12-2018

Exploring the knowledge of algebra for teaching.

Jonathan David Watkins
University of Louisville

EXPLORING THE KNOWLEDGE OF ALGEBRA FOR TEACHING

By

Jonathan David Watkins
B.A., Murray State University, 2003
M.A., University of Louisville, 2011

A Dissertation Submitted to the Faculty of the
College of Education and Human Development of the University of Louisville
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy
in Curriculum and Instruction

Department of Middle and Secondary Education
University of Louisville
Louisville, Kentucky

December 2018

EXPLORING THE KNOWLEDGE OF ALGEBRA FOR TEACHING

By

Jonathan David Watkins
B.A., Murray State University, 2003
M.A., University of Louisville, 2011

A Dissertation Approved on

November 14, 2018

by the following Dissertation Committee:

Susan Peters, Dissertation Co-Director

Jill Adelson, Dissertation Co-Director

Jennifer Bay-Williams

Maggie McGatha

DEDICATION

This dissertation is dedicated to my wife

Diane Adel Watkins

who has been a constant source of love and support throughout my graduate studies

&

to my parents

Dr. David and Mrs. Peggy Watkins

who instilled in me a love of learning and encouraged me to follow my dreams.

ACKNOWLEDGMENTS

There are a number of people who deserve special recognition for helping me throughout my doctoral studies and dissertation journey. First and foremost, I would like to thank Drs. Susan Peters and Jill Adelson for serving as my dissertation co-chairs and for being great teachers, mentors, and friends.

Sue, I will never forget all that you have done for me. Thank you for giving me the opportunity to serve as your first graduate research assistant, allowing me to attend and present at several national conferences with you, providing me guidance throughout my program, and spending countless hours reading and revising my work. In short, thank you for believing and investing in me.

Jill, you have also been such an important mentor in my life. Thank you for expanding my knowledge of advanced statistical methods (including SEM and HLM), allowing me to serve as one of your teaching assistants, and providing valuable feedback related to my statistical methods and analyses on my dissertation study and other research projects. I look forward to our future collaborations.

I would also like to thank Drs. Jenny Bay-Williams and Maggie McGatha for agreeing to serve on my program and dissertation committees. In addition to providing meaningful feedback on my dissertation drafts, Jenny and Maggie have been such an encouragement to me during my doctoral studies. Thank you both for your kindness and support.

There are also many other faculty members and friends who have encouraged and supported me throughout this journey. These faculty members include Drs. Bob Ronau, Bill Bush, Karen Karp, Lateefah Id-Deen, Jeff Valentine, Tom Tretter, Ann Larson, and the late Sam Stringfield. Friends include Kathy Cash, Greg Carmichael, Carrye Wilkins, Raymond Roelandt, and my wife, Diane Watkins. (I know there are others, so I apologize to those I have neglected to include in this list.)

Additionally, I could not have completed this dissertation on the Knowledge of Algebra for Teaching without the assistance of Dr. Robert Floden and the KAT research team at Michigan State University. Thank you so much for allowing me to explore your data and answering all of my questions along the way.

Finally, I would like to thank my high-school calculus teacher, Dana Guess, for making mathematics fun and exciting for me. Ms. Guess, I would not be in mathematics education today if I had not been inspired by your passion for mathematics and desire to help your students succeed. Thank you for choosing to become a public-school teacher and for making a difference in the lives of so many students in Henderson County.

ABSTRACT

EXPLORING THE KNOWLEDGE OF ALGEBRA FOR TEACHING

Jonathan David Watkins

November 14, 2018

For the past few decades, researchers in mathematics education have been exploring the concept of pedagogical content knowledge (PCK)— or knowledge related to teaching content—and applying it to various areas of mathematics, such as algebra. Research related to teacher knowledge of algebra is critical because researchers (e.g., Hill, Rowan, & Ball, 2005) have found correlations between some types of teacher knowledge and student achievement in mathematics; students from around the world are outperforming U.S. students on international assessments of mathematics, including algebra (Organization for Economic Cooperation and Development, 2014, 2016); and algebra plays an integral role in the K-12 mathematics curriculum in the U.S. (National Council of Teachers of Mathematics, 2000).

Given this background, the purpose of this study was to explore the knowledge of algebra for teaching (KAT) by investigating the following research questions: What is the factor structure underlying mathematics teachers' KAT, as measured by an established instrument? Are KAT constructs measured similarly in preservice and inservice teachers? And if so, are there latent mean differences in the KAT of these two groups? These research questions were addressed using multiple-group confirmatory factor analysis—a form of structural equation modeling—to analyze survey

data (n = 1,248) gathered by KAT researchers at Michigan State University. These researchers designed an instrument to measure three types of algebra knowledge, based on their conceptual framework of KAT: knowledge of school algebra; knowledge of advanced mathematics; and mathematics-for-teaching knowledge, which is similar to PCK (Reckase, McCrory, Floden, Ferrini-Mundy, & Senk, 2015).

The analyses suggested that KAT may be a unidimensional construct because a one-factor KAT model fit the data better than a two- or three-factor model. Additionally, the analyses suggested that KAT was measured similarly in preservice and inservice teachers, and that preservice teachers had slightly higher KAT than inservice teachers.

Following the results, there is a discussion of connections between the findings and the research literature and implications of the findings, such as providing more CK- and PCK-focused professional development opportunities for algebra teachers. The researcher concludes with some recommendations for future research and closing remarks.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

CHAPTER 1

INTRODUCTION

Kate and Ashley are freshmen at Anytown High School. They are enrolled in Mr. Stein's Algebra I class and are learning how to solve systems of two linear equations in two unknowns. Kate always has enjoyed mathematics and is able to solve most of the problems in the lesson using the substitution and elimination methods. However, Ashley is struggling greatly with solving systems using these two new methods. She tries to enlist the help of her friend Kate, but to no avail. Kate tells Ashley that she "just gets it," as she tries unsuccessfully to explain her process. When Ashley examines Kate's homework, she notices that her friend has written very little on her paper. "I don't have to write a lot because I do most of the work in my head," Kate explains.

Ashley then asks Mr. Stein—a first-year mathematics teacher—for assistance with the lesson but experiences similar results. She explains the difficulties that she is having to Mr. Stein, but he also seems unable to address Ashley's errors and misunderstandings of the content. Ashley thinks to herself, "I'm sure that Mr. Stein 'knows' algebra, but he just doesn't seem to be able to meet me at my level."

Although the scenario described above is fictitious, it is based on a true story; in fact, it is based on many true stories. Consider a friend, classmate, or even a teacher who is "good" at mathematics but has difficulty using that knowledge to facilitate others' mathematics learning. How would you describe this person's knowledge of mathematics? What about his/her knowledge of mathematics *for teaching*? Prior to the 1980s, most research related to knowledge for teaching mathematics generally focused on subject matter knowledge (SMK) and proxies of teacher knowledge, such as the number of subject matter courses taken in college and years of experience in the classroom (Hill, Sleep, Lewis, & Ball, 2007). Based on the large number of college-level mathematics courses that Mr. Stein had to complete to be eligible to teach mathematics at the secondary level, he very likely would score well on measures that focus on SMK. But clearly, scenarios such as the one described above highlight the fact that these measures may give an incomplete picture of an individual's actual knowledge for teaching. So, what types of knowledge are necessary to be an effective teacher of algebra?

During his 1985 Presidential Address at the American Educational Research Association annual meeting, Lee Shulman revolutionized the way researchers thought about knowledge for teaching with the introduction of the concept of pedagogical content knowledge (PCK), or the content knowledge needed for teaching. According to Shulman (1986), PCK included "the most useful forms of representation…of ideas [from one's content area], the most powerful analogies, illustrations, examples, explanations, and demonstrations—in a word, the ways of representing and formulating the subject that make it comprehensible to others" (p. 9). Since his seminal remarks, a number of researchers have been working to explore and unpack the concept of PCK and apply it to mathematics and specific areas within the discipline (e.g., algebra). For example, the Knowledge of Algebra for Teaching (KAT) research team at Michigan State University has developed a comprehensive framework for KAT, as well as an instrument designed to measure KAT in preservice and inservice teachers. The KAT framework consists of three types of knowledge: knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge; and the KAT instrument includes multiple-choice and open-ended items that are designed to measure these three types of algebra knowledge.

Significance of KAT

There are several reasons that the exploration of teacher knowledge—and more specifically KAT—is significant for the field of mathematics education, especially in the United States. First, students from around the world are outperforming U.S. students on international mathematics assessments, such as the Trends in International Mathematics and Science Study (TIMSS) assessment and the Program for International Student Assessment (PISA) (Mullis, Martin, Foy, & Hooper, 2016; Organization for Economic Cooperation and Development, 2014, 2016). Second, several researchers (e.g., Hill, Rowan, & Ball, 2005) have found correlations between some types of teacher knowledge (e.g., PCK) and student achievement. And third, algebra plays an integral role in the K-12 mathematics curriculum in the U.S. (National Council of Teachers of Mathematics [NCTM], 2000; National Governors Association Center for Best Practices [NGACBP] & Council of Chief State School Officers [CCSSO], 2010). Each of these issues will be discussed in the sections that follow.

International Assessments in Mathematics

TIMSS and PISA are two well-known assessments in mathematics (and other content areas, such as science) and are commonly used to compare student achievement at the international level. TIMSS is conducted every four years by the International Association for the Evaluation of Educational Achievement (IEA) and is administered to a large sample of fourth- and eighth-grade students from around the world (Mullis et al., 2016). PISA is conducted every three years by the Organization for Economic Cooperation and Development (OECD) and is administered to a large sample of 15-year-old students from around the world (OECD, 2014, 2016).

U.S. eighth-graders have consistently earned above average mathematics scores on the TIMSS assessment, which contains four content domains in mathematics: number, algebra, geometry, and data/chance (Mullis et al., 2016). The mean mathematics score for U.S. students on TIMSS 2015 was 518, which was statistically significantly higher than the mean (500) of the combined achievement distribution. Of the 39 countries that participated in the eighth-grade mathematics assessment in 2015, 16 of them (e.g., Singapore, Canada, England, and the U.S.) earned scores that were statistically significantly higher than the mean; 2 of them (Australia and Sweden) earned scores that were not statistically significantly different from the mean; and 21 of them (e.g., Italy, Chile, and South Africa) earned scores that were statistically significantly below the mean (Mullis et al., 2016).

Even though U.S. eighth-graders scored above the mean in 2015, their mean score (518) was much lower than the mean score of several other countries, including Singapore (621), the Republic of Korea (606), Chinese Taipei (599), Hong Kong (594), and Japan (586). Also, only 10% of U.S. eighth-graders who participated in TIMSS 2015 met the advanced international benchmark, which included the ability to solve linear equations; and only 37% met the high international benchmark, which included the ability to simplify and work with algebraic expressions (Mullis et al., 2016).

Despite their above-average performance on TIMSS, U.S. students have consistently earned relatively low scores on PISA assessments. PISA 2012 focused on mathematics—specifically numbers, algebra, and geometry—and was administered to 15-year-old students from 65 participating countries/economies (OECD, 2014). The mean mathematics score for U.S. students on PISA 2012 was 481, which was statistically significantly below the OECD average of 494. Of the 65 participating countries/economies in 2012, 23 of them (e.g., Singapore, Canada, and Germany) earned scores that were statistically significantly above the OECD average; 8 of them (e.g., France and the United Kingdom) earned scores that were not statistically significantly different from the OECD average; and 34 of them (e.g., Sweden, Mexico, Brazil, and the U.S.) earned scores that were statistically significantly below the OECD average. Additionally, the U.S. mean mathematics score was well below the score of the top-performing country/economy: Shanghai-China, which earned a mean score of 613. According to OECD, a score difference of 41 points corresponds to about one academic year of schooling; thus, 15-year-old students in Shanghai-China were on average about three years ahead of U.S. 15-year-olds in mathematics at the time of the study (OECD, 2014).

In 2015, the major focus of PISA was science, but there was still a minor mathematics component. Unfortunately, the mean mathematics score for U.S. students on PISA 2015 dropped to 470, which was again below average and well below the top-performing country/economy: Singapore, which earned a mean score of 564 (OECD, 2016).

Thus, students from around the world are outperforming U.S. students in algebra and other areas of mathematics on widely-recognized international assessments. More specifically, many U.S. eighth-grade students are struggling with basic algebra skills, such as simplifying expressions and solving linear equations (Mullis et al., 2016). And 15-year-old students in the U.S. are performing well-below average in the areas of number, algebra, and geometry (OECD, 2014, 2016).

Researchers in the mathematics-education community are concerned about U.S. students' performance on these types of assessments, as well as students' mathematics achievement in general. Thus, a number of studies have focused on how teacher-related factors affect student achievement, as it is a general assumption in education that teachers directly impact student learning (Eisenberg, 1977).

Teacher Knowledge and Student Achievement

Teacher knowledge is one of the teacher-related factors that has been investigated over the past few decades because researchers believe this factor may play an important role in addressing students' difficulties in mathematics. In particular, several studies (e.g., Baumert et al., 2010; Campbell et al., 2014; Hill et al., 2005; Mohr-Schroeder, Ronau, Peters, Lee, & Bush, 2017) have shown relationships between teacher knowledge and student achievement. For example, Hill and her colleagues found that a group of elementary-school "teachers' mathematical knowledge for teaching positively predicted student gains in mathematics achievement" (Hill et al., 2005, p. 399). And Baumert and his colleagues found similar results with secondary students in Germany. In particular, they found that PCK explained about 39% of the between-class variance in student achievement at the end of the school year (Baumert et al., 2010), which is considerable.

Both of these teacher-knowledge studies used instruments that were specifically designed to measure an individual's knowledge for teaching mathematics (i.e., PCK). For example, Figure 1 shows an item from the instrument used in the Hill and colleagues (2005) study. This item is quite different from content questions generally found in mathematics textbooks and assessments because it requires individuals to make a judgment (based on content knowledge) that generally only mathematics teachers are required to make. For instance, engineers most likely would never need to judge the validity of a unique or student-generated method for multiplying large numbers.

Imagine that you are working with your class on multiplying large numbers. Among your students' papers, you notice that some have displayed their work in the following ways:

Student A	Student B	Student C
35	35	35
× 25	× 25	× 25
125	175	25
+ 75	+ 700	150
875	875	100
		+ 600
		875

Which of these students would you judge to be using a method that could be used to multiply any two whole numbers?

Figure 1. Item measuring content knowledge for teaching mathematics at the elementary-school level. Adapted from "Effects of Teachers' Mathematical Knowledge for Teaching on Student Achievement," by H. C. Hill, B. Rowan, and D. L. Ball, *American Educational Research Journal*, *42*, p. 402. Copyright 2005 by the American Educational Research Association.

Before Shulman introduced the concept of PCK, researchers often measured teacher knowledge using assessments of SMK and/or proxy measures of teacher knowledge, such as number of years of teaching experience. Often these researchers (e.g., Begle, 1972; Eisenberg, 1977) found little or no correlation between teachers' knowledge of mathematics and student achievement. For example, Begle (1972) used the following variables for teacher knowledge of algebra in his study: performance on an advanced algebra assessment, which included a number of abstract-algebra questions; number of years of teaching experience; number of mathematics courses taken in college beyond calculus; and college GPA in mathematics. Ultimately, he—and Eisenberg, who replicated his study—found no correlation between teacher knowledge of algebra and student achievement (Begle, 1972; Eisenberg, 1977).

Based on these studies, it appears that some types of teacher knowledge (i.e., PCK) may contribute more to student achievement in mathematics than other types of teacher knowledge (i.e., SMK). Additionally, proxy measures of teacher knowledge may have led to misleading results in prior research and may not be valid. Therefore, it would be beneficial to have a greater understanding of the various types of teacher knowledge and their effects on mathematics achievement.

Algebra in the K-12 Curriculum

Now that the case for exploring the knowledge for teaching mathematics has been made, what about the case for algebra? That is, why is there a need to focus specifically on knowledge of algebra for teaching? There are several specific reasons to focus on algebra (and KAT), including that (a) algebra comprises a significant portion of the K-12 mathematics curriculum in the U.S. (NCTM, 2000; NGACBP & CCSSO, 2010); (b) mathematics students often experience great difficulty in learning algebra (Blume & Heckman, 2000; National Mathematics Advisory Panel, 2008; RAND Mathematics Study Panel, 2003); and (c) algebra is a "gatekeeper" course and prerequisite for nearly all other areas of mathematics (RAND Mathematics Study Panel, 2003). These reasons will now be discussed briefly.

First, algebra is an integral part of the K-12 mathematics curriculum in the U.S. The National Council of Teachers of Mathematics' (NCTM) seminal work, *Principles and Standards for School Mathematics,* lists algebra as one of five central content goals in mathematics for students in grades K-12 (NCTM, 2000). More recently, the *Common Core State Standards for Mathematics* (CCSSM) (NGACBP & CCSSO, 2010)—which are the standards on which many state standards in the U.S. are based today—includes algebra throughout the K-12 curriculum. In particular, there are standards related to operations and algebraic thinking for students in grades K-5, expressions and equations for students in grades 6-8, and algebra and functions for students in grades 8-12. In fact, algebra and functions comprise two of the six conceptual categories for high-school standards in CCSSM (NGACBP & CCSSO, 2010).

Despite the pervasiveness of algebra in the K-12 curriculum, many students struggle to understand algebraic concepts (Blume & Heckman, 2000). After a nearly two-year investigation that focused on preparing students for algebra, the President's National Mathematics Advisory Panel (2008) concluded that:

> Too many students in middle or high school algebra classes are woefully unprepared for learning even the basics of algebra. The types of errors these students make when attempting to solve algebraic equations reveal they do not have a firm understanding of many basic principles of arithmetic. Many students also have difficulty grasping the syntax or structure of algebraic expressions and do not understand procedures for transforming equations…. (p. 32)

Other reports (e.g., RAND Mathematics Study Panel, 2003) and studies (e.g., OECD, 2014; Mullis et al., 2016) have also documented U.S. students' difficulties in learning and understanding algebra.

Unfortunately, students often pay a high price for struggling with algebraic concepts because courses in algebra often serve as "gatekeepers" in many programs. Thus, "without proficiency in algebra, students cannot access a full range of educational and career options" (RAND Mathematics Study Panel, 2003, p. xx). Similarly, algebra skills are necessary for nearly all areas of mathematics (e.g., geometry, probability, and calculus), as well as many concepts in science (Usiskin, 1995, 2004).

In summary, there are a number of reasons why KAT is an important area for exploration and greater understanding. First, U.S. students are being outperformed in algebra by their peers around the world. Second, there appears to be a connection between some types of teacher knowledge and student achievement in mathematics. And third, algebra in an integral part of the K-12 mathematics curriculum.

Purpose Statement and Research Questions

Given the significance of teacher knowledge and algebra, the purpose of the present study was to explore the various aspects of the knowledge of algebra for teaching (KAT), which will be defined as the set of mathematical knowledge that is necessary to be an effective teacher of algebra. A secondary purpose was to compare and contrast preservice and inservice mathematics teachers' KAT. To address these purposes, I investigated the following three research questions:

1. What is the factor structure underlying mathematics teachers' knowledge of algebra for teaching (KAT), as measured by an established KAT instrument? (This instrument is described in Chapter 2.)
2. Are KAT constructs measured similarly in preservice and inservice teachers?
3. And if so, are there latent mean differences in the KAT of these two groups?

In other words, the main goals of the study were to explore the specific types of knowledge that comprise KAT (via an established KAT instrument) and to determine whether preservice or inservice teachers demonstrated higher levels of KAT (based on their performance on this KAT instrument).

The three research questions were addressed using confirmatory factor analysis (CFA), a form of structural equation modeling (SEM) commonly used to explore latent—or unobserved—variables, such as knowledge. In particular, the first question was addressed by developing a variety of CFA models based on theory, evaluating them for model fit, and using statistical tests to compare them. The second question was addressed by using multiple-groups CFA analyses to determine if the values of model parameters differed across groups (i.e., preservice and inservice teachers), as well as whether or not measures operated the same in those groups (Brown, 2006). And the third question was addressed by testing for latent mean differences between the two groups.

Organization of the Study

Including this introductory chapter, the current study is organized into five chapters. Chapter 1 addressed the rationale for the study and outlined the purpose and research questions for the study. Chapter 2 contains a comprehensive review of the literature on algebra, teacher knowledge, and the KAT framework and study (conducted by the KAT research team). Chapter 3 describes the characteristics of the sample and instrument used in this study and outlines the statistical method (CFA) that was used. Chapter 4 outlines the results of the study, which includes the analyses of several proposed CFA models for KAT and multiple-groups CFA analyses to compare preservice and inservice teachers' KAT. And Chapter 5 contains a discussion of the results (i.e., implications of the study) and concluding remarks.

CHAPTER 2

LITERATURE REVIEW

Because the focus of this study is on middle- and high-school mathematics teachers' knowledge of algebra for teaching, the purpose of this review is to synthesize the literature on both algebra and teacher knowledge. The review contains three main foci: algebra, teacher knowledge, and the Knowledge of Algebra for Teaching (KAT) framework, which frames this study. In the first part, various views of algebra are discussed, as well as student thinking about algebra and teaching methods/strategies related to algebra. In the second part, teacher knowledge is described in terms of content knowledge (CK), general pedagogical knowledge (GPK), and pedagogical content knowledge (PCK). Within the discussions of CK and PCK, explicit connections to algebra are made. And in the third part, a KAT framework is presented, and relationships among the components of this framework (knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge) and the general framework of teacher knowledge (CK, GPK, and PCK) are discussed.

Algebra

In the first part of the review, a brief overview of algebra is presented by exploring several complementary views of algebra (such as algebra as generalized arithmetic and algebra as functions and relationships among quantities). These views are prevalent in the teaching and learning of algebra and their influence can be seen in the various curricular materials available to mathematics teachers. Additionally, researchinformed insights into student thinking about two main areas of algebra—expressions/equations and functions—are discussed. And finally, several teaching methods and strategies that research suggests might be effective for deepening students' understandings in algebra are explored.

Views of Algebra

There are many different but related views of algebra. They include (a) algebra as generalized arithmetic (Bell, 1996; Kaput, 1995; Usiskin, 1999), (b) algebra as symbolic manipulation (Kaput, 1995); (c) algebra as forming and solving equations (Bell, 1996; Usiskin, 1999); (d) algebra as functions and relationships among quantities (Cooney, Beckmann, & Lloyd, 2010; Heid, 1996; Kaput, 1995; Usiskin, 1999); (e) algebra as the study of structure (Kaput, 1995; Usiskin, 1999); and (f) algebra as an activity (Kieran, 1996; Lee, 1997). These various views of algebra are the focus of the discussion that follows.

Algebra as generalized arithmetic. Generalization plays a major role in algebra (Usiskin, 1999). In the algebra as generalized arithmetic view, variables are used to generalize arithmetic patterns. For example, consider the set of integers under addition. The mathematical sentences $-3 + 0 = -3$ and $0 + 7 = 7$ illustrate an important property of integer addition, namely the additive identity property. This property also can be generalized with variables as follows: $a + 0 = 0 + a = a$ for all integers a. Similarly, $4 + 9 = 9 + 4$ and $-5 + 2 = 2 + (-5)$ illustrate the commutative property of integer addition, which can also be generalized with variables as $a + b = b + a$ for all integers a and b. According to Kaput (1995), this view of algebra is popular for several reasons, including that "it explicitly builds on what students presumably know (arithmetic), helps generalize that knowledge, helps build a more general ability to generalize…, and exploits the rich intrinsic structure of the integers as a context for pattern development, formalization, and argument" (p. 7).

Algebra as symbolic manipulation. The algebra as symbolic manipulation view is a commonly-held viewpoint that places emphasis on skills and procedures (Kieran, 2007). These skills and procedures include simplifying expressions, using formal methods (i.e., substitution and elimination) to solve systems of equations, and factoring polynomial and rational expressions. Variables serve as unknowns or constants in the symbolic-manipulation view of algebra (Usiskin, 1999). For example, solving the linear equation $8(x + 2) = 3x + x - 1$ for the unknown variable x involves using several different procedures, including the distributive property of multiplication over addition, and using inverse operations to solve for the unknown, x.

Algebra as forming and solving equations. Although the view of algebra as forming and solving equations overlaps with the previous view (algebra as symbolic manipulation), it extends beyond symbolically solving equations to modeling situations using expressions and equations, as well as engaging in a larger problem-solving process (Bell, 1996). This view includes translating English phrases into algebraic notation (e.g., translating "9 less than twice a number" to $2n - 9$) and using algebraic expressions and equations to represent and solve real-world problems. For example, the expression $3.50 + 2.25m$ could be used to represent the fare of a certain taxi company with an initial pick-up fee of \$3.50 and a charge of \$2.25 per mile. (Note that m represents the number of miles.) If a passenger were charged a fare of \$19.25, solving the equation $3.50 + 2.25m = 19.25$ for m would reveal that the passenger traveled 7 miles ($m = 7$) by taxi.

Algebra as functions and relationships among quantities. If the symbolic-manipulation view reflects a more traditional viewpoint of algebra, the functional view reflects more of a reform viewpoint. The emphasis of this view is on functions and representing functional situations (Kieran, 2007). Functions can be used to show relationships among quantities that vary and are usually defined as single-valued mappings (or correspondences) between one set—called the domain—and another set—called the range (Cooney et al., 2010). Generally, in school algebra, the set of real numbers serves as both the domain and range of functions. In this view of algebra, a variable can take the form of "an argument (i.e., stands for the domain value of a function) or a parameter (i.e., stands for a number on which other numbers depend)" (Usiskin, 1999, p. 10). For example, in the function $f(x) = 3x + 1$, x represents an element of the domain of the function (the first set), whereas $f(x)$ represents the corresponding element of the range (the second set). So, the element $x = 1$ in the domain is mapped to the element $f(1) = 3(1) + 1 = 4$ in the range. This is an example of a linear function, which becomes clear after investigating multiple representations of this function (see Figure 2). The taxi problem from the previous section could also be modeled using functions. Specifically, the linear function $f(m) = 3.50 + 2.25m$ describes the relationship between the number of miles traveled, m, and the total fare, $f(m)$. In this scenario, the fare is a function of the number of miles traveled.

Functions can also model other relationships, including ones that are quadratic, exponential, and trigonometric. For instance, exponential functions are often used to model situations involving exponential growth and decay. In summary, the functional view of algebra "centers on developing experiences with functions and families of functions through encounters with real-world situations whose quantitative relationships can be described by those models” (Heid, 1996, p. 239).

x	$f(x) = y$
–2	–5
–1	–2
0	1
1	4
2	7

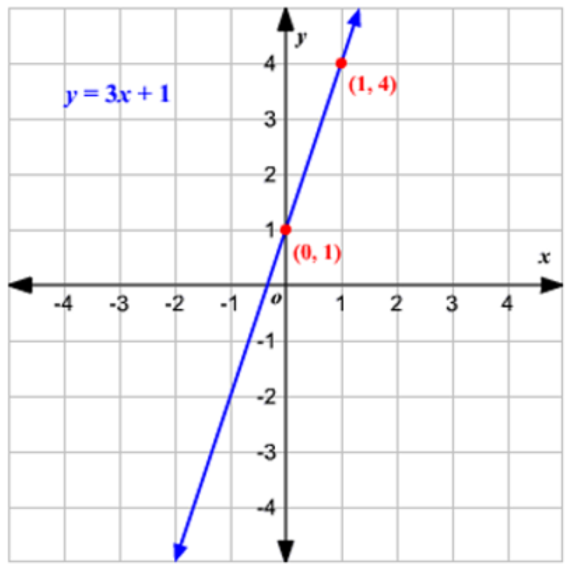


Figure 2. Tabular and graphical representation of the function $f(x) = 3x + 1$.

