planned for next year. Janice’s school is currently overcrowded resulting in large classrooms and strained resources. The school website reports a student to teacher ratio of 17:1, but the actual ratio in practice is closer to 33:1, as indicated by interviews with several teachers. The student population is 87.1% Caucasian, well above state average, with 2.7% Hispanic, 1.7% Asian, and 7.8% African American, all well below the averages for Tennessee schools. Students eligible for free or reduced lunches comprise only 14% of the population compared to a statewide average of 49%. Median

income for families in this community is over \$72,000. With a graduation rate of 98% (state average 97%), this school scored in the top 10% in both reading and math scores on the TN Ready (End of Course) Standardized Tests. Janice has a bachelor's degree in mathematics with a specialty in Secondary Education. During her 10 years of teaching, Janice has taught only remedial algebra or initial algebra courses. Janice had one methods course in her bachelor's program of study.

Teaching Style

Janice employed a lecture and demonstration model as her teaching style. The typical classes I observed were essentially divided in thirds with homework review, direct instruction of new material, and time to work on homework assigned relatively evenly split between the 90 minutes allocated for the block of instruction. Janice always began with a warm-up exercise, and then reviewed answers to homework problems. She transitioned to instruction on new ideas by introducing the sections of the text that were going to be covered during that lesson. The questioning strategies Janice employed utilized factual recall, recognition of procedural efficacy, and neglected the use of open-ended questions designed to probe students' depth of understanding. Generally, Janice would present a new idea using example problems on the board, then work through those problems with the class while explaining the relevant concepts or procedures and making connections with previous learning objectives. For example, when introducing the method of solving quadratic equations by graphing, she was careful to review how they previously discovered they were able to find the axis of symmetry using the coefficients of the equation, then substitution of those numerical values.

One incidence during this introductory period illustrated how Janice sometimes showed a lack of flexibility and inability to adapt quickly in the classroom. Her initial example involved graphing the equation $2x^2 - 5x + 3 = 0$ using a graphing utility projected on a screen in the front of the room. As she proceeded, she remarked,

So typing in two X squared minus five X plus three. And the first one I put up here does not work the way it's supposed to work.

Messed up from the very beginning.

Well this was the equation. Now let me show you what I did wrong. What I would be looking for... if this is, if this is my equation, I'm looking for these two points. The reason I'm struggling here is I can't figure out what these two points are; I wrote it wrong. It does work. You do cross the X axis. The problem is you can't figure out where that is. It's not a good problem for me to give you.

Janice moved to her next example transitioning by saying,

Two X squared minus two and see if it gives you nicer points on where we cross the X axis, which is what happens when you copy the problem wrong.

Here Janice did not realize she could factor the equation as $(2x - 3)(x - 1) = 0$ and missed an opportunity to lay a foundation for the method of solution by factoring which was scheduled for the next lesson. This incident was one of several that indicated a potential deficit in her deeper content knowledge in the area of quadratic equations. This knowledge deficit could be observed in her dependence on establishing standard routines and procedures that conveyed a great deal of basic content knowledge to the students but simultaneously omitted explanation of underlying logic. Janice appeared more confident

and comfortable when routinely accessing this fundamental content knowledge while concurrently avoiding opportunities to construct conceptual understanding. Compared to other mathematics teachers with a math degree, Janice scored nearly one standard deviation (-.893) below the mean score in the area of Knowledge of School Mathematics; knowledge of the topics typically covered in an Algebra 1 course. It is likely that result was a good indicator of Janice's inconsistency in being able to quickly and accurately recognize and correct errors during classroom instruction. On the other hand, some of these difficulties could perhaps be attributed to the alternative curriculum since Janice and her colleagues were implementing a new, untried one-semester curriculum that was modified routinely during area meetings held at regular intervals throughout the semester. This particular occurrence was, however, also a missed opportunity to use that overarching skill of *decompressing* because Janice failed to unpack the connection between locating the x-intercepts by graphing and finding solutions to a quadratic equation that generated those same function intercepts. The notion of decompressing, in the KAT Framework and in this research, includes intentional efforts to expand the student's understanding of a procedure and make connections between the stepwise process and the underlying logic. Janice had indicated this connection as an objective in her pre-instruction plan.

After initially struggling with this problem, the discussion shows Janice recovered and successfully showed how to utilize the graphing utility to find the x-intercepts for a simple quadratic equation, but the opportunity to enrich and deepen the students' learning was lost.

Procedural versus Conceptual Orientation

Janice typically employed an instructional approach intended to teach the students a procedure or process which would lead to a correct answer without emphasis on creating a pathway for students to understand the conceptual underpinning. During homework review and introduction of new concepts or procedures, her questions and responses were rapid fire, essentially verbalizing correct answers and reminding students of previously covered algorithms or conventions. Throughout the unit of instruction on solving quadratic equations, Janice was always very engaged with her students, but it was often a one-way conversation. Janice did a lot of explaining without an exchange of ideas between teacher and students. In one exchange, Janice covered three problems in succession as follows:

Janice: So from my zero, zero left four up three is my point. That's my beginning of my V. From there, down three over one and down three over one the other way. From those points down three over one each way. So the three just makes the V skinnier. [Inaudible], are you paying attention as I reteach yesterday when you weren't here? Okay, same rules apply. So you have your basic square root graph on number five. Again that two is on the inside. It's gonna move right two. Six is gonna move left three. This means I'm moving what?

Student: Left one.

Janice: Left one and?

Student: Up one.

Janice: Okay. Same rules apply. If the number's underneath the square root, then it's considered inside and that's the right or left. And if the number's on the outside, that's up or down. And it's always gonna go that same direction. Any questions? Gotta ask, if you don't understand you can't sit by and not ask. That's how you fall way behind. Yes.

Student: Like, number nineteen, the way you just did it. You just did, you took the number from the square and moved it outside.

Janice: Yeah.

Role as Teacher

Looking at the teacher tasks most often observed in class, and those that were manifested during interviews, there is a concentration in those tasks that involve some action on the teacher's part to directly interact with the student in an explanatory mode. In the table below, the four tasks with the most observed occurrences indicated interactions when Janice asked questions, then clarified problem answers, or corrected student mistakes during the course of direct instruction or reflection on that instruction that occurred in class. As demonstrated in the above exchange, Janice was motivated to cover all the planned material for the lesson, and she sometimes accomplished this objective at the expense of nurturing a learning climate that emphasized the continuity of algebraic concepts over the entire course. This accumulation of teacher tasks listed below showed far less effort to build a mathematical community than to explain material and pursue instructional goals and objectives.

Table 8. Teacher Tasks Used by Janice

Teacher Tasks and Knowledge		Janice
<i>Teacher Tasks</i>		
Analyze Students' Work and Thinking	ASW	19
Design, Modify, Select Mathematical Tasks	DMT	18
Establish and Revise Mathematical Goals	ERG	18
Access, Use Tools, Resources for Teaching	ATT	3
Explain Ideas, Solve Mathematical Problems	ESP	14
Build and Support mathematical Community	BCD	4
<i>Knowledge</i>		
Core Content	CC	42
Representations	REP	3
Content Trajectories	CT	13
Applications and Contexts	AC	4
Language and Conventions	LC	11
Reasoning and Proof	RP	1

As the numbers in the table above indicate, Janice focused heavily on accessing core content knowledge as she utilized the teacher tasks of teaching. That reliance on the factual and procedural aspects of algebraic instruction translated into less emphasis on the long-term application of the solution methods (*decompressing*), connections across the discipline (*bridging*), and higher order thinking on the part of her students (*trimming*). As

a result, students spent most of their time repeating learned facts or processes and much less time reasoning or synthesizing solutions to more challenging, non-standard problems, or in generalizing these approaches to a broader range of problem types.

Janice’s regular dependence on routine procedures and algorithms that followed a specified list of actions was indicative of how she viewed her role as a teacher. One peculiar example of this dependence was the strategy she advocated for factoring a quadratic equation, which contained a quadratic term with something other than a unit leading coefficient. She called this technique the “Bottoms Up” method. In this classroom example, Janice walked the students through a technique to arrive at a factored form of a trinomial that could be used to solve the equation using the zero product property.

The Bottoms Up Method.

Janice: Okay, we have the equation $5x^2 + 11x + 2 = 0$. Now, here are the steps to use to factor.

Step	Result
1.) Multiply the first coefficient by the last number.	$5 \bullet 2 = 10$
2.) Write the factors of that result.	$10 = 1 \bullet 10, 5 \bullet 2$
3.) Circle the pair that add to the middle term.	$1 \bullet 10$
4.) Make two fractions with the circled pair and the first coefficient.	$\frac{1}{5}, \frac{10}{5}$
5.) Reduce these fractions; my letters always go on the bottom.	$\frac{1}{5x}, \frac{2}{x}$
6.) Answers read “bottoms up” Now you have the factors for your equation.	$(5x + 1)(x + 2) = 0.$

Although this is certainly the correct factorization for the quadratic equation, it has no (apparent) connection to the concept of factoring any higher order equation, nor the fundamental ideas behind factorization of a quadratic expression. When I queried Janice about this algorithm, she said she had learned it from another algebra teacher, and further explained,

I know it doesn't tell them what they're doing, but they just have to know how to do this factoring. I would never do this in my algebra 1A / 1B class (taught over two semesters to lower level students).

This behavior and simplistic algorithms continued throughout my observations with similar justification each time. Janice's reliance on procedural behavior left little doubt she wanted her students to be successful, but when she was pressed on whether these same students were prepared for Algebra 2 based on this instruction she agreed they were completely unprepared for the concepts and curriculum they would face next year.

Janice's view of her role as a teacher was enacted through and driven by Assessment-Expectation. It was clear that Janice had a personal interest and investment with her students. She constantly engaged them in conversation and spent time answering their questions during the time allocated for homework. Her concern manifested primarily through this personal interaction on the surface, but also through her sometimes frenetic efforts to ensure any concept or problem type she anticipated would appear on the TNReady test was sufficiently addressed during some period of instruction prior to the testing date. Janice's concern with the students' preparation for the test was so prevalent, it began to blur the distinction between her internal objective to teach core content to mastery and her motivation to ensure her students were able to attain some

measure of success through applying a previously advocated procedure or method to arrive at an answer. Janice encouraged her students to continually rehearse these procedures and practice using their calculators. After admonishing the students to practice these techniques in the days immediately before the TNReady test, she explained:

Janice: It's the first part of your TCAP, only the problems are gonna be much more difficult than what you saw on the TCAP test. So what we're doing this week is I'm making sure that I'm covering the material that I would expect, that you can easily see.

And further during my follow up interview,

Well, I'm scrambling to cover as much as I can before this test on Monday. I'm teaching too much again.

These excerpts demonstrate two important emergent themes in the research; wait-time not provided and instruction motivated by concerns over performance on end of course testing. The students were at this point familiar with the testing requirements that had been imposed over the past two years, and were to a large degree, complicit in the preparation plan because the test served as a portion of their course grade. In the initial exchange above, Janice missed a number of opportunities to probe students' understanding by asking leading or open-ended questions. It became apparent she felt pressed for time and that her objective for instruction was oriented primarily toward preparing the students for their upcoming standardized tests. Janice mentioned the pending test in various contexts on many occasions ranging from classroom instruction to post-instruction interviews. In the graphic below, the accumulated totals of wait-time

opportunities missed, and overt reference to the TNReady test show an inordinate recurrence of these factors in her classroom planning and instruction.

Table 9. Emergent Themes Used by Janice

Emergent Themes	Janice
Wait-Time Not Provided (< 3 seconds)	21
Concessions to Testing Schedule	89

The sheer number of references to preparation for the standardized tests created a confounding factor in trying to realistically analyze Janice's use of her own knowledge of algebra for teaching as she moved through her instructional plan. Janice admitted she would not teach this unit on solving quadratic equations as she did during my observation if she were given the latitude to plan and instruct in a manner that would lead to a mastery learning experience. In our post-unit interview, she said,

Because I really do. I mean, everything I do is based on what do I need, what do they need to do to be successful on that test. When I think about it, over time it's slowly gone that way and I think it's just more and more and more that we're all worried about the test. And that has dictated more and more and more. This would not have happened if it wasn't for testing. I would've taught it if I'd had time to actually teach it. It is, I mean, I would love to say well how dare you do something like that. Well, when my job and my pay is depending on it, then I have to do it.

So we had to give two days to that. Well, the problem was, at that point, we didn't have covered enough of it for it to be beneficial.

So I didn't bring quadratic formula in at the time due to the fact that in the past year's testing it has not been an issue, and... Now, had I not been in the situation where I am with the time constraint then I would think it important, but otherwise I feel like they could get the answers that they need without having that day of learning quadratic formula; Very much revolves around the test.

Evidence of Bridging, Trimming, and Decompressing

Janice did attempt to utilize bridging, trimming, and decompressing skills as she was able in an effort to either elucidate difficult ideas during instruction, or make some connection between previous lessons, current instruction, and future applications. Looking at the use of these deeper, overarching concepts provide an opportunity to examine how she accessed her knowledge for teaching algebra and applied it in her teacher tasks. When we discussed her instructional plan in our pre-instruction interview, and I asked about her expectations based on her students' abilities, she replied

Their graphing is the easiest because they're doing it on the calculator. They actually learn square roots really well. Square roots is easy. They don't even have to be quite as focused with the calculator on that. Um, factoring's probably the next hardest one, but that's because they don't learn factoring very well, not so much the method of zero product property. It's the beforehand that they struggle with, therefore they struggle when you have to do it in the beginning.

And on the quadratic formula,

There won't be time. It won't be necessary. We'll just let it go.

Now, had I not been in the situation where I am with the time constraint then I would think it important, but otherwise I

feel like they could get the answers that they need without having that day of learning quadratic formula.

Here, Janice demonstrated her authentic knowledge of the students' proficiency levels in planning to *trim* the content to meet their needs while still remaining true to the methods of solving quadratic equations. These *trimming* decisions demonstrated a thoughtful and purposeful course of action based on both knowledge of the way her students learned and on content trajectories. She also showed a breadth of knowledge of content in future courses (*bridging*) when she noted there wasn't a critical need for introducing the quadratic formula because they would encounter that solution method in later mathematics instruction.

Janice frequently attempted to make a connection between previous learning and current instruction (*bridging*). When she began the lesson on the factoring method of solution, she asked the class

...you're solving it you're looking for the place where the graph crosses the X axis. What would that be called"?

as a means of *bridging* the instruction with the lesson of the previous day. As she transitioned from graphing to factoring as a solution method, Janice asked the class

We have done this before, and where did I tell you when it said equal to zero that your solutions are? Where are your solutions of a quadratic?

These questioning techniques illustrated Janice's effort to create a thread of continuity through the instruction on solving quadratic equations by *bridging* her instruction with previous knowledge. In utilizing these methods, Janice offered an

opportunity to see her knowledge of algebra for teaching weaving together her teaching tasks, content knowledge and knowledge of students' needs. Janice was regularly observed making an intentional effort to bridge ideas and processes to tie together the common algorithms, procedures, and language conventions the class had learned for obtaining solutions to quadratic equations. She did not, however, use *bridging* to create connections between the first course in algebra and other courses in the discipline.

Because Janice understood her students' capabilities, she made repeated efforts to break down, or *decompress*, concepts or ideas that she anticipated were difficult for her class to comprehend. For example, in her second lesson on factoring trinomials, she led the class through several examples while demonstrating how to change non-standard problems into a workable form as seen in the following exchange.

Janice: Okay, there are two types of trinomial methods. First, if you have the leading coefficient equal to 1, so there is really no number there, and second if you have a number in front so you have to use Bottom's UP. We know our answer is going to end up with two binomials, so we use like a backwards FOIL approach. Factoring is like reverse multiplication. So we have $x^2 - 2x - 8 = 0$, and we see we can factor that as $(x - 4)(x + 2)$ and we just ignore the zero. But what if we have

$$x^2 + 7x + 14 = 2?$$

We have to get it to where it equals zero, so we subtract 2 from both sides and we've got

$$x^2 + 7x + 12 = 0, \text{ which we can now factor.}$$

When she began instruction on solving by factoring, she referred back to that lesson as a bridge to the application of the zero product property by saying:

Janice: Add the A four to the other side, and then what do you do with the equal zero?

Student: [Inaudible]

Janice: Okay, I told you to just throw it out, right?

Student: Yes.

Janice: Okay, what you don't know is what you're really supposed to do.

Student: What, you keep it?

Janice: And that's where today's method comes in.

Student: [Inaudible]

Janice: Wait, I'm going to teach you. That's where today's comes in. In my opinion, the book covered way too much stuff in that one lesson because they wanted to cover how to factor everything and do today's all at once, and we've done it in, what, two weeks now. Okay, I think that was way too fast. So, all I could do at that point was tell you, okay, do this just so you could work the problem and drop the equal zero. Well, now we need to figure out what the equal zero means, okay.

In many cases, Janice showed awareness of the difficulty her students had with algorithms that were initially abstract. Her intentional use of language that simplified the application of the underlying logic was indicative of her focus on providing a pathway that allowed the students to develop a procedure to solve the problem. This attempt at *decompressing* of ideas that Janice routinely employed demonstrated awareness of the challenges students face with new and abstract mathematical procedures. Janice was, unfortunately, not able to unpack the content in ways that connected the conceptual underpinning to the procedural implementation. As an example, when trying to factor a

trinomial with a leading coefficient greater than one, Janice helped her students relate the process to removing a GCF by using this clarifying suggestion:

The idea is if you've still got something common inside of it, then you haven't gone far enough. Chances are, had you gone on with that one, had the five X squared out from, you would've seen the two binomials are the same, and you would've known what to do.

In another instance, using another acronym familiar to the class for the difference of squares, Janice helps the students realize the factoring process can involve additional steps as shown in the following exchange.

Janice: Okay, number 7: $8x^3 + 12x^2 - 2x - 3$.

What method is this?

Student: GCF.

Janice: It's grouping, but when you factor by grouping one of your binomials left is a difference of two squares. So you take it a step further. So that leaves you with three different things. (We have) $(2x + 3)(4x^2 - 1)$.

Okay, let me show you. This, $4x^2 - 1$, is the difference of two squares. It does not happen this way very often, but you have to be aware that when you start factoring, you have to be [inaudible] to go as far as you can go. Just because you factor one way, doesn't mean you're done. So we have to do DOTS method here.

The $4x^2$ is?

$2x$ and $2x$ The 1 is? Square root it... one and one. And the signs are?

Student: Positive, negative.

