8-2019

Characterizing majority rule on various discrete models of consensus.

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CHARACTERIZING MAJORITY RULE ON VARIOUS DISCRETE MODELS OF CONSENSUS

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A Dissertation
Submitted to the Faculty of the
College of Arts and Sciences of the University of Louisville
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy
in
Applied and Industrial Mathematics

Department of Mathematics
University of Louisville
Louisville, Kentucky

August 2019

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A Dissertation Approved on

May 23, 2019

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DEDICATION

In memory of my loving mother Janie Martin.

ACKNOWLEDGEMENTS

First, I would like to acknowledge and thank my advisor Dr. Robert Powers for supporting me in all aspects of my academic and professional life. Dr. Powers is the most patient person I have ever met. He truly makes a difference in math education. I have never met a mathematician who is as clear as he is. When preparing my own lectures, I try my best to model and imitate these qualities to the best of my abilities.

Second, I would like to acknowledge and thank the members of the academic community who have supported me in the pursuit of education and professional goals. In addition to my committee, Dr. McMorris, Dr. Wildstom, and Dr. Gainous; I would also like to specifically acknowledge and thank Dr. Thomas Riedel, Dr. Erika Bratcher, and Dr. Ryan Luke.

Lastly, I would like to acknowledge and thank my friends and family for the support I have received from each of you. Without this support I would have never made it to where I am. Above all else I thank my best friend and partner in life Brandon Thompson. Without his unwavering support and encouragement, none of this would have been possible.

ABSTRACT

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In any social structure, there is often a need to reach decisions, not only within a group but between groups as well, sometimes even urgently so. Each of the individuals constituting these groups has their own preference for the decision to be made. We will discuss the problem of aggregating individual preferences into a collective preference and under what conditions we are required to select a collective majority. In this dissertation we will look at three models of consensus and show the conditions vary based on the model.

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CHAPTER 1
INTRODUCTION

Why does majority rule need to be characterized? The goal of this thesis is to provide an answer to this question. The focus will be on axiomatic or fairness properties that a given voting rule may or may not satisfy.

The 2016 presidential election certainly left many American voters puzzled when the winner had nearly 3 million fewer votes than the runner up. Certainly the choice of method to choose an a alternative left many of the electorate feeling as if the method violated some fairness criterion. In this light, we should choose the method of selecting alternatives based on how 'fair' the method is. However, this asks the question: "How fair can a method of choosing an alternative truly be?".

To answer this question, consider the framework that we require every person in the electorate to completely order the alternatives in terms of preference and the rule selects a completely ordered ranking of the alternatives. A nice fairness condition we consider in this framework is the condition of **weak pareto** which states that if every member of the electorate prefers an alternative x to an alternative y, then the rule should select a ranking of alternatives in which x is preferred to y. Another condition is **Independence of Irrelevant Alternatives**, which states that if we're trying to figure out whether a society prefers an alternative x to y, what people think of the alternative z shouldn't matter.

Both of these conditions seem very reasonable conditions for our rule to satisfy, however according to Arrow's Theorem [3], the only rule satisfying these conditions is a dictatorship. Certainly this seems considerably more 'un-fair' than the situation of the 2016 presidential election. This is just one of many theorems in the realm of mathematical social choice that demonstrates that there is no unanimously 'fair' voting rule to select a candidate from a list of alternatives.

To discuss a rule's virtues and vices amounts to discussing its defining characteristics, that is to say what properties characterizes a rule. For example, in 1952 Kenneth May [12] characterized simple majority rule in terms of anonymity, neutrality, and positive responsiveness. This implies that not only does simple majority rule satisfy these three conditions, but that it is the only such rule satisfying these conditions.

To this day, May's Theorem is a well cited result in the social choice literature. Although the domain in his theorem is very 'simple', it paved the way for many other characterization of majority rule on non-simple domains. However, with larger more complex domains, we need more axioms to accomplish the same goals of providing an axiomatic characterization of majority rule.

Young [16] proved that Borda's rule is the only social choice function, defined on profiles of linear orders, satisfying neutrality, consistency, faithfulness, and cancellation. Fishburn [8] extended Young's theorem to the case where a social choice function takes as input a ballot response profile and outputs a nonempty subset of winning alternatives. Each voter submits a nonempty subset of alternatives called a ballot and the set of all admissible ballots is called the ballot space. A voter's ballot consists of all approved alternatives. Fishburn proved that majority rule is the only social choice function satisfying neutrality, consistency, faithfulness, and cancellation. Alós-Ferrer [2] showed that the axiom of neutrality was not needed for Fishburn's theorem. Moreover, he was able to give a much simpler argument than Fishburn's original proof. This simplicity came at a price. Namely, Alós-Ferrer assumed that the ballot space is the set of all proper subsets of the set of alternatives.

Following further down the path pioneered by May, in this dissertation we will be characterizing majority rule on various discrete models of consensus. It will be our goal to start with the most concrete model and work our way towards the more general models. To do so we will be primarily working with the fairness conditions of faithfulness, consistency, cancellation, neutrality, and non-deviating.

A consensus rule is **faithful** if the society of voters is made up of 1 voter and that voter casts the ballot B then the rule should select that ballot as an output. **Consistency** says that if an alternative x is an acceptable social outcome by two disjoint groups of voters, then x should be an acceptable social outcome for the union of the two groups. Moreover, if another alternative y is not an acceptable outcome for one of the groups, then y should not be part of the social outcome for the union of the two groups. If all alternatives get the same number of votes, then **cancellation** implies that every alternative should belong to the social output. **Neutrality** implies that the labeling of the alternatives does not affect the social outcome. Finally the condition of **non-deviating** implies that if every alternative receives the same approval in two different voting situations, then the rule should select the same set of alternatives for each situation.