Algebra as the study of structure. The focus of the algebra as the study of structure view is on form (i.e., recognizing the form of algebraic expressions) and transformation (Pimm, 1995; Usiskin, 1999). For example, the difference of two perfect squares can be factored as follows: $x^2 – y^2 = (x + y)(x – y)$. This algebraic structure can be used to factor similar expressions, such as $25x^2 – 9$ and $\sin^2 x – \cos^2 x$, or to simplify rational expressions, such as $\frac{x^2-4}{x-2}$. In the structure view of algebra, variables simply serve as arbitrary symbols because the focus is on form rather than functions (variables as arguments), equations (variables as unknowns), or patterns (variables as generalizers) (Usiskin, 1999).

Algebra as an activity. This final view of algebra as an activity does not emphasize a certain aspect of algebra (e.g., symbolic manipulation, representing functional situations); instead, it unifies many of the previous viewpoints by characterizing algebra as "an activity, something you do, an area of action…" (Lee, 1997, p. 187). There are at least three types of activities in algebra: generational, transformational, and global/meta-level (Kieran, 1996). Generational activities include forming or generating expressions and equations in algebra (similar to the "forming and solving equations" view of algebra); transformational activities include enacting rule-based skills, such as performing operations, simplifying, and factoring (similar to the "symbolic manipulation" view of algebra); and global/meta-level activities refer to activities for which algebra can be used as a tool. These include problem solving, modeling, studying functional relationships, and exploring algebraic structures, which incorporates the remaining views of algebra discussed in this section (Kieran, 1996).

Summary. There are a number of different but complementary views of algebra. These viewpoints focus on various aspects of algebra, including generalizing patterns, manipulating symbols, forming and solving equations, representing and exploring functional situations, and recognizing and transforming algebraic structures. However, two common foci also emerge from these viewpoints and help to unify them: expressions/equations and functions. (Note that these are also two of the main areas of focus of the high-school mathematics standards in CCSSM, which was discussed in the previous chapter.) In the sections that follow, student thinking in these areas of algebra will be discussed, as well as relevant teaching methods and strategies.

Student Thinking in Algebra

Over the past few decades, there has been a great deal of research on students' algebraic thinking (e.g., Breidenbach, Dubinsky, Hawks, & Nichols, 1992; Knuth, Stephens, McNeil, & Alibali, 2006; Leinhardt, Zaslavsky, & Stein, 1990; MacGregor & Stacey, 1997; Matz, 1982). These studies have focused on students' understandings of algebraic concepts and their common errors and misconceptions, as well as teaching strategies to address these errors and misconceptions. Research related to student thinking in algebra is of particular interest in this review because it can inform the types of knowledge about students that teachers need to teach algebra. The following discussion on student thinking is divided into the two content foci that emerged from the collective views of algebra: 1) expressions and equations and 2) functions.

Expressions and equations. Research related to student thinking about algebraic expressions and equations includes studies that explore the concept of variable and variable meaning (e.g., Küchemann, 1978; MacGregor & Stacey, 1997; Usiskin, 1999), equivalence and the equal sign (e.g., Asquith, Stephens, Knuth, & Alibali, 2007; Knuth, et al., 2006), and extrapolation techniques, such as linearity and generalization (e.g., Matz, 1982). Student thinking in each of these areas is explored in the sections that follow.

***Variables and variable meaning*.** As suggested by the preceding discussion on views of algebra, variables have many uses in algebra. They can represent generalized numbers, unknowns, arguments, parameters, or arbitrary symbols (Küchemann, 1978; Usiskin, 1999). (See the previous discussion on the various views of algebra for more details.) Additionally, variables can be used incorrectly to represent objects (Küchemann, 1978), which will be discussed in the next paragraph. With these many different uses of variables, it may not be surprising that algebra students often struggle to understand and effectively use variables. Common student misconceptions related to variables include viewing variables as abbreviations or labels (e.g., Küchemann, 1978), viewing all variables as specific values (e.g., Asquith et al., 2007), and being unable to accept expressions containing variables as final answers to problems (e.g., Collis, 1975).

First, students often see variables as abbreviations or labels for actual objects rather than as the number of objects, especially when working with real-world applications (Collis, 1975; Küchemann, 1978; MacGregor & Stacey, 1997). Consider the following word problem:

> Blue pencils cost 5 pence each and red pencils cost 6 pence each. I buy some blue and some red pencils and altogether it cost me 90 pence. If b is the number of blue pencils bought, and if r is the number of red pencils bought, what can you write down about b or r? (Küchemann, 1978, p. 25)

Students who view variables as abbreviations/labels might give the following response: $b + r = 90$, thinking that this equation would indicate that the blue pencils (b) and red pencils (r) together would cost 90 pence. Students also might determine one possible solution, such as $b = 12$ and $r = 5$, and write $12b + 5r = 90$, because 12 blue pencils and 5 red pencils cost 90 pence. And in fact, both of these types of responses were very prevalent among the approximately 3,000 students who participated in Küchemann's (1978) study. This same type of phenomenon has been prevalent in college students' reasoning, as evidenced by the considerable literature surrounding the students and professors problem (e.g., Clement, 1982). (In particular, when asked to use variables to represent the situation in which "there are six times as many students as professors," students often respond with "$6s = p$," in which s and p are incorrectly used as labels for students and professors, respectively.)

Second, students often have difficulty viewing variables as representing varying quantities rather than specific values (Collis, 1975; Küchemann, 1978; Asquith et al., 2007). Consider the following question posed to a sample of middle-school students: "Can you tell which is larger, $3n$ or $n + 6$? Please explain your answer" (Asquith et al., 2007, p. 255). Instead of noticing that $3n$ will be larger than $n + 6$ for some values of n and smaller for other values of n, several students simply evaluated $3n$ and $n + 6$ for a single value of n (such as $n = 1$) and made their decision accordingly ($n + 6$ is larger than $3n$ because $7 > 3$) (Asquith et al., 2007). Küchemann (1978) posed a similar question to the students in his study and observed the same misconception.

And third, some students are unable to accept expressions containing variables (e.g., $3x + 1$) as final answers when asked to simplify algebraic expressions (Collis, 1975; Küchemann, 1978). For example, when novice algebra students are given an expression such as $5x + 9y – 3(x + 3y – 7)$ and asked to simplify it, they often have difficulty accepting $2x + 21$ as their final answer. To these students, $2x + 21$ seems unresolved and has a lack of closure (Collis, 1975; Küchemann, 1978), perhaps due to their previous experience working with numerical equations that do not contain variables and that can generally be simplified to an integer solution (e.g., $5 + 7 \times 3^2 = 68$).

The preceding misconception can lead students to search for ways to remove variables from their expressions (Küchemann, 1978). Consider the following problem: "David is 10 cm taller than Con. Con is h cm tall. What can you write for David's height?" (MacGregor & Stacey, 1997, p. 5). The correct answer is that David is $h + 10$ cm tall. However, several middle- and high-school students who were posed this question found a variety of unique solutions that did not involve the variable h. In particular, several students assigned the variable h a value of 8 based on its position in the alphabet and concluded that David was 18 cm tall; others chose a reasonable height for Con and simply added 10 to the number to find David's height; and a few believed that variables must always equal 1 (possibly confusing $x = 1x$ with $x = 1$) and concluded that David was 11 cm tall (MacGregor & Stacey, 1997).

***Equivalence and the equal sign*.** In addition to their difficulty with variables, many elementary-, middle-, and even high-school students do not fully understand the idea of equivalence or the role of the equal sign in mathematical sentences (Asquith et al., 2007; Baroody & Ginsburg, 1983; Behr, Erlwanger, & Nichols, 1980; Falkner, Levi, & Carpenter, 1999; Kieran, 1981; Knuth et al., 2006; Powell 2015). In particular, students tend to have an operational view of the equal sign rather than a relational one (Asquith et al., 2007; Knuth et al., 2006; Powell 2015). That is, they see the equal sign as a signal to "do something" (i.e., write their answer) as opposed to a relational symbol that shows that the "two sides…should balance or be the same" (Powell, 2015, p. 267).

Students with an operational view of the equal sign often have difficulty understanding equality sentences such as 7 = 3 + 4 and 8 = 8 (Kieran, 1981; Falkner et al., 1999; Powell, 2015). In a study involving sixth-graders, students were asked to talk about these two math sentences (Falkner et al., 1999). Students' comments related to 7 = 3 + 4 included that the equation was written "the wrong way" or "backward" (p. 234). And for 8 = 8, one student indicated that "you just shouldn't write it that way" (p. 235). These students were also asked to solve the problem: 8 + 4 = ☐ + 5, and all of them responded with an answer of 12 (ignoring the "+ 5") or 17 (finding the sum of all three numbers).

Students in middle and high school often demonstrate their limited understanding of the equal sign by writing "equality strings" that are not true (Asquith et al., 2007; Kieran, 1981; Knuth et al., 2006). For example, when solving an equation such as $x + 4 = 11$, students may write the following "equality string": $x + 4 = 11 - 4 = 7$. However, this is not a proper use of the equal sign because it implies that $x + 4 = 7$ when in fact $x = 7$.

Students should develop a relational view of the equal sign to truly understand how to solve non-trivial algebraic equations, such as $5x - 7 = 2x + 8$ (Kieran, 1981). Filloy and Rojano (1989) have labeled linear equations with variables on both sides as non-arithmetical equations and claimed that there is a considerable jump in the level of difficulty (which they have termed "the didactic cut") as students move from arithmetical equations (e.g., $5x - 7 = 3$) to non-arithmetical equations. A relational view may help students understand the various properties of equality that are necessary to solve for x in both types of linear equations. For example, students with a relational view might be more likely than students with an operational view to understand that adding or subtracting the same number on both sides of the equation creates an equivalent equation. In one study, middle-school students who gave a relational definition of the equal sign were more likely to solve a two-step linear equation correctly than students who gave an operational definition (Knuth et al., 2006).

Extrapolation techniques. Although students' difficulties with variables and the equal sign may explain some of their struggles with algebra, many common student errors can actually be attributed to the misuse of extrapolation techniques—i.e., techniques that students use "to bridge the gap between known rules and unfamiliar problems" (Matz, 1982, p. 27). Extrapolation techniques can be divided into at least two categories: linearity and generalization (Matz, 1982).

Linearity. Students begin having experiences related to linearity from early childhood (Van Dooren, De Bock, Janssens, & Verschaffel, 2008). For example, children may notice that one of their stuffed-animal toys has two eyes, two have four eyes, three have six eyes, and so on. Later, they encounter linear situations involving money (e.g., one piece of candy costs 25¢, so three pieces cost 75¢). Students' experiences with linearity can lead them to apply aspects of linearity inappropriately in mathematics, such as during the process of manipulating algebraic expressions and equations (Matz, 1982; Van Dooren et al., 2008). Typical linearity issues include generalized distribution errors, such as $(x+y)^2 = x^2+y^2$; $\sqrt{x+y} = \sqrt{x}+\sqrt{y}$; cancelation errors, such as $\frac{ax+y}{a} = x+y$; and reciprocal errors, such as $\frac{1}{x}+\frac{1}{y}=\frac{1}{z} \rightarrow x+y=z$. All of these types of errors "seem to indicate that students do not realize that not all operators or procedures behave like the linear ones they are most familiar with" (Matz, 1982, p. 33).

Generalization. In addition to linearity errors, the other major extrapolation technique that is misused by students is generalization. In these cases, students try to "[bridge] the gap between known rules and unfamiliar problems by revising a known rule to accommodate particular operators and numbers that appear in a new situation" (Matz, 1982, p. 33). For example, when students are first introduced to the concept of solving quadratic equations, they often fail to realize that $(x - a)(x - b) = c$ implies that $x - a = c$ or $x - b = c$ is only true when $c = 0$ (due to the zero-product property). Therefore, novice students who are asked to solve the quadratic equation $x^2 - 4x - 12 = -7$ may produce the following incorrect work:

$x^2 - 4x - 12 = -7$

$(x - 6)(x + 2) = -7$

$x - 6 = -7 \rightarrow x = 1 \qquad x + 2 = -7 \rightarrow x = 9.$

Section review. In summary, algebra students often struggle to fully grasp the various meanings and uses of variables, hold an operational view of the equal sign (as opposed to a relational view), and misuse extrapolation techniques, as demonstrated by the generalized distribution error $(x + y)^2 = x^2 + y^2$. The discussion will now shift from student thinking about expressions and equations to their thinking about functions. Equations and functions are closely related because equations can sometimes be used to define functions. (For example, the linear equation $y = 2x + 1$ defines a function; in particular, y is a function of x in this case.)

Functions. Research in the area of student thinking about functions includes studies that explore students' limited view of functions (e.g., Breidenbach et al., 1992), their tendency to conflate graphical representations of functions with the visual attributes of the situations being modeled (e.g., Monk, 1992), and their conceptions of functions as actions, processes, and objects (e.g., Dubinsky & Harel, 1992). After a brief discussion on defining the function concept, student thinking related to these areas will be explored.

Definitions of function. Although the definition of function has evolved over the past few centuries (Even, 1993), two general definitions of the concept are widely used today—one related to the idea of correspondence and the other to covariation (Cooney et al., 2010; Leinhardt, et al., 1990; Lloyd & Wilson, 1998). The correspondence definition is the most common type of definition seen in mathematics textbooks and classrooms today (Leinhardt et al., 1990). It is intentionally broad and abstract and defines functions as single-valued mappings (or correspondences) between two sets (Cooney et al., 2010). The mappings are single-valued because each element in the first set (the domain) is mapped to exactly one element in the second set (the range). In addition to explicitly stating this univalence requirement of functions, the correspondence definition of function also implicitly reveals the arbitrary nature of functions (Even, 1993). That is, "functions do not have to exhibit some regularity, be described by any specific expression or particular shaped graph. The arbitrary nature of the two sets means that functions do not have to be sets of numbers" (Even, 1993, p. 96).

In contrast, the covariation definition of function is restricted to contexts that involve mappings from real numbers to real numbers, and the focus is on how quantities vary together (Cooney et al., 2010; Leinhardt et al., 1990). For example, students with a covariation view would most likely look at the table in Figure 2 and notice that as x increases by 1 unit, y increases by 3 units. (On the other hand, students with a correspondence view would be more likely to notice the mappings from x to y, such as $-2 \rightarrow -5$ and $1 \rightarrow 4$.) The covariation definition of function is much more concrete than the correspondence definition and appeals to students' intuitions about dependence, causality, and covariation (Cooney et al., 2010; Leinhardt et al., 1990).

Limited view of functions. Although these two definitions of function apply to a wide variety of situations, a review of the literature reveals that students often have a very restricted view of functions. In particular, many students do not believe that many-to-one correspondences (e.g., constant functions, such as $f(x) = 2$), relations given by more than one formula or rule (e.g., piecewise functions), and relations with arbitrary correspondences are actually functions (Leinhardt et al., 1990; Oehrtman, Carlson, & Thompson, 2008). Additionally, students often believe that only linear graphs or graphs that have a clear pattern can represent functions (Leinhardt et al., 1990).

And students' limited view of functions seems to persist into college. In a study of preservice mathematics teachers, Breidenbach et al. (1992) found that the preservice teachers had difficulty identifying functions written as sets and mappings, equations, graphs, tables, and physical situations. More specifically, many believed that the following did not represent functional situations: $\{(x, 2x + 1)$: x in the set of all integers$\}$ (about 50% of the sample), the graph of an exponential function (about 59%), and a table listing nine distinct names in one column and an amount owed in a second column (about 51%). Also, about 69% of the students believed that $y^4 = x^3$ could not be described using one or more functions (Breidenbach et al., 1992).

Students' (and teachers') limited views of functions can also be seen by observing their use of function notation. In one study, "when asked to express s as a function of t, many high performing precalculus students did not know that their objective was to write a formula in the form of 's = <some expression containing a *t*>'" (Oehrtman et al., 2008, p. 30). And some of the students were unable to describe the meaning of the symbols in a simple linear function, such as $f(x) = 3x$ (Oehrtman et al., 2008).

Students' concept image of functions—or mental pictures and properties associated with the function concept (Tall & Vinner, 1981)—may explain some of the issues that students have with functions and function notation. According to Thompson (1994), "A predominant image evoked in students by the word 'function' is of two written expressions separated by an equal sign" (p. 24). For many students, "$f(x)$" means little more than "the formula to use is…" (Thompson, 1994).

Iconic interpretation. Not only do students have a restricted view of functions and their graphs, but they also have difficulty constructing and interpreting graphs of functions that represent real-world situations (Leinhardt et al., 1990; Monk, 1992; Oehrtman et al., 2008). In particular, students often interpret the shape of a graph as a literal picture of the visual attributes of the situation (Leinhardt et al., 1990). According to Oehrtman and colleagues (2008):

> When dealing with functions as models of concrete situation, there are often topographical structures within the real-world setting itself (e.g., the curves of a racetrack, the elevation of a road traveling across hilly terrain, or the shape of a container being filled with liquid) that students see as being reflected in the function's graph. (p. 29)

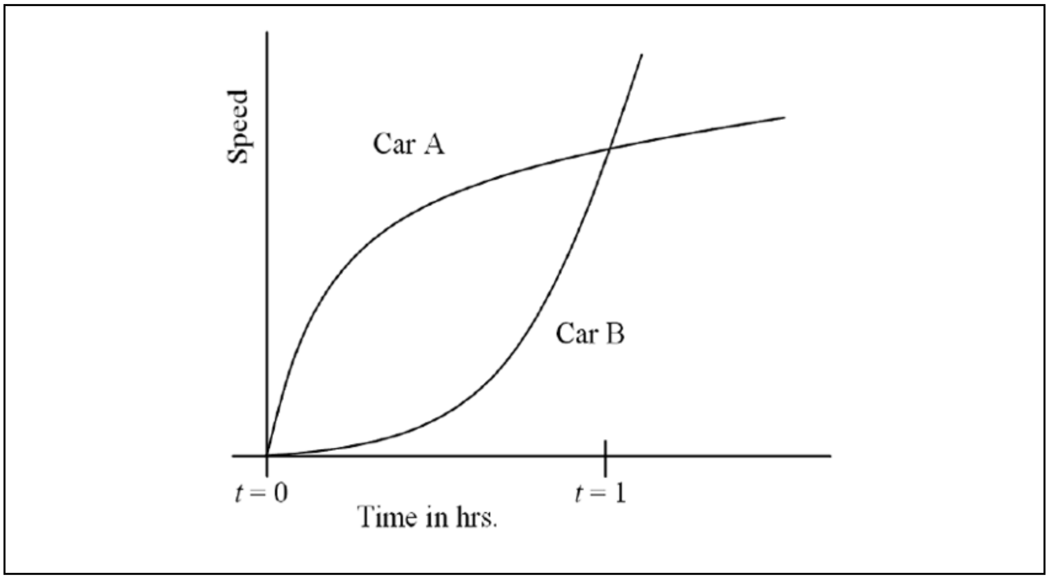


Figure 3. Graph representing the velocity of two cars over time. Adapted from "Students' Understanding of a Function Given by a Physical Model," by S. Monk, 1992, in G. Harel and E. Dubinsky (Eds.), *The Concept of Function: Aspects of Epistemology and Pedagogy*, p. 175. Copyright 1992 by the Mathematical Association of America.

For example, the graphs in Figure 3 show the velocity of two cars over time. When asked to interpret the graphs, many students confuse speed with position and claim that the two cars collide after an hour (at $t = 1$) or that Car B is catching up to or passing Car A around the one-hour mark (Monk, 1992). Thus, students are likely viewing the graph as a picture of the situation and not as a graphical representation of functions that map a set of inputs to a set of outputs (Oerhtman et al., 2008).

Action, process, and object conceptions of functions. In addition to studying students' difficulties with functions and graphs, researchers (e.g., Breidenbach et al., 1992) have developed theories and frameworks to better understand students' various conceptions of functions. In this section, Action-Process-Object-Schema (APOS) theory—a well-known theory related to students' mental construction of processes and objects—will be discussed.

According to APOS theory, students generally begin the study of functions with an *action* conception of functions. Students with an action conception need external clues—such as formulas or recipes—to work with functions (Asiala et al., 1996; Breidenbach et al., 1992) and think about functions one step at a time (Dubinsky & Harel, 1992). At this stage, students can do little more than evaluate a function for a given value or manipulate a function using a step-by-step procedure (Asiala et al., 1996).

After students have had multiple opportunities to perform actions on functions and to reflect upon these actions, their conceptions of function may become *processes* (Asiala et al., 1996; Breidenbach et al., 1992). In contrast to actions, processes are more internal and dynamic. "An individual who has a process conception of a transformation can reflect on, describe, or even reverse the steps of the transformation without actually performing those steps" (Asiala et al., 1996, p. 10). In the context of functions, students with a process conception can visualize a function receiving multiple inputs, performing operations on these inputs, and generating outputs without explicitly carrying out the computations. They can also combine processes—which is necessary for function composition—and reverse processes—which is necessary for finding inverses (Asiala et al., 1996). Additionally, students with a process conception can begin to truly understand what it means for a function to be "one-to-one" or "onto" and can work with functions whose domain and/or range are not numbers (Breidenbach et al., 1992).

Students with a process conception can begin to develop an *object* conception of functions over time. According to Asiala and colleagues (1996):

> When an individual reflects on operations applied to a particular process, becomes aware of the process as a totality, realizes that transformations (whether they be actions or processes) can act on it, and is able to actually construct such transformations, then he or she is thinking of this process as an object. In this case, we say that the process has been encapsulated to an object. (p. 11)

Students with an object conception of functions are able to think more globally about functions. That is, they begin to consider properties and behaviors of functions and families of functions (Arcavi, Drijvers, & Stacey, 2017). These students can also operate on sets of functions in a meaningful way (Thompson, 1994). For example, they should be able to compose two functions, say $f(x)$ and $g(x)$, and reason about how the domain and range of f and g would affect the domain and range of $f \circ g$ and $g \circ f$. Therefore, one goal of instruction should be to help students develop an object conception of functions.

Section review. In summary, algebra students often have a narrow view of functions and incorrectly interpret graphs of real-world situations as literal pictures/ images of the situations. Additionally, students generally have *action* or *process* conceptions of functions, but with practice and experience, they can develop *object* conceptions.

Based on this review of the literature on student thinking about expressions, equations, and functions, it is evident that mathematics teachers need to develop a knowledge base of student thinking to inform their teaching. In the section that follows, research-based teaching methods and strategies in algebra will be discussed and connected to the research on student thinking (outlined in this review).

Teaching Methods and Strategies in Algebra

Mathematics teachers need to be equipped with the tools necessary to help students develop conceptual understandings in mathematics (including algebra) and to overcome any misconceptions they might develop. These tools include (a) building on students' prior knowledge (e.g., Arcavi et al., 2017); (b) choosing a wide range of rich examples, tasks, and questions (e.g., Leinhardt et al., 1990); (c) promoting problem solving (e.g., Star et al., 2015); (d) using multiple representations (e.g., Goldin & Shteingold, 2001); (e) using manipulatives (e.g., Leitze & Kitt, 2000); and (f) incorporating technology (e.g., Yerushalmy & Chazan, 2008). Each of these teaching methods/strategies are explored in the sections that follow.

Prior knowledge. Students enter the mathematics classroom with knowledge that they have acquired both inside and outside of the classroom, and research suggests that when teachers incorporate this knowledge into their instruction, student achievement can increase (see Carpenter & Fennema, 1992). Teachers can build on students' prior out-of-school knowledge by exploring algebraic concepts using familiar real-world contexts, as well as games and puzzles (Arcavi et al., 2017; Kalchman & Koedinger, 2005). For example, students could explore linear functions by considering an authentic situation: going to the movies. More specifically, they could explore how the total cost of going to the movies varies as the number of tickets purchased increases using tables, graphs, and/or equations.

Additionally, teachers can build on students' prior school-based knowledge by considering what students have learned in their previous classes and applying that knowledge to new learning situations. Thus, teachers need "to make public and shared the knowledge carried from math and other subjects and to work from that base…" (Leinhardt et al., 1990, p. 48). For example, before working with rational expressions, a teacher might want to review how to add, subtract, multiply, and divide rational numbers (that do not include variables) and help students to make connections between operating on rational numbers (e.g., $\frac{2}{5} \times \frac{2}{3} = \frac{4}{15}$) and operating on rational expressions (e.g., $\frac{a}{b} \times \frac{a}{c} = \frac{a^2}{bc}$).

Teachers also need to consider students' misconceptions and other inaccuracies in their thinking when building on students' prior knowledge, as misconceptions can greatly impede student learning (National Research Council, 2000). In particular, teachers should consider possible student misconceptions—such as the generalized distribution error, discussed in the previous section on student thinking—and determine how they can best be addressed during instruction. In this case, a teacher may ask students to rewrite $(x + y)^2$ as $(x + y)(x + y)$ and expand it using the distributive property, or she may ask students to determine whether $(x + y)^2 = x^2 + y^2$ is true or false by selecting specific values for x and y and then evaluating the "identity."

Ultimately, teachers need to be aware that all students enter their classroom with existing knowledge that can facilitate their learning, as well as knowledge that can hinder their learning (National Research Council, 2000). Therefore, teachers should strive to build upon students' accurate conceptions and help them address their misconceptions.

Examples, tasks, and questions. Students' prior knowledge is one factor that algebra teachers should consider when choosing which examples, tasks, and questions to explore in the classroom. According to Leinhardt et al. (1990), "The selection of examples is the art of teaching mathematics. Making available for consideration by the student an example that exemplifies or challenges can anchor or critically elucidate a point" (p. 52). In the paragraphs that follow, a brief discussion on high-quality examples, tasks, and questions will be presented.

Examples. When selecting which examples to explore with students, teachers should incorporate examples with both standard and nonstandard forms of expression (Powell, 2015). For instance, when discussing how to solve quadratic equations, teachers should be sure to include some quadratic equations that are in standard form (i.e., $ax^2 + bx + c = 0$) and others that are not. Research suggests that teachers should also consider using solved problems (i.e., problems with their full solution) during instruction (Star et al., 2015). In particular, teachers should present solved problems and ask students to analyze and discuss the steps and solution strategy used to solve each problem. Similarly, teachers can capitalize on students' mistakes by presenting problems with incorrect solutions. "Requiring students to detect errors made by others can raise awareness of their own mistakes" (Arcavi et al., 2017, p. 97). For example, a teacher could present a typical cancellation error, such as $\frac{ax+y}{a} = x + y$, and use it as an opportunity to discuss the rationale behind the mistake (i.e., misconceptions about linearity) and how to avoid it in the future.

Tasks. In addition to exploring a variety of examples and solved problems in class, teachers should strongly consider incorporating tasks of high cognitive demand. In the area of functions, this type of task generally requires interpretation (e.g., classification tasks), construction (e.g., prediction tasks), or possibly both (e.g., translation and scaling tasks) (Leinhardt et al., 1990). When selecting and/or developing tasks, teachers should consider tasks that are aligned with appropriate standards, have multiple entry points, involve exploratory approaches, require the use of multiple representations, are open-ended (i.e., have more than one solution), and make connections to students' prior knowledge, because tasks with these characteristics tend to promote deep learning of mathematical concepts and problem solving (Ronau, Meyer, & Crites, 2014). (Note that problem solving will be discussed in the next section.)