Janice: And we still have the other one to bring down. (So we have)

$$(2x + 3)(2x + 1)(2x - 1).$$

Overall, Janice interpreted the teaching skills of *bridging*, *trimming*, and *decompressing* primarily as tools to encourage students to recall factual information, clarify difficult procedures for students, and facilitate student learning that would lead to finding an appropriate answer to the problems presented in class or on an assessment. As previously noted, the intentional actions used by Janice in her instruction were directed at establishing a repeatable procedure that would result in a solution to standard quadratic equations. Field observations and follow-up interviews indicate that Janice had a deeper understanding of the overarching themes of bridging, trimming, and decompressing than these actions would indicate. Janice held a belief that her students were not capable of conceptually understanding the logic underlying the solution methods. The limitations of the students' proficiency levels inhibited the development of that conceptual understanding. Janice's efforts demonstrated the inherent complexity, and flexibility, of the act of teaching because she realized any attempt to provide exposure to the conceptual underpinnings would sacrifice the students' potential success at the expense of instruction not accessible to the class. Based on the KAT results, Janice possessed a knowledge of teaching techniques and student learning that was nearly the same as the scores of those with the same *experience level*. This was in fact her strongest area of knowledge of algebra for teaching as she scored just marginally below the mean score on the KAT Assessment. Her relative strength in this area, compared with the deficiencies in the other teaching dimensions, was apparent in her assessment of student strengths and weaknesses that translated into direct instruction tailored to meet the proficiency levels of the class.

That direct instruction primarily manifested as repeated focus on procedural competency as a proxy for success in utilizing a solution method. The inevitable consequence of this approach was that the goal of immediate success through procedural proficiency in the short term replaced long term mastery and conceptual understanding resulting in poor preparation for future mathematical work. During the post-unit interview, Janice readily admitted that the focus on common routines during the unit of instruction would result in students that were ill prepared for topics in Algebra 2 when she replied;

Researcher: How much of an impact do you think that has on their Algebra II course?

Janice: Well I think it has a significant impact, especially this semester being worse than ever. Because they just didn't get the time they needed to get to learn the different things. If I had hit, got to factoring on November the 2nd and I still had the rest of till December the 18th, ...I mean, they could learn some big stuff in that. But that's just not what happens. So it has a significant impact. Negative impact on their future learning of what they can learn and succeed in.

The constraints of curriculum guidelines and time had a major impact on Janice's planning and instructional actions including the focus on procedural proficiency. This exchange illustrates her perspective given the current climate.

Researcher: So if you were on a scale of one to five, one being no latitude and five being you can do whatever you want to do when you came into class, where would you say you're at?

Janice: About a one.

Researcher: Given complete autonomy, how would your instruction change for solving quadratic equation?

Janice: Well, there would definitely be a lot more time. I would make sure that they really knew how to factor for real, and could do it well. And then we could do that method and we could do graphing and we'd do square roots. We'd do all of it then. And we would spend several days in time...

Um, we would be able to spend some days on just, let's understand what this is all about. Let's understand, let's pick the method that works best for the different type of problem. Some methods work well for this, some work for that, and I would have time to really show them how they knew what method works well for what type problem.

I never thought about completing the square until this year because technically it has become an Algebra I standard this year that they just threw in. Before that, I would've never thought about it, but yes now, oh I would definitely do quadratic formula. And yeah, I think I probably would do completing the square. Given the opportunity and the time that I would need to teach it well, because it would take more than one day for them to do completing the square and understand it. Um, so given all that, and given motivated students that wanted to learn.

And in response to a question regarding the constricted time frame:

Teacher: Oh I think it'd be a huge difference. It was a huge difference, I mean, I felt like three or four years ago, they were much better prepared for Algebra II than they are now.

Janice was working in a difficult climate, pressured to achieve proficient scores on the end of course standardized test and compelled to teach a first course in algebra in a constricted time frame due to schoolwide schedule changes. Janice indicated in post-unit interviews that she believed her plan was inadequate to prepare her students for later mathematical work, but she also felt she was unable to make appropriate adjustments due to impending test dates, low student algebraic skill levels, and demands of the curriculum guides. She had levels of knowledge in the core topics of algebra, knowledge of teaching strategies, and advanced mathematical knowledge that were lower than the mean scores for in-service teachers of the same experience level and those with similar educational backgrounds and degrees as measured through the KAT Assessment Tool. Janice was fervent in her effort to educate her students, but sometimes had difficulty with adaptation during class when a mathematical challenge arose in part because she felt a strong need to cover all the planned material in her lesson plan. She understood and employed her own mathematical knowledge and the ability to bridge, trim, and decompress material to present algebra in a comprehensible form to her students, yet these actions were confined to the immediate topics at hand and did not make connections across the discipline or the wider curriculum. Janice's activity level was often frenetic but always compassionate and exercised in the context of her deep knowledge of her students' academic needs.

Applications to Research Objectives: Janice

Research Question 1. What knowledge of algebra for teaching do teachers **possess** for teaching a first course in algebra?

Recall that Janice had a lower level of knowledge in three dimensions of KAT (Teaching Knowledge, Knowledge of School Mathematics, and Advanced Knowledge of

Mathematics) than teachers with the same experience level and also those with a college major in mathematics. When addressing the first research question about the level of knowledge of algebra for teaching Janice has for teaching a first course in algebra, it was clear Janice had strengths and weaknesses in all three of these dimensions. Her Teaching Knowledge enabled her to assess the skill levels and learning styles of the students, but proved to be a limiting factor when she presented instruction that was consistently at a lower cognitive demand level thus preventing knowledge construction at mastery levels. Janice's Knowledge of School Mathematics, or content knowledge, placed limitations on the instruction she presented. For example, Janice made mistakes when using classroom examples at times, and she relied heavily on procedural instruction and learning. Finally, her Advanced Mathematical Knowledge was far below her peers in the same experience and college major categories. This limitation prohibited her ability to make good connections outside the narrow focus of graphing and factoring quadratic equations.

Research Question 2. What knowledge of algebra for teaching do teachers draw upon when **preparing for a unit of instruction** on solving quadratic equations?

In the realm of Teaching Knowledge, Janice demonstrated an understanding of the way her students learn and their proficiency levels in algebraic skills and mathematical procedures in general. Throughout the interview and observation process, Janice illustrated this understanding by actively planning for instructional strategies that would meet the perceived needs of her students to develop and utilize routine procedures, and by tailoring instructional plans and activities to make the topics accessible. In her unit design, Janice determined which methods of solution would give her students the best chance for success, and she made intentional moves to create those opportunities for

that success. On the other hand, Janice's limited Knowledge of School Mathematics and Advanced Knowledge of Mathematics was evident in her planning efforts as she considered only procedures and processes that would provide a stepwise algorithm to achieve a correct solution without regard for the underlying logic. As an example, Janice used terms like "Bottom's Up" or "DOTS" both in the interview process and in classroom discussions to describe factoring techniques for trinomials and difference of squares quadratic equations. The more far-reaching effect of this lower knowledge level is that Janice did not access her own knowledge of applications, contexts, or reasoning and proof to enhance the learning experience of her students as they move forward in the mathematics curriculum.

Research Question 3. What knowledge of algebra for teaching is utilized during **instruction** in a unit on solving quadratic equations?

Throughout the unit of instruction Janice accessed her knowledge of students learning and misconceptions, and her content knowledge, formally Teaching Knowledge and Knowledge of School Mathematics. Janice almost exclusively utilized telling and explaining actions during instruction that centered on her own knowledge of content for algebra topics. Her held beliefs about the proficiency levels of her students dictated the instructional strategies utilized for classroom instruction. In making her students' success on assessment her top priority, Janice offered an opportunity to see her knowledge of algebra for teaching weaving together her teaching tasks, content knowledge and knowledge of students' needs. Although the KAT accessed during instruction was limited in its breadth, Janice was able to incorporate her understanding of content to *bridge*, *trim*, and *decompress* the solution methods of graphing and factoring quadratic equations in a

manner that made them comprehensible to her students. Janice's use of these overarching concepts were viewed through her continuing attempts to make the solution methods of solution for quadratic equations tenable for her students through *bridging* only the essential procedural content across the unit, *trimming* the solution methods to reflect primarily the necessary standard routines to achieve a solution, and *decompressing* the connective elements between solution methods by parsing the instruction and compartmentalizing procedures. Janice regularly accessed her Knowledge of Teaching, guided by her content knowledge of solution procedures, to inform her efforts to reformulate and adapt her instruction to meet what she perceived as her students' most critical need; the need for achieving success on assessment through finding a solution for a given quadratic equation.

Individual Case Study Dean

The second subject teacher in this multiple case study, "Dean", is employed at an all-boys private school in an urban setting. The student population consists of grades 7 – 12 and numbers 720 students total. Dean has taught for 13 years at three different schools and is finishing his fifth year in his current position. The demographic composition of the population is 90% Caucasian and 5% each Asian and African American. The student: teacher ratio is 14:1 in all classes. The school has a 100% graduation rate with all students matriculating to four-year post-secondary institutions upon graduation. Scores on the ACT Test average 5 – 6 points above national averages, and there were 18 National Merit Finalists named at the school this year. Dean is currently teaching both Geometry and Algebra 1, but has also taught Calculus and AP Calculus in previous years.

At this school, Final Exams are administered for each course, but there is no external standardized testing.

Teaching Style

Dean's approach to instruction on solving quadratic equations was similar to Janice's approach in its essence. He felt the primary methods that were most accessible and useful to his students were the graphing method and factoring, implementing the square root method as a transition from graphing to factoring methods. It was evident in his preparation and instructional strategies that Dean had a depth of knowledge of the subject matter as well as knowledge of student learning. When asked about his plans for the unit of instruction Dean said:

I'm gonna start with the solutions by graphing. We will first just look at the graph and see where the points on the graph are. We all know the positives to that. It's simple, it's easy, it's visual and it's hard to not see where it crosses. Drawback though is it's not as accurate, and it gives us points that we may not be able to pick off the graph if it's not passing through lattice points. We'll talk about that and I'll teach them how to graph then factor, and then depending on how the students are doing I might take them to complete the square. But mostly I want them to understand what is happening in Algebra I and do some very basic work with it. So I may not even get them to the quadratic formula.

Dean emphasized the connectivity between the visual nature of graphing and the recognition of solutions as points of intersection with the x-axis when using the factoring method of solution. There was a dichotomy in Dean's preferred teaching approach and his perception of the proficiency levels of his students. Because of the high academic

expectations at the school, and the rigorous academic curriculum, Dean held a belief that his students were capable of high level thinking and synthesis within the context of classroom instruction. When pressed on the reasoning for this held belief, Dean later opined that even though he felt the class should be able to achieve mastery of the topic, many of the students in his regular classes did not possess the requisite skills to perform basic algebraic operations. He stressed that it was his intention to use an open-ended questioning and discovery strategy when introducing the idea of finding solutions to quadratic equations, saying,

Um, I do a lot of discovery, you know. So call it, you know, Socratic method of rather than me telling them how to do it or guide them, just give them maybe some worksheet and say, why don't you try putting these numbers in and see what you get, and see what happens. See what kind of shape it makes, see how the numeracy of the equations work before I start just graphing them for themselves. So, instead of starting with graphing, I'll have them do that on their own.

But he later observed that based on his experience and observation,

I feel that most students that I have in Algebra I have not been encouraged to understand what they're actually doing. They've been encouraged just to do, go through rote steps. And come to an answer, and the answer is the ultimate thing to come to. So, and you see it in their work. There's no work shown, or just minimal if you explain it to me. Those kind of things, I need you to explain what they did. There's no real value given towards the procedure, the process of getting there. And I think in math we're trained to think that way. So when I ask them to think that way and understand and tie

the procedure to the concept, there's push back. Call it, they call it "brain hurt." There's push back, you know, so they don't want to do that.

These excerpts reflected Dean's held beliefs that impacted his efforts to incorporate conceptual understanding into instruction while understanding the limitations of his students' proficiency in fundamental skills. Dean repeatedly emphasized in his interview responses that his primary objective was that the students understood the instruction rather than just apply a process or algorithm to achieve an answer to a question. This held belief was an important factor both in planning and implementation of instruction. During our pre-instruction interview, Dean made repeated references to his objective to instill understanding in his students. When asked to prioritize the time needed for covering a topic as opposed to student mastery, Dean responded by saying:

I want them to really ground themselves in understanding. (Priorities are) Most likely, basically, time and the level of understanding. I think probably the latter of the two is more important. The latter drives the former. If they don't understand very well, take more time, it slows down. So it's mostly dependent upon how well they're getting it.

There's really no... it's kind of like the feel. I feel it. If they feel like we're ready to move on, we move on. If not, slow down and go back.

Conceptual versus Procedural Orientation

Dean's questioning methods reflected his objective of constantly evaluating the students' knowledge and assessing their understanding of the concepts and topics being discussed. In almost every case, after working through each example or problem, Dean would intentionally pause and ask the students to re-vocalize or reiterate the solution

method. This approach to his planning and implementation of instruction is reflected in the teacher tasks he used in preparing and presenting the unit of instruction. As we can see in the table below, Dean utilized tasks spread across the spectrum of explaining and adapting instruction. Additionally, however, Dean regularly accessed higher order thinking and knowledge when he frequently employed his knowledge of algebra for teaching in the areas of applications, content trajectories, and reasoning and proof. This use of higher order thinking reflects his emphasis on teaching conceptual understanding in tandem with procedural proficiency.

Table 10. Teacher Tasks Used by Dean

Teacher Tasks and Knowledge		Dean
<i>Teacher Tasks</i>	<i>Code</i>	
Analyze Students' Work and Thinking	ASW	23
Design, Modify, Select Mathematical Tasks	DMT	15
Establish and Revise Mathematical Goals	ERG	20
Access, Use Tools, Resources for Teaching	ATT	3
Explain Ideas, Solve Mathematical Problems	ESP	19
Build and Support Mathematical Community	BCD	12
<i>Knowledge</i>		
Core Content	CC	40
Representations	REP	11
Content Trajectories	CT	16

Applications and Contexts	AC	6
Language and Conventions	LC	12
Reasoning and Proof	RP	7

Dean also accessed his core content knowledge of algebra routinely as indicated in the table above, but the accompanying language he used and cognitive demands placed on the students were indicative of his attempts to continuously examine their degree of mastery and connection with previous knowledge. In each of the following excerpts, Dean was accessing his content knowledge but making connections to previous discussions while encouraging the students to think more deeply about the process. While introducing the notion of the x-intercepts as solutions found through a graphical approach, he said,

In the middle. Right? We know it's gonna be equidistant from the vertex, right? In the same Y value, they must be equidistant from the vertex. It makes a lot of sense, doesn't it?

And when initially showing the basics of a factoring method,

Dean: Okay, well let's start it off anyway. So what should we do first do you think?

Student: zero product property.

Dean: The zero product property? So the zero product property... That basically tells us something very simple. If I take two numbers, and I multiply them, and my answer is zero, what do I know has to be true?

Role as Teacher

Dean utilized a lecture and demonstration approach to teaching with continuous feedback and questioning of his students regarding classroom examples. The physical layout of the room was very compact with all students facing the front. The small size of the room promoted an intimate environment and Dean constantly moved throughout the room interacting with the students.

Dean used questioning techniques that created opportunities like these for students to develop an understanding of both process and concept. For example, when introducing the solution method of graphing, Dean referred the students back to their previous work on discovering the equation for the axis of symmetry for a parabola. Dean used this opportunity to refresh the students' knowledge of the coefficients in a quadratic equation (*bridging*), reinforce the connection between the axis of symmetry and the vertex (*trimming*), and as a springboard to find points on the graph leading to discovery of the intercepts as solutions (*decompressing*). There was however some rigidity in his approach to procedures. For example, as he instructed the students to always use points equidistant from the vertex when graphing the parabola. His repeated reminder for these calculations was:

But we had realized we need to always go equidistant from a vertex. Okay? Even if it's a fraction it's okay. Yes?

This was an interesting dichotomy because Dean emphasized a routine or procedure nested in his efforts to stress a mastery of conceptual underpinning. While Dean continued to emphasize the objective of mastery of both concept and process, he often lapsed into reliance on rote memorization and application of a learned algorithm.

Dean's high expectations of his students sometimes contributed to this dependence on expedience when his instructional agenda occasionally exceeded the available time for his lesson plan. On one such occasion, when he was trying to quickly bring an end to instruction in the final minutes of class, he introduced a trinomial that was not factorable as a basis for an initial discussion of the quadratic formula. After having the students attempt the factorization of the expression then realize the solution was not available because of this condition, he said:

Dean: Okay, but wait, all is not lost. Okay, we have a method to deal with this. So I'm gonna give you a formula. I'm not gonna tell you how I got it. I'm not gonna develop it for you, okay. I'm just gonna give it to you. Take it on faith for now. You'll develop it in Algebra II, okay.

Okay. So now the letters I'm gonna put down in this formula are gonna refer to those letters up there, okay. So don't be too confused about it. It's not that hard. It's simply gonna be a drop, plug and play. Drop and crunch out. So X equals negative B , plus or minus, square up, B square minus four, A , C , all divided by two A . Now, A , B and C here are the A , B and C of here. So I can take our equation, identify A , B and C , and plug 'em in. Write that down. Okay, which means we probably have to do what with this formula?

Student: Save it.

Dean: Memorize it.

At times like this, as Dean would find himself relying on a procedural understanding when students encountered difficulty in finding a solution, his rationale

was to connect the algorithm to a future topic or course to provide a breadth of exposure to algebraic methods. In this instance Dean missed the opportunity to make some conceptual connection with the square root method or establish a foundation for completing the square by providing a formula for problems that he called *just a little too hard*. During our post-unit interview, Dean responded that students in his experience have always liked the quadratic formula because of its surface simplicity. His feeling was:

Um, they like it. They like formulas. They like things they can memorize.

Because Dean taught in a private school, there was no direct connection or application of the Common Core Standards and provisions to administer an end of course standardized test for accountability purposes. For this reason, Dean made no references to any testing or time pressures as factors in planning or implementing his curriculum, and had virtually complete autonomy in both sequencing and content of the instruction he presented. When asked about any external directives or pressures that motivated his instructional strategy, Dean responded:

Researcher: Okay, given complete autonomy how would your instruction in this unit change?

Dean: It wouldn't. I have it. Yeah.

Researcher: And again, do you feel you have the choice to modify instruction if it benefits your students?

Dean: Yes, definitely. I'm encouraged to do that. I don't think I could have more leeway. I'm encouraged to do that. It's the best thing for our students, and the school encourages us to do the best thing for our students.

Dean felt complete freedom to change sequencing or length of units of instruction flexibly in response to the specific needs of his students. Over the course of the first semester, Dean became more clearly aware of the capabilities of the students, and he concurrently believed they had high proficiency levels entering the class, and that they were able to move at an accelerated pace when necessary. That held belief worked against Dean at times. On numerous occasions, Dean missed opportunities to utilize good questioning techniques by allowing wait time for his students' responses. During observation, many of these missed opportunities were not directly related to students' inability to respond, but to Dean's interpretation that they could work quickly and at a higher cognitive demand and be successful. As Table 11 illustrates, across the 3 observations, there were 20 times where Dean asked a question and failed to provide time for students to process the question and respond. On the other hand, unlike Janice, Dean did not communicate to students that assessments were driving instruction because there was no external requirement to administer a high stakes end of course test.

Table 11. Emergent Themes in Dean's Classroom

Emergent Themes	Dean
Wait-Time Not Provided (< 3 seconds)	20
Concessions to Testing Schedule	0

It should be noted that Dean's classes were 45 minutes in length, and the wait-time issues indicated occurred in just three observations of that length. As an example, when transitioning from a review of graphing to solutions using the method of finding square roots, Dean said:

Dean: So we're gonna discuss a couple, two more methods to do this. One, graph, we've done graphing now. I think you guys can do any graph now, yes? [< 3 seconds wait-time] So we've done graphing. Let's discuss a simpler way. Do you guys want an easier way to do this one?