The first and least general mode of consensus that we will work with will be the framework of Alós-Ferrer in which a member of an electorate can approve of any set of alternatives. We will first provide an alternative proof to that of Alós-Ferrer and then provide a more general result by characterizing majority rule with the axioms of neutrality, consistency, faithfulness, and cancellation. Later we provide arguments to show the necessity of each axiom in the characterization.

In Chapter 3 we work on a more general model in which members of an electorate are more restricted in their approval and must choose a ballot of candidates they most prefer. We begin by working with the framework introduced by Fishburn in which the ballots are restricted by size. We later introduce a new model that is even more general. In this model we will present some of the main theorems of this dissertation. In this chapter we will work with the axioms of faithfulness, consistency and cancellation to show for which domains these three axioms uniquely characterize majority rule.

We conclude the results of this dissertation in Chapter 4 by generalizing some of the main results of this dissertation to the latticial[1] framework introduced by Monjardet [13]. We then finish the dissertation with some concluding remarks followed by a discussion of the future of the work discussed in this dissertation.

[1] Latticial refers to a model of social choice based on lattices and semilattices. [13]

CHAPTER 2
COLLECTIVE APPROVAL RULES

Collective approval rules are well studied rules in mathematical social choice. The most well known is the Approval Voting rule. In this chapter we will be considering social choice functions where a "voter" may cast any ballot consisting of whichever candidates (or alternatives) they approve of and the social outcome for these rules will be a nonempty collection of candidates.

2.1 NOTATION AND TERMINOLOGY

The finite set of alternatives is $X = \{x_1, \ldots, x_m\}$ with $m \geq 2$. The set of all subsets of X is denoted by $\mathbb{P}(X)$ and the sets belonging to $\mathbb{P}(X)$ are called **ballots**. The set of natural numbers including 0 is denoted by $\mathbb{N}_0$. A function $\pi : \mathbb{P}(X) \to \mathbb{N}_0$ is called a **ballot response profile** or just simply a **profile** with the interpretation that $\pi(B)$ represents the number of voters that chose the ballot B. The set of all profiles on $\mathbb{P}(X)$ is given by ${\mathbb{N}_0}^{\mathbb{P}(X)}$. Any function of the form

$$f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$$

is called a **collective approval rule** or just simply a **rule**, where

$$P_{ne}(X) = \{A \in \mathbb{P}(X) : A \neq \emptyset\}.$$

A simple example of a collective approval rule that we will be studying is the rule that outputs the entire set of alternatives for every profile. Alós-Ferrer [2] points out that such a rule is implicitly **anonymous**, meaning the names of the voters cannot influence the result.

For any profile $\pi \in {\mathbb{N}_0}^{\mathbb{P}(X)}$ and for any alternative $x \in X$, the number of voters who approve of the alternative x is given by

$$v(x,\pi) = \sum_{B \in \mathbb{P}(X)} \pi(B)\chi_x(B)$$

where

$$\chi_x(B) = \begin{cases} 1 & \text{if } x \in B \\ 0 & \text{otherwise.} \end{cases}$$

The maximum and minimum approval values based on a profile π are

$$\max\, v(\pi) = \max\{v(x,\pi) : x \in X\}$$

and

$$\min\, v(\pi) = \min\{v(x,\pi) : x \in X\}.$$

The **approval voting rule** is the rule $F_A : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ defined as follows: for any profile π,

$$F_A(\pi) = \{x \in X : v(x,\pi) = \max\, v(\pi)\}.$$

Notice that $x \in F_A(\pi)$ means that there is no alternative y that obtained more votes than x. Similarly, the **inverse approval voting rule** is the rule $F_{A^{-1}} : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ defined as follows: for any profile π,

$$F_{A^{-1}}(\pi) = \{x \in X : v(x,\pi) = \min\, v(\pi)\}.$$

Notice that $x \in F_{A^{-1}}(\pi)$ means that there is no alternative y that obtained less votes than x.

For profiles π, π' we define the profile $\pi + \pi'$ by

$$(\pi + \pi')(B) = \pi(B) + \pi'(B)$$

for all $B \in \mathbb{P}(X)$.

Lemma 2.1. *For any profiles π, π' and for every alternative $x \in X$,*

$$v(x, \pi + \pi') = v(x, \pi) + v(x, \pi').$$

Proof. To show this, let $\pi, \pi' \in {\mathbb{N}_0}^{\mathbb{P}(X)}$ be any arbitrary profiles, and let $x \in X$. Then

$$\begin{aligned} v(x, \pi + \pi') &= \sum_{B \in \mathbb{P}(X)} [\pi + \pi'](B)\chi_x(B) \\ &= \sum_{B \in \mathbb{P}(X)} [\pi(B) + \pi'(B)]\chi_x(B) \\ &= \sum_{B \in \mathbb{P}(X)} \pi(B)\chi_x(B) + \sum_{B \in \mathbb{P}(X)} \pi'(B)\chi_x(B) \\ &= v(x, \pi) + v(x, \pi'). \end{aligned}$$

Hence $v(x, \pi + \pi') = v(x, \pi) + v(x, \pi')$ as was desired. $\square$

A rule f is said to satisfy **consistency** (or f is said to be **consistent**) if for any profiles $\pi, \pi' \in {\mathbb{N}_0}^{\mathbb{P}(X)}$,

$$f(\pi) \cap f(\pi') \neq \emptyset \ \Rightarrow \ f(\pi + \pi') = f(\pi) \cap f(\pi').$$

Consistency implies that if an alternative x is an acceptable social outcome by two disjoint groups of voters, then x should be an acceptable social outcome for the union of the two groups. Moreover, if another alternative y is not an acceptable outcome for one of the groups, then y should not be part of the social outcome for the union of the two groups. We now show that the approval voting rule satisfies consistency.