High-quality tasks can take many forms, including model-eliciting activities. These tasks involve simulating real-world situations that involve mathematical thinking and generally require students to interpret situations, construct basic mathematical models (i.e., descriptions or explanations), and make predictions or recommendations based on their models (Lesh & Lehrer, 2003). Model-eliciting activities also stimulate students "to invent, extend, revise, and refine many of the important ideas throughout the mathematics curriculum…" (Van Dooren et al., 2008, p. 335). For example, students could be given a number of different statistics for a group of basketball players (e.g., height, vertical jump, free-throw shooting percentage) and be asked to develop a mathematical model to determine which players should be chosen for a hypothetical team. Model-eliciting task are generally considered high quality because they meet most—if not all—of the criteria listed above (e.g., they have multiple entry points and are open-ended). Additionally, they can be used to help students move from action and process conceptions to develop broader conceptions (i.e., object conceptions). That is, model-eliciting tasks challenge students to think more globally about mathematical concepts.

Questions. Not only do teachers need to carefully select the examples and tasks that they will use in the classroom, but they also need to consider the questions they will pose to students as they work on these examples and tasks. There are at least three types of questions that promote conceptual understandings in algebra: reversibility, flexibility, and generalization questions (Dougherty, 2001; Ronau et al., 2014). Reversibility questions generally ask students to create a problem for a given solution. For example, they could be asked the following question: what are the equations of three lines that pass through the point (1, 2)? Flexibility questions ask students to compare and contrast problems or to solve a problem in multiple ways. For example, they could be asked: how are the functions $f(x) = x^2$ and $g(x) = (x - 3)^2 + 1$ similar, and how are they different? Or they could be asked to solve $x^2 = (x - 3)^2 + 1$ using more than one method (e.g., algebraically and graphically by determining the intersection of the graphs of $f(x)$ and $g(x)$). Generalization questions ask students to use examples to find a pattern, or they ask them to create specific examples of a generalization, such as a rule or pattern. For example, students could be given the graphs of several sets of perpendicular lines and asked the following: what is the relationship between the slopes of perpendicular lines (opposite reciprocal)? They could also be asked to draw the graph of a quadratic equation with no real solutions.

As with high-quality tasks, reversibility, flexibility, and generalization questions can help students develop object conceptions of expressions, equations, and functions by requiring them to think more globally about these concepts. Additionally, these types of questions can broaden students' views of functions, which is quite narrow according to the literature.

Problem solving. Selecting appropriate questions and tasks is an important component of problem-solving instruction, which refers to the entire process of incorporating meaningful tasks into classroom instruction (Lester, 2013; NCTM, 2000). According to Lester and Kehle (2003):

> Successful problem solving involves coordinating previous experiences, knowledge, familiar representations and patterns of inference, and intuitions in an effort to generate new representations and related patterns of inference that resolve some tension or ambiguity (i.e., lack of meaningful representations and supporting inferential moves) that prompted the original problem-solving activity. (p. 510)

To help their students become better problem solvers, teachers should select and develop meaningful problem-solving tasks. But this is only the beginning. Teachers should also listen to their students as they engage in problem solving and use students' ideas as "thinking devices" during classroom discourse, ask good probing questions to stimulate critical thinking, help students apply appropriate problem-solving methods and strategies, and encourage productive struggle (Cai, 2010; DiMatteo & Lester, 2010; Knuth & Peressini, 2001; Lester, 2013). Productive struggle refers to students' efforts to make sense of non-routine problems and understand the underlying mathematical concepts (Hiebert & Grouws, 2007). Productive struggle requires perseverance, as students often have to attempt several approaches/strategies to solve meaningful tasks. Thus, teachers should give students ample time to grapple with tasks and should be careful not to intervene too quickly or do the thinking for their students (Warshauer, 2015). Instead, they should consider probing questions that can facilitate students' learning.

Problem solving is the focal point of several reform curricula (e.g., *Connected Mathematics Project* and *Interactive Mathematics Program*) that have been developed over the past few decades. These curricula generally consist of a sequence of tasks that drive instruction, and the teacher serves as a facilitator (as described above). Reform curricula have been studied extensively, and their use in the mathematics classroom has been linked to increased student achievement (see Senk & Thompson, 2003). In one particular study, a group of students who learned algebra using the *Connected Mathematics Project* curriculum outperformed a group of students who used a more traditional curriculum on open-ended and problem-solving tasks (Post et al., 2008). The students who used the reform curricula also performed as well as the control group on procedural tasks (Post et al., 2008).

Multiple representations. As previously mentioned, when planning instruction, teachers should incorporate tasks and examples that require students to use multiple representations, such as graphs, tables, equations, and verbal descriptions (Ronau et al., 2014). “Effective mathematical thinking involves understanding the relationships among different representations of ‘the same’ concept as well as the structural similarities (and differences) among representational systems” (Goldin & Shteingold, 2001, p. 9). Suggestions for teachers include “copresenting” representations within the same lesson (Kalchman & Koedinger, 2005), exploring the types of information that various representations convey (Star et al., 2015), and asking students to make judgments across representations (Oehrtman et al., 2008). For example, during a lesson on linear equations, students could be asked to explore the equation $y = 3x - 1$, as well as its graphical and tabular representations, and to determine what information is revealed by each representation. (In this case, the equation clearly reveals the line’s slope and *y*-intercept, the graph reveals its *x*- and *y*-intercepts and end behavior, and the table reveals the linear relationship between variables.)

As discussed in the previous section on student thinking about functions, students often find it difficult to construct and interpret graphs of functions that represent real-world situations. However, students may experience greater success in this area if their teachers ask them to use graphical representations on a regular basis, as well as connect the graphs to other representations and make decisions based on multiple representations.

Manipulatives. In addition to graphs, tables, equations, and verbal descriptions, manipulatives are another type of representation that can help students build connections among algebraic concepts. Manipulatives can be physical/concrete (e.g., blocks or tiles) or virtual (e.g., computer applets) and come in many different forms. But they have a common purpose: to provide physical models of abstract concepts that can be manipulated by the learner (Balka, 1993; Suh & Moyer, 2007).

Despite evidence that manipulatives can improve student achievement (Rakes, Valentine, McGatha, & Ronau, 2010; Sowell, 1989; Suh & Moyer, 2007), they are often underutilized in mathematics teaching and learning, especially in middle and secondary education (Leitze & Kitt, 2000; Tooke, Hyatt, Leigh, Snyder, & Borda, 1992). Thus, teachers should look for meaningful ways to incorporate manipulatives into instruction. Balance scales and algebra tiles are two types of manipulatives that are frequently used in algebra classrooms.

Balance scales are designed to help students learn how to solve linear equations (Powell, 2015). Basic balance scales generally have two pans—one for each side of a linear equation—and at least two types of blocks—a unit block that represents 1 and a variable block that represents an unknown *x*. (Note that there are advanced virtual balance scales include a block that represents –1, four pans, and/or a pulley system for negative integers.) Figure 4 shows an example of a concrete balance scale. According to Suh and Moyer (2007):

> Once the beam balances to represent the given linear equation, students can choose to perform any arithmetic operation, as long as they perform the same operation on both sides of the equation, thus keeping the pans balanced. If the equation is not balanced, the beam will slant to one side. The goal…is to get a single *x*-box on one side, with the amount needed for balance on the other side, thus giving the value of *x*…. (p. 160)

There are several benefits of using balance scales to explore the topic of solving linear equations. They include providing students a mental picture and concrete model of the process required to solve linear equation (Vlassis, 2002) and helping them develop a relational view of the equal sign (Rojano & Martínez, 2009). More specifically, balance scales can reinforce the idea that both sides of a linear equation should be balanced or the same (Powell, 2015).

Figure 4. Example of a balance scale.

Algebra tiles also are designed to help students learn how to solve linear equations, but they can also be used to model multiplying monomials and binomials and factoring quadratic expressions (Leitze & Kitt, 2000). Algebra tiles generally consist of small squares (with dimensions of 1×1 and an area of 1 to represent 1), rectangles (with dimensions of $1 \times x$ and an area of x to represent x), and large squares (with dimensions of $x \times x$ and an area of x^2 to represent x^2). (Note that the length of the rectangle is equal to the side length of the small square, and the width is equal to the side length of the large square.) Figure 5 shows an example of a set of algebra tiles.

Working with algebra tiles can help students gain deeper understandings of the concept of variable, number properties and principles, and concepts in geometry (Chappell & Strutchens, 2001). For example, the use of algebra tiles "combines an algebraic and a geometric approach to algebraic concepts using an array-multiplication model similar to that employed in many elementary school classrooms" (Leitze & Kitt, 2000, p. 463). However, teachers should be careful to make connections between students' work with manipulatives (e.g., algebra tiles and balance scales) and the abstract mathematical ideas the materials are meant to promote (Bohan & Shawaker, 1994). In short, if students do not understand how their work with manipulatives is related to the mathematical concepts they are learning, they most likely won't benefit from their use.

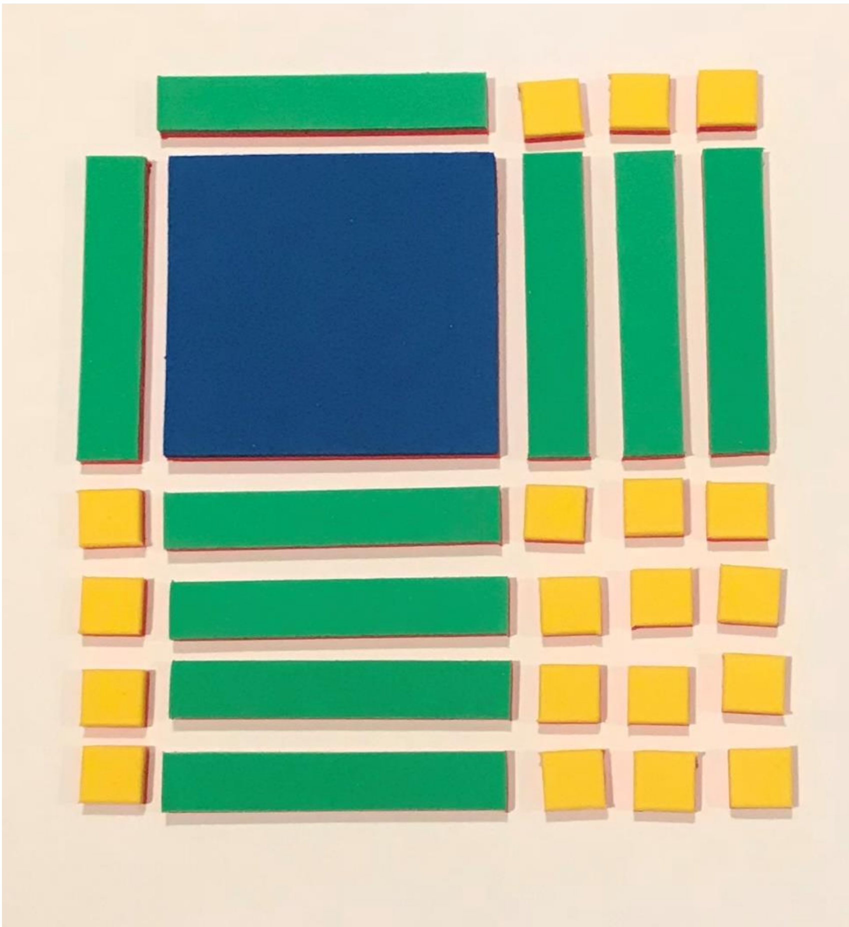


Figure 5. Example of a set of algebra tiles.

Both balance scales and algebra tiles are available in physical and virtual forms. (Virtual manipulatives are freely available online through websites such as the National Library of Virtual Manipulatives: http://nlvm.usu.edu/.) However, teachers need to be aware that there are differences between physical and virtual manipulatives. In particular, physical manipulatives are concrete representations, whereas virtual manipulatives are iconic representations. And based on Bruner's (1966) "enactive-iconic-symbolic" modes of representation, some members of the mathematics-education community in the U.S. and abroad believe students should work with concrete representations before iconic (e.g., virtual) ones. (See Hoong, Kin, & Pien, 2015, for more details.)

Technology. Virtual manipulatives and other digital tools—such as spreadsheets, graphing utilities, and computer algebra systems (CAS)—can greatly enhance mathematics teaching and learning in algebra when used strategically in the classroom (Rakes et al., 2010; Yerushalmy & Chazan, 2008). In particular, spreadsheets and graphing tools can be used to help students discover relationships among quantities; CAS usage can reduce students' cognitive load by performing algebraic manipulations; and all of these tools can be used to support students' conceptual understandings of algebraic concepts through multiple representation (Yerushalmy & Chazan, 2008). Technology available to mathematics teachers and their students include graphing calculators and interactive whiteboards (IWB), computers/laptops, tablets, and smartphones, which can often access online resources such as applets and video clips (Arcavi et al., 2017).

Although teachers should incorporate technology into their algebra instruction, it is important that they are careful and deliberate in their choice of tasks and digital tools. More specifically, they need to choose purposeful tasks in which one or more digital tools are needed to enhance students' conceptual understandings. Additionally, teachers need to "act as a conductor who organizes the teaching in such a way, that the ‘mathematics hidden in the work with the tool’ becomes explicit and subject to discussion” (Arcavi et al., 2017, p. 130).

Section review. In summary, mathematics teachers need to be familiar with research-based teaching strategies and methods and how to use them effectively in the classroom. These strategies/methods include building on students’ prior knowledge, choosing a wide range of rich tasks, promoting problem solving, using multiple representations (including manipulatives), and incorporating technology.

Based on this review of the literature, it is clear that algebra teachers need more than simply content knowledge (i.e., knowledge of algebra and other areas of mathematics) to be effective. They also need to be knowledgeable about their students’ algebraic thinking (including misconceptions), as well as appropriate mathematics teaching strategies to use during instruction. In the next section, frameworks related to teacher knowledge (both general and mathematics-focused) will be discussed and synthesized, and connections to algebra will be made.

Teacher Knowledge

A number of researchers (e.g., Ball, Thames, & Phelps, 2008; Herbst & Kosko, 2014; Kilpatrick et al., 2015; Krauss, Baumert, & Blum, 2008; McCrory et al., 2012; Mohr-Schroeder et al., 2017; Rowland, Huckstep, & Thwaites, 2005) have built upon Shulman’s (1986) broad conceptions of content knowledge (CK) and pedagogical content knowledge (PCK) to develop frameworks specifically focused on the knowledge necessary for teaching mathematics. Although they use different terminology and labels, all of these frameworks of teacher knowledge contain constructs related to CK and PCK, which is the interaction of CK and general pedagogical knowledge (GPK). (See Figure 6 for a Venn diagram of the relationships among CK, PCK, and GPK.)

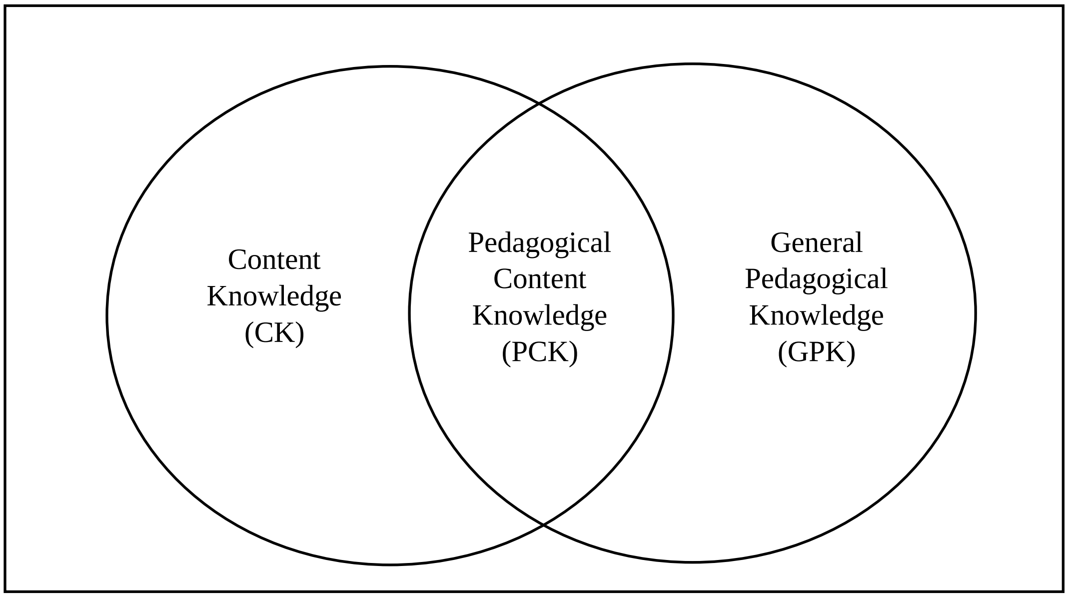


Figure 6. Venn diagram showing the relationships among CK, GPK, and PCK

All three types of knowledge will now be explored, with an emphasis on CK and PCK. More specifically, various frameworks that focus on mathematics teacher knowledge in general (e.g., the Mathematical Knowledge for Teaching and knowledge quartet frameworks) and on teacher knowledge at the secondary level (e.g., the Mathematical Understanding for Secondary Teaching and COACTIV frameworks) will be examined to develop a general consensus about the knowledge necessary to teach algebra.

Content Knowledge

Content knowledge (CK) refers to teachers' knowledge of subject matter, including the content that they teach (Shulman, 1986). This type of knowledge consists of a comprehensive understanding of the facts and concepts of the discipline, as well as a deep understanding of its underlying structures (Shulman, 1986). In mathematics, facts include definitions, formulas, procedures, and foundational information (e.g., multiplication facts), and concepts include general ideas in mathematics, such as the function concept—which was previously discussed in this review. The underlying structures of mathematics include the organization of facts and concepts within the discipline, as well as "the set of ways in which truth or falsehood, validity or invalidity, are established [e.g., reasoning and proof in mathematics]" (Shulman, 1986, p. 9). In short, CK refers to knowledge of the major facts and concepts of a domain, why they are important and true, and how they are related. However, researchers in mathematics education have conceptualized CK in slightly different ways.

In the COACTIV teacher knowledge framework (for secondary mathematics teachers), CK is conceptualized as having a deep understanding of the concepts found in the high-school mathematics curriculum. According to the COACTIV research team, this deep understanding lies somewhere between "the school-level mathematical knowledge that good school students have, and the university-level mathematical knowledge that does not overlap with the content of the school curriculum…" (Krauss et al., 2008, p. 876).

CK also coincides closely with the "foundation" dimension of Rowland, Huckstep, and Thwaites' (2005) knowledge quartet, which is a model of mathematics teachers' knowledge. This dimension of the quartet involves the knowledge and understanding of mathematics that is acquired in academic settings and includes conceptual understandings (i.e., knowing the "whys") in mathematics (Rowland et al., 2005). This dimension also includes other aspects (e.g., teachers' personal beliefs), which are not directly related to CK.

Additionally, CK is closely related to the "mathematical proficiency" perspective outlined in the Mathematical Understanding for Secondary Teaching (MUST) framework (Kilpatrick et al., 2015). The "mathematical proficiency" perspective refers to all of the mathematical knowledge/capabilities that teachers need, and it contains several strands, including conceptual understanding, procedural fluency, strategic competence, and adaptive reasoning—which were originally outlined in the National Research Council's (2001) seminal work, *Adding It Up: Helping Children Learn Mathematics*. If conceptual understanding is "knowing why" in mathematics, then procedural fluency is "knowing how" (i.e., understanding how and when to apply algorithms and procedures, and applying them efficiently). For example, students must know how and when to apply the distributive property of multiplication over addition to express the product of two binomial expressions as a single polynomial expression—e.g., $(x+3)(x-4) = x^2 - 4x + 3x - 12 = x^2 - x - 12$. Next, strategic competence involves using problem-solving strategies to approach mathematical tasks, as well as evaluating the effectiveness of these strategies. And last, adaptive reasoning involves the ability to *adapt* to changes in assumptions and conventions in mathematics. For example, when working with matrices, students need to understand that matrix multiplication is not commutative, consider how this change affects computing with matrices, and be able to adjust—or adapt—to this change.

The preceding sections suggest that although different terminology may be used, there is fairly universal agreement that the content knowledge that mathematics teachers need includes deep understandings of mathematical facts, concepts, and procedures. Although there is less agreement about how to partition CK into subdomains, one widely-recognized model—the Mathematical Knowledge for Teaching (MKT) framework developed by Ball and colleagues (2008)—partitions CK into common content knowledge (CCK), specialized content knowledge (SCK), and horizon content knowledge. Because the latter subdomain's inclusion in the MKT model is provisional (Ball et al., 2008), only the first two subdomains (CCK and SCK) will be discussed. Additionally, connections to similar constructs will be examined.

Common content knowledge (CCK). CCK refers to subject matter knowledge and skills in mathematics that are *not* unique to teaching (Ball et al., 2008). Therefore, this subdomain includes all of the mathematics that is commonly taught and learned in the classroom at the elementary, secondary, and postsecondary levels. In the United States, the K-12 mathematics curriculum generally reflects the content guidelines outlined in two significant standards documents: *Principles and Standards for School Mathematics* (PSSM) (NCTM, 2000) and the *Common Core State Standards for Mathematics* (CCSSM) (NGACBP & CCSSO, 2010). Thus, these documents outline the CCK that U.S. mathematics teachers should have.

Focusing on algebra, PSSM recommends that algebra students learn about patterns, relations, and functions (such as rates of change); variables, expressions, equations, and inequalities (such as the ability to write equivalent forms of expressions and to manipulate equations); and mathematical modeling, such as determining functions that can model quantitative relationships (NCTM, 2000). Similarly, the CCSSM outlines a clear and coherent set of standards related to high-school algebra. For example, high-school algebra students are expected to work with expressions (e.g., simplify expressions and perform operations on polynomials), as well as solve and represent equations and inequalities (NGACBP & CCSSO, 2010). Additionally, standards relating to functions include building and interpreting functions that model relationships in the real-world and creating and analyzing multiple representations of functions (NGACBP & CCSSO, 2010).

These algebra standards are fairly consistent among countries with high-achieving mathematics students. For example, students in Singapore and Japan generally score very well on TIMSS and PISA mathematics assessments, and these countries have algebra standards that are closely aligned to CCSSM. In particular, the national mathematics curriculum of Singapore places emphasis on the following concepts in algebra at the secondary level: ratio and proportion, algebraic expressions and formulas, functions and their graphs, solving equations and inequalities, and solving problems in real-world contexts (Ministry of Education, Singapore, 2012). Similarly, the national mathematics curriculum of Japan places emphasis on linear and quadratic equations, systems of equations, and functional relationships in algebra at the secondary level (Takahashi, Watanabe, & Yoshida, 2008).

There is a great deal of overlap in U.S. algebra standards/goals described in PSSM and CCSSM, as well as international standards. Considered together, these documents give a rich picture of what algebra should look like and what algebra teachers should know to be effective in the mathematics classroom.

Specialized content knowledge (SCK). SCK refers to knowledge and skills in mathematics that *are* unique to teaching (Ball et al., 2008). More specifically, a mathematics teacher might use SCK to explain a rule/procedure or evaluate the merit of an unconventional algorithm or approach (Hill, Schilling, & Ball, 2004). These unconventional approaches can be student-generated or teacher-generated. For example, in algebra, students often learn a variety of techniques for factoring quadratic expressions, such as the "AC" methods (which involves factoring by grouping). Thus, teachers need SCK to determine whether or not unfamiliar approaches to factoring are valid. (Note that this type of knowledge is considered "specialized" because it is knowledge that is needed by teachers but not by mathematics experts in other fields.)

General Pedagogical Knowledge

General pedagogical knowledge (GPK) refers to teachers' knowledge of the processes and practices that are critical to teaching and learning. This type of knowledge involves understanding theories and methods of instruction, student thinking/learning, strategies of classroom management, lesson planning, and assessment (Koehler & Mishra, 2009; König, Blömeke, Paine, Schmidt, & Hsieh, 2011). Building on students' prior knowledge in general (as previously discussed) could be considered GPK because it is related to student thinking and can be applied in subject areas other than mathematics.

Because the focus of this study is knowledge specific to teaching mathematics, GPK will not be discussed in greater detail here. The purpose of this brief introduction of GPK is to provide some context for pedagogical content knowledge, which is knowledge in the intersection of CK and GPK.

Pedagogical Content Knowledge

Pedagogical content knowledge (PCK) refers to the dimension of content knowledge that is directly related to pedagogy (i.e., the interaction of CK and GPK). PCK includes all of the content-specific examples, tasks, representations, and strategies that teachers use to help their students learn; additionally, it includes awareness of students' conceptions and preconceptions about the subject matter (Shulman, 1986). These preconceptions include the misconceptions addressed earlier in this review.

Within the context of mathematics, PCK can be partitioned into several subdomains. In particular, the well-known MKT model (Ball et al., 2008) divides PCK into the following subdomains: knowledge of content and students (KCS), knowledge of content and teaching (KCT), and knowledge of content and curriculum, which Ball and colleagues (2008) suggest is a provisional subdomain. Given their widespread use, the first two subdomains (KCS and KCT) will now be discussed, and connections to similar constructs will be examined.

Knowledge of content and students (KCS). KCS is the combination of knowing about *students* and knowing about the subject matter (Ball et al., 2008). This type of PCK includes anticipating students' thinking (i.e., their conceptions, processes, and connections), interpreting and responding appropriately to their emerging ideas, and developing lessons (i.e., choosing examples and representations) based on students' needs (Ball et al., 2008; Petrou & Goulding, 2011).

In algebra, KCS involves being knowledgeable about students' thinking and misconceptions in algebra (see previous discussion on student thinking about expressions, equations, and function) and using this knowledge to inform instructional decisions. For example, when simplifying rational expressions, such as $\frac{5x+10}{5x}$, students often make "cancelation" errors by canceling terms that are joined with addition or subtraction. In this case, a student might try to cancel "5*x*" from the numerator and denominator and then incorrectly conclude that the expression is equal to 10 or 11. Thus, an experienced teacher who understands this misconception might use a few examples, such as $\frac{5+10}{5} = 3$ (not 10 or 11) and $\frac{5\times10}{5} = \frac{50}{5} = 10$, to demonstrate that it is only appropriate to divide common factors (i.e., the parts joined with multiplication).

KCS is closed related to constructs explicated in other frameworks, including the "contingency" dimension of the knowledge-quartet framework (Rowland, Huckstep, & Thwaites, 2005), the "mathematical context for teaching" perspective outlined in the Mathematical Understanding for Secondary Teaching (MUST) framework (Kilpatrick et al., 2015), and the "knowledge of student misconceptions and difficulties" from the COAVTIV framework (Krauss et al., 2008). Each of these constructs will now be briefly described.

First, the "contingency" dimension of the knowledge-quartet framework involves "the ability to 'think on one's feet'; it is about *contingent* action" (Rowland, Huckstep, & Thwaites, 2005, p. 263). More specifically, it refers to the ability to respond appropriately to students' incomplete thinking in the moment and deviate from the lesson plan, as needed, to meet students' needs (Rowland et al., 2005).

Similarly, the "mathematical context for teaching" perspective outlined in the MUST framework involves accessing, understanding, and assessing students' mathematical thinking (Kilpatrick et al., 2015). In this case, accessing and understanding students' thinking refers to uncovering students' conceptions through their written work and classroom discourse. And assessing students' thinking refers to formatively assessing students' ability to use and connect mathematics concepts and using this information to guide instruction (Kilpatrick et al., 2015).