And

Dean: How do we get that X squared away?

Student: You divide

Dean: [< 2 seconds] not dividing. Take the square of both sides. Yeah. We squared the left side and we squared the right side. Now hold on for a second. I'll put it back on in a second here, okay. So I squared both sides, right? Okay, that's easy.

Evidence of Bridging, Trimming, and Decompressing

As previously indicated, Dean often used richer language during instruction. The vocabulary used during instruction by both student and teacher was notably mathematical in nature. When talking about solutions of quadratic equations, Dean used the idea of

“slicing through the graph” with a horizontal line which could be different than the x-axis, or could be used to find the x-intercepts. Dean says:

Today we're gonna talk about solving these things. We're gonna look at simply how we slice this thing. One little bit. We're gonna look at some, find points where it crosses the X axis. Okay, so when we discuss the solutions of the quadratic we're looking for, almost always, the points where it crosses the X axis. So you guys, look at that graph and tell me, where does it cross the X axis?

Then later, while challenging the students with a fourth degree equation, Dean demonstrated the factorization and repeated use of the zero product property saying:

Dean: So we're gonna have X to the fourth minus five X squared plus four equals zero. And then we have a trinomial here with no GCF or a difference of two squares so we're gonna go with X squared here and X squared there, and since this is a positive sign but then this is a negative sign we know that they're both gonna be zero, or negative. And then since that's five in the middle, um, we're just gonna go with the factor one and four so I have one here and four there. And then we have two difference of two squares, so we'll go X minus one and X plus one. And then here we'll go X minus two and X plus two, which equals zero.

Student: [i n t e r r u p t s] Do you have to split it up with four factors from the X square? Could you have done the X squared equals one and then... ?

Dean: That's a good question. Okay, what you're saying is could I take this over here outside the equation is X square minus one equals zero and then do what?

Student: Add one to both sides.

Dean: So X squared equals one and then...

Student: It'll be positive, negative one.

In this instance, the students were able to make a connection to the square root method quickly and successfully solve the more difficult problem [*bridging*]. Dean accessed his Advanced Mathematical Knowledge to consistently encourage use of more sophisticated language between teacher and students. This emphasis reflected an expectation of higher cognitive thinking and correlated with his use of the overarching concepts of *Bridging*, *Trimming*, and *Decompressing* during planning and classroom activities that focused on construction of both procedural and conceptual knowledge.

Dean looked for opportunities to introduce ideas that would connect his instruction to concepts across the discipline. He used *bridging* to create these connections with concepts beyond the scope of Algebra 1, and establish a basis for deepening the students' understanding between intercepts and solutions for a quadratic equation. Following an exercise in solving a quadratic graphically to find the two points of intersection with the x-axis, Dean demonstrated a graphical representation of a cubic function with three such solutions as would be seen in the Fundamental Theorem of Algebra. In this exchange, several students were able to extend that knowledge to make a correct conjecture about the behavior of a fourth degree polynomial. This demonstration also allowed for clarification on nomenclature and behavior of quadratic functions in general, providing a chance for Dean to again check for mastery learning.

Dean: What intersects the X axis, right? Okay, now again you guys don't know how to graph cubics, which is what that is. Sorry about that. In most cubics, which you're

going to find, oops, is that cubics are gonna behave like this. Okay. So we graph those things and see how they behave, so how many times does it cross the X axis?

Student: Four.

Student: Three.

Student: Three times.

Dean: Three times. One's here, one's here and one's here.

Dean: Very likely, yes. So, make sense, guys? So question for you. If I had a fourth degree polynomial, how many answers would we have?

Student: Four.

In a follow up example, students were able to use their knowledge of graphing these quadratic equations to envision graphs without an x-intercept, allowing Dean to address the eventuality of other types of solutions they would encounter using the imaginary number system. Dean's use of *bridging* extended to applications as well to describe projectile motion and other mathematical models. Even during the planning phases for this unit of instruction Dean included bridging strategies that created a thread of continuity between stages of complexity in factoring polynomials. Dean explains in the following excerpt how he intends to address the transition from simple forms of factorization for single terms or binomials to the higher cognitive demand task of factoring trinomials using familiar terms and properties. In the pre-instruction interview he explained:

Researcher: Okay. You say backwards distribution, um, that's a starting point.

Backwards distribution, kind of reverse multiplication.

Dean: Yes

Researcher: Okay.

Dean: So it's very simple, something they've done before, so they know, so I can connect that previous knowledge to what we're gonna hit with the next which would when you factor a trinomial. You're simply factoring a little bit more. That's all you're doing from a simple GCF.

I go back to discussing finding a common factor with a binomial and factoring just a single binomial, a GCF, out of that binomial. And then I talk about distributing going backwards, and then I talk about factoring out larger things so when I say I have two times X plus five plus five times the quantity X plus five. I treat that as the GCF X plus five and I factor that directly as well. So I go back to factoring using distributive property and from there expand into how to factor a regular trinomial. So I start with one that's already partially factored, and then talk about how we can get to a trinomial from that.

In the above exchange, Dean illustrated his understanding of the use of bridging both within the unit of instruction and as a connection to other mathematical tasks across the domain that students would likely encounter in later courses.

Dean also used his knowledge of his students' skill levels to appropriately present mathematical ideas and procedures in ways that enabled them to construct knowledge of more difficult concepts in a logical progression (*trimming*). At one point during the pre-

instruction interview Dean was questioned on the decision process he used to extend or modify instruction. His response indicated a firm grasp on understanding the capabilities and misconceptions of the students. Dean indicated even though it was difficult even “frustrating” to do so, he recognized the need to pause and ensure a level of understanding as a basis for continuing. In his words:

Of course it is (frustrating). So my, part of my internal thought process is, okay, slow down, put myself in their shoes, guide them.

And in practice in the classroom, Dean moved through increasingly difficult problems, building and constructing a learning opportunities as in the excerpt below.

Dean: Okay. Let's look at a couple then. I'm gonna give you a couple that are already factored and we'll see how to solve them. So if I give you an equation that's already factored. Okay. Now I want you to solve it. So you tell me, what steps would we go through? (trimming) You want to try it yourself first?

Student: Sure.

Dean: Try it yourself first and we'll come back to it and see what we get. So guys, I want you to solve it. So I want you to find the points where it crosses the X axis, okay.

I've already factored, haven't I? So let's go ahead and go over this again. Some of you guys have gotten it and some of you didn't quite catch on yet, but it's okay. So we have a number times a number and equals Y, but we don't want the entire Y. What do we want?

Students: Zero.

Dean: You want it where it crosses the X axis which means Y is equal to zero. Does everyone understand? Promise me you get that. Okay, so we know why we're putting a zero there. Okay. That means we multiply two numbers and we get zero as an answer. What does that tell us?

Student: One of them is zero.

Dean: One or both have to be a zero. In this case we know it's a parabola so probably both. Yes?

Dean followed this exercise with a *decompressing* action that also provided feedback on the students' level of understanding. Here, he ensured clarity in establishing depth of understanding of the function of the variable in the quadratic equation as a single entity representing solution values. Dean repeatedly assessed the students' understanding during this follow-up as he said:

Dean: Yeah. Now, do you guys understand, hold on for a second, do you understand that it can be both of those values? But not at the same time. You understand that? Okay, so let's say for example, let's let X equal negative two. If we do that, it'll be negative two plus two times negative two minus six. So you see that?

Anyone think there would be negative two and six for x at the same time? If you didn't, good. You're my first class that's ever had that, didn't think that. Most of them think it's gonna be one and the other at the same time, but it means that both can be negative two which will make this zero. Or they'll both be positive six which will make the second binomial zero. Okay. And again, if this is zero it doesn't matter what this is does it? If

this is zero it doesn't matter what this is. It could be five thousand for all I care. Yeah.

Okay, cool. Number two. We'll try number two.

Dean assessed the effectiveness of his instruction and modified instructional strategies or sequences based on his assessment of the level of student understanding (*decompressing*) 10 different times during the class observations. Dean adapted his instruction (*trimming*) from examples to representations on 9 separate occasions across the three observations. Additionally, he made efforts to create connections with previous knowledge (*bridging*) a total of 21 times across the observations and two extended interview sessions. Dean's flexible approach reflected not only his Advanced Mathematical Knowledge, but his varied teaching experiences in upper level mathematics. He was able to envision content trajectories and establish a basis for his students to move toward increased understanding of these connections by routinely inserting them in his regular instructional strategies.

Although Dean missed opportunities to deepen his students' learning experience by not providing appropriate wait-time, he was continuously reflective about the classroom instruction and actions needed to remediate misconceptions as well as his own instructional techniques. In the post-instruction interview, Dean was asked to evaluate the effectiveness of his instruction and discuss any modifications needed for future units on solving quadratic equations. In response he explained,

Um, I'm gonna change the sequence next time for sure. I think that when I get to graphing quadratics, um, I actually, I might roll right from factoring quadratics into solving quadratic equations. So, I think I'm gonna go with that route and just keep it back to back where it's just having a break between factoring quadratics to then going

over to linear equations, graphing quadratics then solving quadratics. I think it's too much of a break there. I'm gonna put those back to back. So, maybe a different test, but it's gonna be the topic will be back to back vs. separated, for sure. I think that's gonna change my, I'll have less overlap, less repeat, hopefully less misunderstanding since it'll be a little fresher on their minds, and hopefully a little better understanding overall, it will be, um, reinforced right away vs. having to refresh then reinforce after that. So my sequencing will change. Um, I think I'll spend a little more time on some application of solving quadratics using factoring such as some of the basic area problems you have with length and width and factoring a trinomial area for the length and the width. I'll discuss, I'll do more with that so they can kinda connect with how we can use that as well vs. always going to the quadratic formula. Because some of the boys now want to go to it no matter what even if it doesn't really apply or is the best method to use. So I'll do a little more of that.

Dean had high expectations of his students based in part on the ethos of the school, and in part on his own held beliefs about the requisite knowledge for students entering a first course in algebra. Dean routinely placed higher cognitive demand on the class, yet recognized the need to remediate when it was necessary because of constant feedback he extracted from the students.

Applications to Research Objectives: Dean

Research Question 1. What knowledge of algebra for teaching do teachers **possess** for teaching a first course in algebra?

As previously reported, Dean had higher levels of knowledge in the three dimensions of KAT than Janice, and it was evident in his overall approach to this first course in Algebra. Dean was aware of his students' mathematical abilities based on his observations throughout the year prior to this unit. His Teaching Knowledge was exhibited by his use of appropriate exercises and instructional strategies that placed a higher cognitive demand on the students in response to his held belief in their ability to comprehend the conceptual underpinning to the topics. Dean also demonstrated a higher level of Knowledge of School Mathematics than his peer group as seen in his explanations throughout the unit of instruction covering a broad range of content that also made connections with topics across the Algebra 1 domain via his introduction of solving quadratic equations using both a table and graphical representation. Dean's unusually high level of Advanced Knowledge of Mathematics was most apparent by his use of instruction that made connections between current and future mathematical applications. His emphasis on coupling procedural and conceptual learning demonstrated an understanding of the importance of the interrelationship of these learning tracks for future challenges in the mathematics domain.

Research Question 2. What knowledge of algebra for teaching do teachers draw upon when **preparing for a unit of instruction** on solving quadratic equations?

The KAT accessed during the planning phase of this unit of instruction centered on Dean's stated goal of emphasizing a mastery learning environment. Using his understanding of how his students learn, difficulties they had exhibited earlier in the course, and bridging to connect topics, Dean planned a dual track of instruction to achieve procedural and conceptual understanding. He repeated on several occasions his

commitment to nurturing an environment that encouraged discovery and connection to the logic behind algorithms or processes. Dean connected this knowledge during planning with his knowledge of algebraic course content and the school curriculum for a first course in algebra. His plan for connecting the mechanics of graphing to finding solutions as intercepts highlights the ability to bridge the topics of the unit of instruction. Dean's understanding of advanced mathematics was also accessed during his explanation in the pre-instruction interview when he emphasized ensuring students mastered the task of factoring various types of quadratic equations as a foundation for future mathematical work. Dean had higher knowledge levels across the KAT domain and showed a strong focus on connecting concept and procedure, as well as teaching for understanding through an emphasis on student reasoning through challenging problems. This higher level knowledge across the discipline was Dean's strongest aspect of KAT and served to establish high expectations for the class.

Research Question 3. What knowledge of algebra for teaching is utilized during **instruction** in a unit on solving quadratic equations?

Dean's teaching exhibited his Teaching Knowledge in two primary aspects: 1.) his continued efforts to link procedure with its conceptual foundation, and 2.) the recognition of student misconceptions and decision to modify his plan to accommodate that deficiency. Dean illustrated his ability to use *bridging* within this dimension of knowledge when he extended the notion of graphing a quadratic equation to include real-world scenarios as mathematical models. At the same time, Dean's perception of his students' learning styles occasionally caused him to misinterpret student difficulties because he believed the students either could or should be able to grasp new material

more quickly than their algebraic skills allowed. Dean realized many students had not been encouraged to work toward a mastery understanding and made intentional decisions to structure instruction to accommodate that deficiency. That effort however highlighted a weakness in his instructional actions as he reverted to a procedural approach for expediency on occasion. The language Dean employed was a function of his access to content knowledge at a high level. His use of mathematical terminology to elevate the student's lexicon demonstrated his ability to increase the cognitive demand in the context of the topical discussion. Dean's language when describing the relationships between graphing intercepts and the solutions by factoring for quadratic equations showed a breadth and depth of knowledge in the content and trajectories within the algebra domain. Throughout the unit of instruction, Dean interspersed references and examples to applications of the solution methods for quadratic equations in higher-level math courses. This natural extension of the topic was a clear display of his advanced knowledge of mathematics. Dean's teaching reflects a dynamic interaction between the dimensions of knowledge in the KAT Framework during planning and instruction.

Cross-Case Analysis

The cases of Janice and Dean provide an opportunity to analyze the relationship between the KAT each teacher possessed and KAT accessed during planning and teaching activities. Following the structure of the Individual Case Studies, I discuss in this section both of these relationships in the context of the subject teachers' teaching styles, roles as a teacher, and use of the overarching themes to enact the knowledge dimensions. Because the climate in which the two teachers operate has an impact on both the planning and instructional context relevant to this discussion, I begin with a brief comparison of the teaching environment.

Demographics and Learning Climate

On the surface, the demographic data for the schools employing the subject teachers is strikingly similar. The composition of students' ethnicity, academic achievements, and student to teacher ratios reported are much the same, but deeper analysis reveals some important distinctions in the educational climate. The public school has recently been recognized as one of the top five high schools in the state, and was chosen the best academic school state-wide for systemic improvements by the State Collaborative on Reforming Education. The private school again had a 100% graduation and four-year college admission rate and boasts a state best 18 National Merit Finalists.

Further investigation into the learning climate at each school shows some very important differences. Although the public school lists the student to teacher ratio as 17:1, anecdotal evidence gained from interviews with other teachers at the school indicate the actual ratio is closer to 32:1 for nearly all classes. In the private school,

interviews showed the reported student to teacher ratio of 14:1 is in fact the actual classroom ratio. The faculty at the private school includes 20 members with a Doctoral Degree, and annually awards travel and study grants for the United States and abroad in various academic disciplines to more than 10% of the student body through an endowment to the school.

Teaching Style

At the public school, the decision to revert to a block schedule from a six period day was made prior to the current school year necessitating a restructuring of the curriculum to instruct core content courses in one semester rather than in a year-long period. That decision created anxiety in the subject teacher, Janice, as she had concerns about her ability to adequately cover the expected topics in this first course in algebra. On several occasions Janice indicated she had to modify her course outline because of time constraints. In this excerpt, Janice knew the available time was not sufficient to prepare her students properly. During a post-instruction interview, she said:

Researcher: So, is there a specific situation that you can think of where you had to modify your plans or is just sort of the whole ethos?

Janice: Yeah, I mean, it's just the whole thing. I knew there was no way it was all gonna get covered.

Researcher: Okay. Um, have you had to change that plan?

Janice: Oh, miserably.

Researcher: How does that make you feel?

Janice: It was awful. I mean, there's no way those kids are ready for Algebra II. They don't know how to factor. I taught them how to factor in four days. You can't do that. So it's demeaning to the teacher; it's demeaning to the student. I mean, they didn't learn it, and I know they didn't learn it. We worked, we got to go back a little bit, and we started trying to pull in some more factoring and work on a little more, and they did do a little bit better.

This comment reflected her ongoing struggle with the intersection of her students' needs and their proficiency levels in the context of a restricted instructional time period.

During our interview discussions the impact of this decision to move to a block schedule manifested itself frequently as Janice indicated she felt increased pressure to cover an inordinate amount of material in a restricted time frame, particularly because of the impact of the End of Course test on school ratings and on her own evaluation. When I asked Janice about why she chose to focus primarily on covering the two planned solution strategies, she replied,

(They are) "the ones that are likely to be tested".

Janice had a held belief that her students had a lower proficiency level and predicted they would struggle with the amount of material she needed to cover with them in the time available. She expressed this belief during a post-lesson interview when she said:

They didn't have the knowledge that they needed in order to be able to cover those things on that particular day as we said, you know, like the factoring that they tried, that we had to do so fast, and I knew they didn't know it when we got there.

She made repeated attempts to help them understand the urgency of a sustained effort on their part because of the brevity of instructional time she could allocate to each topic. It was clear from observation that not only Janice, but her students as well, felt the impact of time and test schedules on the classroom instruction. Janice lamented to the class almost apologetically,

Now I would like to say that I have had more time to teach what I need to teach, but in the next couple times I'm not gonna have the time to teach things as I normally would. That's that one semester class. Everything gets put together really fast.

Conversely, Dean had a great deal of latitude in planning and instruction. Because Algebra 1 was taught over the course of the entire year, he also had opportunity to parse instruction over an extended period and revisit previous learning as a bridge to new concepts in the unit of instruction on solving quadratic equations. During observation, Dean frequently referred to introductory topics the class had covered in previous instruction and expressed no stress over his ability to give maximum exposure to his first year algebra students. Here, Dean introduced the concept of solutions by graphing,

Dean: And you put those dots on the graph and you came up with the solution, right? So if we think about our graph, this is kinda, this is beyond the scope of our class. We're not doing this yet. I'm kinda setting you up for [Inaudible] later on. I want you to understand where you're going so you really understand the context behind it. So you tell me if we're, when we're at zero, same problem here, yet, then where are these points going to occur?

I'd be on the X axis, wouldn't I? Yes, exactly. I'd be on the X axis some place...we'll talk about this next semester when we get to that, okay.

Then, Dean referred back to the lesson during the following semester when presenting the method of solution by factoring.

Dean: One or both are zero, right? So that means either this whole thing is zero or this whole thing is zero. Yes. Remember that from last semester? Okay. So what value do I need to put right there to make that thing zero?