Proposition 2.2. *The approval voting rule satisfies the condition of consistency.*

Proof. Let π and π' be profiles such that $F_A(\pi) \cap F_A(\pi') \neq \emptyset$. To show

$$F_A(\pi) \cap F_A(\pi') \subseteq F_A(\pi + \pi'),$$

let $x \in F_A(\pi) \cap F_A(\pi')$. Thus $v(x, \pi) \geq v(y, \pi)$ and $v(x, \pi') \geq v(y, \pi')$ for all $y \in X$. Observe that:

$$\begin{aligned} v(x, \pi + \pi') &= v(x, \pi) + v(x, \pi') \\ &\geq v(y, \pi) + v(y, \pi') \; \forall \; y \in X \\ &= v(y, \pi + \pi'), \end{aligned}$$

thus $x \in F_A(\pi + \pi')$. Now to show

$$F_A(\pi) \cap F_A(\pi') \supseteq F_A(\pi + \pi'),$$

let $z \in F_A(\pi + \pi')$. Thus $v(z, \pi + \pi') = v(z, \pi) + v(z, \pi')$. If $z \in F_A(\pi) \cap F_A(\pi')$ we are done. Then we may assume that $z \notin F_A(\pi)$. Choose $y \in F_A(\pi) \cap F_A(\pi')$ such that $v(z, \pi) < v(y, \pi)$ and so, $v(y, \pi) - v(z, \pi) > 0$. Given $v(z, \pi + \pi') \geq v(y, \pi + \pi')$ for all $y \in X$ it follows that:

$$\begin{aligned} v(z, \pi) + v(z, \pi') &\geq v(y, \pi) + v(y, \pi') \\ v(z, \pi') &\geq (v(y, \pi) - v(z, \pi)) + v(y, \pi') \\ v(z, \pi') &> v(y, \pi') \;\; \text{since } v(y, \pi) - v(z, \pi) > 0 \end{aligned}$$

contrary to $y \in F_A(\pi')$. Hence it follows that $F_A(\pi) \cap F_A(\pi') \supseteq F_A(\pi + \pi')$. Thus $F_A(\pi) \cap F_A(\pi') = F_A(\pi + \pi')$ and hence the approval voting rule satisfies consistency.

□

Example of rules we can consider that are not consistent is the class of mean based rules studied by Duddy and Piggins [6]. For any profile π, the mean approval of π is

$$\bar{v}(\pi) = \text{mean } v(\pi) = \sum_{x \in X} \frac{v(x, \pi)}{|X|}.$$

An example of a mean based rule is $F_{\text{mean}} : {\mathbb{N}_0}^{\mathbb{P}(X)} \rightarrow P_{ne}(X)$ defined by: for all profiles π

$$F_{\text{mean}}(\pi) = \{x \in X : v(x, \pi) \geq \bar{v}(\pi)\}.$$

To see that the rule F_{mean} is not consistent, consider the following example:

Example 2.1. *The set of alternatives is* $X = \{x_1, x_2, x_3\}$. π *is a profile where two voters approve of the alternative* x_1*, four voters approve of the alternative* x_2*, and three voters approve of the alternative* x_3. π' *is a profile where four voters approve of the alternative* x_1*, two voters approve of the alternative* x_2*, and three voters approve of the alternative* x_3*. The mean approvals of* π *and* π' *are* $\bar{v}(\pi) = 3$ *and* $\bar{v}(\pi') = 3$*. Thus* $F_{mean}(\pi) = \{x_2, x_3\}$ *and* $F_{mean}(\pi') = \{x_1, x_3\}$*. Since both outputs have a commonly selected alternative, consistency would imply that* $F_{mean}(\pi + \pi') = \{x_3\}$*. Consider the profile* $\pi + \pi'$*. In this profile there are eighteen voters approving of each of the three alternatives. Hence* $\bar{v}(\pi') = 6$ *and thus* $F_{mean}(\pi + \pi') = \{x_1, x_2, x_3\}$*. It follows that* F_{mean} *is not a consistent rule.*

For any ballot $B \in \mathbb{P}(X)$, the profile where one voter chooses B is denoted by π_B, so $\pi_B(B) = 1$ and $\pi_B(B') = 0$ for all $B' \neq B$. A rule f satisfies **faithfulness** (or f is said to be **faithful**) if, for all nonempty ballots B,

$$f(\pi_B) = B.$$

If there is just one voter and that voter submits the ballot B which is not the empty set, then faithfulness implies that the social outcome should be B. This is a very natural assumption we would like a rule to satisfy since if there is only one ballot to aggregate into a nonempty subset of alternatives, then it should simply be that ballot. An example of a rule that is not faithful would be the constant function which selects the entire set of alternatives for every possible profile.

Proposition 2.3. *The approval voting rule satisfies the condition of faithfulness.*

Proof. To show that the approval voting rule satisfies faithfulness, let $B \in \mathbb{P}(X)$ which is nonempty. Consider π_B. Since $\max v(\pi) = 1$, and $v(x, \pi) = 1$ if and only if $x \in B$, it follows that $F_A(\pi_B) = B$ and hence the approval voting rule satisfies faithfulness. $\square$

A rule f satisfies **cancellation** (or f is said to be **cancellative**) if, for any $\pi \in {\mathbb{N}_0}^{\mathbb{P}(X)}$,

$$v(x,\pi) = v(y,\pi) \text{ for all } x, y \in X \;\Rightarrow\; f(\pi) = X.$$

That is to say, if all alternatives get the same number of votes, then cancellation implies that every alternative should belong to the social output.