Likewise, "knowledge of student misconceptions and difficulties," which the COACTIV researcher team described as a subdimension of PCK (Krauss et al., 2008), includes the ability to "detect, analyze (e.g., give cognitive reasons for a given problem), or predict a typical student error or comprehension difficulty" (Krauss et al., 2008, p. 876).

Considering all of these various constructs, KCS can be summarized as the ability to anticipate and understand students' thinking and to respond to it appropriately "in the moment." It also involves formatively assessing students' thinking and modifying instruction to increase their understanding.

Knowledge of content and teaching (KCT). KCT is the combination of knowing about *teaching* and knowing about the subject matter (Ball et al., 2008). This type of PCK involves making decisions related to instructional design, such as the choice and sequencing of examples and activities (Ball et al., 2008) and the creation of representations and examples—a strand of the MUST model's "mathematical activity" perspective. KCT also includes evaluating "the instructional advantages and disadvantages of representations used to teach a specific idea and [identifying] what different methods and procedures afford instructionally" (Ball et al., 2008, p. 401).

In algebra (and other areas of mathematics), KCT involves using the teaching methods and strategies that were outlined earlier in this review. These strategies include choosing rich sets of examples/tasks, promoting problem solving, using multiple representations (of functions, etc.), and using manipulatives (e.g., algebra tiles). For example, when preparing a lesson on solving linear equations, an experienced teacher might sequence her examples as follows (in a logical progression from least to most complex): one-step equations with the variable on the left side of the equation, such as $2x = 10$; one-step equations with the variable on the right side of the equation, such as

$-8 = y + 7$; two-step equations, such as $-3x + 5 = 11$; multi-step equations involving distribution, such as $9 = 5(x + 7)$; and multi-step equations with variables on both sides of the equation, such as $-2(x + 1) = 4x + 6$.

Like KCS, KCT is also closely related to constructs explicated in other frameworks. They include the "transformation" and "connection" dimensions of the knowledge-quartet framework (Rowland et al., 2005), the "mathematical context for teaching" perspective of the MUST model (Kilpatrick et al., 2015), and the "knowledge of mathematical tasks" and "knowledge of mathematics-specific instructional strategies" from the COACTIV framework (Krauss et al., 2008). Each of these constructs will now be briefly described.

The "transformation" and "connection" dimensions of the knowledge-quartet framework correspond directly to KCT. The "transformation" dimension includes the choice and use of examples and representations, whereas the "connection" dimension includes the sequencing of examples and topics, as well as the overall coherence of the planning and teaching (Rowland et al., 2005).

Likewise, the "mathematical context for teaching" perspective of the MUST model involves the teacher's ability "to identify foundational or prerequisite concepts that enhance the learning of a concept…[and] how the concept fits learning trajectories" (Kilpatrick et al., 2015, p. 27). This perspective has several strands and is broader than KCT; however, KCT is very similar to this perspective's strand related to knowing and using the curriculum.

Lastly, there are two PCK subdimensions identified in the COACTIV framework—"knowledge of mathematical tasks" and "knowledge of mathematics-specific instructional strategies" (Krauss et al., 2008)—that closely resemble KCT. "Knowledge of mathematical tasks" includes the ability to select and implement appropriate tasks (as discussed earlier in this review), as well as the ability to produce multiple solutions to a problem. And "knowledge of mathematics-specific instructional strategies" includes the ability to "explain mathematical situations or to provide useful representations, analogies, illustrations, or examples to make mathematical content accessible to students" (Krauss et al., 2008, p. 876).

Considering all of these various constructs, KCT can be described as the ability to use knowledge of the curriculum to select and create examples and representations. It also involves understanding the advantages and disadvantages of examples and representations and how to sequence them for maximum impact in the classroom.

Section Review

In summary, teacher knowledge can be categorized into three large domains: content knowledge (CK), general pedagogical knowledge (GPK), and pedagogical content knowledge (PCK), which is knowledge at the intersection of CK and GPK. And these domains can be further divided into subdomains. In particular, CK includes common content knowledge (CCK) and specialized content knowledge (SCK), and PCK includes knowledge of content and students (KCS) and knowledge of content and teaching (KCT). (See Figure 7 for a visual representation of these relationships.)

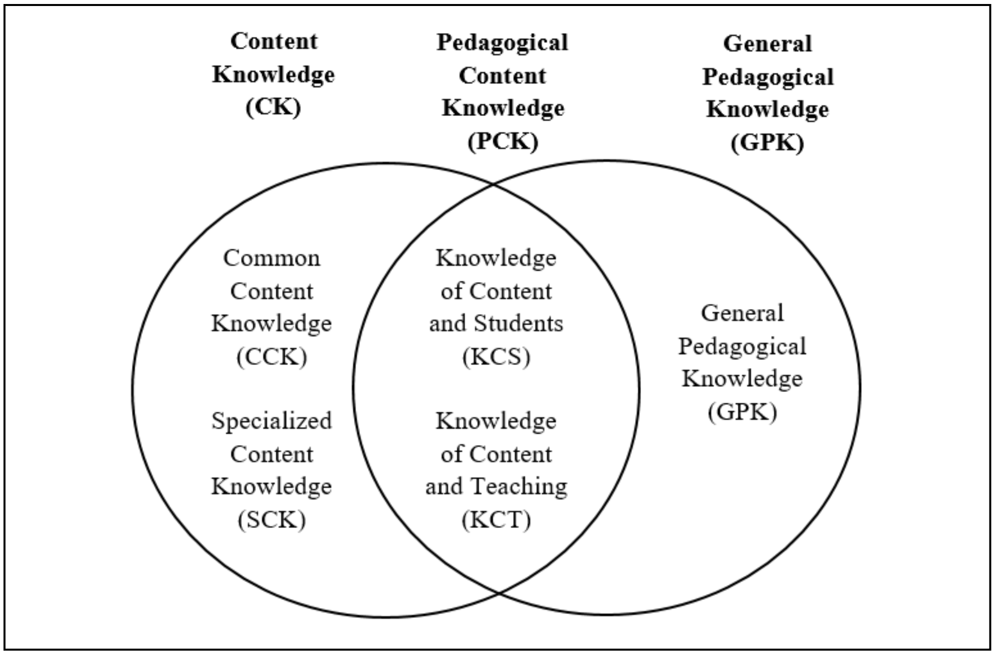


Figure 7. Venn diagram showing the relationships among knowledge types.

Preservice and Inservice Teachers' Mathematical Knowledge for Teaching

A number of studies related to preservice and inservice mathematics teachers' knowledge for teaching have revealed that many mathematics teachers may have some deficiencies in their CK and PCK. In one international study, the Teacher Education and Development Study in Mathematics (TEDS-M), researchers investigated the CK and PCK of preservice mathematics teachers from 17 countries (including the U.S.). The TEDS-M assessment included CK items related to number and operation, algebra and functions, geometry and measurement, and data and chance. It also included PCK items related to mathematics curricular knowledge, knowledge of planning for mathematics teaching and learning, and knowledge of enacting mathematics for teaching and learning (Tatto et al., 2012).

TEDS-M found that the mean CK score for lower secondary (up to Grade 10) preservice teachers in the U.S. was 468, which was statistically significantly below the international mean of 500. However, the mean CK score for lower and upper secondary preservice teachers (i.e., middle- and high-school preservice teachers in the U.S.) was 553, which was statistically significantly above the international mean. Also, about 87.1% of the U.S preservice secondary teachers in the sample reached at least the first CK benchmark on the TEDS-M assessment, which indicated that they were "likely to evaluate algebraic expressions correctly, and solve simple linear and quadratic equations [correctly]…" (Tatto et al., 2012, p. 142). However, only 44.5% reached the second CK benchmark, which indicated that they "were likely to correctly answer questions about functions…, to read, analyze, and apply abstract definitions and notation, and to make and recognize simple arguments" (Tatto et al., 2012, p. 144). Therefore, the TEDS-M study suggests that many middle- and high-school preservice mathematics teachers in the U.S. have some major gaps in their CK.

Additionally, the mean PCK score for lower secondary preservice teachers in the U.S. (471) was statistically significantly below the international mean of 500, whereas the PCK score for lower and upper secondary preservice teachers (542) was statistically significantly above the international mean. However, only 61.0% of the U.S. preservice secondary teachers reached the PCK benchmark, which indicated that they were likely to be familiar with at least some mathematics curricula, to be able to plan and enact mathematics instruction, and to analyze simple student errors (Tatto et al., 2012). Thus, the TEDS-M study suggests that middle- and high-school preservice mathematics teachers in the U.S. could improve in the area of PCK, as well.

In another study related to secondary teachers' knowledge for teaching, 152 prospective teachers completed an open-ended questionnaire related to the function concept—as previously discussed in the section on student conceptions of algebra—and demonstrated limited understandings of the concept (Even, 1993). The prospective teachers also tended to focus on rules (e.g., the vertical line test) rather than conceptual understandings of the concept (Even, 1993). Similarly, Huang and Kulm (2012) administered the KAT instrument (which is described in the next section) and several open-ended items related to functions to 115 preservice teachers and found that most of them had limited knowledge of algebra (and functions) for teaching. In particular, these preservice teachers struggled with the appropriate use of graphical representations and experienced difficulty manipulating symbols, solving some types of equations (including quadratic equations), and reasoning in algebra (Huang & Kulm, 2012).

In one of the few teacher-knowledge studies of inservice mathematics teachers at the secondary level, Cankoy (2010) investigated how 58 teachers explained division by zero ($a \div 0$) and why any nonzero number raised to the zero power is one ($a^0 = 1$). When describing division by zero, about 81% of the inservice teachers gave procedural explanations—such as stating that “it is a rule”—whereas 10% gave more conceptual responses that involved patterns or the idea of limits. (The remaining 9% did not respond.) When describing $a^0 = 1$, about 76% of the teachers gave procedural explanations, whereas only 24% gave conceptual responses that involved algebraic manipulation—such as $1 = \frac{a^4}{a^4} = a^{4-4} = a^0$—or patterns (Cankoy, 2010).

There is also relatively little research that compares preservice and inservice mathematics teachers’ knowledge for teaching. In Germany, the COACTIV research team (Krauss et al., 2008) surveyed 198 inservice mathematics teachers, 90 preservice teachers (at the end of their education programs), and 30 secondary students (18-19 years old) who were enrolled in advanced mathematics courses. Participants completed 35 open-ended items related to CK and PCK in mathematics, including items focused on algebra content and teaching algebra. As the researchers expected, the inservice teachers statistically significantly outperformed preservice teachers and secondary students on both CK and PCK items. Also, preservice teachers outperformed secondary students on both item types. Thus, they concluded that CK and PCK continue to grow during preservice teacher training and professionalization (Krauss et al., 2008).

In a related study, the COACTIV research team (Kleickmann et al., 2013) compared cross-sectional data from four groups of mathematics teachers: beginning preservice teachers (n = 117), third-year preservice teachers (n = 126), student teachers (n = 539), and inservice teachers (n = 198). Participants completed open-ended items related to CK and PCK in mathematics. After controlling for several covariates (such as GPA), researchers found the following for academic-track teachers: the CK of third-year preservice teachers was 0.58 standard deviations (*SD*) higher than the CK of beginning preservice teachers, and the CK of student teachers was 0.44 *SD* higher than the CK of third-year preservice teachers. Inservice teachers scored about the same as student teachers on CK items. Thus, the data suggest that CK develops primarily during preservice teacher training (Kleickmann et al., 2013).

Additionally, after controlling for covariates, the PCK of third-year preservice teachers was 0.60 *SD* higher than the PCK beginning preservice teachers; the PCK of student teachers was 0.53 *SD* higher than the PCK of third-year preservice teachers; and the PCK of inservice teachers was 0.46 SD higher than the PCK of student teachers. “As observed for CK…, the first phase of teacher education seems to play an important role in the development of PCK. However, the learning opportunities offered in the induction phase also seem to foster the development of PCK" (Kleickmann et al., 2013, p. 99).

In a third study comparing preservice and inservice mathematics teachers, researchers at UCLA investigated algebra teacher subject matter knowledge and pedagogical content knowledge using two knowledge-mapping tasks, the algebra items from the MKT (Hill et al., 2008) instrument, and a Student Response Analysis (SRA) task (Buschang, Gregory, Delacruz, & Baker, 2012). Participants included 13 subject-matter experts (i.e., students in a doctoral program in mathematics), 10 PCK experts (i.e., teachers with National Board Certification or who were serving as teacher trainers), 17 novice teachers (i.e., teachers with no more than two years of experience), and 46 experienced teachers (i.e., teachers with more than two years of experience and who did not meet the PCK expert requirement). Despite the relatively small sample size, the PCK-expert teachers scored statistically-significantly higher than both novice and experienced teachers on the MKT task. Additionally, the experienced teachers slightly outperformed the novice teachers, but the result was not significant. Another noteworthy trend was that the PCK-expert teachers scored higher than novice and experienced teachers on the SRA task—which was designed to measure teachers' PCK, and more specifically, their knowledge of content and students, or KCS. Also, as expected, the novice and experienced teachers scored higher than the subject-matter experts on the SRA task (Buschang et al., 2012).

In summary, the preservice and inservice teachers who participated in these studies generally had limited conceptions of some algebraic concepts (such as functions), had issues related to planning and enacting mathematics instruction, and tended to use procedural rather than conceptual explanations. Additionally, inservice teachers generally had higher PCK than preservice teachers, but they had about the same CK as preservice teachers who were student teaching.

Knowledge of Algebra for Teaching Framework

The knowledge of algebra for teaching (KAT) framework was developed by researchers at Michigan State University (including Robert Floden, Joan Ferrini-Mundy, Raven McCrory, Mark D. Reckase, and Sharon L. Senk) to describe the mathematical knowledge necessary to be an effective teacher of algebra. The KAT framework consists of three types of knowledge: knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge. The first two types of knowledge describe the subject matter knowledge that mathematics teachers need to teach algebra, whereas the third type of knowledge describes the content knowledge that is unique to teaching algebra. Each type will be discussed in the sections that follow.

Knowledge of School Algebra

Knowledge of school algebra refers to teachers having a deep understanding of the actual algebra topics that they will teach (e.g., solving linear and quadratic equations) and can be classified as CCK. (Recall that CCK involves subject matter knowledge and skills in mathematics that are *not* unique to teaching.) In the United States, school algebra equates to the mathematics courses conventionally referred to as algebra I and II (McCrory et al., 2012). The scope of the content and proficiencies associated with school algebra was previously outlined in the CCK and algebra sections.

Knowledge of Advanced Mathematics

Knowledge of advanced mathematics can also be classified as CCK and refers to a deep understanding of advanced topics in algebra and other areas of mathematics that most secondary teachers are not expected to teach. In essence, algebra teachers need to know more mathematics than they are expected to teach because it can help them deepen their understanding of school algebra by giving them “some perspective on the trajectory and growth of mathematical ideas beyond school algebra” (McCrory et al., 2012, p. 597). A few of the college-level courses that are relevant to secondary algebra teachers include calculus, linear algebra, mathematical analysis (both real and complex), and abstract algebra (McCrory et al., 2012).

Content from a typical abstract algebra course can exemplify some of the relationships between knowledge of school algebra and knowledge of advanced mathematics. Abstract algebra courses generally focus on rings and fields, which are foundational concepts for prospective secondary mathematics teachers (Conference Board of the Mathematical Sciences, 2012). Rings are sets with two binary operations—addition and multiplication—and the following properties: addition is commutative and associative; there is an additive identity; every element has an additive inverse; and multiplication is associative and distributes over addition. Fields are rings with several additional properties, namely that multiplication is commutative, there is a multiplicative identity, and every nonzero element has a multiplicative inverse. These two algebraic structures are important because they can help teachers think more abstractly about the set of integers under addition and multiplication (which is an example of a ring), as well as the set of real numbers and complex numbers under addition and multiplication (which are examples of fields).

Mathematics-for-Teaching Knowledge

Mathematics-for-teaching knowledge is “knowledge that is mathematical, that is intuitively useful for teaching, and that is unlikely to be taught explicitly, except to teachers” (McCrory et al., 2012, p. 598). This type of knowledge can be classified a combination of SCK and PCK (Reckase et al., 2015). (Recall that SCK refers to content knowledge that is unique to teaching, whereas PCK refers to content knowledge related to student thinking—or KCS—and teaching—or KCT.) Mathematics-for-teaching knowledge includes

> …such things as the mathematical entailments of representations and manipulatives, what makes a particular concept mathematically difficult, how mathematical errors reflect specific misconceptions, and how advanced knowledge is connected to school knowledge. It also includes mathematics needed to identify mathematical goals within and across lessons, to choose among algebraic tasks or texts, to select what to emphasize with curricular trajectories in mind, and to enact other tasks of teaching. (Reckase et al., 2015, p. 252)

KAT Research

In addition to developing the KAT framework discussed in the previous section, researchers involved in the KAT project also designed an instrument (i.e., an algebra assessment) to measure preservice and inservice mathematics teachers’ knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge. (The development of this KAT instrument is described in Reckase et al., 2015.)

The KAT instrument has two forms (20 questions per form with five questions appearing on both forms) and includes both multiple-choice and open-ended items. Each item addresses one of the three types of knowledge described in the KAT framework (e.g., mathematics-for-teaching knowledge); one of the following major topics in algebra: functions and their properties or expressions, equations, and inequalities; and one of the following subtopics: core concepts and procedures, representations, applications, or reasoning and proof (Reckase et al., 2015). (See the Method section for more details about the KAT instrument.)

After developing their instrument, project researchers administered the two forms of the KAT assessment to 841 inservice, preservice, and former mathematics teachers. Preliminary factor analyses of the data indicated modest support for distinct dimensions of knowledge of algebra for teaching (KAT) but not for the three-dimensional structure proposed in the KAT framework (Reckase et al., 2015). In particular, parallel analysis, which involves comparing the eigenvalues obtained from a sample to eigenvalues calculated from random data (McCoach, Gable, & Madura, 2013), suggested two, not three, dimensions of KAT. And cluster analysis showed two main clusters—one which contained most of the "teaching knowledge" items and another which contained most of the "school knowledge" items. The "advanced mathematics knowledge" items were fairly scattered (Reckase et al., 2015).

In the following chapter, I discuss the research methodology that I used to further analyze the factor structure of the data gathered during the final phase of the KAT project. In particular, I explain how confirmatory factor analysis (CFA) was used to explore several theories related to the knowledge of algebra for teaching, including the three-factor theory proposed by KAT researchers. I also discuss how multiple-groups CFA was used to compare the KAT of preservice and inservice mathematics teachers.

CHAPTER 3

METHOD

As previously mentioned, the purpose of this study was to explore the various theories related to the knowledge of algebra for teaching (KAT), as well as to compare and contrast KAT in preservice and inservice mathematics teachers. To address these purposes, I investigated the following research questions:

1. What is the factor structure underlying mathematics teachers' KAT, as measured by an established KAT instrument?
2. Are KAT constructs measured similarly in preservice and inservice teachers?
3. And if so, are there latent-mean differences in the KAT of these two groups?

These research questions were addressed using confirmatory factor analysis (CFA), which is a component of structural equation modeling (SEM). Both SEM and CFA are discussed in the following section.

Structural Equation Modeling and Confirmatory Factor Analysis

SEM is an extension of the general linear model (GLM) and refers to an entire family of statistical techniques that allows analysis of both observed variables (i.e., variables that are measured) and latent variables (i.e., variables that are not directly measured and generally correspond to factors that are not observable, such as attitude and knowledge). The main goals of SEM are "(1) to understand patterns of covariances among a set of observed variables and (2) to explain as much of their variance as possible with the researcher's model" (Kline, 2011, p. 10). SEM models can be decomposed into measurement models, which relate observed variables with latent variables, and structural models, which relate latent variables to other latent variables (Byrne, 2012). One of the major advantages of SEM over more traditional statistical techniques is that SEM accounts for the measurement error in all observed variables (Adelson, 2012).

CFA is a form of SEM that involves only measurement models. It is used when researchers have theories related to the structure of one or more constructs, or factors. (Note that the terms "construct" and "factor" are used interchangeably.) CFA requires researchers to use their theories to create apriori models that specify the number of factors, how the factors are related to observed variables such as items on a survey, and whether or not factors should be correlated (Thompson, 2004). Statistical software, such as *Mplus*, can then be used to analyze these models.

Use of CFA in the Study

In the present study, I addressed the first research question (related to the factor structure underlying mathematics' teachers KAT) by developing, evaluating, and comparing a variety of theory-driven CFA models. In particular, I specified a three-factor CFA model (knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge), based on the KAT framework; a two-factor CFA model (knowledge of algebra and mathematics-for teaching knowledge), as proposed by the KAT research team; and a one-factor CFA model (knowledge of algebra for teaching), based on research that has questioned the multidimensionality of the teacher-knowledge construct (e.g., Kahan, Cooper, & Bethea, 2003) and/or the ability to empirically measure distinct dimensions of teacher knowledge (e.g., Herbst & Kosko, 2014; Mohr-Schroeder et al., 2017; Schilling, Blunk, & Hill, 2007). (These models are discussed in more detail in the *Hypothesized CFA Models* section.)

I addressed the second question (related to the measurement of KAT constructs in preservice and inservice teachers) by using a special type of CFA—multiple-groups CFA. Multiple-groups analysis allows researchers to examine whether the values of model parameters differ across groups (e.g., inservice and preservice teachers), as well as whether or not measures operate the same in those groups (Brown, 2006). After considering these measurement issues, I was able to address the third question (related to possible differences in preservice and inservice mathematics teachers' KAT) by testing for latent mean differences between the two groups.

Participant Characteristics

As previously mentioned, I analyzed data gathered by the researchers involved with the KAT project. The data set included survey responses from 1,248 middle- and high-school mathematics teachers, undergraduate and graduate students (many preservice mathematics teachers), mathematicians, and mathematics teacher educators. (Note that the original dataset contained 1,265 participants, but 17 were not included in this study due to missing and/or miscoded data. For example, a few participants had data values that were outside the range of possible answer choices.)

Approximately 45% (n = 563) of the participants were enrolled in college (as undergraduate or graduate students) when they completed the KAT survey, and about 42% (n = 523) were not enrolled (e.g., college graduates). The remaining 13% (n = 162) were missing data on this item. Additionally, about 63% (n = 780) of the participants had earned or were working toward a major or minor in mathematics, and about 58% (n = 720) were teaching or had recent teaching experience in mathematics at the middle- or high-school level. Table 1 gives a more detailed description of the sample.

Table 1
*Description of the Study Participants (*N *= 1248)*

	Frequency	Percent
College/university status		
Not enrolled	523	41.9
Freshman (1st year)	15	1.2
Sophomore (2nd year)	34	2.7
Junior (3rd year)	86	6.9
Senior (4th year)	147	11.8
Graduate student	239	19.2
College student (Other/ unknown status)	42	3.4
Unknown (missing data)	162	13.0
Mathematics background		
Mathematics major	661	53.0
Mathematics minor	119	9.5
Not a math major or minor	250	20.0
Unknown (missing data)	218	17.5
Teaching status		
Inservice teacher	677	54.2
Former teacher (last 5 years)	43	3.4
Preservice teacher/no teaching	343	27.5
Unknown (missing data)	185	14.8

Note that 42 participants indicated that they were currently attending college but did not give their year of study or chose “other.” Also, 218 participants had missing and/or miscoded data for one or both of the following questions: What is/was your major in college? What is/was your minor in college? Thus, these participants’ mathematics backgrounds are unknown. Additionally, 185 participants had missing and/or miscoded data for one or both of the following questions: Are you currently employed as a middle or high school teacher? Have you taught in middle or high school in the last five years? Thus, their teaching status is unknown.

Sampling Procedures

Although not explicitly stated by the KAT research team (Reckase et al., 2015), the sample described above is a convenience sample. More specifically, the sample includes teachers who participated in professional development activities with KAT researchers, advanced students in teacher preparation programs at several institutions, graduate students in mathematics, and faculty members from the Michigan State University mathematics department (M. Reckase, personal communication, April 10, 2017). KAT researchers made an effort to include a variety of groups—such as undergraduate students, mathematicians, and middle- and high-school mathematics teachers—because they believed these groups would have unique knowledge profiles. “For example, the mathematicians and undergraduate mathematics majors [should] be high on the advanced mathematics dimension and low on the teaching knowledge dimension, whereas the middle school mathematics teachers (many of whom would not have majored in mathematics) [should] be low on advanced knowledge but high on teaching knowledge” (Reckase et al., 2015, p. 252).

Survey of Knowledge for Teaching Algebra

Study participants completed one of two forms of the Survey of Knowledge for Teaching Algebra, which I will refer to as the KAT assessment. Each form of the KAT assessment contains 20 questions: 17 multiple-choice and three open-ended items. (Four multiple-choice and one open-ended item are included on both forms, for a total of 35 unique items.)

Table 2

Characteristics of Assessment Items (Type of Knowledge Assessed)

	Form 1	Form 2
Multiple-choice		
Item 1	School	School
Item 2	School	School
Item 3	School	School
Item 4	School	Teaching
Item 5	Teaching	Advanced
Item 6	Teaching	School
Item 7	School	Advanced
Item 8	School	Teaching
Item 9	Advanced	School
Item 10	School	Teaching
Item 11	Teaching	Teaching
Item 12	Advanced	School
Item 13[a]	School	School
Item 14[a]	School	School
Item 15[a]	Teaching	Teaching
Item 16[a]	Advanced	Advanced
Item 17	School	Advanced
Open-ended		
Item 18[a]	Teaching	Teaching
Item 19	School	School
Item 20	Advanced	Advanced

[a]Denotes items that appear on both forms of the KAT assessment

As discussed in the previous chapter, each item is designed to address one of the three types of knowledge outlined in the KAT framework: knowledge of school algebra, knowledge of advanced mathematics, or mathematics-for-teaching knowledge; one of two algebra topics: functions and their properties or expressions, equations, and inequalities; and one of four subtopics: core concepts and procedures, representations, applications, or reasoning and proof (Reckase et al., 2015). Table 2 shows the type of KAT knowledge intended to be assessed by each of the test items. In summary, there are 11 items on Form 1 and nine items on Form 2 that assess knowledge of school algebra, four items on Form 1 and five items on Form 2 that assess knowledge of advanced mathematics, and five items on Form 1 and six items on Form 2 that assess mathematics-for-teaching knowledge. (Note that multiple-choice questions 13-16 and open-ended question 18 appear on both forms.)

The KAT assessment is still in use today, so its items are protected and only available to researchers who sign a strict confidentiality agreement. A few of the KAT assessment items have been released and are discussed below. (See Figures 8-10.) Specific details about the development of the items that appear on the KAT assessment can be found in Reckase and colleagues (2015).

Which of the following situations can be modeled using an exponential function?

i. The height h of a ball t seconds after it is thrown into the air.
ii. The amount of money A in a jar after w weeks, if each week \$$d$ are put in the jar.
iii. The value V of a car after t years if it depreciates $d\%$ per year.

A. i only

B. ii only

C. iii only

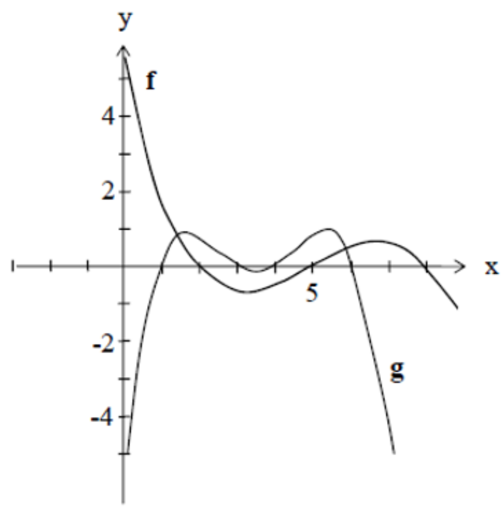
D. i and ii only

E. ii and iii only

Figure 8. Released item that assesses knowledge of school algebra. Copyright 2006, Knowing Mathematics for Teaching Algebra (KAT) Project, NSF REC-0337595, Division of Science and Mathematics Education, Michigan State University.