Dean had high expectations of his students that reflected the ethos of the school and stated during our initial interview that he preferred to use a teaching approach that encourages discovery rather than procedural fluency. When asked about what his students should expect to learn in the unit on solving quadratic equations, Dean said:

The primary thing I want you to know is that when they are finding the solution points for quadratic, they're finding the points where the graph, where the equation crosses the Y axis, or the X axis. Pardon me. So they find the roots to the solutions where it crosses the X axis, that it doesn't, it's connected and related to the entire graph. It's just a slice of it, or a slice where it happens to be crossing the X axis. Where we can also apply that and slice that parabola wherever we want to slice it.

Dean used mathematically mature language and connection to establish a foundation for multiple concepts and topics that would follow the initial introduction to graphing a quadratic equation and observing points of intersection. For example, as demonstrated in the previous excerpt, when talking about his introduction to solutions by graphing, Dean wanted the class to see the application in a much broader sense so they

could develop an appreciation for applications of the graphical solution method in real-world scenarios.

Procedural versus Conceptual Orientation

These varying levels of knowledge were manifested in the instructional strategies and the teacher's approach to planning and implementation of the unit of instruction on solving quadratic equations. Janice's instruction was heavily dependent on developing procedural skills that enabled the students to achieve some level of success by arriving at a solution. Dean's instruction was more conceptually oriented, though there was also a strong reliance on proficiency in an algorithm or procedure for completing a solution process. Janice would use language such as:

Well, that rule's actually a little different. I would personally take the answers and multiply them out until I found the one that works.

or:

The middle number goes to the largest of my two. Do I have to do bottoms up or do I get to go straight to the answer?

Here "bottoms up" is a step-wise factoring procedure for quadratic equations with leading coefficients not of unit size. Regarding the use of a procedural approach for solving quadratic equations, Dean responded:

I want them to really ground themselves in understanding what's happening.

but later, during a post-instruction interview, he admitted:

I would say across the board by the time I've gotten to quadratic formula that's been the easiest to learn. Kids learn it very quickly because it's a formula.

Both teachers had a firm grasp on the skill levels of their students based on their observations throughout the year and taught to that level, though Dean used more appropriate mathematical language. During a pre-instruction interview he said:

So they find the roots to the solutions where it crosses the X axis, that it doesn't, it's connected and related to the entire graph. It's just a slice of it, or a slice where it happens to be crossing the X axis. Where we can also apply that and slice that parabola wherever we want to slice it.

Then later during a lesson on graphical solutions, he explained to his students:

Today we're gonna talk about solving these things. We're gonna look at simply how we slice this thing.

Testing Focus and Lack of Wait-Time

The advent of Common Core Standards and subsequent End of Course Testing requirements as an accountability measure have been a part of the educational climate conversation for nearly a decade. The impact on classrooms is varied and remains a matter of debate, but was a critical factor in this research. In the public school system in Tennessee, students are given a standardized test called the TNReady Test at the end of their algebra and geometry courses as a means of measuring their proficiency levels. These data are utilized as a ranking mechanism for the schools, and have been used as a contributing factor in teacher evaluations. Though the public school where Janice teaches has perennially ranked in the top ten percent of the schools in the state, she repeatedly referenced the impending test throughout observations and interviews. Prior to instruction, Janice addressed the impact of the test by indicating that her plans for

instruction were in essence built around the material she expected to see assessed on the TNReady test. When specifically asked which methods for solving quadratic equations she planned to teach, her response, as previously noted, was:

Which ones are gonna be tested.

The situation at the private school was dramatically different because Dean had no requirement imposed to administer a standardized test as a metric for accountability to the educational system. Dean structured his planned instruction to meet the needs of the curriculum and lead to a course end exam that contributed to the student's final grade. The absence of an End of Course standardized test allowed Dean to focus on appropriate sequencing and depth of coverage for each topic rather than following a prescribed course of action dictated by the curriculum guide utilized at Janice's school. The following table shows the cumulative counts for references to a test in some form along with another emerging theme; wait-time.

Table 12. Comparison of Emergent Themes

Emergent Themes	Janice	Dean
Wait-Time Not Provided	21	20
Concessions to Testing Schedule	89	0

The overwhelming occurrence of concessions to the testing issue on the part of Janice overshadowed the application of her mathematical knowledge in a productive way and created a difficulty in accurately assessing her true motivations for using various teacher tasks or instructional strategies.

On the other hand, both teachers frequently failed to provide appropriate wait-time strategies in their instruction. As shown in the above table, both Janice and Dean regularly failed to take advantage of opportunities to deepen students' learning or expand on a complex issue. For Janice, the difficulty with wait-times stemmed primarily from an urgency to cover the topics she anticipated on the TNReady Test. Janice was focused on completing her daily agenda so that she could keep to the schedule indicated on the curriculum guide. At times, Janice missed a teaching opportunity because a student posed a question that was not specifically on the topic for discussion that day. At those times she would simply give an answer without exposition or time to ascertain whether there was clarity on the part of the student. Dean similarly missed a number of teaching opportunities, but his lack of wait-time was associated with his held belief that the students had a higher level of algebraic skills than was evident. Dean's teaching style incorporated many points of feedback throughout the period of instruction, and he believed the class was capable of mastery learning because of the consistent participation he observed. In both cases, the two teachers failed to use good questioning techniques and appropriate strategies to provide more depth and robust connections to previous or future learning. Dean perceived higher ability in his students to work quickly, and saw no need to provide wait-time. It may also be that, as a consequence of his elevated Advanced Knowledge of Mathematics, he had unrealistic expectations of his students. Janice, on the

other hand, had low expectations of her students, and saw no value in wait-time either because her students were not capable of deeper thought processes, or would be unable (in the context of her sense of urgency to cover all the points in the lesson plan) to formulate an answer in a reasonable amount of time. Janice expressed her rationale for using simplistic methods based on her perception of the students' lower algebraic skills during the post-unit interview when she responded, saying:

And so I just found that they can do it. It's less, when you get in some of the methods that have much more steps to them, they get bogged down on the details of the steps and they just get upset and they don't even try. Because originally that's where I was at. I used to try that. Then I tried the other methods.

Role as a Teacher

Because an understanding of Algebra 1 is foundational for future success in subsequent math courses, it is essential students attain a depth and breadth of the core content and develop facility with the use of many new algebraic skills. This places an enormous burden on the instructor to emphasize mastery of the often abstract content of algebra as a basis for connection to other applications across the discipline. The observations in the cutaway of the 6 X 6 matrix of KAT tasks and knowledge below showed that the two subject teachers relied heavily on explanatory tasks and their own content knowledge more often than other mathematical knowledge for teaching.

Janice			Dean		
Tasks of Teaching	Knowledge Categories		Tasks of Teaching	Knowledge Categories	
	Core Content	Content Trajectory		Core Content	Content Trajectory
Analyze Students' Work and Thinking	8	5	Analyze Students' Work and Thinking	14	1
Design, Modify, Select Mathematical Tasks	11	4	Design, Modify, Select Mathematical Tasks	4	6
Establish and Revise Mathematical Goals	10	4	Establish and Revise Mathematical Goals	10	4
Explain Ideas, Solve Mathematical Problems	9		Explain Ideas, Solve Mathematical Problems	7	

Figure 6. Concentration of Teacher Tasks

For Janice, these cells show 2 of 3 observed tasks of teaching she chose during classroom interactions were for explaining or telling. That trend is critical in reflecting her reliance on a procedural approach. It was also indicative of her assessment of the low proficiency level of the class, and the expectations she had for these students given the constraints of time as a function of TNReady testing. During the post-unit interview, Janice expressed her frustration saying,

...at that point that I'm preparing I'm giving them whatever trick and whatever anything I can come up with to help them get an answer for that test.

Though Janice utilized these expedient instructional strategies frequently, it was clear she was conflicted about the impact external factors had on her instruction. Janice was realistic about those constraints, and indicated that, given sufficient time, her instructional model would include much more conceptual learning and connections across the domain. During post-unit interviews she said:

Janice: That I hate that my world is defined off of a test and that I don't feel like you can get a true feeling of how these different things could be taught, or yeah, I mean I don't think you get a good feeling of how they really can do when we're constrained as much as we are.

Dean's use of these tasks of teaching and content-related knowledge account for about 1 of 2 of the observed interactions with his class. Although representing only half of the tasks of teaching Dean utilized during this unit, this concentration of effort in these actions indicated a stronger dependence on core content knowledge than higher level thinking such as applications or reasoning and proof. Dean previously expressed a stated objective of using a "discovery" approach rather than a procedural approach. At the same time, Dean explained his occasional use of standard routines by saying:

I feel that most students that I have in Algebra I have not been encouraged to understand what they're actually doing. They've been encouraged just to do, go through rote steps. And come to an answer, and the answer is the ultimate thing to come to.

This realization created a pedagogical conflict for Dean as he attempted to balance his commitment to teaching for mastery with the level of algebraic skills the students brought to this first course in algebra. While managing this dichotomy, in about

1 of 4 cases Dean accessed higher order thinking and categories of knowledge including applications, language and conventions, and reasoning and proof in utilizing the tasks of teaching. Though there is not a strict hierarchy of categories indicated in the KAT Project study, these latter categories require syntactic knowledge of algebra and correlate with Dean's higher level of content knowledge and advanced knowledge of mathematics for clarifying misconceptions resulting from algebraic use of terminology and symbols not previously encountered by first year algebra students. Additionally, the applications of algebraic methods along with emphasis on reasoning and proof, call for a depth of knowledge across the curriculum and reflected Dean's higher level of Advanced Knowledge of Mathematics. At the core of Dean's instructional paradigm was his belief that his students needed to develop new cognitive pathways and learning skills reflecting a mastery orientation. During various interviews, he repeatedly made comments that demonstrated his commitment to student understanding though he occasionally encountered students' resistance to this instructional objective.

These responses to internal and external factors by both subject teachers demonstrate the complexity of teaching a first course in algebra. Each of the teachers accurately assessed the skill levels of their students and utilized their respective knowledge of algebra for teaching to access relevant tasks of teaching and knowledge categories they felt met the needs of their students. The choices made and instructional strategies employed illustrated the orientation of each teacher toward the teaching model they perceived as most appropriate given the diversity of the educational climates encountered and their respective comfort level with procedurally or conceptually based instruction. In the following section, the ways these teachers held and *used* their

mathematical knowledge across the knowledge dimensions of KAT will be analyzed in the context of the three overarching categories of *Bridging*, *Trimming*, and *Decompressing*.

Evidence of Bridging, Trimming, Decompressing Across KAT Dimensions in Instruction

Bridging, Trimming, and Decompressing describe the ways teachers operate below the surface actions of teacher tasks in the everyday classroom. The manner in which these three themes manifest in the algebra classroom illustrates the teacher's ability to establish and nurture a continuity and coherence across the mathematics domain by making connections, clarifying difficult concepts while maintaining mathematical integrity, and unfolding topics to create an atmosphere where conceptual and procedural understanding can interact and mature on parallel tracks. This kind of teaching is not the sole domain of Algebra 1, but is a critical component of successful instruction as students make the transition into secondary mathematics.

When comparing the use by the two subject teachers of these strategic actions, it is important and instructive to note that the classroom observation time for Dean was actually only one-half that for Janice. Though each classroom teacher was observed for three lessons, Janice taught 90 minute blocks of instruction while Dean had classes of 47 minutes length. Extrapolating the data using that context, Dean used Bridging at a rate four times more often than Janice. Dean utilized bridging to make connections within the unit on solving quadratic equations and also to demonstrate the continuity of application from prior learning to anticipated future mathematical work. Janice was primarily observed using bridging as a means of connecting different methods of solution within the unit, but failing to explain any connection beyond the first year algebra curriculum.

Similarly, Dean applied the trimming strategy more than twice as often as Janice to ensure transparency within the procedural understanding context and as a step in a piecewise progression of increasing difficulty toward conceptual mastery. Janice essentially used trimming as a means of addressing the current algebraic skill set of her students. Her simplifying instructions and rigidly guided practice of a standard routine was directed at short term memorization efforts that, when repeated, would bring them to a more complete grasp of solution methods and procedures. Finally, Dean applied decompressing at a pace greater than three times Janice's usage. A distinguishing characteristic in Dean's instructional plan was his intentional effort to emphasize the conceptual underpinnings of solution methods to encourage the students to merge their procedural and conceptual understanding. This deliberate action is important in establishing a foundation for future learning because decompressing helps students make sense of an idea and realize it has a usefulness beyond the current instruction (McCrory et al, 2012). Janice utilized decompressing to help her students make sense of an algorithm or procedure by structuring examples that encouraged discussion of the algebraic properties associated with the solution methods. Her application of decompressing was tempered by her held belief about the efficacy of her students as validated by her statement that *"at that point that I'm preparing I'm giving them whatever trick and whatever anything I can come up with to help them get an answer for that test"*.

Effective use of bridging, trimming, and decompressing is indicative of a breadth and depth of knowledge across the discipline. The disparity of knowledge in all three dimensions of the KAT Assessment showed Dean had more knowledge of algebra for teaching in every category. Whether that disparity was a direct result of academic

background or teaching experience could not be directly ascertained, but the results of Dean's implementation of mathematically mature language and more challenging instruction seem to indicate a depth of knowledge of both algebraic concepts and mathematical work anticipated in future courses in the discipline. These findings support a conclusion that levels of KAT have a direct impact on the learning experience classroom teachers are equipped to provide. As KAT grows through experience and further education or training, the instructional tools at the disposal of the classroom teacher also increase, resulting in more productive instruction for both current and future learning opportunities. The observation of this research effort was that Dean's higher level of knowledge in the important dimensions defined by the KAT Framework manifested in a more challenging and beneficial learning climate for his students.

Summary of Findings: Applications to Research Objectives

This case study of two high school teachers of a first course in algebra indicates that a teachers' KAT may be related to their planning and teaching. The teacher with the stronger content knowledge used bridging, trimming, and decompressing more frequently, and in ways qualitatively different than the teacher with lower levels of knowledge in the three dimensions of knowledge defined in the KAT Project; *Teaching Knowledge*, *Knowledge of School Mathematics*, and *Advanced Mathematical Knowledge*. This study also found that there are teacher actions motivated both by external factors and deficiencies in their pedagogical skill set that are independent of their content knowledge and that may influence the learning environment. In these two cases, both teachers failed on numerous occasions to provide wait time for students, missing opportunities for higher level thinking, student discourse, and development of learning pathways that promote

mastery understanding at the conceptual level. One of the teachers' focus on high stakes assessment greatly influenced her decision making and actions in the classroom. While strong KAT can support and underpin a teacher's ability to bridge, trim, and decompress, this research observed other factors also influence their practice. In the following chapter, the findings are discussed and interpreted through the lens of the KAT dimensions of knowledge and their application vis à vis the overarching themes. Implications and recommendations for future research that proceed from these findings are also discussed in the concluding chapter.

CHAPTER 5

DISCUSSION AND CONCLUSIONS

In his seminal article, “Those Who Understand: Knowledge Growth in Teaching,” Shulman (1986) introduced a new paradigm to address the knowledge needed by classroom teachers that included a combination of content knowledge, pedagogical knowledge, and curricular knowledge. Since that time, the burgeoning area of research into the mathematical knowledge for teaching needed by classroom teachers has produced numerous studies (Ball, 1991; Ball & Bass, 2000; Ball & Bass, 2003; Ball, Thames, & Phelps, 2008; Ferrini-Mundy et al, 2005; Hill & Ball, 2009; Ma, 1999; Marcus & Chazan, 2010) that have contributed to the evolution of a codification of that knowledge and clarification of how that knowledge translates to classroom practices. In an effort to investigate how the mathematical knowledge for teaching algebra specifically manifests itself in the classroom, the Knowledge of Algebra for Teaching (KAT) Project team at Michigan State University (McCrory et al, 2012) has conducted an ongoing effort for more than a decade to develop a framework to measure and evaluate the intersection of tasks for teaching and mathematical knowledge as classroom teachers use their KAT in practice. This study uniquely extends the work of the KAT Project team by applying their framework to a particular unit of instruction in a first course in algebra by observing the single topic of solving quadratic equations to investigate the extent of knowledge for

teaching algebra teachers possess and how they access that knowledge in planning and instruction.

In this chapter, the research objectives and methodology will be briefly summarized and addressed as they apply to the specific goals of the study. In the sections that follow, findings are discussed in the context of the established research framework, emergent themes that impacted the interpretation of the teachers' actions and decisions are also discussed with focus on comparison and contrasts between the subject teachers, and limitations of the study are addressed. Implications and potential future research areas are included and, finally a summary of the study is discussed in the final section.

Purpose of the Study

The KAT Project at Michigan State University (Ferrini-Mundy et al, 2005) with a focus on the practice of teaching in a first course in algebra had a dual objective to 1.) “expand the conceptualization of knowledge for teaching algebra at the secondary level”, and, 2.) “provide a tool for studying that knowledge” (p.12) that would facilitate future research. The objective of the current research was to investigate the intersection of the particular mathematical knowledge teachers have that pertains to teaching a first course in algebra and the ways in which they access and use that knowledge when planning and instructing a unit of instruction on solving quadratic equations. This particular knowledge investigated, defined as Knowledge of Algebra for Teaching by the KAT Project falls into three dimensions; Teaching Knowledge, Knowledge of School Mathematics, and Advanced Knowledge of Mathematics. In this study, the goal was to extend the base of knowledge established in the work by the KAT Project Team by applying the framework

they developed to actual classroom practice. The goals of the current study, expressed in the specific research questions, were threefold:

1. What knowledge of algebra for teaching (KAT) do teachers possess for teaching a first course in algebra?
2. What knowledge of algebra for teaching do teachers draw upon when preparing for a unit of instruction on solving quadratic equations?
3. What knowledge of algebra for teaching is utilized during instruction in a unit on solving quadratic equations?

The construct for this investigation, illustrated by the KAT Matrix, was the observation of how tasks of teaching from the specified framework were used by the participating teachers throughout the unit of instruction, and the relationship of these tasks to defined categories of knowledge that the subject teachers referenced as they employed these actions. Interpreting these results as manifestations of the dimensions of knowledge listed above, and identifying how they demonstrated application of the overarching themes of Bridging, Trimming, and Decompressing enabled the development of an explanation of how the teachers put their mathematical knowledge for teaching algebra into practice.

Methodology

Because the focus of the study was the cognitive processes and subsequent decisions of the two subject teachers, a mixed methods design was utilized (Merriam, 2009; Yin, 2014) with individual case-studies and a cross-case analysis to develop a robust explanation of teacher actions and decisions as they occurred in the changing dynamic of the algebra classroom. Data was collected using an assessment instrument

designed within the KAT Framework (McCrory et al, 2012) to collect demographic and academic data and also a metric of KAT Knowledge levels for the subject teachers, and through semi-structured interviews, classroom observation, and classroom artifacts.

The assessment was scored using the assessment tools provided by the KAT Project and their normalized data, and then descriptive statistics were calculated for comparison to the existing database of the 1,000 previously tested pre-service and in-service teachers from their project data. Interviews were conducted pre-unit instruction and following completion of the unit, and also as follow-up interviews to classroom observations. Classroom sessions were also audio-taped then transcribed for analysis against the a priori codes consisting of the teacher tasks and knowledge categories. The results of that coding exercise were accumulated in the cells of the KAT Matrix previously described in the methodology chapter to enable a visual interpretation of the pattern of activities most frequently used by the classroom teachers.