Proposition 2.4. *The approval voting rule satisfies the condition of cancellation.*

Proof. We will consider a profile π such that $v(x,\pi) = v(y,\pi)$ for all $x, y \in X$. Observe that since each of the alternatives received the same number of approvals, $v(x,\pi) = \max v(\pi)$ for every alternative $x \in X$. By the definition of approval voting it follows that

$$F_A(\pi) = X$$

and hence approval voting satisfies the condition of cancellation. $\square$

To understand the role of cancellation in characterizing approval voting, we consider a refinement of approval voting. Define $F_w : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ by

$$F_w(\pi) = \{x \in F_A(\pi) : \; w(x,\pi) \leq w(y,\pi) \;\forall\; y \in F_A(\pi)\}$$

where

$$w(x,\pi) = \sum_{B \in \mathbb{P}(X)} |B| \; \pi(B) \chi_x(B).$$

Notice that $F_w(\pi) \subseteq F_A(\pi)$ for any profile π, hence we say F_w is a refinement of approval voting. If $X = \{x_1, x_2, x_3\}$ and $\pi = \pi_{\{x_1\}} + \pi_{\{x_2,x_3\}}$, then $F_A(\pi) = X$. Notice that $w(x_1,\pi) = 1$, and $w(x_2,\pi) = w(x_3,\pi) = 2$. Thus $F_w(\pi) = \{x_1\}$. Thus the rule F_w is not cancellative.

A **permutation** on X is a bijective function $\sigma : X \to X$. For each $B \in \mathbb{P}(X)$ we use the notation $\sigma[B]$ to denote the image of B under the permutation σ. For example if $B = \{x_1, x_2\}$, and $\sigma = (x_2 \; x_3)$ then

$$\sigma[B] = \sigma[\{x_1, x_2\}] = \{\sigma[x_1], \sigma[x_2]\} = \{x_1, x_3\}.$$

A rule f satisfies **neutrality** (or f is said to be **neutral**) if, for any profiles π and π' and for any permutation σ of X,

$$\pi'(\sigma[B]) = \pi(B) \text{ for all } B \in \mathbb{P}(X) \;\Rightarrow\; \sigma\left[f(\pi)\right] = f(\pi').$$

In this case we denote the profile π' as $\sigma[\pi]$. Hence f satisfies neutrality if and only if $f\left(\sigma[\pi]\right) = \sigma[f(\pi)]$ for all profiles π and for all permutations σ of X. Neutrality implies that the labeling of the alternatives does not affect the social outcome.

Before we can show that approval voting is neutral, we first introduce a lemma.

Lemma 2.5. *If π and π' are two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$ then $v(x, \pi) = v(\sigma[x], \pi')$ for all $x \in X$.*

Proof. Let π and π' be two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. For any $x \in X$ notice that

$$\begin{aligned} v(x,\pi) &= \sum_{B\in\mathbb{P}(X)} \pi(B)\chi_x(B) \\ &= \sum_{B\in\mathbb{P}(X)} \pi'(\sigma[B])\chi_x(B) \\ &= \sum_{\sigma[B]\in\mathbb{P}(X)} \pi'(\sigma[B])\chi_{\sigma[x]}(\sigma[B]) \\ &= v(\sigma[x],\pi'). \end{aligned}$$

Hence $v(x, \pi) = v(\sigma[x], \pi')$ for all $x \in X$. $\square$

Proposition 2.6. *The approval voting rule satisfies the condition of neutrality.*

Proof. Let π and π' be two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. Then by Lemma 2.5 it follows that $v(x, \pi) = v(\sigma[x], \pi')$ for all $x \in X$. Now consider that:

$$\begin{aligned} x \in F_A(\pi) &\iff v(x,\pi) \geq v(y,\pi) \; \forall \; y \in X \\ &\iff v(\sigma[x],\pi') \geq v(\sigma[y],\pi') \; \forall y \in X \text{ by Lemma 2.5} \\ &\iff \sigma(x) \in F_A(\pi') \end{aligned}$$

Thus the approval voting rule satisfies neutrality. □

To consider a rule which is not neutral, we give a different refinement of approval voting. Consider the rule

$$min(F_A) : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$$

defined by: for all profiles π

$$min(F_A)(\pi) = \min \; F_A(\pi)$$

where

$$\min \; F_A(\pi) = \{x_i : x_i \in F_A(\pi) \text{ and } i < j \; \forall \; x_j \in F_A(\pi) \setminus \{x_i\}\}.$$

Notice that min $F_A(\pi)$ is the unique element belong to the approval voting output having minimum index. Consider the profile $\pi_{\{x_1,x_2\}}$, $min(F_A)\left(\pi_{\{x_1,x_2\}}\right) = \{x_1\}$. Now observe that given the permutation $\sigma = (x_1 \; x_2)$, we have $\sigma\left[\pi_{\{x_1,x_2\}}\right] = \pi_{\{x_1,x_2\}}$. But

$$x_2 = \sigma(x_1) \notin f\left(\sigma\left[\pi_{\{x_1,x_2\}}\right]\right) = f\left(\pi_{\{x_1,x_2\}}\right) = \{x_1\}$$

and thus the rule $min(F_A)$ is not neutral.

The last well studied condition we will discuss is the condition of non-deviating. A rule f is said to satisfy the condition of **non-deviating** (or f is said to be **non-deviating**) if π and π' are profiles such that $v(x,\pi) = v(x,\pi')$ for every alternative $x \in X$ then $f(\pi) = f(\pi')$. If every alternative receives the same approval in two separate societies then the rule should select the same social outcome for each of the societies.

Proposition 2.7. *The approval voting rule satisfies the condition of non-deviating.*

Proof. To show that approval voting satisfies the condition of non-deviating, let π and π' be two profiles such that $v(x, \pi) = v(x, \pi')$ for all $x \in X$. Then max $v(\pi) =$ max $v(\pi')$ and $F_A(\pi) = F_A(\pi')$ by the definition of approval voting. Thus approval voting rule satisfies the condition of non-deviating. □

The conditions of consistency, faithfulness, cancellation, neutrality and non-deviating were introduce in the 1970s and dealt with by authors such as Smith [15], Young [16], Fine and Fine [7], Fishburn [9], and Xu [11]. Finally, we will consider a less studied condition. Introduced by Duddy and Piggins [6], the condition of discerning is relatively new to the social choice literature.