Figure 8 shows an example of a released multiple-choice item from the KAT assessment that is intended to assess knowledge of school algebra. Recall that knowledge of school algebra refers to a deep understanding of the algebra topics that are taught in the middle- and high-school mathematics classroom. This type of knowledge can be classified as common content knowledge (CCK).

The graphs of two real-valued functions, *f* and *g*, are shown below. Each mark on the axes represents one unit.



How many solutions does the equation $f^3 - 2f^2g + fg^2 = 0$ have on the interval [0, 8]?

A. 2

B. 3

C. 5

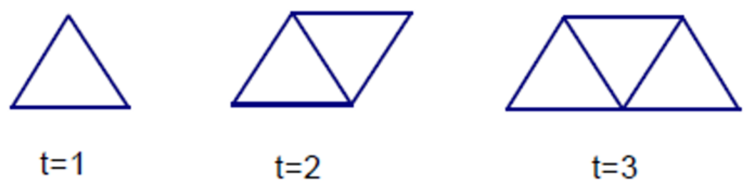
D. 6

E. 7

Figure 9. Released item that assesses knowledge of advanced mathematics. Copyright 2006, Knowing Mathematics for Teaching Algebra (KAT) Project, NSF REC-0337595, Division of Science and Mathematics Education, Michigan State University.

Figure 9 shows an example of a released item that is intended to assess knowledge of advanced mathematics. Recall that knowledge of advanced mathematics refers to a deep understanding of advanced topics in algebra and other areas of mathematics (e.g., calculus and linear algebra) that are not generally taught to middle- and high-school students. This type of knowledge can also be classified as CCK.

Amy is building a sequence of geometric figures with toothpicks, by following a specific pattern (making triangles up and down alternatively). Below are the pictures of the first three figures she builds. Variable t denotes the position of a figure in the sequence.



In finding a mathematical description of the pattern, Amy explains her thinking by saying:

"First, I use three sticks for each triangle:

But then I see that I am counting one stick twice for each of the triangles except the last one, so I have to take those away."

If f represents the total number of toothpicks used in a picture, which of the following equivalent formulas most closely matches Amy's explanation?

A. $f = 2t + 1$

B. $f = 2(t + 1) - 1$

C. $f = 3t - (t - 1)$

D. $f = 3t + 1 - t$

Figure 10. Released item that assesses mathematics-for-teaching knowledge. Copyright 2006, Knowing Mathematics for Teaching Algebra (KAT) Project, NSF REC-0337595, Division of Science and Mathematics Education, Michigan State University.

And Figure 10 shows an example of a released multiple-choice item that is intended to assess mathematics-for-teaching knowledge. Recall that teaching knowledge refers to mathematical knowledge that is especially useful for teaching and encompasses both specialized content knowledge (SCK) and pedagogical content knowledge (PCK).

Hypothesized CFA Models

The items on the KAT assessment served as indicators of latent constructs (e.g., mathematics-for-teaching knowledge) in three hypothesized CFA models. In these models, observed variables (i.e., the KAT assessment items) are represented by rectangles, and latent variables (including measurement error) are represented by ovals or circles, which is standard practice in SEM (Kline, 2011). Additionally, correlations are indicated with curved double-headed arrows, and causal paths are indicated with straight single-headed arrows. For example, in Figure 11, the arrows from the Advanced Mathematics Knowledge latent construct (circle) to the Q5, Q7, Q16, Q17, and Q20 indicators (rectangles) show that the KAT researchers predicted that teachers' knowledge of advanced mathematics would affect their responses to these test items. The error terms (small circles) show that measurement error also would affect their responses. (Note that the "1"s in the model indicate the scaling of the latent variables, which is discussed in the model-identification section.)

As previously mentioned, I addressed my first research question (related to the factor structure underlying mathematics' teachers KAT) by creating, evaluating, and comparing three theory-driven CFA models. Each of these models will be described in the sections that follow.

Three-Factor KAT Model

First, I specified a three-factor CFA model, based on the KAT framework. This model included three latent variables: knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge. The information given in Table 2 was used to assign indicators to each latent variable. Figure 11 shows the three-factor CFA model that corresponds to Form 2 of the KAT assessment. (A similar model was created for Form 1 of the assessment.)

Two-Factor KAT Model

Next, I specified a two-factor CFA model, as proposed by the KAT research team. In particular, Reckase and colleagues (2015) hypothesized that "the Advanced Mathematics items scale is an upper extension of the School Algebra scale" (p. 265). This model included two latent variables: knowledge of algebra and mathematics-for-teaching knowledge. Thus, all indicators of knowledge of advanced mathematics were reassigned to the knowledge of algebra variable. Figure 12 shows the two-factor CFA model that corresponds to Form 2 of the KAT assessment. (Again, a similar model was created for Form 1 of the assessment.)

One-Factor KAT Model

Finally, I specified a one-factor CFA model, based on research (e.g., Kahan et al., 2003; Schilling et al., 2007) that questions the ability to empirically measure distinct dimensions of teacher knowledge. More specifically, a number of studies have shown high correlations between CK and PCK. This model only included one latent variable: knowledge of algebra for teaching. Thus, all indicators were assigned to this latent variable.

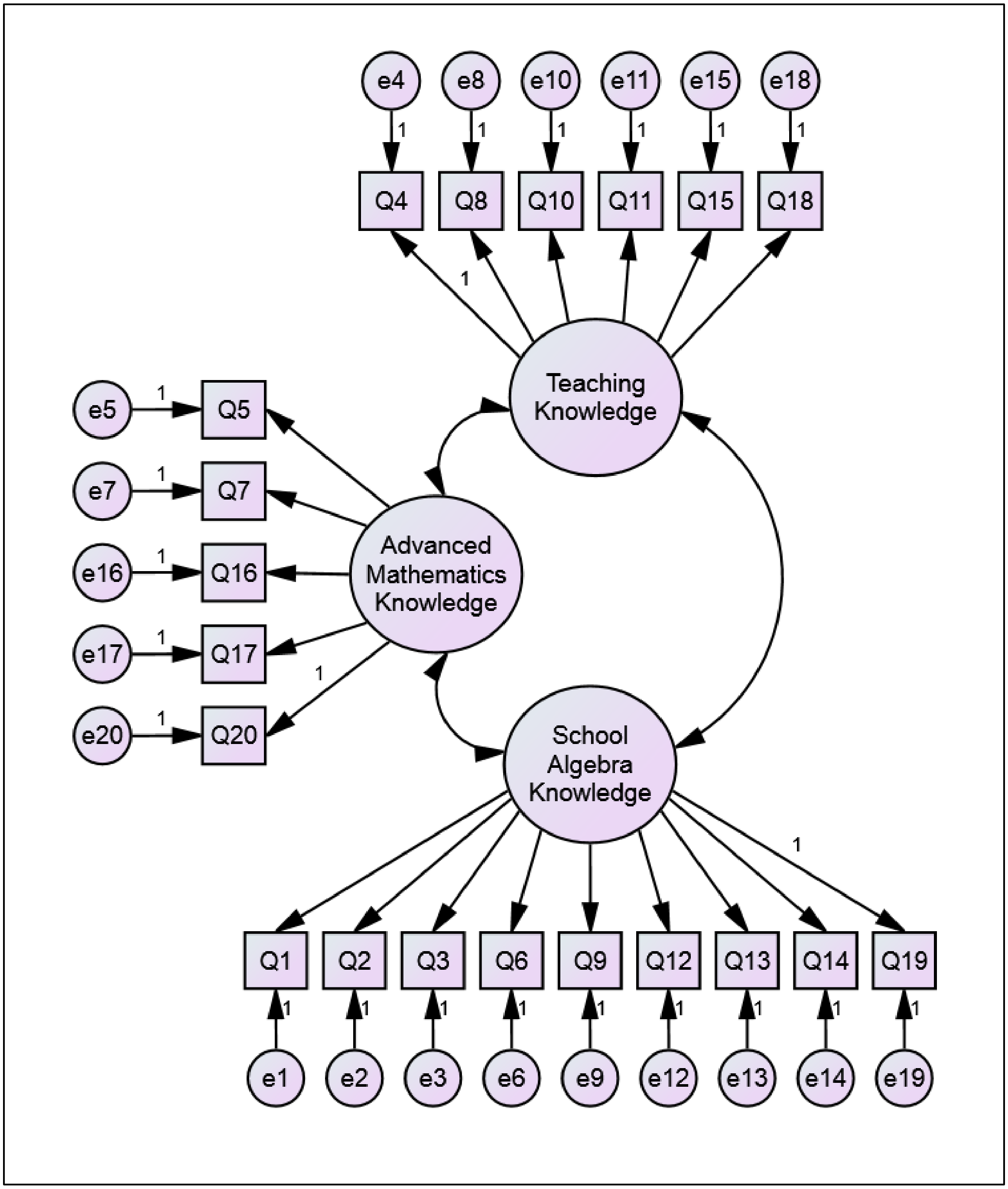


Figure 11. Three-factor CFA model for Form 2 of the KAT assessment.

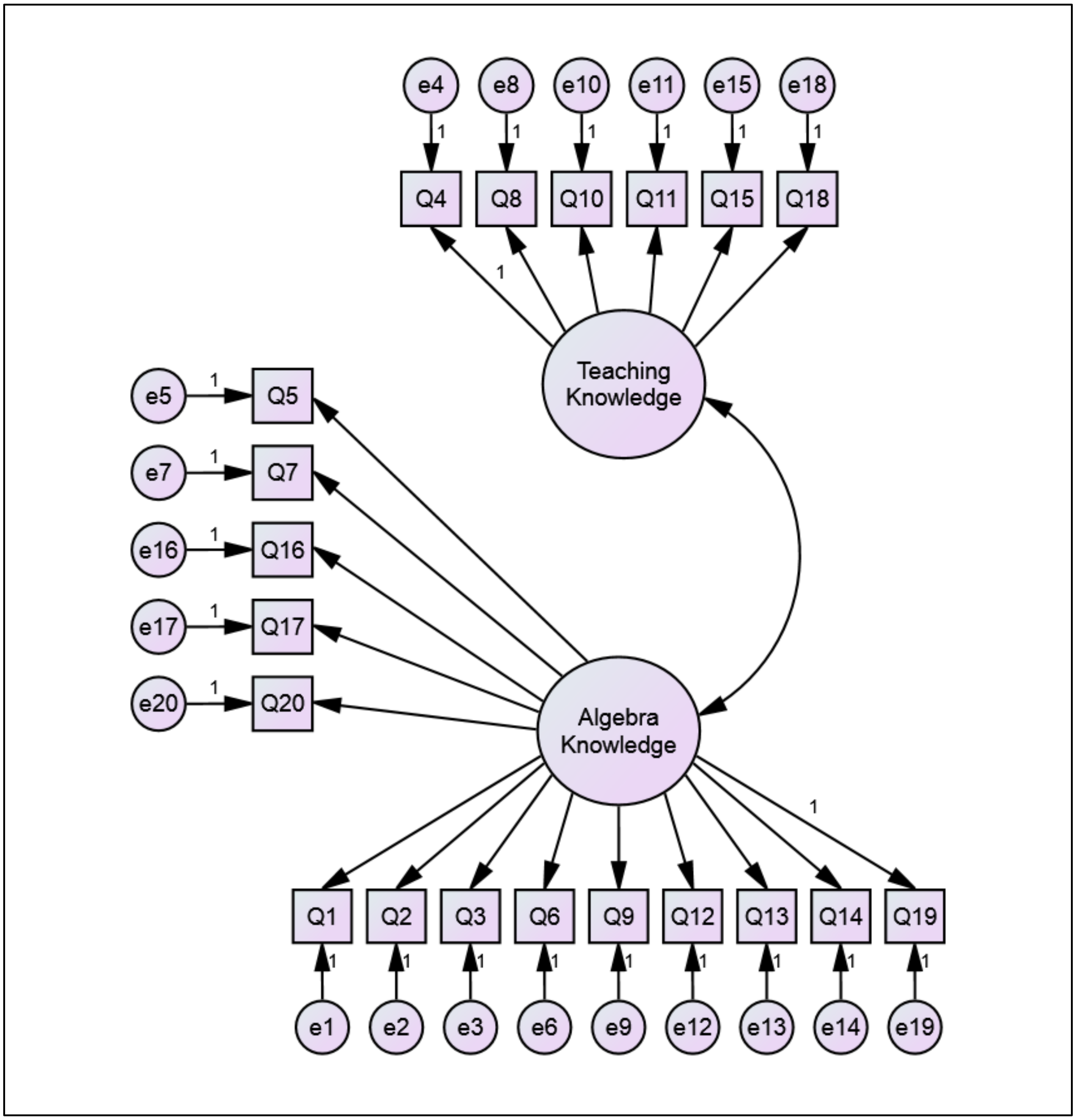


Figure 12. Two-factor CFA model for Form 2 of the KAT assessment.

Identification

After developing these CFA models, I evaluated model identification. In the broadest sense, "the issue of identification focuses on whether or not there is a unique set of parameters consistent with the data" (Byrne, 2012, p. 32). CFA models can be empirically underidentified (if the number of degrees of freedom is less than 0), just-identified (df = 0), or overidentified (df > 0), which is preferable.

To ensure that my hypothesized CFA models were overidentified, I calculated each model's degrees of freedom, which is simply the difference between the number of knowns (i.e., observations) and the number of unknowns in a model. If there are *m* observed variables in a model, then there are $\frac{m(m+1)}{2}$ knowns (i.e., the number of unique elements in the variance-covariance matrix). For example, the three-factor CFA model in Figure 11 has $\frac{20(21)}{2} =$ 210 knowns because there are 20 items on each form of the KAT assessment. The number of unknowns (in CFA) is the sum of the error variances, factor variances, correlations, and paths (that are not fixed). Thus, the three-factor model in Figure 11 has 20 (error variances) + 3 (factor variances) + 3 (correlation) + 17 (paths) = 43 unknowns and 210 – 43 = 167 degrees of freedom. Using a similar procedure, I confirmed that my two-factor and one-factor CFA models were also overidentified, with 169 and 170 degrees of freedom, respectively.

I also had to scale all latent variables in the models (both error terms and factors), which involved setting specific unstandardized residual path coefficients to 1 (see Figures 11 and 12). By fixing the paths between error terms and indicators to 1, I assigned each error term "a scale related to variance in the indicator…that is unexplained by the factor this indicator is supposed to reflect…" (Kline, 2011, p. 127). I also scaled all latent variables using the marker variable strategy (see Brown, 2006) because the latent variables were unobserved and had no units of measurement. By fixing the path between each factor and one of its indicators to 1, I essentially assigned each factor the metric of that particular indicator (Brown, 2006). For example, the Advanced Mathematics Knowledge factor in Figure 11 shares the same metric as the Q20 indicator in that model.

Clearly, marker variables should be closely related to their respective factors and have highly reliable scores (Kline, 2011), which is the case for the marker variables used in this study. (Note that the specific marker variables used in this study are available upon request; however, they are not given in this chapter because the specific items on the KAT assessment are not available to the public. Also note that the actual marker variables used in this study vary from those shown in Figures 11-12.)

Model Estimation

After evaluating model identification, I used the statistics software Mplus (Version 8; Muthén & Muthén, 2017) to run my actual analyses. I chose this particular software package because it includes the option to use weighted least squares with adjusted mean and variance (WLSMV), which is the preferred estimation method for analyzing a model that contains dichotomous variables (e.g., multiple-choice items that are either correct or incorrect; Byrne, 2012) and also allows for analysis of a multiple-groups CFA (which was needed to compare preservice and inservice teachers' KAT).

Model Fit

Using the output produced by Mplus, I evaluated the fit of the hypothesized models, when appropriate. (Note that the latent variable covariance matrix was not positive definite for some models, so model fit was irrelevant and therefore not considered in these cases. This is described in greater detail in the Chapter 4 section on the analysis of the three-factor model.) Model fit refers to the process of determining how well the models fit the sample data. As is often the case in statistics, there is an art and science to evaluating model fit; that is, researchers have to consider multiple model fit indices (which often give conflicting information) and make judgments about whether there is adequate model fit. Some of the more commonly-used fit indices include the Tucker-Lewis Index (TLI), Comparative Fit Index (CFI), Root Mean Square Error of Approximation (RMSEA), and model chi-square (Byrne, 2012; Kline, 2011). The TLI and CFI are "goodness-of-fit" statistics with a range of 0 (bad) to 1 (good). These indices compare the hypothesized model to the baseline ("worst") model and add a penalty for each parameter estimated. Generally speaking, a TLI/CFI value of at least .90 indicates acceptable fit, and a value of at least .95 indicates good fit (Byrne, 2012).

On the other hand, the RMSEA is a "badness-of-fit" statistic (with no upper bound) that "takes into account the error of approximation in the population…" (Byrne, 2012, p. 73). As a rule of thumb, RMSEA indicates good fit for values between 0 (best) and .05, marginal fit for values between .05 and .08, mediocre fit for values between .08 and .10, and poor fit for values greater than .10 (Byrne, 2012).

Lastly, the model chi-square is another "badness-of-fit" statistic that "tests the difference in fit between a given overidentified model and whatever unspecified model would imply a covariance matrix that perfectly corresponds to the data covariance matrix" (Kline, 2011, p. 200). Unfortunately, this fit index is negatively influenced by large samples, so it does not always provide valuable information. As a general rule, a chi-square *p*-value of greater than .05 indicates good model fit (Kline, 2011).

Model Respecification and Comparison

After evaluating model fit, I considered respecifications (i.e., modifications) to the models, based on theory and empirical evidence. More specifically, I examined the modification indices produced by Mplus to see if any respecifications—such as correlating the error variances of any pairs of indicators—seemed logical and appropriate.

(Modification indices show where correlations and paths can be added to a model to improve overall fit.)

Afterward, I had planned to compare my CFA models using chi-square difference tests, which often are used to compare nested models. In this study, the three models are nested because it is possible to move between the models by simply removing (or adding) latent variables. For example, I removed one latent construct (knowledge of advanced mathematics) from the three-factor KAT model to create the two-factor model. Unfortunately, several of the models had latent variable covariance matrices that were not positive definite, so comparing the models using this method was not possible.

Multiple-Group Analysis

Because I was interested in conducting a CFA with multiple groups (to address my second and third research questions), I tested my “best” hypothesized model for configural invariance, measurement invariance, and latent mean differences, respectively, as outlined in Byrne (2012) and Muthén and Muthén (2011). (Note that the order of these tests is important because a model should demonstrate configural invariance before it is tested for measurement invariance, and it should demonstrate at least partial measurement invariance before it is used to investigate latent mean differences.) Each of these tests is briefly described below.

Testing for Configural Invariance

The test for configural invariance is used to reveal how well the number of factors and factor-loading pattern in a baseline model represent the data in both groups of interest (preservice and inservice teachers in this study). To conduct this test,

> no equality constraints are imposed on any of the parameters. Thus, the same parameters that were estimated in the baseline model for each group separately are again estimated in this multigroup model. In essence, then, you can think of the model being tested here as a multigroup representation of the baseline model. (Byrne, 2012, p. 206)

Testing for Measurement Invariance

The test for measurement invariance is used to determine whether or not equality constraints can be imposed across groups on any or all of the factor loadings. More specifically, the configural model is compared to an invariance model with constrained factor loadings/thresholds. When this test results in a statistically significant change in χ^2 ($\Delta \chi^2$; $p < .05$) or a change in CFI (ΔCFI) greater than .01 (Cheung & Rensvold, 2002), it indicates non-invariance and suggests that equality constraints should not be imposed on one or more of the factor loadings. In this case, modification indices can be examined to determine which factor loadings may not be equal across groups. After equality constraints are removed for any "troublesome" factor loadings, models can be tested for partial measurement invariance.

Testing for Latent Mean Differences

Lastly, the test for latent mean differences is used to investigate whether or not there are significant differences in group means. (In this study, I was investigating differences in preservice and inservice teachers' knowledge of algebra for teaching.) This test involves fixing the factor means of one group to zero while labeling the factor means of the other group; this allows the parameters to be freely estimated for the latter group.

Conclusion

In summary, I analyzed KAT-assessment data gathered from researchers at Michigan State; the assessment contains items related to school algebra, advanced mathematics, and teaching knowledge. I used confirmatory factor analysis (CFA) to explore several theory-driven models related to the knowledge of algebra for teaching (KAT) to examine the structure of the data. Additionally, I used multiple-group CFA to compare and contrast the KAT of preservice and inservice mathematics teachers. In the next chapter, I outline the major findings from my analyses.

CHAPTER 4

RESULTS

Findings from this study are presented in this chapter. In particular, I begin by briefly discussing relevant descriptive statistics, such as the proportion of teachers who correctly answered the various types of KAT questions. Then, I address each of the three research questions. Recall that this study attempts to answer the following research questions:

1. What is the factor structure underlying mathematics teachers' knowledge of algebra for teaching (KAT), as measured by an established KAT instrument?
2. Are KAT constructs measured similarly in preservice and inservice teachers?
3. And if so, are there latent-mean differences in the KAT of these two groups?

Descriptive Statistics

Before addressing my research questions, I examined a number of descriptive statistics for the KAT assessment data. Recall that there are two forms of the KAT assessment, and each form contains 17 multiple-choice items and 3 open-ended items. Table 3 displays the proportion of correct answers given on the multiple-choice items by group (full sample, preservice teachers, and inservice teachers) and by question type (school algebra, advanced mathematics, and mathematics-for-teaching knowledge).

Table 3

Proportion of Correct Answers on MC Items by Group and Question Type

	Form 1	Form 2
Group / Question Type	*M* (*SD*[a])	*M* (*SD*[a])
Full sample	*n* = 637	*n* = 611
School Algebra	.640 (.177)	.624 (.142)
Advanced Mathematics	.421 (.069)	.417 (.129)
Teaching Knowledge	.486 (.148)	.600 (.152)
Preservice teachers	*n* = 160	*n* = 183
School Algebra	.670 (.195)	.642 (.159)
Advanced Mathematics	.477 (.085)	.455 (.146)
Teaching Knowledge	.494 (.123)	.623 (.124)
Inservice teachers	*n* = 405	*n* = 272
School Algebra	.613 (.177)	.654 (.139)
Advanced Mathematics	.379 (.064)	.429 (.112)
Teaching Knowledge	.473 (.169)	.608 (.179)

[a]Standard deviation of the proportions for all questions of the given question type

From this table, it is clear that participants in this sample had the least difficulty with the multiple-choice items related to school algebra knowledge and the most difficulty with the multiple-choice items related to advanced mathematics knowledge. The proportion of correct answers on the multiple-choice items related to mathematics-for-teaching knowledge fell between the proportions of correct answers for the other two categories. More specifically, the full sample correctly answered: 64.0% of the school algebra items on Form 1 and 62.4% on Form 2; 48.6% of the teaching items on Form 1 and 60.0% on Form 2; and 42.1% of the advanced mathematics items on Form 1 and 41.7% on Form 2. Note that the results were very similar in all groups (i.e., the full sample, preservice teachers, and inservice teachers) and across both forms of the assessment, with the exception of the items related to mathematics-for-teaching knowledge.

Additionally, on the multiple-choice items, the preservice teachers in the sample generally outperformed the inservice teachers on all question types and across both forms. The only exception was Form 2 multiple-choice items related to school algebra knowledge for which inservice teachers slightly outperformed preservice teachers (65.4% vs. 64.2%).

Table 4 displays the average score on the open-ended items by group and by question type. Recall that there is only one open-ended item per question type on each form of the KAT assessment. Participants earned a score between 0-4 for each open-ended item, for which a score of 0 indicated an entirely incorrect solution and a score of 4 indicated a complete and correct solution.

Table 4
Average Score on Open-Ended Items by Group and Question Type

	Form 1	Form 2
Group / Question Type	*M (SD*[a]*)*	*M (SD*[a]*)*
Full sample	*n* = 637	*n* = 611
School Algebra	1.88 (1.34)	2.05 (1.31)
Advanced Mathematics	1.78 (1.37)	2.28 (1.53)
Teaching Knowledge	1.46 (1.83)	1.40 (1.54)
Preservice teachers	*n* = 160	*n* = 183
School Algebra	2.13 (1.22)	2.15 (1.32)
Advanced Mathematics	1.71 (1.27)	2.36 (1.46)
Teaching Knowledge	1.88 (1.90)	1.73 (1.70)
Inservice teachers	*n* = 405	*n* = 272
School Algebra	1.72 (1.37)	2.06 (1.36)
Advanced Mathematics	1.70 (1.36)	2.20 (1.58)
Teaching Knowledge	1.17 (1.72)	1.28 (1.48)

[a]Standard deviation of the scores for each open-ended item

Unlike the multiple-choice items, participants in all groups and across both forms generally had the most difficulty with the open-ended item related to teaching knowledge. There was less consistency across forms on the least-difficult items, as the participants who took Form 1 had the least difficulty with the open-ended item related to school algebra knowledge whereas the participants who took Form 2 had the least difficulty with the open-ended item related to advanced mathematics knowledge.

Similar to the finding for the multiple-choice items, the preservice teachers in the sample outperformed the inservice teachers on all question types and across both forms on the open-ended items. However, in general, the scores for all open-ended items were fairly low, considering that participants could earn up to 4 points and the average scores on these items for the full sample were between 1.40 and 2.28.