Emerging themes that were identified became critical factors in the interpretation of the teachers' decision-making process as they illustrated both internal and external influences on the actions and instructional strategies they utilized during both planning and classroom instruction. The impact of these emerging themes presented an unexpected filter through which the teachers viewed their planning and instruction, and proved to be a confounding factor in analyzing the teachers' true levels of knowledge of algebra for teaching.

The graphical display of the cells for the resulting matrix presented a means for investigating the density of actions and focal centers of knowledge that was accessed by the teachers. That data enabled correlation with the dimensions of knowledge measures

from the initial assessment tool. Within those accumulated occurrences, patterns of use of the overarching themes could be observed.

Finally, overarching themes of bridging, trimming, and decompressing of subject matter that overlaid the three dimensions of knowledge previously discussed were applied to the tasks of teaching to analyze in more depth the way these teachers held and used their knowledge of algebra for teaching to enrich and deepen their students learning experiences and develop pathways for success in future mathematical work.

These actions were conducted for each individual subject teacher and analyzed with respect to the academic background, teaching experiences, learning climate, and knowledge levels for each participant. Subsequently, patterns of similarity of action and distinct reactions were observed and analyzed in an effort to ascertain factors that influenced the decisions and strategies for each subject teacher.

Discussion and Implications

Knowledge of Algebra for Teaching (KAT)

The evidence in numerous research efforts (Ball et al. 2001; Ball & Bass, 2003; Ball, Thames, & Phelps, 2008; Fennema & Franke, 2002; Ferrini-Mundy, 2009; Hill, Rowan, & Ball 2005; Lampert, 1989; Ma, L., 1999) suggests that as teachers are adaptive, flexible, and innovative, manage student misconceptions, parse content into manageable packets, construct multiple representations, facilitate mathematical discussions, and bridge various topics and content, that they draw upon their Mathematical Knowledge for Teaching (MKT), which is contextualized for specific levels and situations. As Ball and her colleagues stated, “what teachers and students are

able to do *together* (emphasis added) with mathematics in classrooms is at the heart of mathematics education...” (Ball et al., 2001, p.433). In the current study, a specific subset of MKT that applies to teaching a first course in algebra, was a primary focal point.

KAT Dimensions of Knowledge

Using the KAT Assessment Form 1, three dimensions of knowledge were assessed for the subject teachers. These are: Teaching Knowledge, Knowledge of School Mathematics, and Advanced Knowledge of Mathematics. Those categories of knowledge cover pedagogical knowledge, content knowledge, knowledge of student learning, and curricular knowledge that enables teachers to effectively plan and prepare for instruction based on students’ proficiencies, curriculum requirements, and anticipated misconceptions. There were substantial differences in the level of KAT possessed by the two classroom teachers in this study. Although the two participating teachers had similar academic preparations, Dean had higher knowledge levels in each area. Dean scored substantially above the mean in all of the metrics used in the assessment tool when compared to teachers with the same level of experience. Janice conversely scored below the mean in every category of the assessment. It was not the focus of this research to establish any causal link for these disparities, but there may be contributing factors that were revealed during data collection and observation. Because Dean had a Master’s of Education, it is possible his increased academic exposure resulted in higher knowledge levels. An alternative explanation may also be found in his prior teaching experience; Dean had previously taught both Calculus and AP Calculus. It is feasible that the preparation for these classes and the actual instruction in those courses contributed to his

elevated knowledge levels. Janice had taught only Pre-Algebra and Algebra 1 in her teaching experience and completed only her Bachelor's Degree with certification in secondary mathematics education. Regardless of the factors that led to the differences in KAT, the impact of this disparity was clearly observable.

Dean had a deeper educational background than Janice that was reflected in their relative scores on the KAT Assessment. Janice scored at below average levels on the three knowledge dimensions in the Knowledge of Algebra for Teaching Assessment with Knowledge of School Mathematics as her lowest score. This was troubling since that category is indicative of an understanding of the topics that comprise the school curriculum of material students are expected to learn in a first course in algebra. Dean scored above average in all three categories with a notable result in Advanced Knowledge of Mathematics nearly two standard deviations above the mean score. That result should indicate a depth and breadth of understanding of topics across the domain and an ability to make connections to advanced ideas anticipated in future math courses. The disparity in these knowledge dimensions resulted in observable differences in the content and manner of instruction in this unit of instruction on solving quadratic equations.

The most obvious difference in KAT was illustrated in the disparity of Advanced Knowledge of Mathematics levels. Compared to teachers with a major in mathematics, there were nearly 3 standard deviations between the scores for Janice and Dean. That difference was apparent in the depth of understanding Dean emphasized at both the procedural and conceptual levels. His language was more mathematically mature and examples more challenging throughout the unit of instruction. Janice was effective at

creating connections within the unit of instruction, but created no continuity with future mathematical work. Her low levels of this advanced knowledge potentially inhibited her ability to construct a learning environment that raised the cognitive demand on her students beyond the stepwise routines for standard problems. These differences were reflected throughout the planning and instructional phases of this unit as seen in the paragraphs below on KAT accessed in planning and instruction.

KAT Knowledge Levels as a Lens for Instructional Approach

This disparity of knowledge manifested in the both the planning and instructional approach used by the two participating teachers. Janice primarily utilized a procedural approach throughout the unit of instruction. Whether her expectations or assessment of the students' algebraic skills may have been influenced by past experience with this group of students, or other external factors impacted those instructional decisions, Janice's routine practice reflected a predisposition to rely on memorization and repetition of a prescribed algorithm to achieve success in the form of a solution. On the other hand, Dean, though he occasionally reverted to memorized processes, made a concerted effort to place emphasis on the underlying logic or conceptual underpinning in his planning and instruction.

Further, Dean's approach to the unit of instruction reflected higher cognitive thinking, use of language, and multiple connections to future mathematical work at the conceptual level. Janice demonstrated a propensity to present instruction that was far less demanding and placed lower cognitive demand on the students. Her approach was tailored toward expedient success through algorithmic repetition or routine manipulation.

The disparity in the knowledge of levels in the three dimensions as described in the KAT Project was observable in the ways the two teachers accessed and utilized that knowledge to make planning and instructional decisions. In the context of the research objective of assessing the KAT each teacher possessed, the findings strongly suggest these differences in knowledge of algebra for teaching had an impact on the planning and implementation of instruction in this unit on solving quadratic equations. The data suggests teachers' KAT is one of the indicators of their ability to effectively enact particular classroom preparation and instructional strategies that reflect their knowledge across the three dimensions.

KAT Accessed for Planning

The transition from arithmetic to algebra places exceptional cognitive demands on both student and teacher as they navigate the difficulties associated with the introduction of symbolic and syntactic representations that are wholly new to the first year algebra student. The challenge of moving from a concrete to an abstract construction requires a new lexicon and a restructuring of conceptual understanding to accommodate manipulation of expressions, variables, equations, and functional representations. The RAND Mathematics Study Panel (2003), called a first course in algebra a “unique and formidable gatekeeper” (p.47) because it serves as a sort of initiation into a level of mathematical instruction that provides the basis for learning further skills across the mathematics domain.

Accessing KAT in Planning to Meet Student Needs

The Algebra 1 classroom teacher has the daunting task of introducing the novice student to that abstract and syntactic environment that defines a first course in algebra.

By default, there is an enormous amount of content that is unfamiliar to incoming students that must be presented in a comprehensible manner while maintaining mathematical integrity. The classroom teacher must ensure the mathematically immature students in this introductory course can negotiate the transition from numeracy to symbolic representation with understanding. For these reasons, it was not surprising that the two subject teachers in this study focused a substantial portion of their planning for instruction in tasks of teaching that centered on core content and content trajectories within the context of this first course in algebra while using tasks that employed explaining and telling.

In the reform movement, teachers have been discouraged from telling and encouraged to use a more student-centered, activity based approach that focuses on open-ended questions engaged by their students. As a more reasoned approach, Chazan and Ball (1995) suggest a refinement to that constricting didactic strategy, recognizing there are many forms of telling, including clarification, definition of unfamiliar terminology, and responding to the evolving dynamics of the classroom to address the needs of the students. The two classroom teachers in this study utilized different forms of telling based on their knowledge of the students' proficiencies and skill level. Janice intentionally structured lesson plans around constructing routine practices that could be rehearsed repeatedly to facilitate the solution of standard quadratic equations by following a stepwise progression. Dean created plans that provided opportunity for explanation of procedure situated in the underlying logic of the algorithm. He also embedded examples that could potentially extend the algorithms in ways that established pathways to learning for future mathematical challenges. The reality, however, was that Dean was often

unable to execute these plans and reverted to a procedural emphasis. Ultimately, though both Janice and Dean expressed that their objectives during planning included instilling the ability in their students to extend their reasoning ability to any standard or non-standard form of a quadratic equation, they spent most of their instructional time explaining or telling them how to approach the task of finding a solution.

Though tasks of teaching utilized by both Janice and Dean accessed their own core content knowledge, the way these tasks manifested in planning for classroom instruction differed in many respects. Janice's unit plan primarily employed a procedural approach to instruction, with emphasis on learning and following a prescribed sequence of actions to achieve a correct solution when graphing or factoring a quadratic equation. Although Dean relied on procedures at times during instruction, he had planned for deliberate actions to clarify the logical underpinnings of those algorithms. Throughout the period of interview and observation, Janice repeatedly lamented the use of an approach dependent on rote memorization but indicated she felt constrained by the limited instructional time available.

The unit plans each teacher constructed reflected both their expectations of their students and the pedagogical strategies indicative of their Teaching Knowledge. Janice relied on any technique accessible to the students that would give them a solution without regard for the logical underpinning. Her dependence on this strategy was likely counter-productive in the end as her students had difficulty recalling those procedures. That plan also intersected with Janice's knowledge of the topics in a first course in algebra, or Knowledge of School Mathematics. Janice accessed her own content knowledge available, but at times had a difficult time with adaptation or explanation of

misconceptions. Dean had unrealistic expectations of his students that impacted his planning activities. He sometimes planned instruction that was accelerated or complicated for the time allotted based on that belief. That strategy occasionally resulted in a modification of instruction to a procedural approach independent of any conceptual connection. The glaring difference in the knowledge accessed for planning was that Dean used his Advanced Knowledge of Mathematics to plan for the incorporation of more complex examples and language. This exposure likely prepared his students for future mathematical experiences across the domain.

KAT Accessed in Planning to Meet Teacher and Student Expectations

Janice, whose Knowledge of School Mathematics (the normal curriculum for an Algebra 1 class) was nearly one standard deviation below peers with a math major (-.893), focused on content or its internal trajectory through the course nearly 70% of the time spent in planning or instruction. Dean, with a level of Knowledge of School Mathematics comparable to his peers (std. dev.107), allocated just over 50% of his planning and instruction to a focus on core content or the content trajectory within the unit of instruction. The most unexpected observation of this density of tasks of teaching versus categories of knowledge was that Dean, with a very high level of Advanced Knowledge of Mathematics (1.934 standard deviations above the mean) was not more diverse or advanced in his own instruction.

The balance of planning and instructional time available to Janice was spent introducing and reinforcing representations or language conventions that served as memory anchors for the class as an entity. She gave 10% of the time to building representations of the content such as a template for factoring or stepwise routines for

finding solutions by graphing. The remaining 20% of her time was focused on establishing language conventions that were used as prompts to steer the efforts of students toward a solution. These conventions included terms like “Bottoms Up” or “DOTS” for factoring techniques as well as “FOIL” when factoring quadratic expressions with unit leading coefficients. Dean divided his remaining time relatively equally between building representations across the explanatory tasks (22%) and making efforts to construct conceptual learning pathways by emphasizing Applications and Contexts, or Reasoning and Proof knowledge categories (27%) through his use of bridging and decompressing.

When considered as a cumulative effort, neither Janice nor Dean regularly accessed their knowledge in the areas of applications or reasoning and proof at the higher cognitive demand end of the tasks versus knowledge spectrum. Throughout several interviews and three classroom observations each, their combined total for these categories placing higher cognitive demand on the students was only 18/165 actions or roughly 11% of their combined planning and instructional time. Both Janice and Dean had a clear understanding of their students’ skill levels and their instructional needs, and consequently tailored the classroom instruction to fit the expectations they held for the class.

In this area (accessing KAT for planning) of the investigation, levels of the subject teachers’ knowledge in the three dimensions of knowledge indicated in the KAT Instrument were surprisingly not a factor in differentiating the type of instructional tasks utilized by the subject teachers. It is possible they both felt the cognitive demands of a first course in algebra dictate a heavier reliance on instruction centered on content to

establish a base of understanding in anticipation of the mathematical work required in mathematics courses at a higher level. That reliance likely dictated the pedagogical decisions made by the two subject teachers in this study. The implication of this instructional pattern is that, although knowledge dimensions reliably correlated to classroom instructional approach, the mathematical maturity of the students coupled with the complexity of the topic were of greater importance to the classroom teacher during planning and instruction.

KAT Accessed in Instruction

The core issue in this study was the investigation of how teachers *use* their knowledge of algebra for teaching in the *actual practice of teaching*. The tasks of teaching utilized and categories of mathematical knowledge accessed in applying those tasks are indicators of how teachers hold and employ their knowledge in the most effective manner based on classroom conditions and evolving student needs. Within this context of the dimensions of knowledge: Teaching Knowledge, Knowledge of School Mathematics, and Advanced Knowledge of Mathematics, elements of pedagogical content knowledge, mathematical knowledge, and horizon, or curricular, knowledge exist in concert to convey the complex skill set needed to successfully teach a first course in algebra. Overlaying these dimensions of knowledge are three overarching themes: Bridging, Trimming, and Decompressing (McCrary et al, 2012) that weave the dimensions of knowledge together to reveal a clear, robust picture of how classroom teachers ultimately apply their knowledge in practice. Observing those overarching themes in this study as they permeated the continuous access and application of teachers' knowledge of algebra for teaching gave insight into the ways these two subject teachers

flexibly used their own knowledge and held beliefs. The patterns that emerged from that observation elucidated the important impact of the differing levels of knowledge in each of the three dimensions.

Accessing KAT in Instruction to Meet Student Needs

Realizing the student composition for both teachers were students in a first course in algebra, there were limitations to the complexity of instruction the situation could support due to inherent student deficiencies. Both teachers demonstrated Teaching Knowledge as they addressed student needs during their instruction, using explication and telling as a primary tool to introduce new topics. Dean was able to extend that instruction to some degree by making stronger connections to future mathematical work. The differences in instruction mirrored those in planning; Janice taught from a procedural standpoint, offering a stepwise process to reach a solution that would translate into success on the end of course test. Dean made frequent attempts to incorporate a conceptual understanding during instruction while at times relying on a procedure to complete the task during classroom discussions. They each accessed content knowledge in the dimension of Knowledge of School Mathematics as they taught the methods of solutions for quadratic equations, and each focused primarily on conveying topical knowledge in the content with varying degrees of enhancement. Dean's higher level of Advanced Knowledge of Mathematics enabled him to construct a thread of continuity between this course in algebra and future coursework in the mathematics curriculum.

In the next section, the enactment of KAT dimensions of knowledge viewed through the lens of these overarching concepts, observed during the instructional phase of this unit, emerges as the primary contributing factor in explaining the impact of differing

levels of KAT on classroom interactions and instructional strategies. Instructional practices by each subject teacher were also determined to be heavily influenced by the emerging themes discussed in the previous chapter, adding a complexity to the observation of the learning environment. Those emerging themes provided an additional lens through which to view the resulting analysis of the knowledge accessed for the purposes of instruction.

Evidence of Bridging, Trimming, Decompressing Across KAT Dimensions in Instruction

Bridging, Trimming and Decompressing overlay the knowledge dimensions in the KAT Framework. Both of the subject teachers used these overarching themes, but with varying skill. While Dean was able to use bridging to make connections across the mathematics domain, Janice established narrow connections only within the unit of instruction. At various times, both teachers utilized trimming to adapt and restructure instruction to make it comprehensible to their students. Janice presented simplified language that helped establish standard routines as did Dean, but Dean also used trimming to incorporate ties to the underlying logic and applications beyond the unit. Decompressing was evident at different levels and in differing applications. Janice knew what decompressing entailed, but was only able to unpack procedures absent the accompanying conceptual underpinning, while Dean was able to provide depth to the instruction that created pathways to higher cognitive skills.

Impact of Emergent Themes on KAT Accessed During Instruction

Two themes that emerged during interviews and classroom observations had substantial impact on the interpretation of how the knowledge of algebra for teaching was used in planning and instruction. These were: Wait-Time Issues and Testing Preparation Issues. The effect each of these themes had on that analysis was important in very different ways. While allowing appropriate wait-time, or “think-time” (Stahl, 1994, p.2) proved to be difficult for both Janice and Dean, the issue of test preparation and perceived pressure associated with testing was only apparent in the case of Janice, the subject teacher at a public school.

Students need time to process information and formulate appropriate responses during various times in the classroom. Rowe (1986), who developed the concept of wait-time, found that extending the pause between questioning and student responses resulted in dramatic differences in the quality and substance of student responses and also decreased the teacher-centered action of telling. Other research such as that by Stahl (1994), determined that an increase in wait-time also improved scores on academic achievement tests. From the perspective of the teacher, wait-time increases also have a positive effect on behavior. The classroom teacher uses a more varied and complex questioning strategy and expectations of student performance increase when wait-time after student responses as well as before providing correct answers (Rowe, 1986; Stahl, 1994; Fletcher-Wood, 2013).

Impact of Wait-Time Not Provided on Instruction

During the observation of classroom instruction, both Janice and Dean failed on numerous occasions to allow sufficient wait-time after posing a question. In each case,

the held beliefs of the teachers motivated this strategy in ways that are described in detail below.

In the case of Janice, two factors came into play: there was a limited time period to cover a substantial amount of material, and Janice had lower expectations of her students based on previous experience in the classroom. During the three classroom observations, Janice missed an opportunity to encourage higher order thinking, or clarify student misconceptions, on 21 separate occasions. At an average pace of seven times each class, Janice posed a question or problem, then immediately provided the correct answer without explanation. In doing so, Janice missed the opportunity to establish higher cognitive demands and nurture more sophisticated information processing. The dimension of knowledge defined as Teaching Knowledge by the KAT Project team refers to knowledge outside academic courses that includes understanding how students learn, their common misconceptions, and knowing what to emphasize to promote continuity of curricular trajectories. Janice had a score in this dimension well below the mean score ($-.595$ std. dev. below the mean) on the KAT Assessment when compared to teachers with a college major of mathematics. In many ways, her instruction was procedural in nature and enacted through telling in ways that did not encourage student exploration. Combined with the imposition of a restructured curriculum due to a school-wide scheduling change to a block schedule, this deficiency in Teaching Knowledge was likely a major contributor in her lack of understanding of the effective use of appropriate wait-time.