A rule f is said to be **discerning** if for all profiles π and all $x \in X$,

$i)$ If $v(x, \pi) > v(y, \pi)$ for all $y \in X \setminus \{x\}$ then $x \in f(\pi)$.

$ii)$ If $v(x, \pi) < v(y, \pi)$ for all $y \in X \setminus \{x\}$ then $x \notin f(\pi)$.

The first condition implies that if there is an alternative which is approved of more than any other alternative, then that alternative should be included in the social outcome. The second condition implies that if there is an alternative which is approved of less than any other alternative, then that alternative should not be included in the social outcome.

Proposition 2.8. *The approval voting rule is discerning.*

Proof. To show that approval voting is discerning, consider a profile π such that $v(x, \pi) > v(y, \pi)$ for all $y \in X \setminus \{x\}$. Thus $v(x, \pi) =$ max $v(\pi)$, hence it follows by definition of approval voting that $x \in F_A(\pi)$.

Now consider a profile π such that $v(x, \pi) < v(y, \pi)$ for all $y \in X \setminus \{x\}$. Thus $v(x, \pi) \neq$ max $v(\pi)$, hence it follows by definition of F_A that $x \notin F_A(\pi)$. Hence the approval voting rule is discerning. □

For any profile π, let

$$Null(\pi) = \{x \in X \ : \ v(x, \pi) = 0\}.$$

A profile π is said to be **trivial** if $Null(\pi) = X$. Examples of such profiles include the profile generated by no voters and profiles generated by voters only approving of the empty set of alternatives. We first consider the social output of these profiles.

Lemma 2.9. *If ρ is a trivial profile and a rule $f : \mathbb{N}_0{}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies faithfulness, consistency, and is non-deviating then $f(\rho) = X$.*

Proof. By way of contradiction we will suppose that $f(\rho) \neq X$. Since $f(\rho) \neq \emptyset$, let $x \in f(\rho)$ and $y \notin f(\rho)$. Since f is faithful, $f\left(\pi_{\{x,y\}}\right) = \{x, y\}$. Observe that since f is non-deviating,

$$f\left(\rho + \pi_{\{x,y\}}\right) = f\left(\pi_{\{x,y\}}\right) = \{x, y\}.$$

But by consistency we have that,

$$f\left(\rho + \pi_{\{x,y\}}\right) = f(\rho) \cap f\left(\pi_{\{x,y\}}\right) = \{x\}$$

a contradiction. Thus we have that $f(\rho) = X$ as was desired. □

Now we can state our first characterization of approval voting.

Theorem 2.10. *A rule $f : \mathbb{N}_0{}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies faithfulness, consistency, and is non-deviating if and only if f is the approval voting rule.*

Proof. We have shown that the approval voting rule satisfies faithfulness, consistency, and the non-deviating condition. It is left to show that approval voting is the only rule satisfying these conditions. So let $f : \mathbb{N}_0{}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfy faithfulness, consistency, and the condition of non-deviating. We will show $f(\pi) = F_A(\pi)$ for all possible profiles $\pi \in \mathbb{N}_0{}^{\mathbb{P}(X)}$.

First let $\pi \in {\mathbb{N}_0}^{\mathbb{P}(X)}$ be a trivial profile. By Lemma 2.9, we have that $f(\pi) = X$ which coincides with approval voting. Now let $\pi \in {\mathbb{N}_0}^{\mathbb{P}(X)}$ be any fixed nontrivial profile. Since π is nontrivial, the integer $\ell = \max v(\pi)$ is strictly greater than 0. For each integer j in the interval $[1, \ell]$ let

$$B_j = \{x \in X : j \leq v(x, \pi)\}.$$

Observe that

$$B_1 = X \setminus Null(\pi) \text{ and } B_\ell = F_A(\pi).$$

Also notice that

$$B_{j_1} \subseteq B_{j_2} \text{ if } j_1 \geq j_2.$$

Now construct a new profile π^* as follows:

$$\pi^* = \pi_{B_1} + \pi_{B_2} + \cdots + \pi_{B_l} \Rightarrow f(\pi^*) = f\left(\pi_{B_1} + \pi_{B_2} + \cdots + \pi_{B_\ell}\right)$$

Since f is faithful it follows that $f\left(\pi_{B_j}\right) = B_j$ for $j = 1, \cdots, \ell$. Therefore,

$$\bigcap_{j=1}^{l} f\left(\pi_{B_j}\right) = \bigcap_{j=1}^{l} B_j = B_\ell \neq \emptyset$$

Since f satisfies consistency we get

$$f(\pi^*) = \bigcap_{j=1}^{l} f\left(\pi_{B_j}\right) = B_\ell.$$

Thus $f(\pi^*) = F_A(\pi)$.

For each $x \in X \setminus Null(\pi)$, $x \in B_j$ if and only if $j = 1, 2, \ldots, v(x, \pi)$ and so $v(x, \pi^*) = v(x, \pi)$. Since f is non-deviating, we have that $f(\pi^*) = f(\pi)$. Hence $f(\pi) = F_A(\pi)$ and we're done. $\square$

Now we will consider the role of cancellation in characterizing approval voting.

Lemma 2.11. *If $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies consistency and cancellation, then f is non-deviating.*

Proof. Let $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ be a rule satisfying consistency and cancellation. Let π and π' be profiles such that $v(x, \pi) = v(x, \pi')$ for all $x \in X$. For each alternative $x_i \in X$, let

$$\ell_i = \max\, v(\pi) - v(x_i, \pi) \geq 0.$$

Construct the profile ρ as follows,

$$\rho = \sum_{x_i \in X} \ell_i \cdot \pi_{\{x_i\}}.$$

Now consider that for the profile $\pi + \rho$, we have that for each $x \in X$,

$$\begin{aligned} v(x, \pi + \rho) &= v(x, \pi) + v(x, \rho) \\ &= v(x, \pi) + \max\, v(\pi) - v(x, \pi) \\ &= \max\, v(\pi). \end{aligned}$$

Thus it follows by cancellation that $f(\pi + \rho) = X$. Now consider the profile $\pi' + \rho$. Since $v(x, \pi) = v(x, \pi')$ for all $x \in X$, we have that

$$\begin{aligned} v(x, \pi' + \rho) &= v(x, \pi') + v(x, \rho) \\ &= v(x, \pi) + \max\, v(\pi) - v(x, \pi) \\ &= \max\, v(\pi). \end{aligned}$$

Thus it follows by cancellation that $f(\pi' + \rho) = X$.