In addition to calculating these descriptive statistics, I examined the inter-item correlations for the two forms of the KAT assessment. Tables 5 and 7 present the inter-item correlations, means, and ranges for Form 1 and Form 2, respectively, using the entire sample. Additionally, Tables 6 and 8 present the same information for the samples of preservice and inservice teachers. Based on these tables, it is evident that the items on both forms show divergent validity because their intercorrelations are relatively low—in particular, they are less than .5 in most cases, and only one intercorrelation is greater than .7 (see Kline, 2011). (Divergent validity is a type of construct validity that tests whether items that should not be related are actually unrelated.) However, the items on both forms do not show convergent validity because, in general, the intercorrelations among the items that are supposed to measure the same construct (e.g., mathematics-for-teaching knowledge) are not consistently higher than their intercorrelations among the items that

Table 5

Summary of Inter-Item Correlations, Means, and Ranges for KAT Assessment Form 1 (Full Sample)

Type (Item)	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	T1	T2	T3	T4	T5	A1	A2	A3	A4
S1[a] (1)	–																			
S2 (2)	.26	–																		
S3 (3)	.34	.41	–																	
S4 (4)	.08	.17	.21	–																
S5 (7)	.23	.41	.41	.19	–															
S6 (8)	.38	.52	.39	.25	.41	–														
S7 (10)	.30	.42	.44	.16	.37	.43	–													
S8 (13)	.36	.52	.42	.19	.53	.57	.49	–												
S9 (14)	.15	.20	.15	.24	.16	.26	.15	.33	–											
S10 (17)	.28	.55	.63	.25	.57	.43	.56	.61	.28	–										
S11 (19)	.35	.49	.46	.17	.48	.43	.45	.58	.36	.67	–									
T1[b] (5)	.22	.29	.24	.20	.40	.30	.24	.39	.19	.26	.37	–								
T2 (6)	.37	.42	.42	.23	.46	.34	.39	.38	.31	.34	.37	.18	–							
T3 (11)	.32	.19	.22	.03	.23	.18	.25	.37	.13	.44	.26	.18	.26	–						
T4 (15)	.01	.32	.08	.10	.34	.24	.30	.23	.11	.32	.35	.34	.12	.19	–					
T5 (18)	.29	.36	.36	.27	.36	.43	.41	.51	.26	.62	.54	.36	.30	.24	.33	–				
A1[c] (9)	.03	.16	.22	.15	.28	.38	.23	.13	.13	.25	.32	.27	.22	.05	.23	.39	–			
A2 (12)	.31	.21	.14	.16	.26	.33	.17	.31	.14	.20	.29	.23	.21	.23	.25	.28	.21	–		
A3 (16)	.36	.52	.44	.19	.47	.59	.45	.60	.29	.60	.60	.41	.37	.24	.36	.57	.38	.34	–	
A4 (20)	.45	.53	.37	.24	.57	.51	.45	.60	.28	.71	.64	.35	.39	.22	.28	.62	.38	.37	.68	–
M	.77	.76	.88	.31	.71	.36	.76	.54	.60	.72	1.78	.37	.51	.72	.34	1.88	.36	.38	.52	1.46
Range	0-1[d]	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-4

Note. Shaded cells indicate correlations that are not statistically significantly different from zero. For all other correlations, $p < .05$ (two-tailed). [a]School Algebra Knowledge item (11 total). [b]Teaching Knowledge item (5 total). [c]Advanced Mathematics Knowledge item (4 total). [d]For multiple-choice items, 0 = incorrect response and 1 = correct response.

Table 6

Summary of Inter-Item Correlations, Means, and Ranges for KAT Assessment Form 1 (Preservice and Inservice Teachers)

Type (Item)	S1	S2	S3	S4	S5	S6	S7	S8	S9	S10	S11	T1	T2	T3	T4	T5	A1	A2	A3	A4
M	.75	.75	.86	.29	.64	.32	.74	.51	.61	.65	1.70	.33	.55	.72	.30	1.72	.33	.34	.47	1.17
S1[a] (1)	–	.25	.36	.01	.25	.30	.30	.30	.12	.32	.33	.16	.35	.35	.04	.25	.04	.23	.30	.43
S2 (2)	.24	–	.42	.22	.40	.42	.42	.54	.27	.56	.51	.18	.42	.25	.32	.31	.15	.16	.51	.51
S3 (3)	.16	.50	–	.26	.44	.39	.54	.41	.23	.63	.51	.36	.57	.28	.13	.37	.21	.12	.44	.35
S4 (4)	.39	.13	.13	–	.26	.24	.15	.18	.22	.23	.21	.20	.26	.06	.21	.32	.17	.16	.22	.28
S5 (7)	.02	.34	.02	.11	–	.41	.39	.60	.23	.56	.50	.37	.56	.32	.37	.34	.31	.26	.46	.55
S6 (8)	.49	.68	.31	.28	.22	–	.42	.58	.29	.43	.38	.19	.30	.23	.21	.37	.32	.26	.61	.51
S7 (10)	.27	.45	.12	.21	.33	.37	–	.54	.20	.60	.48	.17	.48	.25	.38	.39	.25	.13	.47	.50
S8 (13)	.45	.43	.34	.13	.19	.49	.35	–	.40	.63	.64	.37	.48	.46	.29	.53	.20	.36	.61	.70
S9 (14)	.16	.09	.03	.26	.06	.16	.11	.11	–	.37	.48	.21	.31	.19	.14	.35	.13	.18	.38	.41
S10 (17)	.14	.64	.69	.30	.41	.34	.41	.54	.13	–	.70	.22	.44	.55	.32	.57	.22	.19	.60	.71
S11 (19)	.33	.37	.12	.23	.33	.42	.34	.39	.08	.53	–	.33	.40	.36	.37	.54	.33	.31	.60	.69
T1[b] (5)	.35	.50	-.11	.15	.46	.51	.31	.38	.17	.15	.51	–	.21	.25	.28	.28	.21	.23	.36	.30
T2 (6)	.42	.41	.14	.33	.26	.37	.23	.21	.28	.17	.33	.21	–	.35	.23	.32	.27	.25	.45	.50
T3 (11)	.23	.03	.06	-.01	.05	.05	.21	.23	-.03	.21	.00	.03	.15	–	.22	.34	.04	.30	.29	.34
T4 (15)	-.23	.22	-.22	-.12	.17	.32	.06	.05	.13	.15	.30	.39	.00	.13	–	.36	.32	.27	.42	.28
T5 (18)	.26	.48	.25	.16	.16	.49	.43	.41	.07	.66	.58	.43	.27	.04	.22	–	.38	.27	.58	.63
A1[c] (9)	-.02	.18	.21	.16	.13	.38	.11	-.06	.09	.31	.33	.31	.16	.07	.15	.34	–	.22	.43	.48
A2 (12)	.44	.16	.27	.17	.01	.38	.11	.24	.05	.06	.24	.21	.14	.13	.01	.24	.25	–	.35	.30
A3 (16)	.45	.52	.34	.26	.38	.53	.40	.53	.15	.54	.52	.49	.21	.09	.22	.56	.23	.32	–	.72
A4 (20)	.43	.50	.31	.31	.35	.44	.28	.38	.16	.57	.51	.38	.26	.07	.18	.49	.20	.37	.49	–
M	.77	.78	.93	.34	.84	.36	.74	.59	.52	.83	1.71	.42	.43	.71	.42	2.13	.39	.44	.59	1.88
Range	0-1[d]	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-4

Note. Summary of data for preservice teachers on lower diagonal and summary of data for inservice teachers on upper diagonal. Shaded cells indicate correlations that are not statistically significantly different from zero. For all other correlations, $p < .05$ (two-tailed).
[a]School Algebra Knowledge item (11 total). [b]Teaching Knowledge item (5 total). [c]Advanced Mathematics Knowledge item (4 total). [d]For multiple-choice items, 0 = incorrect response and 1 = correct response.

Table 7

Summary of Inter-Item Correlations, Means, and Ranges for KAT Assessment Form 2 (Full Sample)

Type (Item)	S1	S2	S3	S4	S5	S6	S7	S8	S9	T1	T2	T3	T4	T5	T6	A1	A2	A3	A4	A5
S1[a] (1)	–																			
S2 (2)	.34	–																		
S3 (3)	.41	.22	–																	
S4 (6)	.44	.35	.33	–																
S5 (9)	.34	.21	.21	.31	–															
S6 (12)	.30	.33	.46	.40	.16	–														
S7 (13)	.44	.33	.37	.47	.26	.48	–													
S8 (14)	.23	.34	.35	.26	.23	.32	.27	–												
S9 (19)	.36	.23	.30	.40	.10	.39	.45	.25	–											
T1[b] (4)	.25	.21	.25	.27	.21	.35	.37	.27	.25	–										
T2 (8)	.26	.18	.25	.27	.24	.17	.30	.18	.25	.22	–									
T3 (10)	.21	.15	.25	.24	.15	.15	.19	.10	.24	.27	.17	–								
T4 (11)	.48	.37	.41	.38	.24	.44	.46	.30	.29	.27	.18	.30	–							
T5 (15)	.27	.14	.31	.34	.14	.28	.35	.25	.49	.16	.18	.18	.25	–						
T6 (18)	.33	.28	.36	.32	.15	.37	.30	.16	.45	.30	.16	.17	.22	.24	–					
A1[c] (5)	.31	.28	.40	.37	.34	.44	.53	.38	.36	.38	.25	.12	.33	.42	.35	–				
A2 (7)	.37	.34	.27	.37	.33	.42	.41	.27	.29	.29	.25	.21	.41	.30	.33	.47	–			
A3 (16)	.61	.41	.48	.52	.34	.54	.54	.35	.43	.32	.27	.22	.52	.46	.43	.52	.45	–		
A4 (17)	.43	.30	.49	.35	.15	.46	.45	.36	.47	.38	.22	.18	.40	.41	.38	.61	.37	.52	–	
A5 (20)	.44	.24	.31	.46	.23	.35	.39	.25	.59	.32	.26	.16	.30	.34	.52	.44	.40	.55	.44	–
M	.67	.63	.61	.76	.85	.33	.55	.60	2.28	.63	.50	.73	.78	.36	2.05	.27	.51	.57	.31	1.40
Range	0-1[d]	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-1	0-4

Note. Shaded cells indicate correlations that are not statistically significantly different from zero. For all other correlations, $p < .05$ (two-tailed). [a]School Algebra Knowledge item (9 total). [b]Teaching Knowledge item (6 total). [c]Advanced Mathematics Knowledge item (5 total). [d]For multiple-choice items, 0 = incorrect response and 1 = correct response.

Table 8

Summary of Inter-Item Correlations, Means, and Ranges for KAT Assessment Form 2 (Preservice and Inservice Teachers)

Type (Item)	S1	S2	S3	S4	S5	S6	S7	S8	S9	T1	T2	T3	T4	T5	T6	A1	A2	A3	A4	A5
M	.69	.64	.67	.79	.86	.36	.58	.64	2.20	.61	.53	.74	.84	.32	2.06	.31	.52	.56	.33	1.28
S1[a] (1)	–	.42	.58	.43	.30	.37	.38	.23	.50	.29	.33	.27	.63	.23	.42	.23	.39	.56	.53	.54
S2 (2)	.27	–	.32	.41	.28	.39	.37	.42	.40	.15	.32	.19	.54	.19	.34	.38	.49	.48	.38	.37
S3 (3)	.15	.07	–	.41	.34	.54	.42	.34	.41	.29	.31	.15	.53	.31	.41	.28	.30	.62	.45	.48
S4 (6)	.39	.19	.16	–	.21	.45	.59	.17	.52	.33	.27	.06	.28	.30	.47	.24	.33	.48	.17	.45
S5 (9)	.28	.16	.06	.31	–	.28	.50	.41	.19	.28	.14	.14	.37	.17	.20	.30	.42	.35	.04	.18
S6 (12)	.27	.28	.56	.46	.16	–	.48	.29	.46	.33	.21	.16	.38	.33	.42	.42	.38	.67	.45	.24
S7 (13)	.49	.29	.38	.51	.04	.58	–	.22	.55	.28	.33	.01	.38	.31	.37	.54	.46	.44	.46	.43
S8 (14)	.25	.20	.46	.36	.18	.38	.19	–	.25	.31	.12	.12	.28	.31	.28	.36	.42	.46	.39	.23
S9 (19)	.39	.18	.43	.40	.14	.57	.49	.29	–	.38	.27	.22	.32	.40	.52	.37	.36	.48	.51	.67
T1[b] (4)	.19	.31	.11	.24	.16	.36	.46	.22	.24	–	.29	.20	.12	.34	.29	.42	.33	.30	.39	.32
T2 (8)	.22	.12	.14	.12	.29	.12	.30	.11	.16	.05	–	.02	.19	.11	.15	.19	.37	.26	.35	.20
T3 (10)	.14	.10	.15	.37	.23	.13	.30	.03	.24	.24	.19	–	.32	-.05	.12	.05	.16	.12	.13	.09
T4 (11)	.34	.21	.42	.47	.27	.44	.46	.23	.43	.31	.03	.35	–	.32	.35	.25	.45	.64	.44	.46
T5 (15)	.21	-.02	.32	.29	.26	.25	.40	.15	.50	.08	.17	.33	.18	–	.15	.37	.35	.35	.38	.27
T6 (18)	.35	.34	.35	.28	.24	.48	.33	.33	.56	.43	.29	.19	.19	.30	–	.26	.40	.43	.39	.51
A1[c] (5)	.26	.24	.54	.47	.19	.58	.55	.34	.56	.34	.25	.11	.33	.55	.62	–	.53	.44	.63	.38
A2 (7)	.41	.27	.22	.52	.27	.47	.35	.20	.35	.31	.16	.34	.46	.29	.32	.51	–	.48	.35	.32
A3 (16)	.51	.30	.30	.53	.48	.66	.56	.20	.54	.31	.34	.27	.42	.49	.53	.61	.45	–	.52	.59
A4 (17)	.27	.22	.53	.44	.29	.60	.42	.33	.54	.42	.02	.10	.28	.35	.46	.65	.43	.50	–	.41
A5 (20)	.30	.12	.25	.61	.27	.60	.47	.40	.68	.36	.32	.25	.29	.44	.62	.66	.48	.63	.56	–
M	.75	.64	.63	.77	.88	.32	.57	.57	2.36	.69	.51	.73	.75	.44	2.15	.28	.55	.64	.34	1.73
Range	0-1[d]	0-1	0-1	0-1	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-1	0-1	0-4	0-1	0-1	0-1	0-1	0-4

Note. Summary of data for preservice teachers on lower diagonal and summary of data for inservice teachers on upper diagonal. Shaded cells indicate correlations that are not statistically significantly different from zero. For all other correlations, $p < .05$ (two-tailed).

[a]School Algebra Knowledge item (9 total). [b]Teaching Knowledge item (6 total). [c]Advanced Mathematics Knowledge item (5 total). [d]For multiple-choice items, 0 = incorrect response and 1 = correct response.

are supposed to measure other constructs. (Convergent validity is a type of construct validity that tests whether items that should be related are actually related.)

RQ 1: Factor Structure Underlying Mathematics Teachers' KAT

As described in Chapter 3, I addressed the first research question by developing, evaluating, and comparing three CFA models: a three-factor model (knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge) based on the KAT framework (McCrory et al., 2012); a two-factor model (knowledge of algebra and mathematics-for-teaching knowledge) as proposed by the KAT research team (Reckase et al., 2015); and a one-factor CFA model (knowledge of algebra for teaching) based on research (e.g., Kahan et al., 2003; Schilling et al., 2007) questioning the ability to empirically measure distinct dimensions of teacher knowledge. I discuss each of these models in the sections that follow.

Three-Factor Model

The three-factor model contains three latent factors: knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge. (See Chapter 3 for a detailed description of this model.) Upon running this model in Mplus using WLSMV estimation, I received error messages for both datasets (Form 1 and Form 2), which indicated that the latent variable covariance matrices were not positive definite. That is, each of the covariance matrices produced a nonpositive determinant, which indicated that the variance was negative or zero (but this is not possible). When this occurs, the results of the analysis are meaningless. Upon further inspection, I found very high estimated correlations (.94, .99, and .99) among the three factors on Form 1, and the estimated correlations among the factors on Form 2 were all greater than 1 (1.004, 1.063, and 1.086). According to psychometrician and Mplus developer Bengt Muthén, “A correlation estimate of 1 means that…factors are indistinguishable. A correlation estimate higher than 1 means that the model does not make sense for the data because correlations should not be higher than 1” (Muthén, 2011). In these types of situations, Muthén recommends disregarding the troublesome model and exploring other models (Muthén, 2011).

Therefore, I concluded that the three-factor model did not make sense for either dataset (Form 1 or 2). Given the error messages received during the analyses, it would have been inappropriate to consider fit statistics for this model, so the next logical step was to explore my two-factor model.

Two-Factor Model

The two-factor model contains two latent factors: knowledge of algebra—which combines knowledge of school algebra and advanced mathematics—and mathematics-for-teaching knowledge. (See Chapter 3 for a detailed description of this model.) Upon running this model, I again received error messages for both datasets indicating that the latent variable covariance matrices were not positive definite. Again, I found that the estimated correlations among the latent factors were all greater than 1. In particular, the estimated correlation between the two factors was 1.007 for the Form 1 data and 1.074 for the Form 2 data.

Therefore, I concluded that the two-factor model also did not make sense for either of the datasets. The final step was to explore my one-factor model.

One-Factor Model

The one-factor model contains only one latent factor: knowledge of algebra for teaching. (See Chapter 3 for a detailed description of this model.) This model ran without any errors for both datasets, so I evaluated the fit of the models. Table 9 displays the values of several model fit indices (e.g., RMSEA, CFI, and TLI) that were produced by Mplus during the analysis phase. Note that there are fit indices for the full sample, preservice teachers, and inservice teachers. (I will only discuss the fit indices for the full sample in this section. Fit indices for the subgroups are addressed in the discussion of Research Question 2.)

Table 9
Model Fit Statistics for One-Factor Model by Group

Group / Model Fit Statistics	Form 1	Form 2
Full sample		
Chi-square *p*-value	< .001[d]	< .001[d]
RMSEA[a]	.031	.033
CFI[b]	.981	.975
TLI[c]	.979	.972
Preservice teachers		
Chi-square *p*-value	.402	.231
RMSEA	.012	.021
CFI	.995	.991
TLI	.994	.990
Inservice teachers		
Chi-square *p*-value	.001[d]	< .001[d]
RMSEA	.030	.040
CFI	.984	.965
TLI	.983	.961

[a]Root Mean Square Error of Approximation. [b]Comparative Fit Index. [c]Tucker-Lewis Index
[d]Model fit statistic that suggested poor model fit. (All others suggested very good model fit.)

Based on the model fit statistics, the one-factor model had very good fit for the Form 1 data (RMSEA = .031; CFI = .981; TLI = .979) and for the Form 2 data (RMSEA = .033; CFI = .975; TLI = .972) because the CFI and TLI values were greater than .95

and the RMSEA values were less than .05. Note that the chi-square *p*-values in Table 9 suggested very poor model fit in both cases ($p < .001$ for Form 1 and Form 2), but this was expected because this fit index is negatively influenced by large samples (Byrne, 2012). Therefore, I relied more heavily on the other measures of model fit.

Next, I examined the modification indices (M.I.) to determine whether any model respecifications would be logical or appropriate. Generally speaking, adding paths and correlations with larger M.I. values are more likely to improve model fit than adding those with smaller M.I. values (Kline, 2011). In this study, all of the M.I. values were less than 3.84 (the critical value of χ^2 with 1 *df* at $p < .05$) for both datasets, indicating that overall model fit would not statistically significantly improve if a fixed parameter in the model were freely estimated (Brown, 2006). This suggested that no model respecifications were necessary, which was not surprising given that both models had very good fit. (Refer to Table 9.)

Originally, I had planned to compare my various CFA models using the chi-square difference test, which is used to compare nested models. However, because the two- and three-factor models had latent variable covariance matrices that were not positive definite, I was not able to compare the models using this method. Ultimately, I concluded that the one-factor model was best because it exhibited good model-data fit (for both forms), and the analysis of the other models resulted in errors that suggested they were poor-fitting models and that the latent constructs could not be distinguished.

RQ 2: Measurement of KAT in Preservice/Inservice Teachers

I addressed the second research question (related to whether KAT constructs were measured similarly in preservice and inservice teachers) by conducting a multiple-group

CFA. This involved testing my best hypothesized model (i.e., the one-factor model) for configural invariance, measurement invariance, and latent mean differences, respectively. (See Chapter 3 for more details on these tests.)

Test for Configural Invariance

Recall that testing for configural invariance reveals how well the factor-loading pattern and the number of latent constructs in the baseline model represent the data in all of the groups of interest. To test for configural invariance, I first fit the one-factor model separately for my two groups—preservice and inservice teachers—and examined their fit indices. As shown in Table 9, the RMSEA, CFI, and TLI values indicated very good model fit for both groups of teachers and across both forms. Additionally, the chi-square *p*-values suggested good model fit for preservice teachers across both forms.

Then, I fit a multigroup baseline model by allowing all parameters to be freely estimated except factor means, which were fixed to zero in both groups, and scale factors, which were fixed to one in both groups. This configural model had very good fit for the Form 1 data (RMSEA = .033; CFI = .977; TLI = .975) and for the Form 2 data (RMSEA = .036; CFI = .971; TLI = .969), as the CFI and TLI values were greater than .95 and the RMSEA values were less than .05. This was expected because the previously-described work revealed the baseline model fit each group well.

Test for Measurement Invariance

Because configural invariance was supported, the next step was to test for measurement invariance. Recall that testing for measurement invariance is used to determine whether equality constraints can be imposed across groups on any or all of the factor loadings/thresholds. I fit an invariance model by holding factor loadings and thresholds equal across both groups and fixing factor means to zero and scale factors to one for preservice teachers and allowing factor means and scale factors to be freely estimated for inservice teachers. This invariance model had very good fit for the Form 1 data (RMSEA = .032; CFI = .978; TLI = .977) and for the Form 2 data (RMSEA = .034; CFI = .973; TLI = .972).

Additionally, the corrected chi-square difference test and ΔCFI test both indicated that the invariance model should be chosen over the configural/baseline model for both forms. In particular, $\Delta\chi^2(18) = 16.243$, $p = .58$, for Form 1, and $\Delta\chi^2(18) = 13.279$, $p = .77$, for Form 2; and ΔCFI = .001 for Form 1 and ΔCFI = .002 for Form 2. (Recall from Chapter 3 that a statistically nonsignificant $\Delta\chi^2$ and a ΔCFI < 0.1 indicate measurement invariance.)

Based on these analyses, I concluded that KAT constructs were measured similarly in the preservice and inservice teachers in this study. This allowed me to progress to the final phase of multiple-group CFA analysis, which involved comparing the latent means of the groups of interest.

RQ 3: Differences in the KAT of Preservice/Inservice Teachers

Because measurement invariance was supported, I was finally able to test for scalar invariance—that is, to compare the latent means of the preservice and inservice teachers' knowledge of algebra for teaching. This involved fixing the factor mean of one group to zero and allowing the mean to be freely estimated in the other group. (Note that I fixed the factor mean of preservice teachers to zero in this study.)

By analyzing the Mplus output generated during the test for measurement invariance, I found a latent mean difference of approximately –0.216 (S.E. = 0.064, $p =$

.001) in knowledge of algebra for teaching between preservice and inservice teachers for Form 1, and a latent mean difference of approximately –0.101 (S.E. = 0.055, $p = .065$) in knowledge of algebra for teaching between preservice and inservice teachers for Form 2. Because I chose to estimate the latent means of the inservice teachers, this indicated that inservice teachers had significantly ($p < .05$) lower knowledge of algebra for teaching than preservice teachers for Form 1, but the difference in these two groups of teachers was not statistically significantly different from zero ($p > .05$) for Form 2.

I also calculated standardized effect sizes (i.e., Cohen's *d* values) to determine the magnitude of the latent mean differences. Following the procedures outlined in Hancock (2001), I found an effect size of $d = 0.369$ for Form 1 and $d = 0.200$ for Form 2. (This indicated that the latent means for preservice teachers were 0.369 standard deviations and 0.200 standard deviations higher than for inservice teachers, respectively.) According to Cohen (1988), an effect size of 0.2 is generally classified as a "small" effect, and an effect size of 0.5 is generally classified a "medium" effect. So, the latent mean differences in KAT between the preservice and inservice teachers in this study could be classifed as a "small" to "medium" effect.

Ultimately, I concluded that the preservice teachers had slightly higher knowledge of algebra for teaching than inservice teachers. This conclusion was based on three major findings. First, inservice teachers had a lower latent mean than preservice teachers for both forms, even though the difference wasn't statistically significant for Form 2. Second, the latent mean differences between preservice and inservice teachers could be classified as a "medium" effect size for Form 1. And third, the descriptive statistics discussed at the beginning of this chapter revealed that the preservice-teacher group outperformed the inservice-teacher group on all question types (school algebra, advanced mathematics, and mathematics-for-teaching knowledge) and across both forms.

Conclusion

In summary, I found that the one-factor (knowledge of algebra for teaching) model for KAT fit the data better than the two-factor (knowledge of algebra and mathematics-for-teaching knowledge) and three-factor (knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge) models across both forms of the assessment. That is, in response to Research Question 1, the analyses suggested that KAT may be a unidimensional (rather than a multidimensional) construct. Additionally, in response to Research Question 2, I found that KAT was measured similarly in preservice and inservice teachers (using the tests for configural and measurement invariance), and in response to Research Question 3, I found that preservice teachers had slightly higher KAT than inservice teachers.

CHAPTER 5

DISCUSSION

The goal of this chapter is to reflect on the major findings presented in Chapter 4. It includes a brief summary of my KAT study, as well as a discussion of important connections between the findings and the research literature, implications of the findings, important conclusions drawn from the data, and recommendations for future research.

Summary of the Study

The main purpose of the present study was to explore the factor structure of the knowledge of algebra for teaching (KAT) via an instrument developed by researchers at Michigan State University, and a secondary purpose was to compare and contrast the KAT of preservice and inservice mathematics teachers. To address these purposes, I investigated the following research questions:

1. What is the factor structure underlying mathematics' teachers KAT, as measured by an established KAT instrument?
2. Are KAT constructs measured similarly in preservice and inservice teachers?
3. And if so, are there latent mean differences in the KAT of these two groups?

These research questions were addressed using confirmatory factor analysis (CFA), a form of structural equation modeling (SEM) commonly used to explore latent—

or unobserved—variables. More specifically, the first question was addressed by developing a variety of CFA models, based on theory and research; evaluating them for model fit; and using statistical tests to compare them. The second question was addressed by using multiple-groups CFA analyses to determine if the values of model parameters differed across groups (i.e., preservice and inservice teachers), as well as whether or not the measures operated the same in those groups. And the third question was addressed by testing for latent mean differences between the two groups of teachers.

Upon analyzing the data—which included survey responses from 1,248 middle- and high-school mathematics teachers, undergraduate and graduate students (including many preservice mathematics teachers), mathematicians, and mathematics teacher educators—I found that a unidimensional model of KAT best fit the data across both forms of the assessment. That is, a one-factor (knowledge of algebra for teaching) model for KAT fit the data better than a two-factor (knowledge of algebra and mathematics-for-teaching knowledge) or three-factor (knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge) model. Also, I found that KAT was measured similarly in preservice and inservice teachers, and that preservice teachers had slightly higher KAT than inservice teachers.

Linking Findings to the Literature

The two major findings in this study (i.e., the lack of support for a multidimensional model of KAT and the performance of preservice teachers relative to inservice teachers on KAT-related assessment items) are surprising and unexpected, especially given the research literature on algebra, teacher knowledge, and KAT. In the following sections, I compare my research findings with the theories and findings that were outlined in my literature review (in Chapter 2).

Lack of Support for a Multidimensional KAT Model

As previously mentioned, my analyses suggested that KAT is a unidimensional construct. However, there are several reasons why this finding is surprising. First, it does not support the model developed by the KAT research team. These researchers hypothesized that KAT is composed of three distinct factors: knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge (McCrory et al., 2012), but my analyses suggested only one factor.

Additionally, most models related to mathematical knowledge for teaching (e.g., Ball et al., 2008; Kilpatrick et al., 2015; Kleickmann et al., 2013; Krauss et al., 2008; McCrory et al., 2012; Rowland et al., 2005) are multidimensional. These models include the Mathematical Knowledge for Teaching (MKT) framework, which consists of six domains of teacher knowledge, including common content knowledge (CCK), specialized content knowledge SCK), knowledge of content and students (KCS), and knowledge of content and teaching (KCT) (Ball et al., 2008). Other models include the COACTIV and Mathematical Understanding for Secondary Teaching (MUST) models, as well as the knowledge-quartet framework. The COACTIV model consists of two factors related to teacher knowledge, content knowledge (CK) and pedagogical content knowledge (PCK) (Kleickmann et al., 2013; Krauss et al., 2008). In contrast, the MUST and knowledge quartet frameworks are distinctly different from the previous models, but they are still multidimensional. In particular, the MUST model consists of three perspectives: mathematical proficiency, mathematical activity, and mathematical context for teaching (Kilpatrick et al., 2015); and the knowledge quartet consists of four dimensions of teacher knowledge: foundation, transformation, connection, and contingency (Rowland et al., 2005). (See Chapter 2 for additional information about these frameworks.)