Dean also had a below average knowledge level ($-.328$ std. dev. below the mean) in Teaching Knowledge compared to other math majors, even though he had a good

Knowledge of School Algebra and an excellent Advanced Knowledge of Mathematics (1.934 std. dev. above the mean). In Dean's case, his expectations for his students' performance was unrealistically high as he believed they were capable of a faster pace of assimilation than typical classes in a first course in algebra. For this reason, he frequently stated a question, then immediately gave the solution as he thought the class was already aware of the correct answer. During three observations of a 45 minute duration each, Dean missed the opportunity 20 times to check for understanding, enrich the learning experience, or clarify potential misconceptions. In comparison, this rate is actually twice the average Janice demonstrated as her classes were 90 minutes in length. The common factor in these cases is a lower level of Teaching Knowledge which includes pedagogical content knowledge (KAT Project, Ferrini-Mundy et al, 2005). Both Rowe (1994) and Cashin (1995) suggest that some teacher characteristics such as lack of confidence or anxiety about loss of positional control of the classroom may contribute to the lack of wait-time. Cashin also adds the admonition that establishing a culture in which wait-time is expected is necessary at an early stage to avoid the eventuality of dealing with a classroom in which all the students learn "when the instructor asks a question, she does not expect an answer" (p.5). These comments by Rowe and Cashin suggest a deficiency in Teaching Knowledge is a probable cause in the wait-time issues outlined above for the two subject teachers.

Impact of High Stakes Testing on Instruction

The issue of anxiety over high stakes testing, however, presented an even more challenging obfuscation of the true application of KAT in Janice's classroom. In the current climate of standards-based, high stakes testing and accountability in our public

schools, many teachers operate under a system that utilizes testing results as a contributing factor in teacher evaluation. In a 2015 report for the National Council on Teacher Quality by Doherty & Jacobs, 43/50 states report using objective measures of student achievement as part of the evaluation process. Until the current moratorium to review this policy in Tennessee, Janice worked in a school district that weighed the testing results as 35% of her evaluation for purposes of contract renewal. Concurrently, her school changed to a block schedule from a seven period year-long course of instruction at the beginning of the school year, thus necessitating a restructuring of the Algebra 1 curriculum. Against this backdrop, it was not surprising that student outcomes on the TNReady Standards-Based Test were of increased interest to Janice during the interview process. Dean, however, experienced no impact vis á vis high stakes testing for accountability.

References to the TNReady Test became so prevalent that it was sometimes difficult to ascertain the actual level of KAT Janice possessed, and the ways she accessed that knowledge for purposes of planning and instruction. Based on the frequency with which Janice referred to the content and structure of the impending TNReady test, there is strong evidence that this emerging theme played an important role in her planning and structuring of instruction during the unit on solving quadratic equations. Janice's preoccupation with the testing schedule inhibited her ability to modify the length or sequence of instruction to address student needs and also limited the complexity of the instruction she could provide. These results correlate with lower Teaching Knowledge and Knowledge of School Mathematics that Janice possessed. Although this external factor possibly obscured some of the ways Janice used her KAT, her overall patterns of

instruction and classroom management were still deemed to be reliable indicators of the essence of her KAT and the means by which she accessed and deployed this knowledge during this unit of instruction. In the opinion of this study, Janice, though pre-occupied with the TNReady, exhibited an inability to plan or implement teaching at moderate or high cognitive demand levels.

Implications and Future Research

At the outset of this study, the research objectives were to examine the actions taken by classroom teachers to facilitate the learning of methods for solving quadratic equations through observation of their cognitive processes in planning and conducting their instruction. Utilizing the KAT Framework, the study created an objective overlay to accumulate the diverse teacher tasks employed and relate these tasks to categories of the teacher's own knowledge for teaching a first course in algebra. The results of this study add to the existing knowledge in that context, but may fall short of giving a full representation of the dynamics present in the classrooms of the two subject teachers.

Marcus and Chazan (2010) suggest another dimension useful for interpreting teacher actions in the classroom; a sensitivity to the needs of their students that translated into bridging, trimming, or decompressing the mathematics in what may initially be viewed as poor pedagogical choices or incorrect application of processes as a means of expedience to lead their students to some degree of success. Their findings suggest the occurrences of this sensitivity are “places where one can find teachers’ knowledge in action and where one find the results of learning from teaching” (pp. 185-186). In these two cases, the teachers were sensitive to the level of knowledge they thought their students possessed. For example, in the case of Janice, she believed that a focus on

procedures better prepared students for their high stakes assessment. Consideration of those sensitivities may lead to the addition of another layer of analysis when using the KAT Framework, a dimension that places more emphasis on negotiating the complexities of abstraction in algebra that students in an introductory course frequently encounter.

Secondly, the implications of the emerging themes was dichotomous. On the one hand, the practice of allowing appropriate wait-time continues to be an area of growth for many classroom teachers and should be the subject of continuing emphasis through education and professional development. The issue of anxiety with test results is more complicated in the current high stakes climate. Accountability for providing depth of understanding in any topic is critical to all classroom teachers, but policy makers should be sensitive to the learning environment and provide clarity of expectations. Additionally, the presence of this confounding factor should be considered as part of any future investigation in application of teacher knowledge where end-of-course tests are administered for school accountability reports.

Additionally, although the different schedules, block scheduling versus a year-long scheduling, presented an opportunity to observe teachers in diverse learning climates, it also may have presented a confounding factor because pacing and sequencing decisions are necessarily different in these disparate circumstances. More research is needed to validate or extend these findings when the participants operate under the same scheduling scheme. Like many other contextual factors, including assessment decisions, scheduling may influence the way teachers draw on their KAT as they adapt their planning and instruction to fit time constraints. It is, then, possible that the subject teachers in this study modified their instructional plans in ways that were not transparent

to the research based on the length of time available in their respective scheduling routines.

Since the development of the KAT Framework, and prior to this study, there has been a paucity of subsequent research to isolate an individual topic in algebra, or any another course, as a context for analyzing teacher knowledge in action in the classroom. Li (2011) investigated how subject matter knowledge and pedagogical knowledge was manifested when teaching algebraic routines in solving quadratic equations with the quadratic formula, while Aritgue et al (2001) examined the epistemological, cognitive, and didactic dimensions of teacher knowledge necessary to teach a first course in algebra, emphasizing that different nuances and strategies would emerge as various topics were taught. Neither of these previous studies attempted to provide a robust picture of the interaction between knowledge of algebra for teaching and decisions across the spectrum of planning and executing a unit of instruction in this area of solving quadratic equations.

The results of this study, consistent with earlier research in Algebra I, indicate that a teacher's knowledge in the dimensions of Teaching Knowledge, Knowledge of School Mathematics, and Advanced Knowledge of Mathematics influence planning and teaching decisions when teaching about quadratic functions. As the body of knowledge develops and grows, the understanding of this impact on classroom instruction will also be better understood and predicted. This understanding is useful in determining where teacher preparation through education courses or professional development should be targeted to address deficiencies in teacher knowledge. As a continuing area of investigation, more research is needed to examine the way this knowledge of teaching

algebra is manifested across the curriculum, and the entire domain of mathematics instruction at the secondary level.

The result of the disparity in the knowledge levels of each of these teachers was observable and measurable based on the way each of them held and used their knowledge. The implication of these results is twofold: 1.) knowledge in the KAT dimensions can be assessed and should be addressed in educational as well as professional development opportunities, and 2.) teacher preparation programs should be cognizant of the levels of knowledge in the dimensions described in this study that pre-service teachers acquire, and make efforts to remediate deficiencies prior to completion of the education program.

Finally, it is possible the previous experiences of these two teachers in this study were a critical factor in formulating their instructional approach. In their study on teaching and learning algebra cited above, Artigue et al (2001) expressed a belief that knowledge for teaching algebra was an evolutionary process and dependent on personal coursework, internal belief systems, and experiences in actual practice. Janice had spent her entire teaching career instructing introductory algebra courses, while Dean taught not only algebra and geometry, but had previously taught both Calculus and AP Calculus. The diversity and advanced level of the courses Dean taught could have enabled him to develop a broader curricular knowledge, and refine his advanced knowledge of mathematics enabling him to present instruction in a more global sense as part of the larger mathematical domain.

Although there is no real evidence in research that past experiences outside of teaching increase the ability to employ knowledge of algebra for teaching, Dean had prior

experiences requiring high cognitive function and management of time constraints, whereas Janice had no other professional experience. Previous research on the effects of prior experience (NCATE, 2010) has focused on student achievement as a metric for the impact of prior experience, noting that past experiences produce no measurable increase in student performance on assessments. That perspective may be too restrictive to allow an effective and accurate portrayal of the impact of experiences on the complex task of teaching a first course in algebra. The Principles and Standards for School Mathematics (NCTM, 2000), suggest that being “proficient in a complex domain such as mathematics entails the ability to use knowledge flexibly, applying what is learned in one setting appropriately in another” (p. 20). Additional research efforts focused on the domain of teachers’ mathematical knowledge in practice could refine these findings and may indicate there is value in providing diverse teaching experiences.

Limitations

There are several limitations to this study. First, it was limited to a single unit of instruction. Ball and Bass (2003) note that the nuances of appropriate knowledge as well as individual student needs vary greatly from one academic level to another. As a result, the findings may not accurately reflect teacher practices across the entire algebra curriculum.

While data were gathered on planning and teaching, there is much planning and in-the-moment teaching decisions that are not captured in the data. It is likely the subject teachers drew on their KAT in preparing for this instruction in solving quadratic equations in ways that were not observed prior to the initiation of data collection. This

limitation could have impacted some of the observations made during the collection and analysis of data, and bears consideration in future research.

In the interest of diversity in learning environments, two subject teachers were selected in different scheduling schemes. Janice taught in a public school that had reverted to a block scheduling plan at the beginning of the academic year, while Dean was teaching in a school using an eight period, year-long scheduling plan. This design successfully provided opportunities to observe different learning climates, but also created a limitation in comparing the actions of the teachers in this unit. The block schedule required coverage of the curriculum in one semester, and also created a planning challenge to Janice because it necessitated a restructuring of her previous year-long curriculum. In the year-long schedule, Dean was able to provide different periods of review even though each of his class periods were only half the length of Janice's class period. As with the focus on a single topic, the reader can likely associate patterns observed here with their individual circumstances, but generalization of the findings is limited by these conditions.

Summary

There are many facets to the knowledge teachers of mathematics bring to the classroom teaching experience. That knowledge, grounded in past personal experiences, academic course work, education in preparation, and held beliefs about the teaching vocation, inevitably impacts the manner in which teachers approach their classroom management and instruction. In this study, a snapshot of that knowledge was examined in a relationship to the teaching of quadratic functions in a first course in algebra. The results underline and validate anecdotal and empirical beliefs that teachers' knowledge of

their craft and their students matters and strongly influences the way they use that knowledge in practice.

Although the KAT Framework emphasizes the connection between knowledge across the content, pedagogical, and curricular domains and the actions teachers take in planning and delivering instruction in the algebra classroom, the complexities of the learning environment sometimes transcend the seemingly obvious. The application of certain tasks of teaching are predictably observable based on the depth of knowledge within the dimensions of knowledge utilized in the KAT Framework, but there are unexpected nuances that can dramatically impact the way instruction is structured and delivered. Because Dean had exceptionally high Advanced Knowledge of Mathematics, the high expectations he held of his students' proficiencies were skewed causing him to alter his instruction through an absence of appropriate wait-time that would allow students more time to construct mastery learning skills and deepen conceptual understanding, while Janice held such low expectations she routinely neglected both wait-time and connections to conceptual underpinning in favor of developing standard routines. Though Dean had deeper content knowledge, and a breadth of experience, he frequently reverted to rote memorization or procedural skill emphasis for expediency, while Janice relied heavily on procedural skills because she had external pressure from a revised curriculum and expectations by administration of achievement on a high-stakes test. Finally, though there was substantial disparity between the knowledge levels across all three dimensions in the KAT Framework, the knowledge accessed by each teacher was nearly identical throughout the unit of instruction, perhaps due to the abstract nature of the algebra content.

As Ferrini-Mundy et al (2005) said, “In order to make more effective decisions in both policy and practice, we need to understand better how teachers effectively draw on knowledge in teaching, what kinds of knowledge seem most important for teachers to use, and how to assess this knowledge” (p.11). In the continuum of research on MKT, and particularly the subset of that domain known as KAT, the objective of this research was to further the discussion of what knowledge is important, and what that knowledge looks like in actual practice inside the classroom. Moving forward, this study provides a springboard to further the investigation into the critical responsibility of determining the levels of appropriate knowledge teachers possess, and the repercussions of that knowledge on classroom instruction.

REFERENCES

- Ames, C., & Archer, J. (1988). Achievement goals in the classroom: Students' learning strategies and motivation processes, *Journal of Educational Psychology*, 80(3), 260-267.
- Anderman, E., Eccles, J., Yoon, K., Roeser, R., Wigfeld, A., & Blumenfeld, P. (2001). Learning to value mathematics and reading: Relations to mastery and performance-oriented instructional practices. *Contemporary Educational Psychology*, 26, 76-95.
- Anderson, L. M., & Holt-Reynolds, D. (1995). *Prospective teachers' beliefs and teacher education pedagogy: Research based on a teacher educator's practical theory* (Research Report 95-6). East Lansing, MI: National Center for Research in Teacher Learning, Michigan State University.
- Artigue, M. Assule, T., Grugeon, B., & Lenfant, A. (2001). Teaching and learning algebra: Approaching complexity through complementary perspectives, in H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.), *The Future of the Teaching and Learning of Algebra, Proceedings of the 12th ICMI Study Conference* (pp.21-32. Melbourne, Australia: The University of Melbourne.
- ASCD Common Core Adoption by State (2013). Retrieved July, 17, 2013 from <http://www.ascd.org/common-core-state-standards/common-core-state-standards-adoption-map.aspx>.

Ball, D. (1991). Research on teaching mathematics: Making subject matter part of the equation. In J. Brophy (Ed.), *Advances in Research on Teaching: Vol. 2, Teachers' Knowledge of Subject Matter As It Relates to Their Teaching Practices* (pp. 1-48). Greenwich, CT: JAI Press.

Ball, D., & Bass, H. (2000). Interweaving Content and Pedagogy in Teaching and Learning to Teach: Knowing and Using Mathematics. In J. Boaler (Ed.), *Multiple Perspectives on the Teaching and Learning of Mathematics* (pp.83-104). Westport, CT: Ablex Publishing.

Ball, D. & Bass, H. (2003). Toward a practice-based theory of mathematical knowledge for teaching. In B. Davis & E. Simmt (Eds.). *Proceedings of the 2002 Annual Meeting of the Canadian Mathematics Education Study Group* (pp. 3-14). Edmonton, AB: CMESG/GCEDM.

Ball, D. L., Hill, H. C., & Bass, H. (2005). Knowing mathematics for teaching: Who knows mathematics well enough to teach third grade, and how can we decide? *American Educator*, 29(1), 14-17, 20-22, 43-46.

Ball, D., Lubienski, S., & Mewborn, D. (2001). Research on teaching mathematics: The unsolved problem of teachers' mathematical knowledge. In V. Richardson (Ed.). *Handbook of Research on Teaching* (4th ed.) (433-456). New York: McMillan.

Ball, D., Thames, M., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, 59(5).

Bassey, M. (1981). Pedagogic research: on the relative merits of search for generalisation (sic) and study of single events. *Oxford Review of Education*, 7, 73-93.

Borko, H., Frykholm, J., Pittman, M., Eiteljorg, E., Nelson, M., Jacobs, J., Koellner-Clark, K., 2005. Preparing teachers to foster algebraic thinking, *ZDM*, 37(1), 43-52.

Breiteig, T., Grevholm, B. & Kislenko, K. (2005). Beliefs and attitudes in mathematics teaching and learning. *Vurdering i matematikk-Hvorfor og hvordan*, 129-138.

Butterfield, B., & Chinnapan, M. (2010). Walking the talk: Translation of mathematical content knowledge to practice. Retrieved July 2, 2011, from <http://ro.uow.edu.au/edupapers/173>.

Calderhead, J. (2002). Knowledge Bases of Teaching. Retrieved June 23, 2011 from <http://www.answers.com/library/Education+Encyclopedia-cid-13576171>.

Cashin, W., (1995). Answering and asking questions. Idea paper no. 31, Center for Faculty Evaluation & Development, January, 1995.

Center for Research in Mathematics and Science Teacher Education (CRiMSTeD). Diagnostic Teacher Assessments for Middle School (DTAMS) project, University of Louisville. Retrieved February 15, 2015 from <http://louisville.edu/education/research/centers/crmstd>

Chazan, D. & Ball, D. (2005). Beyond Exhortations Not to Tell: The Teachers' Role in Discussion-Intensive Mathematics Classes.

Chouinard, R., Karsenti, T., & Roy, N. (2007). Relations among competence beliefs, utility value, achievement goals, and effort in mathematics. *British Journal of Educational Psychology*, 77, 501-517.

- Cline, H. & Mandinach, E., (2000). The Corruption of a Research Design: A Case Study of a Curriculum Innovation Project
- Mestre, J. (2000). Progress in Research: The Interplay Among Theory, Research Questions, and Measurement Techniques in A. Kelly and R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education*, (pp.169-190). Mahwah, NJ: Lawrence Earlbaum Associates, Inc.
- Cornell Center for Teaching Excellence, retrieved January 2016.
- Dewalt, K. and Dewalt, B. (2002) *Participant Observation*. Lanham, MD. Rowman & Littlefield
- Dewey, J. (1920). *Reconstruction in Philosophy*. New York: Holt.
- Didiş, M. Baş, S., & Erbaş, A. (2011). Students' reasoning in quadratic equations with one unknown. *The Seventh Congress of the European Society for Research in Mathematics Education, University of Rzeszow, Poland, February, 2011* (pp. 1-10).
- Ding, M. (2007). Knowing mathematics for teaching: A case study of teacher responses to students' errors and difficulties in teaching equivalent fractions (Doctoral Dissertation, Texas A&M, August, 2007).
- Doherty, K. & Jacobs, S. (2015). State of the States: Evaluating Teaching, Leading and Learning, November 2015, Washington, D.C.: National Council on Teacher Quality.
- Fennema, E. & Franke, M. (1992). Teachers' knowledge and its impact. In D. Grouws (Ed.), *Handbook of Research on Mathematics and Learning* (pp. 147-164), Reston, VA: National Council of Teachers of Mathematics.

Ferrini-Mundy, J., Floden, R., McCrory, R., Burrill, G., Sandow, D. (2005). *Knowledge for teaching school algebra: Challenges in developing an analytic framework*. East Lansing, MI: Michigan State University, Knowledge for Teaching Algebra Project.

The Final Report of the National Mathematics Advisory Panel on Mathematics (2008). Washington, DC: U.S. Department of Education.

Fletcher-Wood, H., (2013). Before you say anything – count to three...retrieved from <http://improvingteaching.co.uk/2013/08/17/increasing-wait-time/> March 14, 2016.

Fries-Gaither, J. (2008). Questioning techniques: Research-based strategies for teachers. Retrieved from <http://beyondpenguins.ehe.osu.edu/issue/energy-and-the-polar-environment/questioning-techniques-research-based-strategies-for-teachers>, June, 2016.