Now notice by consistency we have that

$$f(\pi + \pi' + \rho) = f(\pi) \cap X = f(\pi)$$

since $f(\pi' + \rho) = X$. Similarly since $f(\pi + \rho) = X$ we have

$$f(\pi' + \pi + \rho) = f(\pi') \cap X = f(\pi').$$

Since the operation of profile addition is commutative, we have that

$$f(\pi) = f(\pi + \pi' + \rho) = f(\pi' + \pi + \rho) = f(\pi')$$

and thus f is non-deviating. $\square$

We now state the characterization of approval voting given by Alós-Ferrer [2] and observe that it is a consequence of Theorem 2.10 and Lemma 2.11.

Theorem 2.12 (Alós-Ferrer 2006)**.** *A rule $f : \mathbb{N}_0^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies faithfulness, consistency, and cancellation if and only if f is the approval voting rule.*

Our goal now is to provide a new characterization of approval voting using the condition of discerning. As a first step toward this goal we need the following lemma.

Lemma 2.13. *If $f : \mathbb{N}_0^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies neutrality, consistency, and is discerning, then for any non-trivial profile π,*

$$Null(\pi) \cap f(\pi) = \emptyset.$$

Proof. By way of contradiction, we will suppose there exists $y \in Null(\pi) \cap f(\pi)$. If $|Null(\pi)| = 1$ then we have a contradiction to discerning, so we may assume $|Null(\pi)| \geq 2$. Let $\sigma = (y\ x)$ for $x \in Null(\pi) \setminus \{y\}$, and define a new profile π' as follows; $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. Notice that $\pi = \pi'$ and so by neutrality;

$$f(\pi) = f(\pi') = \sigma[f(\pi)].$$

Since x was chosen arbitrarily, it follows that $Null(\pi) \subseteq f(\pi)$. Suppose $Null(\pi) = \{y, x_1, x_2, \ldots, x_j\}$. Since π is nontrivial there exists $z \in X \setminus Null(\pi)$. Define $\sigma_i = (z\ x_i)$ for $x_i \in \{x_1, \ldots, x_j\} = Null(\pi) \setminus \{y\}$. For each σ_i define a new profile π_i as follows: $\pi_i(\sigma_i[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. By neutrality, $y = \sigma_i(y) \in f(\pi_i)$ for each i. Hence it follows by consistency that

$$y \in f(\pi) \cap \left(\bigcap_{i=1}^{j} f(\pi_i) \right) = f(\pi + \pi_1 + \pi_2 + \cdots + \pi_j)$$

Now consider that:

$$v(t, \pi + \pi_1 + \cdots + \pi_j) = \begin{cases} 0 & \text{if } t = y \\ v(z, \pi) & \text{if } t \in \{z\} \cup Null(\pi) \setminus \{y\} \\ (j+1) \cdot v(t, \pi) & \text{otherwise.} \end{cases}$$

But $v(y, \pi + \pi_1 + \cdots + \pi_j) < v(x, \pi + \pi_1 + \cdots + \pi_j)$ for all $x \in X \setminus \{y\}$. Hence by discerning $y \notin f(\pi + \pi_1 + \pi_2 + \cdots + \pi_j)$ a contradiction, and so the assumption of $y \in Null(\pi) \cap f(\pi)$ is false. Hence

$$Null(\pi) \cap f(\pi) = \emptyset.$$

as was desired. □

When considering a profile such as π_B we now know that, under the conditions in the previous lemma, $f(\pi_B) \subseteq B$. Now we will show when $f(\pi_B) = B$.

Lemma 2.14. *If $f : \mathbb{N}_0^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies neutrality, consistency, and is discerning then f is faithful.*

Proof. Let $B \in P_{ne}(X)$. By Lemma 2.13 we have that $f(\pi_B) \subseteq B$; in addition we have $f(\pi_B) \neq \emptyset$. First we will consider that $|B| = 1$. We may assume that $B = \{x\}$ for some $x \in X$. It follows from above we have that $f(\pi_{\{x\}}) = \{x\}$ and we are done.

We now consider that $|B| > 1$. Let $x \in B$ such that $x \in f(\pi_B)$. Consider the permutation $\sigma = (x\ y)$ for any $y \in B \setminus \{x\}$. Let π' be the profile defined by $\pi'(\sigma[C]) = \pi_B(C)$ for all $C \in \mathbb{P}(X)$. Hence $\pi_B = \pi'$ and thus $f(\pi_B) = f(\pi')$. By neutrality we have that;

$$\sigma[f(\pi_B)] = f(\pi') = f(\pi_B).$$

Thus it follows that $y = \sigma(x) \in f(\pi_B)$. Since y was chosen arbitrarily, it follows that $f(\pi_B) = B$. □

We now state our final characterization of approval voting as a consequence of Theorem 2.12 and Lemma 2.14.

Theorem 2.15. *A rule* $f : \mathbb{N}_0^{\mathbb{P}(X)} \to P_{ne}(X)$ *satisfies neutrality, consistency, cancellation, and is discerning if and only if* f *is the approval voting rule.*

2.3 INDEPENDENCE OF AXIOMS

In the previous section, we characterized the approval voting rule with 3 conditions in Theorem 2.10, then using that characterization we expanded our conditions going to Theorem 2.12 then finally to Theorem 2.15. At first it may seem like the final characterization is a less concise or noteworthy characterization than the first or even the second. In fact, it may appear that not all the conditions in Theorem 2.15 are necessary to characterize approval voting. However, for each of the four conditions we will provide a rule violating that condition, but satisfying the other three.