Not only are there numerous examples of multidimensional models of mathematical knowledge for teaching in the research literature, but also several of these models (e.g., Hill et al., 2004; Krauss et al., 2008) have been empirically-supported. In particular, Hill and colleagues (2004) were able to at least partially support their MKT model using factor analyses. More specifically, their analyses revealed several distinct factors, including CCK, SCK, and KCS (Hill et al., 2004). Additionally, the COACTIV research team was able to find empirical support for their two-factor (CK and PCK) model for secondary mathematics teachers' knowledge for teaching (Krauss et al., 2008). And during their exploratory analyses with an initial sample, the KAT research team found empirical support for at least two distinct dimensions of KAT (Reckase et al., 2015).

Although most models of mathematical knowledge for teaching are multidimensional, a few researchers have developed unidimensional ones. For example, Kahan and colleagues (2003) developed a one-factor—mathematical content knowledge (MCK)—model for teacher knowledge, which they used to explore the relationships between teachers' MCK and their teaching. However, even these researchers acknowledged that their knowledge construct does not capture all elements of PCK: "We are well aware that teacher factors other than MCK are at work in teaching in practice….[and] our focus on the role of MCK is not intended to exclude other factors" (Kahan et al., 2003, p. 231).

Even though it is widely accepted that mathematical knowledge for teaching is multidimensional, this study is not the first to find empirical support for a one-factor model of teacher knowledge in mathematics. For example, Mohr-Schroeder and colleagues (2017) developed a three-factor model of teachers' knowledge for teaching geometry, which closely resembles the KAT framework. In particular, their Geometry Assessment for Secondary Teachers (GAST) includes items from the following three subdomains of their framework: knowledge of school geometry, knowledge of advanced geometry, and geometry PCK; however, their analyses suggested that a one-factor model fit the data better than a three-factor model (Mohr-Schroeder et al., 2017). Similarly, Herbst and Kosko (2014) developed a four-factor framework of mathematical knowledge for teaching geometry (MKT-G), which included the following MKT subdomains: CCK, SCK, KCT, and KCS. Similar to the GAST research team, these researchers developed an instrument to measure their framework's four subdomains of teacher knowledge, but their analyses also suggested a single measure of knowledge for teaching geometry (Herbst & Kosko, 2014).

Performance of Preservice Teachers on KAT Assessment

In addition to finding support for a one-factor KAT model, my analyses also suggested that preservice teachers had slightly (but statistically significantly) higher KAT than inservice teachers. This finding is also surprising because it contradicts many of the findings in the literature (e.g., Buschang et al., 2012; Kleickmann et al., 2013; Krauss et al., 2008). More specifically, the COACTIV research team found that the PCK of the

German inservice teachers who participated in their study was 0.46 standard deviations higher than the PCK of the student teachers (Kleickmann et al., 2013). In contrast, the KAT of the preservice teachers in this study was 0.369 standard deviations higher than the KAT of inservice teachers for Form 1 and 0.200 standard deviations higher for Form 2. Additionally, Buschang and colleagues (2012) found that expert teachers (e.g., teacher trainers) outperformed both preservice and experienced teachers on an algebra task designed to measure PCK. Generally speaking, the limited number of studies in this area have found that inservice mathematics teachers have higher PCK than preservice teachers, and that inservice teachers generally have the same or higher CK as preservice teachers (e.g., Buschang et al., 2012; Kleickmann et al., 2013; Krauss et al., 2008).

However, even though the preservice teachers in this study demonstrated slightly higher KAT than the inservice teachers, both groups struggled on the assessment. More specifically, the sample of preservice and inservice teachers answered fewer than half of the multiple-choice items related to advanced mathematics correctly (across both forms), fewer than half of the multiple-choice items related to teaching knowledge correctly (for Form 1), and about two-thirds of the multiple-choice items related to school algebra knowledge correctly (across both forms). Additionally, their scores on all open-ended items were fairly low, considering that participants could earn up to 4 points per item and the average scores on these items were between 1.17 and 2.36 for the sample of preservice/inservice teachers. (See Tables 3-4 in Chapter 4 for more details.)

Although it is somewhat alarming that the preservice and inservice teachers in this study struggled on the KAT assessment, the result was not completely unexpected given the literature on knowledge for teaching mathematics (e.g., Cankoy, 2010; Even, 1993;

Huang & Kulm, 2012; Tatto et al., 2012). For example, Huang and Kulm (2012) administered the KAT assessment and several open-ended items related to functions to a group of preservice teachers and found that most of them had limited knowledge of algebra for teaching.

Discussion Questions

So, why did the data in this study suggest a unidimensional model for KAT? And why did preservice teachers score higher than inservice teachers on KAT-related items? In the next section, I discuss several plausible explanations for these major findings.

Plausible Explanations and Implications

This final discussion is divided into four parts. First, I explore several possible explanations for the major findings in the study. Then, I discuss implications of the findings, as well as recommendations for future research and a few closing thoughts about KAT.

Plausible Explanations for the Findings

Unidimensional KAT construct. Given the theoretical and empirical support for multidimensional models of teacher knowledge in mathematics, it is possible—but not likely—that KAT is a unidimensional construct. Although the data did not support three distinct factors of KAT (knowledge of school algebra, knowledge of advanced mathematics, and mathematics-for-teaching knowledge), as theorized by the KAT research team, it is plausible that there are at least two distinct factors: one related to CK (i.e., algebra knowledge) and one related to PCK (i.e., algebra-for-teaching knowledge). In fact, KAT researchers found empirical support—through cluster and parallel analyses—for two dimensions of KAT during their analyses of an initial sample (Reckase et al., 2015).

A more reasonable explanation for this finding is that it is extremely difficult to create assessment items (especially multiple-choice items) that can measure teaching knowledge because this type of knowledge may be closely connected to content knowledge. According to the KAT research team, "Often it is possible to answer the Teaching items using solid mathematics skills without having acquired the knowledge through classroom experience or pedagogy-related coursework or professional development" (Reckase et al., 2015, p. 263). After piloting the KAT assessment with an initial sample, researchers reclassified several of the items that were designed to measure mathematics-for-teaching knowledge because their analyses suggested that the items actually measured school algebra knowledge (Reckase et al., 2015).

Although it may be easier to create and distinguish between items that measure school algebra and advanced mathematics knowledge, it is possible that these two types of items actually measure the same construct. In fact, after their initial analyses, the KAT research team hypothesized that "the Advanced Mathematics items scale is an upper extension of the School Algebra scale" (Reckase et al., 2015, p. 265). That is, both types of items appear to measure the same construct, algebra knowledge.

Performance of preservice teachers on KAT assessment. Given that inservice teachers often score higher than preservice teachers on assessments related to PCK (Buschang et al., 2012; Kleickmann et al., 2013; Krauss et al., 2008), it is also surprising that the preservice teachers in this study answered more questions correctly on the KAT assessment than the inservice teachers. However, there are reasonable explanations for this finding, as well.

First, recall that the KAT assessment includes a set of advanced mathematics items, which includes questions from calculus, abstract algebra, analysis, and linear algebra. Because the preservice teachers had most likely taken advanced courses in mathematics more recently than the inservice teachers, they may have been more prepared to answer these types of questions. In fact, Table 3 (in Chapter 4) shows that preservice teachers outperformed inservice teachers on advanced mathematics items more than they outperformed them on the school algebra and mathematics-for-teaching knowledge items. For example, the preservice teachers who took Form 1 of the KAT assessment answered 47.7% of the advanced mathematics questions correctly, whereas the inservice teachers only answered 37.9% of the advanced mathematics questions correctly—a difference of nearly 10%. However, the difference in the proportion of correct answers for the two groups was negligible—only 2.1% (for Form 1) and 1.5% (for Form 2)—on items that were designed to measure teaching knowledge.

Also recall that middle- and high-school mathematics preservice teachers (as a group) in the U.S. have performed well on some recent assessments of teacher knowledge, including the TEDS-M (Tatto et al., 2012). In this international study (which included preservice teachers in 17 countries), researchers found that the CK and PCK of mathematics preservice teachers in the U.S. were statistically significantly above the international mean (Tatto et al., 2012). Therefore, it is possible that colleges of education are producing stronger teacher candidates than in the past, which could help explain why the preservice teachers in this study outperformed the inservice teachers.

Additionally, it is possible that the inservice teachers' views of algebra and teaching practices could have affected their performance on the KAT assessment. In particular, many of the inservice teachers in the sample may have had limited views of algebra, such as the *algebra as symbolic manipulation* (Kaput, 1995) and/or the *algebra as forming and solving equations* (Bell, 1996; Usiskin, 1999) viewpoints. Also, they may have taught (and likely still teach) in a procedural manner, focusing on rules and procedures rather than conceptual understandings in algebra. Recall that Cankoy (2010) found that the inservice mathematics teachers in his study overwhelmingly gave rule-based explanations for several concepts, including division by zero. It is possible that the inservice teachers would have performed better on the KAT assessment if they had more sophisticated views of algebra, including the *algebra as functions and relationships among quantities* viewpoint (Cooney et al., 2010; Heid, 1996; Kaput, 1995; Usiskin, 1999), and focused on teaching conceptually by implementing some of the teaching strategies discussed in Chapter 2. These strategies include building on students' prior knowledge (e.g., Arcavi et al., 2017), promoting problem solving (e.g., Star et al., 2015), using multiple representations (e.g., Goldin & Shteingold, 2001), using manipulatives (e.g., Leitze & Kitt, 2000), and incorporating technology into instruction (e.g., Yerushalmy & Chazan, 2008).

Another plausible explanation for this finding is that the sample of preservice and inservice teachers in this study may not be representative of the population of all preservice and inservice teachers. According to a member of the KAT researcher team, the sample of preservice teachers included a number of advanced college students who were studying mathematics education at Michigan State University (M. Reckase, personal communication, April 10, 2017). Michigan State has one of the strongest mathematics education programs in the U.S. (Reyes, Glasgow, Teuscher, & Nevels, 2007), so its preservice teachers may have higher KAT than many preservice teachers. Additionally, most of the inservice teachers in the sample were recruited while they were participating in professional development activities (M. Reckase, personal communication, April 10, 2017), so they also may not be representative of the population of inservice teachers. In particular, the sample may have included many novice teachers who were seeking additional training, or it may have included many motivated and innovative teachers who regularly take advantage of PD opportunities.

Also, the sample of inservice teachers may not be representative of *algebra* teachers because we do not know which mathematics courses the inservice teachers actually taught. It is possible that some of these teachers taught courses other than algebra, such as statistics and geometry. During their study, Herbst and Kosko (2014) found that geometry teachers had more MKT-G than other high-school mathematics teachers, so it is very possible that algebra teachers have more KAT than other high-school math teachers, as well.

Implications of the Findings

Based on the findings in this study, it is evident that we still do not have a clear and complete understanding of KAT (and more generally, teacher knowledge in mathematics). Although it may be useful to conceptualize KAT as a three-dimensional construct—composed of school algebra knowledge, advanced mathematics knowledge, and mathematics-for-teaching knowledge—there is currently no empirical support for a three-factor model. However, it is important that we continue to explore KAT and related constructs for several reasons, including that students from around the world are outperforming U.S. students on international mathematics assessments (Mullis et al., 2016; OECD, 2014, 2016); researchers (e.g., Hill et al., 2005) have found correlations between some types of teacher knowledge (e.g., PCK) and student achievement; and algebra plays an integral role in the K-12 mathematics curriculum in the U.S. (NCTM, 2000; NGACBP & CCSSO, 2010).

Additionally, the findings suggest that we most likely need to provide more CK- and PCK-focused professional development (PD) opportunities for inservice algebra teachers. Recall that the preservice teachers in this study outperformed the inservice teachers across both forms of the KAT assessment. Therefore, inservice algebra teachers would likely benefit from PD focused on reviewing important concepts in algebra, discussing appropriate strategies for teaching these concepts, and considering how students think about these concepts (as discussed in Chapter 2).

Moreover, preservice teachers may benefit from additional experiences related to KAT in their content and method courses during their undergraduate studies. As previously mentioned, although the preservice teachers in the study outperformed the inservice teachers, both groups struggled on the KAT assessment. This suggests that there may be major gaps in the KAT of both preservice and inservice teachers, which is consistent with the literature on mathematics teachers' knowledge (e.g., Cankoy, 2010; Even, 1993; Huang & Kulm, 2012; Tatto et al., 2012).

Recommendations for Future Research

Given the findings in this study (as well as studies by the KAT research team), there is still much to learn about KAT. Recommendations for future research include replicating the current study with a new sample of preservice and inservice teachers; developing and empirically evaluating new theoretical models for KAT, such as a two-factor model with CK- and PCK-related components; and creating and testing new KAT-related items, especially items that can measure mathematics-for-teaching knowledge. Also, researchers should consider developing and testing a variety of item types, including constructed-response (CR) and video-analysis questions. The latter question type requires students to analyze video clips of classroom instruction and has been used by Kersting (2008) and other researchers as a novel approach to measuring teaching knowledge. Different item types should be explored because they may reveal aspects of KAT that cannot be revealed using multiple-choice (i.e., dichotomous) items. According to Martinez (1999), "The range of cognitions [e.g., knowledge, procedures, and schemas] that can be elicited by CR items is greater than the range of multiple-choice items" (p. 209). Therefore, it is possible that CR (and other types of) items may be able to detect distinct dimensions of KAT.

Additional recommendations include developing and evaluating the effectiveness of CK- and PCK-focused professional development experiences for inservice algebra teachers, as well as innovative algebra activities for preservice teachers to explore in their content and methods courses. Potential areas of exploration could include designing high-quality tasks, incorporating manipulatives and technology into instruction, promoting problem solving, and analyzing student thinking in algebra (as discussed in Chapter 2). Perhaps, engaging in these types of experiences would increase the KAT of both preservice and inservice teachers, which in turn could possibly lead to increases in student achievement in the algebra classroom.

Concluding Remarks

Although algebra comprises a significant portion the K-12 mathematics curriculum in the U.S. (NCTM, 2000; NGA CBP & CCSSO, 2010), our students continue to struggle with even the most basic algebra skills (Blume & Heckman, 2000; Mullis et al., 2016; National Mathematics Advisory Panel, 2008; OECD, 2014, 2016; RAND Mathematics Study Panel, 2003). This study and others (e.g., Cankoy, 2010; Even, 1993; Huang & Kulm, 2012; Tatto et al., 2012) have found that our algebra teachers may have limited knowledge of algebra for teaching. This finding is rather alarming because recent studies in the area of mathematical knowledge for teaching (e.g., Baumert et al., 2010; Campbell et al., 2014; Hill et al., 2005) have found relationships between teacher knowledge and student achievement. Unfortunately, the results of this study also reveal that we still know relatively little about mathematics teachers' knowledge of algebra. In particular, the KAT research team hypothesized that KAT is composed of three distinct factors, but this study only found empirical support for a one-factor model. If we can gain deeper understandings of KAT and how to help our preservice and inservice teachers develop this knowledge, we may begin to see better teaching in the algebra classroom and increases in our students' overall achievement in algebra.

REFERENCES

Adelson, J. L. (2012). Examining relationships and effects in gifted education research: An introduction to structural equation modeling. *Gifted Child Quarterly*, *56*, 47-55.

Arcavi, A., Drijvers, P., & Stacey, K. (2017). *The learning and teaching of algebra: Ideas, insights, and activities*. New York, NY: Routledge.

Asiala, M., Brown, A., Devries, D. J., Dubinsky, E., Mathews, D., & Thomas, K. (1996). A framework for research and curriculum development in undergraduate mathematics education. In J. Kaput, A. H. Schoenfeld, E. Dubinksy, & T. Dick (Eds.), *Conference Board of the Mathematical Sciences issues in mathematics education: Research in collegiate mathematics education II* (Vol. 6, pp. 1-32). Providence, RI: American Mathematical Society.

Asquith, P., Stephens, A. C., Knuth, E. J., & Alibali, M. W. (2007). Middle school mathematics teachers' knowledge of students' understanding of core algebraic concepts: Equal sign and variable. *Mathematical Thinking and Learning*, *9*, 249-272.

Balka, D. S. (1993). Making the connections in mathematics via manipulatives. *Contemporary Education*, *65*, 19-23.

Ball, D. L., Thames, M. H., & Phelps, G. (2008). Content knowledge for teaching: What makes it special? *Journal of Teacher Education*, *59*, 389-407.

Baroody, A. J., & Ginsburg, H. P. (1983). The effects of instruction on children's understanding of the "equals" sign. *The Elementary School Journal*, *84*, 198-212.

Baumert, J., Kunter, M., Blum, W., Brunner, M., Voss, T., Jordan, A.,…Tsai, Y.-M. (2010). Teachers' mathematical knowledge, cognitive activation in the classroom, and student progress. *American Educational Research Journal*, *47*, 133-180.

Begle, E. G. (1972). *Teacher knowledge and student achievement in algebra*. Stanford, CA: School Mathematics Study Group.

Behr, M. J., Erlwanger, S., & Nichols, E. (1980). How children view the equals sign. *Mathematics Teaching*, *92*, 13-15.

Bell, A. (1996). Problem-solving approaches to algebra: Two aspects. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 167-185). Dordrecht, The Netherlands: Kluwer.

Blume, G. W., & Heckman, D. S. (2000). Algebra and functions. In E. A. Silver & P. A. Kenney (Eds.), *Results from the seventh mathematics assessment of the National Assessment of Educational Progress* (pp. 269-306). Reston, VA: National Council of Teachers of Mathematics.

Bohan, H. J., & Shawaker, P. B. (1994). Using manipulatives effectively: A drive down Rounding Road. *The Arithmetic Teacher*, *41*, 246-248.

Breidenbach, D., Dubinsky, E., Hawks, J., & Nichols, D. (1992). Development of the process conception of function. *Educational Studies in Mathematics*, *23*, 247-285.

Brown, T. A. (2006). *Confirmatory factor analysis for applied research*. New York, NY: Guilford Press.

Bruner, J. S. (1966). *Toward a theory of instruction*. Cambridge, MA: Harvard University Press.

Buschang, R. E., Chung, G. K. W. K, Delacruz, G. C., & Baker, E. L. (2012). Validating measures of algebra teacher subject matter knowledge and pedagogical content knowledge. *Educational Assessment, 17*, 1-12.

Byrne, B. M. (2012). *Structural equation modeling with Mplus: Basic concepts, applications, and programming*. New York, NY: Routledge.

Cai, J. (2010). Helping elementary students become successful mathematical problem solvers. In D. V. Lambdin & F. K. Lester Jr. (Eds.), *Teaching and learning mathematics: Translating research for elementary teachers* (pp. 9-14). Reston, VA: National Council of Teachers of Mathematics.

Campbell, P. F., Nishio, M., Smith, T. M., Clark, L. M., Conant, D. L., Rust, A. H.,…Choi, Y. (2014). The relationship between teachers' mathematical content and pedagogical knowledge, teachers' perceptions, and student achievement. *Journal for Research in Mathematics Education, 45*, 419-459.

Cankoy, O. (2010). Mathematics teachers' topic-specific pedagogical content knowledge in the context of teaching a^0, 0! and a ÷ 0. *Educational Sciences: Theory & Practice, 10*, 749-769.

Carpenter, T. P., & Fennema, E. (1992). Cognitively Guided Instruction: Building on the knowledge of students and teachers. *International Journal of Educational Research, 17*, 457-470.

Chappell, M. F., & Strutchens, M. E. (2001). Creating connections: Promoting algebraic thinking with concrete models. *Mathematics Teaching in the Middle School*, *7*, 20-25.

Cheung, G. W., & Rensvold, R. B. (2002). Evaluating goodness-of-fit indexes for testing measurement invariance. *Structural Equation Modeling*, *9*, 233-255.

Clement, J. (1982). Algebra word problem solutions: Thought processes underlying a common misconception. *Journal for Research in Mathematics Education*, *13*, 16-30.

Cohen, J. (1988). *Statistical power analysis for the behavioral sciences* (2nd ed.). Hillsdale, NJ: Erlbaum.

Collis, K. F. (1975). *The development of formal reasoning*. Newcastle, Australia: University of Newcastle.

Conference Board of the Mathematical Sciences. (2012). *The mathematical education of teachers II*. Providence, RI: American Mathematical Society.

Cooney, T. J., Beckmann, S., & Lloyd, G. M. (2010). *Developing an essential understanding of functions for teaching mathematics in grades 9-12*. P. S. Wilson & R. M. Zbiek (Eds.). Reston, VA: National Council of Teachers of Mathematics.

DiMatteo, R. W., & Lester, F. K., Jr. (2010). The role of problem solving in the secondary school mathematics classroom. In J. Lobato & F. K. Lester Jr. (Eds.), *Teaching and learning mathematics: Translating research for secondary teachers* (pp. 7-12). Reston, VA: National Council of Teachers of Mathematics.

Dougherty, B. J. (2001). Access to algebra: A process approach. In H. Chick, K. Stacey, J. Vincent, & J. Vincent (Eds.). *The future of the teaching and learning of algebra, Proceedings of the 12th ICMI Study Conference* (pp. 207-212). Victoria, Australia: The University of Melbourne.

Dubinsky, E., & Harel, G. (1992). The nature of the process conception of functions. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 85-106). Washington, DC: Mathematical Association of America.

Eisenberg, T. A. (1977). Begle revisited: Teacher knowledge and student achievement in algebra. *Journal for Research in Mathematics Education, 8,* 216-222.

Even, R. (1993). Subject-matter knowledge and pedagogical content knowledge: Prospective secondary teachers and the function concept. *Journal for Research in Mathematics Education*, *24,* 94-116.

Falkner, K. P., Levi, L., & Carpenter, T. P. (1999). Children's understanding of equality: A foundation for algebra. *Teaching Children Mathematics*, *6*, 232-236.

Filloy, E., & Rojano, T. (1989). Solving equations: The transition from arithmetic to algebra. *For the Learning of Mathematics*, *9*, 19-25.

Goldin, G., & Shteingold, N. (2001). Systems of representations and the development of mathematical concepts. In A. A. Cuoco & F. R. Curcio (Eds.), *The roles of representation in school mathematics* (pp. 1-23). Reston, VA: National Council of Teachers of Mathematics.

Hancock, G. R. (2001). Effect size, power, and sample size determinants for structured means modeling and MIMIC approaches to between-groups hypothesis testing of means on a single latent construct. *Psychometrika*, *66*, 373-388.

Heid, M. K. (1996). A technology-intensive functional approach to the emergence of algebraic thinking. In N. Bednarz, C. Kieran, & L. Lee (Eds.), *Approaches to algebra: Perspectives for research and teaching* (pp. 239-255). Dordrecht, The Netherlands: Kluwer.

Herbst, P., & Kosko, K. (2014). Mathematical knowledge for teaching and its specificity to high school geometry instruction. In J. J. Lo, K. R. Leatham, & L. R. Van Zoest (Eds.), *Research trends in mathematics teacher education* (pp. 23-45). Cham, Switzerland: Springer International.

Hiebert, J., & Grouws, D. A. (2007). The effects of classroom mathematics teaching on students' learning. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 371-404). Charlotte, NC: Information Age.

Hill, H. C., Rowan, B., & Ball, D. L. (2005). Effects of teachers' mathematical knowledge for teaching on student achievement. *American Educational Research Journal*, *42*, 371-406.

Hill, H. C., Schilling, S. G., & Ball, D. L. (2004). Developing measures of teachers' mathematics knowledge for teaching. *The Elementary School Journal*, *105*, 11-30.

Hill, H. C., Sleep, L., Lewis, J. M., & Ball, D. L. (2007). Assessing teachers' mathematical knowledge: What knowledge matters and what evidence counts? In

F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 111-152). Charlotte, NC: Information Age.

Hoong, L. Y., Kin, H. W., & Pien, C. L. (2015). Concrete-Pictorial-Abstract: Surveying its origins and charting its future. *The Mathematics Educator, 16*, 1-19.

Huang, R., & Kulm, G. (2012). Prospective middle grade mathematics teachers' knowledge of algebra for teaching. *Journal of Mathematical Behavior*, *31*, 417-430.

Kahan, J. A., Cooper, D. A., & Bethea, K. A. (2003). The role of mathematics teachers' content knowledge in their teaching: A framework for research applied to a study of student teachers. *Journal of Mathematics Teacher Education*, *6,* 223-252.

Kalchman, M., & Koedinger, K. R. (2005). Teaching and learning functions. In M. S. Donovan & J. D. Bransford (Eds.), *How students learn: History, mathematics, and science in the classroom (pp. 351-393)*. Washington, DC: The National Academies Press.

Kaput, J. J. (1995). A research base supporting long term algebra reform? In D. T. Owens, M. K. Reed, & G. M. Millsaps (Eds), *Proceedings of the seventh annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (Vol. 1, pp. 71-94). Columbus, OH: ERIC Clearinghouse for Science, Mathematics, and Environmental Education.

Kersting, N. (2008). Using video clips of mathematics classroom instruction as item prompts to measure teachers' knowledge of teaching mathematics. *Educational and Psychological Measurement*, *68*, 845-861.

Kieran, C. (1981). Concepts associated with the equality symbol. *Educational Studies in Mathematics*, *12*, 317-326.

Kieran, C. (1996). The changing face of school algebra. In C. Alsina, J. Alvarez, B. Hodgson, C. Laborde, & A. Pérez (Eds.), *Eighth International Congress on Mathematical Education: Selected lectures* (pp. 271-290). Seville, Spain: S.A.E.M. Thales.

Kieran, C. (2007). Learning and teaching algebra at the middle school through college levels. In F. K. Lester Jr. (Ed.), *Second handbook of research on mathematics teaching and learning* (pp. 707-762). Charlotte, NC: Information Age.

Kilpatrick, J., Blume, G., Heid, M. K., Wilson, J., Wilson, P., & Zbiek, R. M. (2015). Mathematical understanding for secondary teaching: A framework. In M. K. Heid, P. S. Wilson, & G. W. Blume (Eds.), *Mathematical understanding for secondary teaching: A framework and classroom-based situations* (pp. 9-30). Charlotte, NC: Information Age & National Council of Teachers of Mathematics.

Kleickmann, T., Richter, D., Kunter, M., Elsner, J., Besser, M., Krauss, S., & Baumert, J. (2013). Teachers' content knowledge and pedagogical content knowledge: The role of structural differences in teacher education. *Journal of Teacher Education*, *64*, 90-106.

Kline, R. B. (2011). *Principles and practice of structural equation modeling* (3rd ed.). New York, NY: Guilford Press.

Knuth, E.J., & Peressini, D. D. (2001). Unpacking the nature of discourse in mathematics classrooms. *Mathematics Teaching in the Middle School*, *6*, 320-325.

Knuth, E. J., Stephens, A. C., McNeil, N. M., & Alibali, M. W. (2006). Does understanding the equal sign matter? Evidence from solving equations. *Journal for Research in Mathematics Education*, *37*, 297-312.

Koehler, M. J., & Mishra, P. (2009). What is technological pedagogical content knowledge? *Contemporary Issues in Technology and Teacher Education*, *9*, 60-70.

König, J. Blömeke, S., Paine, L.,Schmidt, W. H., & Hsieh, F. (2011). General pedagogical knowledge of future middle school teachers: On the complex ecology of teacher education in the United States, Germany, and Taiwan. *Journal of Teacher Education*, *62*, 188-201.