Guba, E.G. (1981). Criteria for addressing the trustworthiness of naturalistic inquiries. *Educational Communication and Technology Journal*, 29, 75-91.

Hill, H. & Ball, D. (2009) R&D: The Curious – and Crucial – Case of Mathematical Knowledge for Teaching. *Phi Delta Kappan*, 91(2), pp.68-71.

Hill, H. C., Blunk, M. L., Charalambous, C. Y., Lewis, J. M., Phelps, G. C., Sleep, L., et al. (2008). Mathematical knowledge for teaching and the mathematical quality of instruction: An exploratory study. *Cognition and Instruction*, 26, 430-511.

Hill, H., Rowan, B., & Ball, D. (2005). Effects of teachers' mathematical knowledge on student achievement. *American Educational Research Journal*, 42(2), 371-406.

Hill, H.C., Schilling, S.G., & Ball, D.L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *Elementary School Journal*, 105(1), 11-30.

Hoch, M., & Dreyfus, T. (2005). Students' difficulties with applying a familiar formula in an unfamiliar context. In Chick, H. & Vincent, J. (Eds.). *Proceedings of the 29th Conference of the International Group for the Psychology of Mathematics Education*, vol. 3 (145-152). Melbourne, Australia: Practice in Mathematics Education.

Iszak, A. (2008). Knowledge for teaching fraction multiplication. *Cognition and Instruction*, 26, 95-143.

Iszak, A., Jacobson, E., de Araujo, Z. Orrill, C. (2012). Measuring mathematical knowledge for teaching fractions with drawn quantities. *Journal for Research in Mathematics Education*, 43(4), 391-427.

Kelly, M., (2016). Wait time. Retrieved from <http://712educators.about.com/od/teachingstrategies/g/Wait-Time.htm> March 14, 2016.

Kleve, B. (2009). Aspects of a teacher's mathematical knowledge in a lesson on fractions. In M. Joubert (Ed.), *Proceedings of the British Society for Research into Learning Mathematics*, 29(3), November, 2009.

Kotsopoulos, D. (2007). Unravelling student challenges with quadratics: A cognitive approach. *Australian Mathematics Teacher*, 63(2), 19-24.

Lampert, M. (1989). Choosing and using mathematical tools in classroom discourse. In J. Brophy (Ed.), *Advances in Research on Teaching: A Research Annual*. Greenwich, CT: JAI Press.

- Lampert, M. (1990). When the problem is not the question and the solution is not the answer: Mathematical knowing and teaching. *American Educational Research Journal*, 27(1), 29-63.
- Li, X. (2011). Mathematical knowledge for teaching algebraic routines: a case study of solving quadratic equations. *Journal of Mathematics Education*, Vol. 4, No. 2, pp.1-16.
- Lima, R. (2008). *Procedural embodiment and quadratic equations*. Paper presented at the International Congress on Mathematical Education, Monterrey, Mexico, July 6-13, 2008.
- Lima, R., & Tall, D. (2010). An example of the fragility of a procedural approach to solving equations (Draft), 1-20. Retrieved June 13, 2013 from homepages.warwick.ac.uk/staff/David.Tall/pdfs/dot2010x-lima.
- Lincoln, Y. and Guba, E.G. (1985). *Naturalistic Inquiry*. , Beverly Hills, CA: Sage Publications.
- Ma, L. (1999). *Knowing and teaching elementary mathematics: Teachers' understanding of fundamental mathematics in China and the United States, Anniversary Edition, 2010*. Abingdon, Oxon: Routledge Publishing.
- Marcus, R., & Chazan, D. (2010). What Experienced Teachers Have Learned from Helping Students Think About Solving Equations in the One-Variable-First Algebra Curriculum. In R. Leikin & R. Zazkis (Eds.), *Mathematics Teacher Education: Vol. 5. Learning through teaching mathematics: Development of teachers' knowledge and expertise in practice* (pp.169-187). Springer Netherlands.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M., Senk, S. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices. *Journal for Research in Mathematics Education*, 43(5), 584-615.

McEwan, E.K. (2004). *Seven strategies of highly effective readers*. Thousand Oaks, CA: Corwin Press.

Merriam, S. (2009). *Qualitative Research: A Guide to Design and Implementation*. San Francisco, CA: Jossey-Bass.

Mestre, J. (2000). Progress in Research: The Interplay Among Theory, Research Questions, and Measurement Techniques in A. Kelly and R. Lesh (Eds.), *Handbook of Research Design in Mathematics and Science Education*, (pp.151-168). Mahwah, NJ: Lawrence Earlbaum Associates, Inc.

Miles, M. and Huberman, M. (1994). *Qualitative Data Analysis: An Expanded Sourcebook* (2nd ed). Thousand Oaks, CA: Sage Publications.

Mosteller, Frederick, (1995).The future of children CRITICAL ISSUES FOR CHILDREN AND YOUTHS. Vol. 5, No. 2 – Summer/Fall 1995.

National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*.

National Council of Teachers of Mathematics. (2014). *Principles to Actions: Ensuring Mathematical Success for All*.

National Council for Accreditation of Teacher Education (2010). What Makes a Teacher Effective? Retrieved May 11, 2016 from

<http://www.ncate.org/public/researchreports/teacherpreparationresearch/whatmakesateacher-effective/tabid/361/default.aspx>

Niss, M. (1994). Mathematics in society. In R. Biehler, R. Scholz, R. Straber, & B. Winkelmann (Eds.). *Didactics of Mathematics as a Scientific Discipline* (pp. 367-378). Dordrecht, Netherlands: Kluwer Academic Publishers.

OECD (2014), “A Profile of Student Performance in Mathematics”, in *PISA 2012 Results: What Students Know and Can Do* (Vol. 1, Revised edition, February, 2014): *Student Performance in Mathematics, Reading and Science*, OECD Publishing.

Patton, M. (2002). *Qualitative Research and Evaluation Methods, 3rd Edition*. Thousand Oaks, CA: Sage Publications.

Phan, H. (2009). Reflective thinking, effort, persistence, disorganization, and academic performance: A meditational approach. *Electronic Journal of Research in Educational Psychology*, 7(3), 927-952.

Pitts, J.M. (1994). Personal understandings and mental models of information: a qualitative study of factors associated with the information-seeking and use of adolescents (Doctoral Dissertation, Florida State University, 1994).

Pressley, M. et al, (1992). Beyond direct explanation: transactional instructional of reading comprehension strategies. *The Elementary School Journal*, 92(5), 513-555.

RAND Mathematics Study Panel. (2003). *Mathematical Proficiency for All Students: Toward a Strategic Research and Development Program in Mathematics Education*.

Rand Mathematics Study Panel, D. Ball, Chair. Office of Educational Research and Improvement: U. S. Department of Education.

Rauff, J. (1994). Constructivism, factoring, and beliefs. *School Science and Mathematics*, 94(8), 421-426.

Romberg, T. (1992). Perspectives on Scholarship and Research Methods in D. Grouws (Ed.), *Handbook Research on Mathematics Teaching and Learning*, (pp.49-64). Reston, VA: National Council of Teachers of Mathematics.

Rowe, M., (1986). Wait time: slowing down mat be a way of speeding up! *Journal of Teacher Education*. 1996, 37-43.

Schilling, S. & Hill, H. (2007). Assessing measures of mathematical knowledge for teaching: A validity approach. *Measurement: Interdisciplinary Research and Perspectives*, 5(2-3), 70-80.

Schoenfeld, A. (2002). A highly interactive discourse structure. *Social Constructivist Teaching*, 9, 131-169.

Shenton, A. (2004). Strategies for ensuring trustworthiness in qualitative research papers. *Education for Information*, 22, 63-75.

Shulman, L. (1986). Those Who Understand: Knowledge Growth in Teaching. *Educational Researcher*, 15(2), 1-22

Shulman, L. S. (1987). *Knowledge and teaching: Foundations of the new reform*. Harvard Educational Review, 57(1), 1-22.

- Silverman, D. (2001). *Interpreting qualitative data: methods for analysing talk, text, and interaction* (2nd ed.). London: Sage Publications.
- Sowder, J. (2007). *The mathematical education and development of teachers*. Retrieved April 14, 2011 from http://www.sci.sdsu.edu/CRMSE/old_site/jsowderpaper.htm.
- Spangler, D. (1992). Assessing Students' Beliefs About Mathematics. *The Mathematics Educator*, 3(1). 19-23.
- Stahl, R., (1994). Using think-time and wait-time skillfully in the classroom. ERIC Digest, ED370885, May, 1994.
- Stake, R. E. (1995). *The art of case study research*. Thousand Oaks, CA: Sage Publications.
- Stake, R.E. (2006). *Multiple Case Study Analysis*., New York: The Guilford Press.
- Stylianides, A. & Ball, D. (2004). *Studying the mathematical knowledge needed for teaching: The case of teachers' knowledge of reasoning and proof*. Paper prepared for the 2004 Annual Meeting of the American Educational Research Association, San Diego, CA.
- Tschannen-Moran, M., Woolfolk, A. & Hoy, W. (1998). Teacher efficacy: Its meaning and measure. *Review of Educational Research*, 68(2), 202-248.
- Vaiyavutjamai, P. & Clements, M. (2006). Effects of Classroom Instruction on Students' Understanding of Quadratic Equations. *Mathematics Education Research Journal*, 18(1), 47-77.

Vaiyavutjamai, P., Ellerton, N., & Clements, M. (2005) *Students' attempts to solve two elementary quadratic equations: A study in three nations*. Retrieved June, 17, 2013 from www.merga.net.au/documents/RP852005.pdf

Yin, R. K. (2003). *Case study research: Design and methods* (3rd ed.), *Applied Social Research Methods Series* (Vol. 5). Thousand Oaks, CA: Sage.

Yin, R.K. (2014) *Case Study Research: Design and Methods* (5th Ed.). Los Angeles: Sage Publications.

Zemelman, S., Daniels, H., Hyde, A. (2005). *Best Practice: Today's Standards for Teaching and Learning in America's Schools* (3rd ed.). Portsmouth, NH: Heinemann Publishing.

APPENDICES

Survey of Knowledge for Teaching Algebra

Michigan State University
2006

Assessment Questions

Form 1

- * Please mark your choices and write your responses in the Answer Booklet
- * Please return this form to the administrator together with your Answer Booklet after completion

Copyright 2006, Knowing Mathematics for Teaching Algebra (KAT) Project, NSF REC-0337595, Division of Science and Mathematics Education, Michigan State University.
Not for reproduction or use without written consent of KAT.

1. At a storewide sale, shirts cost \$8 each and pants cost \$12 each. If S is the number of shirts and P is the number of pants bought, which of the following is a meaning for the expression $8S + 12P$?
- A. The number of shirts and pants bought
 - B. The cost of 8 shirts and 12 pants
 - C. The cost of P shirts and S pants
 - D. The cost of S shirts and P pants

2. Susan was trying to solve the equation $2x^2 = 6x$.

First she divided both sides by 2.

$$x^2 = 3x$$

Then she divided both sides by x :

$$x = 3$$

Gustavo said, "You can't divide both sides by x ." Susan responded, "If you can divide both sides by 2, why can't you divide by x ?" They asked their teacher to explain.

Which of the following explanations is correct?

- A. Since x is a variable it can vary, you may not be dividing both sides by the same number.
- B. You can't cancel x because it does not represent a real number.
- C. You can only divide by whole numbers when solving equations.
- D. It is better to take the square root of both sides after dividing by 2, that way you won't have to worry about dividing by x .
- E. If you divide both sides by x , then you might be dividing by 0, and would miss the solution $x = 0$.

3. Given the function f , defined by $f(x) = 3x^2 + 2x - 4$, which of the following equals $f(x + a)$?

A. $3x^2 + 3a^2 + 2x + 2a - 4$

B. $3x^2 + 6xa + 3a^2 + 2x - 4$

C. $3(x + a)^2 + 2(x + a) - 4$

D. $3(x + a)^2 + 2x - 4$

E. $3x^2 + 2x - 4 + a$

3. Given the function f , defined by $f(x) = 3x^2 + 2x - 4$, which of the following equals $f(x + a)$?

A. $3x^2 + 3a^2 + 2x + 2a - 4$

B. $3x^2 + 6xa + 3a^2 + 2x - 4$

C. $3(x + a)^2 + 2(x + a) - 4$

D. $3(x + a)^2 + 2x - 4$

E. $3x^2 + 2x - 4 + a$

4. Let $f(x) = \log_2 x^2$. Which of the following functions have the same graph as $y = f(x)$?

i. $y = 2\log_2 x$

ii. $y = 2\log_2 |x|$

iii. $y = 2|\log_2 x|$

A. i only

B. ii only

C. iii only

D. i and ii only

E. i, ii, and iii

5. Students are given the following problem:

Find the number of the real roots of the equation $9^x - 3^x - 6 = 0$

Peter denotes $y = 3^x$ and gets the equation $y^2 - y - 6 = 0$, which has 2 different roots.
He concludes that the given equation also has 2 different roots.

Which of the following is true about Peter's solution?

- A. Peter's conclusion and his arguments are correct.
- B. Peter's original approach to the problem (substitution of $y = 3^x$) is not correct.
- C. Peter factors wrong.
- D. The quadratic equation $y^2 - y - 6 = 0$ does not have 2 different roots.
- E. Peter does not take into account the range of the function $y = 3^x$.

6. Which of the following can be represented by areas of rectangles?

- i. The equivalence of fractions and percents, e.g. $\frac{3}{5} = 60\%$
- ii. The distributive property of multiplication over addition: For all real numbers a , b , and c , we have $a(b + c) = ab + ac$
- iii. The expansion of the square of a binomial: $(a + b)^2 = a^2 + 2ab + b^2$

- A. ii only
- B. i and ii only
- C. i and iii only
- D. ii and iii only
- E. i, ii, and iii

7. A student is asked to give an example of a graph of a function $y = f(x)$ that passes through the points A and B (see Figure 1). The student gives the answer shown in Figure 2. When asked if there is another answer the student says: “No, this is the only function.”

Figure 1

Figure 2

Which of the following best evaluates the student's answer of “No” to the second question?

- A. The student is right, because that is the only way a line will pass through both points.
- B. The student is right, because this function is of the form $f(x) = mx + b$.
- C. The student is right, because his graph passes the vertical line test.
- D. The student is wrong, because graphing is not an appropriate way to solve this problem.
- E. The student is wrong, because there are infinitely many functions that pass through points A and B.

8. The given graph represents speed vs. time for two cars. (Assume the cars start from the same position and are traveling in the same direction.) Use this information and the graph below to answer.

What is the relationship between the *position* of car A and car B at $t = 1$ hour?

- A. The cars are at the same position.
- B. Car A is ahead of car B.
- C. Car B is passing car A.
- D. Car A and car B are colliding.
- E. The cars are at the same position and car B is passing car A.

9. How many solutions exist for the equation $\tan(x) = x^2$?
- A. No solutions
 - B. Exactly one solution
 - C. Exactly two solutions
 - D. More than two solutions, but a finite number of solutions
 - E. An infinite number of solutions

10. A textbook includes the following theorem:

If line l_1 has slope m_1 and line l_2 has slope m_2 then $l_1 \perp l_2$ if and only if

$m_1 m_2 = -1$ (i.e. “slopes are negative reciprocals”).

(McDougal Littell, Algebra 2)

Three teachers were discussing whether or not this statement generalizes to all lines in the Cartesian plane.

Mrs. Allen: The statement of the theorem is incomplete: it doesn't provide for the pair of lines where one is horizontal and one is vertical. Such lines are perpendicular.

Mr. Brown: The statement is fine: a horizontal line has slope 0 and a vertical line has slope ∞ and it's OK to think of 0 times ∞ as -1 .

Ms. Corelli: The statement is fine; horizontal and vertical lines are not perpendicular.

Whose comments are correct?

- A. Mrs. Allen only
- B. Mr. Brown only
- C. Ms. Corelli only
- D. Mr. Brown and Ms. Corelli.
- E. None are correct.

11. In a first year algebra class, which of the following is **NOT** an appropriate way to introduce the concept of slope of a line?
- A. Talk about the rate of change of a graph of a line on an interval.
 - B. Talk about speed as distance divided by time.
 - C. Toss a ball in the air and use a motion detector to graph its trajectory.
 - D. Apply the formula $\text{slope} = \frac{\text{rise}}{\text{run}}$ to several points in the plane.
 - E. Discuss the meaning of m in the graphs of several equations of the form $y = mx + b$.

12. Consider the statement below.

For all a and b in S , if $ab = 0$, then either $a = 0$ or $b = 0$.

For which of the following sets S is the above statement true?

- i. the set of real numbers
 - ii. the set of complex numbers
 - iii. the set of 2×2 matrices with real number entries
-
- A. i only
 - B. ii only
 - C. iii only
 - D. i and ii only
 - E. i, ii and iii

13. Some high school students were asked to prove that the following statement is true:

When you multiply any 3 consecutive whole numbers, your answer is always a multiple of 6.

Below are three answers.

Kate's answer

A multiple of 6 must have factors of 3 and 2.
 If you have three consecutive numbers, one will be a multiple of 3 as every third number is in the three times table.
 Also, at least one number will be even and all even numbers are multiples of 2.
 If you multiply the three consecutive numbers together the answer must have at least one factor of 3 and one factor of 2.

Leon's answer

$$1 \times 2 \times 3 = 6$$

$$2 \times 3 \times 4 = 24 = 6 \times 4$$

$$4 \times 5 \times 6 = 120 = 6 \times 20$$

$$6 \times 7 \times 8 = 336 = 6 \times 56$$

Maria's answer

n is any whole number

$$n \times (n + 1) \times (n + 2) = (n^2 + n) \times (n + 2)$$

$$= n^3 + n^2 + 2n^2 + 2n$$

Canceling the n 's gives $1 + 1 + 2 + 2 = 6$

Which are valid proofs?

- A. Kate's only
- B. Maria's only
- C. Kate's and Leon's
- D. Leon's and Maria's
- E. Kate's and Maria's

Adapted from Core Plus Mathematics Project.

Copyright 2006, Knowing Mathematics for Teaching Algebra (KAT) Project, NSF REC-0337595, Division of Science and Mathematics Education, Michigan State University. Not for reproduction or use without written consent of KAT.

14. How many real solutions does the following equation have?

$$\sqrt{x-2} = \sqrt{1-x}$$

- A. none
- B. one
- C. two
- D. four
- E. infinitely many

15. Which of the following (taken by itself) would give substantial help to a student who wants to expand $(x + y + z)^2$?