First we will discuss the independence of neutrality in the characterization of approval voting in Theorem 2.15. Let $|X| \geq 3$ and suppose $\leq$ is a linear order on X. For any subset $Y \subseteq X$, let $min(Y)$ be the unique element belonging to Y such that $min(Y) \leq y$ for all $y \in Y$. Consider the function $g : \mathbb{N}_0^{\mathbb{P}(X)} \to P_{ne}(X)$ defined by

$$g(\pi) = \begin{cases} X & \text{if } F_A(\pi) = X \\ \{min\,(F_A(\pi))\} & \text{otherwise.} \end{cases}$$

Proposition 2.16. *The rule* g *satisfies consistency.*

Proof. Suppose π and π' are profiles such that $g(\pi) \cap g(\pi') \neq \emptyset$. If $g(\pi) = g(\pi') = X$, then $g(\pi + \pi') = X$ and thus $g(\pi) \cap g(\pi') = g(\pi + \pi')$.

Now suppose $g(\pi) \cap g(\pi') \neq X$, then without loss of generality we may assume $g(\pi) \neq X$. Thus $g(\pi) = \{u\}$ where $u = min(F_A(\pi))$. Observe that $g(\pi) \subseteq F_A(\pi)$

for all π. So,

$$u \in g(\pi) \cap g(\pi') \subseteq F_A(\pi) \cap F_A(\pi') = F_A(\pi + \pi').$$

Since $u \in F_A(\pi + \pi')$ it follows that $z \leq u$ where $z = min\,(F_A(\pi + \pi'))$. Also consider that $z \in F_A(\pi + \pi')$ implies that $z \in F_A(\pi) \cap F_A(\pi')$. In particular, $z \in F_A(\pi)$ and so $u \leq z$, hence $u = z$. It follows that $g(\pi + \pi') = \{z\} = \{u\} = g(\pi) = g(\pi) \cap g(\pi')$ and hence the rule g satisfies consistency. □

Proposition 2.17. *The rule g is discerning.*

Proof. Suppose π is a profile such that $v(x, \pi) > v(y, \pi)$ for all $y \in X \setminus \{x\}$. Then by definition of the approval voting rule, $F_A(\pi) = \{x\}$. Hence it follows that $g(\pi) = min(F_A(\pi)) = min(\{x\}) = \{x\}$.

Now suppose that π is a profile such that $v(x, \pi) < v(y, \pi)$ for all $y \in X \setminus \{x\}$. Then by definition of the approval voting rule, $x \notin F_A(\pi)$. Consider that $g(\pi) \subseteq F_A(\pi)$. Since $x \notin F_A(\pi)$ it follows that $x \notin g(\pi)$. Thus the rule g is discerning. □

Proposition 2.18. *The rule g satisfies cancellation.*

Proof. Suppose π is a profile such that $v(x, \pi) = v(y, \pi)$ for all $x, y \in X$. Since approval voting satisfies cancellation, $F_A(\pi) = X$. It follows that $g(\pi) = X$. Hence the rule g satisfies cancellation. □

Proposition 2.19. *The rule g does not satisfy neutrality.*

Proof. By way of contradiction suppose that g satisfies neutrality. Let x_1, x_2 be two alternatives such that $x_1 \leq x_2$. Consider the profile $\pi_{\{x_1,x_2\}}$ and the permutation $\sigma = (x_1\ x_2)$. Notice that $\sigma\left[\pi_{\{x_1,x_2\}}\right] = \pi_{\{x_1,x_2\}}$. Hence by neutrality $\sigma[x_1] = \{x_2\} \in g\left(\pi_{\{x_1,x_2\}}\right)$, but $g\left(\pi_{\{x_1,x_2\}}\right) = \{x_1\}$ contrary to assumption that g satisfies neutrality. □

We will now show that the discerning condition can not be dropped from Theorem 2.15. Consider the function $f_X : \mathbb{N}_0{}^{\mathbb{P}(X)} \to P_{ne}(X)$ defined by

$$f_X(\pi) = X.$$

Since every alternative is always in the social output, it is trivial that f_X is consistent, neutral and cancellative. We now show that f_X violates the discerning condition.

Proposition 2.20. *The rule f_X is not discerning.*

Proof. Let π be a profile such that $v(x,\pi) < v(y,\pi)$ for all $y \in X \backslash \{x\}$. By definition of f_X, $x \in f_X(\pi)$. Hence f_X is not discerning. $\square$

The next condition we will discuss the independence of in the characterization of approval voting in Theorem 2.15 is consistency. Recall the function we defined in Section 2.1, $F_{\text{mean}} : \mathbb{N}_0{}^{\mathbb{P}(X)} \to P_{ne}(X)$ defined by

$$F_{\text{mean}}(\pi) = \{x \in X \ : \ v(x,\pi) \geq \bar{v}(\pi)\}$$

where

$$\bar{v}(\pi) = \text{mean } v(\pi) = \sum_{x \in X} \frac{v(x,\pi)}{|X|}.$$

Proposition 2.21. *The rule F_{mean} satisfies faithfulness.*

Proof. For $B \in \mathbb{P}(X)$ with $B \neq \emptyset$ consider the profile π_B. If $x \in B$ then $v(x, \pi_B) = 1$, otherwise $v(x, \pi_B) = 0$. Hence it follows that

$$\bar{v}(\pi) = \frac{|B|}{|X|} \leq 1.$$

It follows from the definition of F_{mean} that $F_{\text{mean}}(\pi_B) = B$ and thus F_{mean} satisfies faithfulness. $\square$