Krauss, S., Baumert, J., & Blum, W. (2008). Secondary mathematics teachers' pedagogical content knowledge and content knowledge: Validation of the COACTIV constructs. *ZDM Mathematics Education*, *40*, 873-892.

Küchemann, D. (1978). Children's understanding of numerical variables. *Mathematics in School*, *7*, 23-26.

Lee, L. (1997). *Algebraic understanding: The search for a model in the mathematics education community* (Unpublished doctoral dissertation). Université du Québec à Montréal: Montreal, Quebec, Canada.

Leinhardt, G., Zaslavsky, O., & Stein, M. K. (1990). Functions, graphs, and graphing: Tasks, learning, and teaching. *Review of Educational Research*, *60*, 1-64.

Leitze, A. R., & Kitt, N. A. (2000). Using homemade algebra tiles to develop algebra and prealgebra concepts. *The Mathematics Teacher*, *93*, 462-466, 520.

Lesh, R., & Lehrer, R. (2003). Models and modeling perspectives on the development of students and teachers. *Mathematical Thinking and Learning*, *5*, 109-129.

Lester, F. K., Jr. (2013). Thoughts about research on mathematical problem-solving instruction. *The Mathematics Enthusiast*, *10*, 245-278.

Lester, F. K., Jr., & Kehle, P. (2003). From problem solving to modeling: The evolution of thinking about research on complex mathematical activity. In R. Lesh & H. M. Doerr (Eds.), *Beyond constructivism: Models and modeling perspectives on mathematics problem solving, learning, and teaching* (pp. 501-517). Mahwah, NJ: Erlbaum.

Lloyd, G. M., & Wilson, M. (1998). Supporting innovation: The impact of a teacher's conceptions of functions on his implementation of a reform curriculum. *Journal for Research in Mathematics Education*, *29*, 248-274.

MacGregor, M., & Stacey, K. (1997). Students' understanding of algebraic notation: 11-15. *Educational Studies in Mathematics, 33,* 1-19.

Martinez, M. E. (1999). Cognition and the question of test item format. *Educational Psychologist*, *34*, 207-218.

Matz, M. (1982). Towards a process model for high school algebra errors. In D. Sleeman & J.S. Brown (Eds.), *Intelligent Tutoring Systems* (pp. 25-50). New York, NY: Academic Press.

McCoach, D. B., Gable, R. K., & Madura, J. P. (2013). *Instrument development in the affective domain: School and corporate applications* (3rd ed.). New York, NY: Springer.

McCrory, R., Floden, R., Ferrini-Mundy, J., Reckase, M. D., & Senk, S. L. (2012). Knowledge of algebra for teaching: A framework of knowledge and practices. *Journal for Research in Mathematics Education*, *43,* 584-615.

Mohr-Schroeder, M., Ronau, R. N., Peters, S., Lee, C. W., & Bush, W. S. (2017). Predicting student achievement using measures of teachers' knowledge for teaching geometry. *Journal for Research in Mathematics Education*, *48*, 520-566.

Monk, S. (1992). Students' understanding of a function given by a physical model. In G. Harel & E. Dubinsky (Eds.), *The concept of function: Aspects of epistemology and pedagogy* (pp. 175-193). Washington, DC: Mathematical Association of America.

Ministry of Education, Singapore. (2012). *Mathematics syllabus: Secondary one to four*. Retrieved from https://www.moe.gov.sg/docs/default-source/document/education/syllabuses/sciences/files/mathematics-syllabus-sec-1-to-4-express-n(a)-course.pdf

Mullis, I. V. S., Martin, M. O., Foy, P., & Hooper, M. (2016). *TIMSS 2015 international results in mathematics*. Chestnut Hill, MA: TIMSS & PIRLS International Study Center, Boston College.

Muthén, B. O. (2011, March 18). Correlation between factors [Web log comment]. Retrieved from http://www.statmodel.com/discussion/messages/9/380.html?1512523418

Muthén, L. K., & Muthén, B. O. (2011). *Regression analysis, exploratory factor analysis, confirmatory factor analysis, and structural equation modeling for categorical, censored, and count outcomes* [PowerPoint slides]. Retrieved from the Mplus website: https://www.statmodel.com/download/Topic%202-v20%20 [Compatibility%20Mode]1.pdf

Muthén, L. K., & Muthén, B. O. (2017). Mplus (Version 8) [Computer software]. Available from https://www.statmodel.com/

National Council of Teachers of Mathematics. (2000). *Principles and standards for school mathematics*. Reston, VA: Author.

National Governors Association Center for Best Practices & Council of Chief State School Officers. (2010). *Common Core State Standards for Mathematics*. Washington, DC: Author.

National Mathematics Advisory Panel. (2008). *Foundations for success: Final report of the National Mathematics Advisory Panel*. Washington, DC: U.S. Department of Education.

National Research Council. (2000). *How people learn: Brain, mind, experience, and school*. J. D. Bransford, A. L. Brown, & R. R. Cocking (Eds.). Washington, DC: National Academy Press.

National Research Council. (2001). *Adding it up: Helping children learn mathematics*. J. Kilpatrick, J. Swafford, & B. Findell (Eds.). Washington, DC: National Academy Press.

Oehrtman, M., Carlson, M. P., & Thompson, P. W. (2008). Foundational reasoning abilities that promote coherence in students' function understanding. In M. P. Carlson & C. Rasmussen (Eds.), *Making the connection: Research and teaching in undergraduate mathematics education* (pp. 27-42). Washington, DC: Mathematical Association of America.

Organization for Economic Cooperation and Development. (2014). *PISA 2012 results: What students know and can do – student performance in mathematics, reading, and science* (Rev. ed., Vol. 1). Paris, France: OECD.

Organization for Economic Cooperation and Development. (2016). *PISA 2015 results: Excellence and equity in education* (Vol. 1). Paris, France: OECD.

Petrou, M., & Goulding, M. (2011). Conceptualising teachers' mathematical knowledge in teaching. In T. Rowland & K. Ruthven (Eds.), *Mathematical knowledge in teaching* (pp. 9-25). Dordrecht, The Netherlands: Springer.

Pimm, D. (1995). *Symbols and meanings in school mathematics*. London: Routledge.

Post, T. R., Harwell, M. R., Davis, J. D., Maeda, Y., Cutler, A., Andersen, E.,…Norman, K. W. (2008). Standards-based mathematics curricula and middle-grades students' performance on standardized achievement tests. *Journal for Research in Mathematics Education*, *39*, 184-212.

Powell, S. R. (2015). The influence of symbols and equations on understanding mathematical equivalence. *Intervention in School and Clinic*, *50*, 266-272.

Rakes, C. R., Valentine, J. C., McGatha, M. B., & Ronau, R. N. (2010). Methods of instructional improvement in algebra: A systematic review and meta-analysis. *Review of Educational Research*, *80*, 372-400.

RAND Mathematics Study Panel. (2003). *Mathematical proficiency for all students: Toward a strategic research and development program in mathematics education*. Santa Monica, CA: RAND.

Reckase, M. D., McCrory, R., Floden, R. E., Ferrini-Mundy, J., & Senk, S. L. (2015). A multidimensional assessment of teachers' knowledge of algebra for teaching:

Developing an instrument and supporting valid inferences. *Educational Assessment*, *20*, 249-267.

Reyes, R., Glasgow, R., Teuscher, D., & Nevels, N. (2007). Doctoral programs in mathematics education in the United States: 2007 status report. *Notices of the American Mathematical Society*, *54*, 1283-1293.

Rojano, T., & Martínez, M. (2009). From concrete modeling to algebraic syntax: Learning to solve linear equations with a virtual balance. In S. L. Swars, D. W. Stinson, & S. Lemons-Smith (Eds.), *Proceedings of the 31st annual meeting of the North American Chapter of the International Group for the Psychology of Mathematics Education* (pp. 235-243). Atlanta, GA: Georgia State University.

Ronau, R. N., Meyer, D., & Crites, T. (2014). *Putting essential understanding of functions into practice in grades 9-12*. T. Crites & B. J. Dougherty (Eds). Reston, VA: National Council of Teachers of Mathematics.

Rowland, T., Huckstep, P., & Thwaites, A. (2005). Elementary teachers' mathematics subject knowledge: The knowledge quartet and the case of Naomi. *Journal of Mathematics Teacher Education*, *8*, 255-281.

Schilling, S. G., Blunk, M., & Hill, H. C. (2007). Test validation and the MKT measures: Generalizations and conclusions. *Measurement: Interdisciplinary Research and Perspectives*, *5*, 118-128.

Senk, S. L., & Thompson, D. R. (2003). *Standards-based school mathematics curricula: What are they? What do students learn?* Mahwah, NJ: Lawrence Erlbaum Associates.

Shulman, L. S. (1986). Those who understand: Knowledge growth in teaching. *Educational Researcher, 15*, 4-14.

Sowell, E. (1989). Effects of manipulative materials in mathematics instruction. *Journal for Research in Mathematics Education*, *20*, 498-505.

Star, J. R., Caronongan, P., Foegen, A., Furgeson, J., Keating, B., Larson,…Zbiek, R. M. (2015). *Teaching strategies for improving algebra knowledge in middle and high school students*. Washington, DC: National Center for Education Evaluation and Regional Assistance, Institute of Education Sciences, U.S. Department of Education.

Suh, J., & Moyer, P. S. (2007). Developing students' representational fluency using virtual and physical algebra balances. *Journal of Computers in Mathematics and Science Teaching*, *26*, 155-173.

Takahashi, A., Watanabe, T., & Yoshida, M. (2008). *English Translation of the Japanese Mathematics Curriculum in the Course of Study: Grades 1-9*. Madison, NJ: Global Education Resources.

Tall, D., & Vinner, S. (1981). Concept image and concept definition in mathematics with particular reference to limits and continuity. *Educational Studies in Mathematics*, *12*, 151-169.

Tatto, M. T., Peck, R., Schwille, J., Bankov, K., Senk, S. L., Rodriguez, M.,…Rowley, G. (2012). *Policy, practice, and readiness to teach primary and secondary mathematics in 17 countries: Findings from the IEA Teacher Education and Development Study in Mathematics (TEDS-M).* Amsterdam, the Netherlands: International Association for the Evaluation of Educational Achievement.

Thompson, B. (2004). *Exploratory and confirmatory factor analysis*. Washington, DC: American Psychological Association.

Thompson, P. W. (1994). Students, functions, and the undergraduate curriculum. In E. Dubinsky, A. H. Schoenfeld, & J. J. Kaput (Eds.), *Research in collegiate mathematics education. I* (Vol. 4, pp. 21-44). Providence, RI: American Mathematical Society.

Tooke, D. J., Hyatt, B., Leigh, M., Snyder, B., & Borda, T. (1992). Why aren't manipulatives used in every middle school mathematics classroom? *Middle School Journal*, *24*, 61-62.

Usiskin, Z. (1995). Why is algebra important to learn? *American Educator, 19*, 30-37.

Usiskin, Z. (1999). Conceptions of school algebra and uses of variables. In B. Moses (Ed.), *Algebraic thinking, Grades K-12: Readings from NCTM's school-based journals and other publications* (pp. 7-13). Reston, VA: National Council of Teachers of Mathematics.

Usiskin, Z. (2004). A significant amount of algebra. *Nieuw Archief voor Wiskunde*, *5*, 147-152.

Van Dooren, W., De Bock, D., Janssens, D., & Verschaffel, L. (2008). The linear imperative: An inventory and conceptual analysis of students' overuse of linearity. *Journal for Research in Mathematics Education*, *39*, 311-342.

Vlassis, J. (2002). The balance model: Hindrance or support for the solving of linear equations with one unknown. *Educational Studies in Mathematics, 49*, 341-359.

Warshauer, H. K. (2015). Strategies to support productive struggle. *Mathematics Teaching in the Middle School*, *20*, 390-393.

Yerushalmy, M., & Chazan, D. (2008). Technology and curriculum design: The ordering of discontinuities in school algebra. In L. D. English (Ed.), *Second handbook of international research in mathematics education* (pp. 806-837). New York, NY: Routledge.

CURRICULUM VITAE

Jonathan David Watkins

6705 Northwind Way
Crestwood, Kentucky 40014
jonathan.watkins@louisville.edu
(502) 494-1625 (Cell)

251 Belknap Academic Bldg.
University of Louisville
Louisville, Kentucky 40292
(502) 852-8099 (Office)

EDUCATION

Ph.D., Curriculum & Instruction (4.0 GPA)
Emphasis: Mathematics Education
Dissertation: Exploring the Knowledge of Algebra for Teaching
University of Louisville, Louisville, KY — Expected December 2018

M.A., Mathematics (3.92 GPA)
University of Louisville, Louisville, KY — December 2011

B.A., Mathematics and Spanish (3.97 GPA)
Murray State University, Murray, KY — December 2003

PROFESSIONAL CERTIFICATIONS

CRLA Certified Advanced Mathematics Tutor (Level II)
College Reading and Learning Association, Oak Creek, WI — Earned 2012

Professional Certification for Teaching Mathematics, Grades 8-12
Education Professional Standards Board, Frankfort, KY — 2004-2011

Professional Certification for Teaching Spanish, All Grades
Education Professional Standards Board, Frankfort, KY — 2004-2011

ACADEMIC AND PROFESSIONAL EXPERIENCES

Adjunct Faculty, Department of Mathematics
Bellarmine University, Louisville, KY — 2014-Present

- Teach elementary mathematics content course
- Teach review course for the Praxis CASE mathematics exam
- Frequently engage students in problem-solving investigations

- Coordinate course scheduling with School of Education and Department of Mathematics

Assistant Director & Instructor, REACH Math Resources
Resources for Academic Achievement (REACH)
University of Louisville, Louisville, KY 2013-Present

- Coordinate and teach intervention algebra courses (GEN 103/104)
- Supervise and evaluate teaching assistants and peer tutors
- Provide support for at-risk students
- Tutor math students in the REACH Math Resource Center, which is the mathematics tutoring center at the University
- Promoted from Coordinator to Assistant Director in 2017

University Supervisor, College of Education & Human Development
University of Louisville, Louisville, KY 2013-2015

- Supervised prospective mathematics teachers during their field and student-teaching experiences
- Conducted formal observations and provided constructive feedback to the prospective teachers
- Collaborated with math methods course instructors

Local Site Research Coordinator, The National Writing Project
University of Louisville, Louisville, KY 2013

- Served as a key facilitator in data collection efforts at the Local Writing Project site and participating schools
- Conducted training workshops for teachers serving as proctors

Graduate Research Assistant, CEHD
University of Louisville, Louisville KY 2012-2013

- Assisted Dr. Susan A. Peters with her CAREER grant in statistics education from the National Science Foundation
- Contributed to a comprehensive literature review of research related to statistics and statistics education
- Co-developed statistics activities for use with inservice teachers during professional development
- Facilitated professional-development sessions

Manager (Graduate Student Assistant), REACH Math Resources
University of Louisville, Louisville, KY 2011-2012

- Supervised and evaluated peer tutors
- Created and maintained master tutoring schedule
- Reconciled payroll

Mathematics Faculty, Commonwealth Honors Academy
Murray State University, Murray, KY 2010, 2011

- Wrote curriculum proposal for introductory applied statistics course (HON 106)
- Taught HON 106 to gifted high-school students attending CHA
- Co-led personal development seminar

Graduate Teaching Assistant, Department of Mathematics
University of Louisville, Louisville, KY 2009-2011

- Co-taught mathematics for liberal arts course (MATH 105)
- Co-taught college algebra course (MATH 111)

Mathematics and Spanish Teacher
Murray High School, Murray, KY 2004-2007

- Wrote curriculum for mathematics and Spanish courses
- Taught mathematics courses, including algebra, geometry, precalculus with trigonometry, and general mathematics
- Taught basic Spanish courses
- Served as assistant coach of the Quick Recall and Future Problem Solving teams

Spanish Instructor, Summer Challenge Program
Murray State University, Murray, KY 2004

- Developed basic Spanish course for elementary students
- Taught the course on Saturdays during the summer

UNIVERSITY COURSES TAUGHT OR CO-TAUGHT

University of Louisville, Louisville, KY

EDTP 313: P-5 Math Methods (Co-taught with Dr. Karen Karp)

EDTP 408/423/608/623: Middle/High School Mathematics Methods (Co-taught with Dr. Lateefah Id-Deen)

EDAP/ECPY 764: Structural Equation Modeling (Co-taught with Dr. Jill Adelson)

GEN 103/104: Special Topics in College Mathematics (Currently teaching)
Required intervention courses for students who do not achieve the college-readiness standard for the entry-level MATH course required for their major. Courses allow for individualized instruction via computer-assisted learning. Topics are determined by pre-assessments of algebra skills and may include factoring, rational expressions and equations, radical expressions, and quadratic equations.

MATH 105: Introduction to Contemporary Mathematics (Co-taught with several different faculty members in the Department of Mathematics)
Use of mathematical modeling to solve practical problems. Applications include management science, social choice, population growth, and personal finance.

MATH 111: College Algebra (Co-taught with several different faculty members in the Department of Mathematics)

Bellarmine University, Louisville, KY

MATH 523: Foundations of Mathematics
This course is an investigation at the Master's level of topics from the elementary and middle school mathematics curriculum. It is not a study of how to do mathematics, but why the way we do mathematics works. The NCTM (National Council of Teachers of Mathematics) Standards guide the course through a study of problem-solving; set theory; ancient numeration systems; numeration in various bases; the four basic operations; fractions and decimals; real numbers; number theory; ratios, rates, and proportions; probability and statistics; and geometry and measurement.

Murray State University, Murray, KY

HON 106: Topics in Science, Engineering, and Technology—Introduction to Statistics (Taught at the Commonwealth Honors Academy)
Basic concepts in statistics are introduced. Topics include descriptive vs. inferential statistics, correlation and regression, experimentation vs. observation, probability and probability distributions, sampling distributions.

PUBLICATIONS (REFEREED)

Peters, S. A., Bennett, V. M., Young, M., & **Watkins, J. D.** (2016). A fair and balanced approach to the mean. *Mathematics Teaching in the Middle School*, *21*, 364-372.

Peters, S. A., **Watkins, J. D.**, & Bennett, V. M. (2014). Middle and high school teachers' transformative learning of center. In K. Makar, B. de Sousa, & R. Gould (Eds.), *Sustainability in statistics education. Proceedings of the Ninth International Conference on Teaching Statistics (ICOTS-9)*, Flagstaff, Arizona, USA. The Hague-Leidschenveen, The Netherlands: International Statistical Institute.

Peters, S. A., **Watkins, J. D.**, & Bennett, V. M. (2014, April). *Middle-school and secondary teachers' transformative learning of center*. Paper presented at the 2014 National Council of Teachers of Mathematics Research Conference, New Orleans, LA.

PRESENTATIONS

Watkins, J. D. (2018, March). *Exploring the Knowledge of Algebra for Teaching: Preliminary findings*. Presented at the 2018 Spring Research Conference, University of Louisville, Louisville, KY.

Watkins, J. D., & Boyd, K. E. (2018, March). *Intervention in algebra*. Presented at the Innovative Educators Summit, Isle of Palms, SC. (Invited presentation)

Peters, S. A., & **Watkins, J. D.** (2018, February). *Activities to facilitate middle and secondary mathematics teachers' learning of distribution within professional development.* Presented at the 22nd annual Association of Mathematics Teacher Educators Conference, Houston, TX. (Refereed)

Watkins, J. D., & Wilkins, C. Y. (2018, February). *Redesigning mathematics courses using the Emporium model*. Presented at the 2018 National Association for Developmental Education Conference, National Harbor, MD. (Refereed)

Watkins, J. D., & Boyd, K. E. (2017, March). *An emporium approach to intervention courses in algebra*. Presented at the Redesign: Technology and Innovation in Mathematics & English Conference, Charleston, SC. (Invited presentation)

Watkins, J. D. (2016, November). *Using games of chance to teach probability*. Presented at the Kentucky Council of Teachers of Mathematics Conference, Louisville, KY.

Watkins, J. D. (2016, September). *An emporium approach to intervention courses in algebra* [Webinar]. Retrieved from https://blog.hawkeslearning.com/webinars-ondemand/

Watkins, J. D., & Adelson, J. L. (2016, April). *Development and validation of the Self-Perceptions in Algebra Scales*. Presented at the 2016 American Educational Research Association Annual Meeting and Exhibition, Washington, DC. (Refereed)

Watkins, J. D. (2015, April). *Games of chance: Exploring CCSSM probability standards in motivating contexts.* Presented at the 2015 National Council of Teachers of Mathematics Annual Meeting and Exposition, Boston, MA. (Refereed)

Peters, S. A., & **Watkins, J. D.** (2015, April). *Distribution and average: Deepening Common Core statistical knowledge for teaching*. Presented at the 2015 National Council of Teachers of Mathematics Annual Meeting and Exposition, Boston, MA. (Refereed)

Watkins, J. D. (2015, April). *Self-perceptions in Algebra Scales*. Presented at the Graduate Research Symposium, University of Louisville, Louisville, KY.

Watkins, J. D., & Wilkins, C. Y. (2015, March). *Rethinking intervention courses in algebra: An emporium approach*. Presented at the 27th annual International Conference on Technology in Collegiate Mathematics, Las Vegas, NV.

Watkins, J. D. (2015, March). *Development and validation of the Self-Perceptions in Algebra Scales*. Presented at the 2015 Spring Research Conference, University of Louisville, Louisville, KY.

Watkins, J. D. (2014, April). *Investigating the relationships between homework and achievement in mathematics*. Presented at the 2014 Spring Research Conference, Cincinnati, OH.

Watkins, J. D. (2014, April). *Investigating the relationships between homework and achievement in mathematics with hierarchical linear modeling*. Presented at the 6th annual Graduate Research Symposium, University of Louisville, Louisville, KY.

Peters, S. A., & **Watkins, J. D.** (2014, February). *Activities to facilitate middle and secondary mathematics teachers' transformative learning of statistics within professional development.* Presented at the 18th annual Association of Mathematics Teacher Educators Conference, Irvine, CA. (Refereed)

Peters, S. A., & **Watkins, J. D.** (2013, November). *Standard(s) statistics: Engaging with CCSS Mathematical Practices and statistical content.* Presented at the National Council of Teachers of Mathematics Regional Conference and Exposition, Louisville, KY. (Refereed)

Watkins, J. D. (2013, April). *Investigating the relationships between gender and mathematics-related constructs with MANOVA and multiple-groups CFA.* Presented at the ECPY (Educational and Counseling Psychology) Fun Friday Lecture Series, University of Louisville, Louisville, KY.

Watkins, J. D. (2013, April). *Investigating the relationships between gender and mathematics-related constructs with MANOVA and multiple-groups CFA.* Presented at the 2013 Spring Research Conference, University of Kentucky, Lexington, KY.

Watkins, J. D. (2013, March). *An examination of gender differences in competency and extrinsic-value beliefs about mathematics, and enjoyment of mathematics in U.S. eighth-grade students*. Presented at the 5th annual Graduate Research Symposium, University of Louisville, Louisville, KY.

Watkins, J. D. (2013, February). *Functions: Moving beyond the vertical line test*. Presented at the 2013 Kentucky Center for Mathematics Conference, Lexington, KY.

GRANTS

Graduate Student Council Travel Award ($350)
University of Louisville, Louisville, KY 2016
Conference travel support grant to present at 2016 American Educational Research Association Annual Meeting and Exhibition, Washington, DC

College of Education and Human Development Research and Faculty Development Travel Grant ($100)
University of Louisville, Louisville, KY 2016
Conference travel support grant to present at 2016 American Educational Research Association Annual Meeting and Exhibition, Washington, DC

Graduate Student Council Travel Award ($350)
University of Louisville, Louisville, KY 2015
Conference travel support grant to present at 2015 National Council of Teachers of Mathematics Annual Meeting and Exposition, Boston, MA

College of Education and Human Development Research and Faculty Development Travel Grant ($100)
University of Louisville, Louisville, KY 2015
Conference travel support grant to present at 2015 National Council of Teachers of Mathematics Annual Meeting and Exposition, Boston, MA

Graduate Student Council Travel Award ($350)
University of Louisville, Louisville, KY 2014
Conference travel support grant to present at the 18th annual Association of Mathematics Teacher Educators Conference, Irvine, CA

PROFESSIONAL SERVICE

Contractor, College of Education and Human Development (CEHD)
University of Louisville, Louisville, KY 2016
Completed alignment work: MET/PRAXIS and UofL mathematics courses

Consultant, Hawkes Learning
Charleston, SC 2016
Reviewed upcoming mathematics for elementary teachers textbook

Assessment Reader, General Education Curriculum Committee
University of Louisville, Louisville, KY 2016
Scored open-ended mathematics items using grading rubrics

Reviewer, National Council of Teachers of Mathematics
Reston, VA 2015
Reviewed proposals for NCTM Research Conference

Contractor, Human Resources Research Organization (HumRRO)
Louisville, KY 2014
Completed alignment work: NAEP and ACT WorkKeys

Reviewer, *The Journal of Educational Research*
Taylor & Francis, London, England 2013
Reviewed submissions to the journal

Member, CEHD Conceptual Framework Committee
University of Louisville, Louisville, KY 2012-2013
Updated conceptual framework for the CEHD

Contractor, HumRRO
Louisville, KY 2012
Completed alignment work: Qatar and U.S. Mathematics Standards

Technology Specialist/Committee Member
2012 Spring Research Conference, CEHD
University of Louisville, Louisville, KY 2011-2012
Assisted with conference planning and logistics

PROFESSIONAL MEMBERSHIPS

NATIONAL
American Educational Research Association (AERA)
Association of Mathematics Teacher Educators (AMTE)
National Council of Teachers of Mathematics (NCTM)
Special Interest Group – Research in Mathematics Education (SIG-RME)

STATE AND LOCAL
Greater Louisville Council of Teachers of Mathematics (GLCTM)
Kentucky Council of Teachers of Mathematics (KCTM)

OTHER WORK EXPERIENCES

Legislative Assistant, House of Representatives
Kentucky General Assembly, Frankfort, KY 2007-2009

Activities Director, Commonwealth Honors Academy
Murray State University, Murray, KY 2006-2007

Residential Counselor, Commonwealth Honors Academy
Murray State University, Murray, KY 2001-2002

OTHER NOTABLE EXPERIENCES

Volunteer, Cochran Elementary School
Jefferson County Public Schools, Louisville, KY 2012
Worked with elementary teachers and students during mathematics instruction

Volunteer, Lead2Read, South Heights Elementary
Henderson County Schools, Henderson, KY 2007-2008
Helped at-risk elementary students build confidence in their reading ability

Peer-Led Team Learning (PLTL) Supervising Teacher
Murray High School, Murray, KY 2006-2007
Collaborated with a Murray State mathematics professor to bring math majors into the high-school classroom to assist with small-group learning tasks

HONORS AND AWARDS

Faculty Favorite Award
University of Louisville, Louisville, KY 2017-2018

Finalist, 2017 Learning Impact Award
IMS Global Learning Consortium 2017
Award recognizes outstanding achievement by an institution (University of Louisville) and partnering organization (Hawkes Learning)

Research Presentation Award (3rd Place Overall)
Graduate Research Symposium
University of Louisville, Louisville, KY 2013

Outstanding Male Graduate, Class of 2003
Murray State University, Murray, KY 2003

Dr. Frank Julian Outstanding Residential College Leader
Murray State University, Murray, KY 2002

Residential College Council Member of the Year
Murray State University, Murray, KY 2002