- i. See what happens in an example, such as $(3 + 4 + 5)^2$.
- ii. Use $(x + y + z)^2 = ((x + y) + z)^2$ and the expansion of $(a + b)^2$.
- iii. Use the geometric model shown below.

	x	y	z
x	x^2	xy	xz
y	xy	y^2	yz
z	xz	yz	z^2

- A. ii only
- B. iii only
- C. i and ii only
- D. ii and iii only
- E. i, ii and iii

16. If $f(x) = ax^3 + bx^2 + cx + d$, what is the slope of the line tangent to this curve at $x = 2$?

A. $8a + 4b + 2c$

B. $8a + 4b + 2c + d$

C. $12a + 4b + c$

D. $12a + 4b + c + d$

17. Given the table to the right, determine $f(g(3))$.

A. 4

B. -1

C. 0

D. 1

E. 5

x	$f(x)$	$g(x)$
-2	0	5
-1	6	3
0	4	2
1	-1	1
2	3	-1
3	-2	0

18. a) On a test a student marked both of the following as non-functions

(i) $f: \mathbf{R} \Rightarrow \mathbf{R}, f(x) = 4$, where $\mathbf{R}$ is the set of all the real numbers.

(ii) $g(x) = x$ if x is a rational number, and $g(x) = 0$ if x is an irrational number.

For each of (i) and (ii) above, decide whether the relation is a function, and write your answer in the Answer Booklet.

b) If you think the student was wrong to mark (i) or (ii) as a non-function, decide what he or she might have been thinking that could cause the mistake(s). Write your answer in the Answer Booklet.

19. Solve the inequality $(x - 3)(x + 4) > 0$ in **two** essentially different ways. Show your work in the Answer Booklet.

20. Let $\mathbf{A} = \begin{bmatrix} p & q \\ r & s \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} t & u \\ v & w \end{bmatrix}$. Then $\mathbf{A} \Delta \mathbf{B}$ is defined to be $\begin{bmatrix} pt & qu \\ rv & sw \end{bmatrix}$.

Is it true that if $\mathbf{A} \Delta \mathbf{B} = \mathbf{O}$, then either $\mathbf{A} = \mathbf{O}$ or $\mathbf{B} = \mathbf{O}$ (where $\mathbf{O}$ represents the zero matrix)? Justify your answer and show your work in the Answer Booklet.

----- END OF FORM 1 -----

Item 18: Adopted from Ruhama Evens paper.

Items 19 and 20: Copyright 2006, Knowing Mathematics for Teaching Algebra (KAT) Project, NSF REC-0337595, Division of Science and Mathematics Education, Michigan State University. Not for reproduction or use without written consent of KAT.

Survey of Knowledge for Teaching Algebra

Start here **Answer Booklet**

Directions

1. This booklet contains two parts:
Part I Background questionnaire – read each question and mark your choices directly in the booklet;
Part II Answer sheets for the assessment questions – refer to Form 1 or Form 2 for the assessment questions, then mark your choices and write your responses correspondingly in this booklet.
2. When making choices, use black ink or a number 2 pencil to fill the circles completely.
3. If you decide to change your answer to a question, completely erase your first mark and fill in your new choice.
4. You have 60 minutes to complete the entire survey.
5. When you complete the survey, please return this Answer Booklet as well as the Assessment Questions form to the administrator.

©2006 Knowing Mathematics for Teaching Algebra (KAT) Project
National Science Foundation Grant REC-0337595
210 North Kedzie Lab
Division of Science and Mathematics Education
Michigan State University
East Lansing, MI 48824

PLEASE DO NOT MARK IN THIS AREA

[illegible]

SERIAL #

PART I BACKGROUND QUESTIONNAIRE

1. Are you currently attending college/university as an undergraduate or graduate student?

- ☐ No -- Go to question 2
- ☐ Yes -- What year are you in your college/university preparation?
 - ☐ 1st year (Freshman)
 - ☐ 2nd year (Sophomore)
 - ☐ 3rd year (Junior)
 - ☐ 4th year (Senior)
 - ☐ Graduate student
 - ☐ Other

2. What is/was your major in college?

- ☐ Mathematics
- ☐ Other

3. What is/was your minor in college?

- ☐ Mathematics
- ☐ Other

4. Including the current semester, which of the following *types* of courses have you taken?
Mark all that apply.

Mathematics Courses

- ☐ Calculus
- ☐ Linear Algebra (e.g., vector spaces, matrices, dimensions, eigenvalues, eigenvectors)
- ☐ Abstract Algebra (e.g., group theory, field theory, ring theory; structuring integers, ideals)
- ☐ Advanced Geometry and/or Topology
- ☐ Real and/or Complex Analysis
- ☐ Number Theory and/or Discrete Mathematics
- ☐ Differential Equations and/or Multivariate Calculus

Mathematics Education Courses

- ☐ Methods of teaching mathematics (planning mathematics lessons, using curriculum materials and manipulatives, organizing and delivering mathematics lessons, etc.)
- ☐ Psychology of learning mathematics (how students learn, common student errors or misconceptions, cognitive processes, etc.)
- ☐ Assessment in mathematics instruction (developing and using tests and other assessments)

5. Are you currently employed as a middle or high school teacher?

- ☐ Yes -- Go to question 6
- ☐ No -- Have you taught in middle or high school in the last five years?
 - ☐ Yes -- Go to question 6
 - ☐ No -- Skip the rest of this questionnaire and proceed to Part II: Assessment Questions

6. Which of the following algebra courses have you taught in the last five years? Mark all that apply.

- ☐ Pre-algebra
- ☐ Remedial algebra
- ☐ First year algebra
- ☐ Second year algebra
- ☐ Advanced algebra
- ☐ Algebra in an integrated program
- ☐ Other course focused on algebra
- ☐ I have not taught any kind of algebra courses in the last five years

7. For how many years have you taught algebra, i.e., any of the courses listed above?

- ☐ 0
- ☐ 1-2 years
- ☐ 3-6 years
- ☐ 7-10 years
- ☐ More than 10 years

8. Which of the following mathematics classes have you taught in the past? Mark all that apply.

- ☐ Calculus
- ☐ AP calculus
- ☐ AP statistics
- ☐ International Baccalaureate mathematics classes

9. Which of the following mathematics books/programs have you taught from in the past? Mark all that apply.

- ☐ CMP (*Connected Mathematics Project*), Prentice Hall
- ☐ *Discovering Algebra*, Key Curriculum Press
- ☐ CPMP (*Core Plus Mathematics Project*) *Contemporary Mathematics in Context: A Unified Approach*, Glencoe/McGraw-Hill
- ☐ IMP (*Interactive Mathematics Program*), Key Curriculum Press
- ☐ MiC (*Mathematics in Context*), Holt, Rinehart & Winston
- ☐ UCSMP (*University of Chicago School Mathematics Project*), Scott Foresman/Addison Wesley/Prentice Hall

10. Have you received certification from the National Board for Professional Teaching Standards (NBPTS)?

- ☐ Yes
- ☐ No

PLEASE DO NOT MARK IN THIS AREA

○○○○○○○○○○○○○○○○○○○○○☒

Please read the assessment questions from Form 1 or Form 2 that accompanies this booklet.

There are a total of 20 assessment questions. Items 1 to 17 are multiple-choice questions, each has only one correct answer. Mark your choice for each question in the answer sheet provided below. Items 18 to 20 are open-ended questions. Write down your responses in the spaces provided in the next few pages.

Please indicate which assessment form you are using by filling in the corresponding circle:

Form 1 Form 2 ☐

Fill in your answers to the multiple-choice questions:

1. ☐ A ☐ B ☐ C ☐ D ☐ E

2. ☐ A ☐ B ☐ C ☐ D ☐ E

3. ☐ A ☐ B ☐ C ☐ D ☐ E

4. ☐ A ☐ B ☐ C ☐ D ☐ E

5. ☐ A ☐ B ☐ C ☐ D ☐ E

6. ☐ A ☐ B ☐ C ☐ D ☐ E

7. ☐ A ☐ B ☐ C ☐ D ☐ E

8. ☐ A ☐ B ☐ C ☐ D ☐ E

9. ☐ A ☐ B ☐ C ☐ D ☐ E

10. ☐ A ☐ B ☐ C ☐ D ☐ E

11. ☐ A ☐ B ☐ C ☐ D ☐ E

12. ☐ A ☐ B ☐ C ☐ D ☐ E

13. ☐ A ☐ B ☐ C ☐ D ☐ E

14. ☐ A ☐ B ☐ C ☐ D ☐ E

15. ☐ A ☐ B ☐ C ☐ D ☐ E

16. ☐ A ☐ B ☐ C ☐ D ☐ E

17. ☐ A ☐ B ☐ C ☐ D ☐ E

PLEASE DO NOT MARK IN THIS AREA

○○○○○○○○○○○○○○○○○○○○○

a) Mark one response for each of (i) and (ii)

- (i) f is a function ☐ Yes ☐ No
- (ii) g is a function ☐ Yes ☐ No

b)

Write your answer to Question 19:

Write your answer to Question 20:

PERF

PAGE 7

Appendix 3

Interpretation of Knowledge of Algebra for Teaching
(KAT) IRT Scale Scores

Knowledge of Algebra for Teaching Project
Michigan State University

May 2009

IRT Results for Knowledge of Algebra for Teaching Assessment

The Knowledge of Algebra for Teaching (KAT) project constructed two assessment forms, each designed to assess three dimensions of teacher's knowledge for teaching secondary school algebra. Draft items were drawn from other assessment tools or written for the project by mathematicians, mathematics educators, or experienced secondary school teachers. Items were revised on the basis of pilot tests and substantive reviews by mathematicians. From the resulting pool, items were selected to construct two forms with comparable content coverage. A validation study used responses from 1170 preservice and in-service mathematics teachers to gather evidence about construct validity. The validity study sample includes experienced and prospective teachers from a variety of colleges, high schools, and middle schools across the country. Although the sample includes teachers with a range of experience and mathematical preparation, it was not selected to be representative of any particular population.

Each form of the assessment contains 20 items: 17 multiple choice and three open-ended questions. In the validity study, both forms had reliability (Cronbach alpha) of .84

IRT analysis was used to create a basis for computing scale scores for overall test performance and for performance on each of three dimensions: knowledge of school algebra, advanced knowledge of mathematics, and teaching knowledge. Descriptions of these three dimensions are provided in Figure 1. Item parameters estimates were calibrated using PARSCALE 4.1 (Muraki & Bock, 2003). IRT ability scores are reported, rather than raw scores, because IRT scores are more stable and less affected by item difficulty level, thus minimizing bias when scores are compared across different test forms.

For the total test score, test items were rescaled so that the ability scores of the sample in the validity study had a mean of 50 and standard deviation of 10. For the subscale scores, a separate rescaling was done so that the ability scores for each subscale also have a mean of 50 and a standard deviation of 10.

The score reports on the following pages use the item parameter estimates from the validity study, so the scores for each group can be compared with the validity study sample. Form 1 was used with the DCMi sample.

Table 1 displays the total test scale and each subscale mean and standard deviation for the KAT validity study. Tables 2 and 3 compare the same samples, broken down by career stage (preservice or in service), then by college major. Note that college major has a strong association with mean score, both for the overall test and for each subscale.

Table 1 Means and standard deviations of the scaled scores for KAT

Scaled Scores	KAT		
	Cases	Mean	Std. Deviation
Teaching Knowledge	1170	50.0	10.0
School Knowledge	1170	50.0	10.0
Advanced Knowledge	1170	50.0	10.0
Total Score	1170	50.0	10.0

Table 2 Means and standard deviations of the scaled scores for KAT by teaching service

Cluster	Service	KAT		
		Cases	Mean	Std. Deviation
Teaching	In-service	681	48.6	10.0
Knowledge	Pre-service	409	50.9	9.5
School	In-service	681	48.7	10.1
Knowledge	Pre-service	409	50.8	9.4
Advanced	In-service	681	49.4	10.7
Knowledge	Pre-service	409	50.4	9.1
Total	In-service	681	48.7	9.4
Score	Pre-service	409	50.8	10.2

Table 3 Means and standard deviations of the scaled scores for KAT by college major

Cluster	Service	KAT		
		Cases	Mean	Std. Deviation
Teaching	Math	714	53.0	9.0
Knowledge	Non-Math	428	44.8	9.3
School	Math	714	52.8	9.4
Knowledge	Non-Math	428	45.1	9.0
Advanced	Math	714	52.3	8.8
Knowledge	Non-Math	428	46.2	10.6
Total	Math	714	52.5	9.6
Score	Non-Math	428	45.7	9.2

Figure 1

Three Dimensions of Teachers' Mathematical Knowledge for Teaching

Knowledge of School Algebra:

In this category we include knowledge of mathematics in the intended algebra curriculum for middle and high school. This is the knowledge we expect students to learn in school algebra. The algebra standard described in *Principles and standards for school mathematics* (NCTM, 2000) is an overview of the big ideas in middle and high school algebra curricula. More specific grade-level algebra expectations on which we build our conception of Knowledge of School Algebra is provided in standards issued by various states and in textbooks and other instructional materials. See Usiskin and Dossey (2004, p. 17) for a list of topics in one popular algebra series.

Advanced Knowledge of Mathematics:

This category includes other mathematical knowledge, in particular college level mathematics, which gives a teacher perspective on the trajectory and growth of mathematical ideas beyond school algebra. The *Mathematical Education of Teachers* (CBMS, 2001), gives examples of advanced knowledge, which is both broader and deeper than what is taught in school algebra. Some general areas of mathematics that are presumed to provide breadth and depth of understanding of school algebra are calculus, linear algebra, number theory, abstract algebra, real and complex analysis, and mathematical modeling. According to Usiskin, Peressini, Marchisotto and Stanley (2001) knowing alternate definitions, extensions and generalizations of familiar theorems, and a wide variety of applications of high school mathematics are also characteristics of an advanced perspective of mathematics.

Teaching Knowledge:

This includes mathematical knowledge specific to teaching algebra that may not be taught in advanced mathematics courses or known by some mathematicians. It includes such things as what makes a particular concept difficult to learn and what misconceptions lead to specific mathematical errors. It also includes mathematics needed to identify mathematical goals within and across lessons, to choose among algebraic tasks or texts, to select what to emphasize with curricular trajectories in mind, and to enact other tasks of teaching. The knowledge referred to here may fall into the category of pedagogical content knowledge or it may be pure mathematical content applied in teaching.

References:

- Conference Board of the Mathematical Sciences. (2001). *The mathematical education of teachers* (Vol. 11). Providence, RI: American Mathematical Society.
- National Council of Teachers of Mathematics. (2000). *Principles and Standards for School Mathematics*.

Appendix 4

Semi-Structured Interview Questions

- Section 1. Teacher decisions and preferences for instruction in the various solution techniques for quadratic equations.
- Section 2 Comparison of the teacher's conception regarding teaching and learning of each of these methods for solving quadratic equations.
- Section 3 Questions to explore the teacher's decision process in planning and presenting instruction and efforts to influence student behavior leading to improved performance on the end of unit assessment.
- Section 4 Factoring as a method of solution for quadratic equations. (This section and the next two include both questions about teacher knowledge and student conceptions for use of this method).
- Section 5 Completing the square as a method of solution.
- Section 6 Using the quadratic formula.

Within each category, the following questions will be used to explore the decisions made during planning and instruction more deeply. Additionally, as other issues arise, these questions may spur discussion to investigate and elucidate other ideas

or strategies that contribute to teacher decisions and actions. Further probing questions were constructed to pursue a thick description of the teachers' thinking as the planning and teaching process intersected with student conceptions and difficulties.

Pre-Unit Interview Questions

Section 1. Planning and Preparation

- If I were a student in your class, what would you want me to take away from this unit of instruction on solving quadratic equations?
- What methods for solving quadratic equations have you used in your instruction?
- What methods do you prefer to use when you teach this unit of instruction? Can you explain why?
- During your planning and preparation, how do you decide when and in what manner to present the different methods for solving the quadratic equations?
- When you were preparing for this unit of instruction, what factors influenced your decisions about which methods to teach to your students?

Section 2. Comparison of the methods of solution.

- What method of solution you prefer over the others? Why do you prefer that method?

- Are students able to learn one method more quickly or easily than the others?

What do you think the reason(s) are for that ability?

- Which methods do you expect your students to use in solving the following quadratic equation? Why do you think that would be their method of choice?

$$x^2 - 10x + 25 = 0$$

Section 3. Questions regarding planning, presentation, and strategies to improve performance

- Help me understand your thought process when you begin your planning for teaching your students how to solve quadratic equations.
- Describe how much latitude you feel you have in making these decisions about instructing this unit on quadratic equations.
- How do you allocate class time to each of the different methods during your planning and before you actually begin the teaching for solving quadratic equations?
- When you have to modify these plans, how does that make you feel?
- Describe a situation where you have had to modify your plans?
- What percentage of the instructional time for the entire unit do you allocate to
 - a. Factoring?
 - b. Completing the Square?
 - c. Using the Quadratic Formula?
- Explain your thought process in making those allocations.
- If you gave your students complete freedom to choose, how do you think they would approach teaching solving quadratic equations?

- What do you think is the motivation for those decisions by your students?
- Given complete autonomy, how would your instruction in this unit change?
- Why do you think you would make those specific changes?
- Do you feel you have the choice to modify instruction if it benefits your students?

Post-Unit Interview Questions

Section 4. Factoring as a solution technique

- Describe in words the approach you use when teaching the students to factor a quadratic equation.
- Do you prefer any specific techniques of factoring (undoing, the box method, or others)? Please explain your preference for this (these) technique(s).
- A student proposes the following solution to a quadratic equation:

$$x^2 - 4x - 5 = 0$$

$$x^2 - 4x = 5$$

$$x(x-4) = 5$$

$$x = 5 \text{ or } x = 9$$

How would you address this error?

Section 5. Completing the Square

- Explain how you teach your students to use the method of completing the square as a solution technique.
- If this is not one of the methods you normally use, can you explain why?
- Do you prefer one way of teaching this method over another? Explain what makes this method your preference?

Section 6. Using the Quadratic Formula.

- Describe how you initially introduce the idea of using the quadratic formula to solve a quadratic equation.
- How do your students react to this method of solution?
- Tell me why you think they react in that way.
- Describe a typical first lesson when you are teaching your students to use the quadratic formula.
- What kind of difficulties do your students have trying to use the quadratic formula?
- Describe how you help your students overcome these mistakes / misunderstandings.

Appendix 5

Pre-Lesson Questionnaire.

Please take just 10 -15 minutes to answer these questions and send them back to me via e-mail. Thank you.

Describe your lesson objectives for the day.

How do you intend to present these ideas?

What kinds of examples or problems will you use (please provide some examples here or I can get them from the lesson plan)?

If you have a quiz planned, please include a copy.

Describe any student difficulties you encountered yesterday.

Explain how those difficulties impacted your plans for today.

Please explain what difficulties you anticipate your students might have with today's lesson.

CURRICULUM VITA

NAME: Michael Lee Raney

ADDRESS: Department of Mathematics
4001 Harding Road
Nashville, TN 37111

DOB: Covington, Kentucky – December 21, 1952

EDUCATION
& TRAINING: B.S., Engineering
United States Military Academy
1970-74

B.S., Logistics and Staff Management
Command & General Staff College
1984-1985

M.S., Applied Mathematics
Rensselaer Polytechnic Institute
1981-1983

M.S., Operations Research & Statistical Analysis
Rensselaer Polytechnic Institute
1981-1983

PROFESSIONAL
SOCIETIES: National Council of Teachers of Mathematics
Association for Supervisors of Curriculum Development
Mathematical Association of America
National Education Association