Proposition 2.22. *The rule F_{mean} is discerning.*

Proof. Suppose π is a profile such that $v(x,\pi) > v(y,\pi)$ for all $y \in X \setminus \{x\}$. We first show that,

$$v(x,\pi) = \max\, v(\pi) > \bar{v}(\pi).$$

Since $\max\, v(\pi) \geq v(x,\pi)$ for each $x \in X$ and $\max\, v(\pi) > v(y,\pi)$ it follows that

$$\sum_{x\in X} \max\, v(\pi) > \sum_{x\in X} v(x,\pi)$$

and we get

$$\frac{\sum_{x\in X} \max\, v(\pi)}{|X|} > \frac{\sum_{x\in X} v(x,\pi)}{|X|}.$$

By definition of $\bar{v}(\pi)$ we have that

$$v(x,\pi) = \max\, v(\pi) > \bar{v}(\pi).$$

Therefore, by the definition of F_{mean}, $x \in F_{\text{mean}}(\pi)$. Now consider the case that π is a profile such that $v(x,\pi) < v(y,\pi)$ for all $y \in X \setminus \{x\}$. By a similar argument we have that

$$v(x,\pi) = \min\, v(<)\bar{v}(\pi)$$

and thus it follows that $x \notin F_{\text{mean}}(\pi)$. Thus F_{mean} is discerning. $\square$

Proposition 2.23. *The rule F_{mean} satisfies neutrality.*

Proof. Let π and π' be two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. Then by Lemma 2.5 it follows that $v(x,\pi) = v(\sigma[x],\pi')$ for all $x \in X$ and thus it follows that $\bar{v}(\pi) = \bar{v}(\pi')$. Therefore,

$$\begin{aligned} x \in F_{\text{mean}}(\pi) &\iff v(x,\pi) \geq \bar{v}(\pi) \\ &\iff v(\sigma[x],\pi') \geq \bar{v}(\pi') \\ &\iff \sigma(x) \in F_{\text{mean}}(\pi') \end{aligned}$$

Thus, the rule F_{mean} satisfies neutrality. $\square$

Proposition 2.24. *The rule F_{mean} satisfies cancellation.*

Proof. Suppose π is a profile such that $v(x,\pi) = v(y,\pi) = k$ for all x, y in X. Observe that

$$\begin{aligned}\bar{v}(\pi) &= \sum_{x\in X} \frac{v(x,\pi)}{|X|} \\ &= \frac{|X| \cdot k}{|X|} \\ &= v(x,\pi).\end{aligned}$$

Since $v(x,\pi) = \bar{v}(\pi)$ it follows that $F_{\text{mean}}(\pi) = X$. Hence F_{mean} satisfies cancellation. □

We showed in Example 2.1 that F_{mean} is not consistent when $|X| = 3$. The next proposition shows that this example can be extended to the case where $|X| \geq 3$.

Proposition 2.25. *If $|X| \geq 3$, then F_{mean} is not consistent.*

Proof. By way of contradiction, assume F_{mean} is consistent. Suppose $|X| = n \geq 3$ and π, π' are profiles such that

$$v(x,\pi) = \begin{cases} 2 + (n-3) & \text{if } x = x_1 \\ 4 + (n-3) & \text{if } x = x_2 \\ 3 + (n-3) & \text{otherwise} \end{cases}$$

and,

$$v(x,\pi') = \begin{cases} 4 + (n-3) & \text{if } x = x_1 \\ 2 + (n-3) & \text{if } x = x_2 \\ 3 + (n-3) & \text{otherwise.} \end{cases}$$

Since $v(x, \pi + \pi') = v(x, \pi) + v(x, \pi')$ for all $x \in X$. It follows that

$$v(x, \pi + \pi') = \begin{cases} 6 + 2(n-3) & \text{if } x = x_1 \\ 6 + 2(n-3) & \text{if } x = x_2 \\ 6 + 2(n-3) & \text{otherwise.} \end{cases}$$

Given that $\bar{v}(\pi) = \bar{v}(\pi') = n$, and $\bar{v}(\pi + \pi') = 2n$. It follows that $F_{\text{mean}}(\pi + \pi') = X$. Since

$$F_{\text{mean}}(\pi) \cap F_{\text{mean}}(\pi') = X \setminus \{x_1, x_2\}$$

it follows that $F_{\text{mean}}(\pi + \pi') \neq F_{\text{mean}}(\pi) \cap F_{\text{mean}}(\pi')$. Hence F_{mean} is not consistent.

$\square$

The last condition we will discuss the independence of in the characterization of approval voting in Theorem 2.15 is cancellation. Recall the rule $F_w : \mathbb{N}_0^{\mathbb{P}(X)} \to P_{ne}(X)$ defined in Section 2.1 by

$$F_w(\pi) = \{x \in F_A(\pi) : \; w(x, \pi) \leq w(y, \pi) \; \forall \; y \in F_A(\pi)\}$$

where

$$w(x, \pi) = \sum_{B \in \mathbb{P}(X)} |B| \; \pi(B) \chi_x(B).$$

We showed this rule violated cancellation; we will now show it satisfies the conditions of neutrality, consistency, and discerning.

Proposition 2.26. *If π and π' are two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$ then $w(x, \pi) = w(\sigma[x], \pi')$ for all $x \in X$.*

Proof. Let π and π' be two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. First consider that for any permutation σ, $|B| = |\sigma[B]|$ for all B. For any $x \in X$

notice that

$$\begin{aligned}
w(x,\pi) &= \sum_{B\in\mathbb{P}(X)} |B|\pi(B)\chi_x(B) \\
&= \sum_{B\in\mathbb{P}(X)} |\sigma[B]|\pi'(\sigma[B])\chi_x(B) \\
&= \sum_{\sigma[B]\in\mathbb{P}(X)} |\sigma[B]|\pi'(\sigma[B])\chi_{\sigma[x]}(\sigma[B]) \\
&= w(\sigma[x],\pi').
\end{aligned}$$

Hence $w(x,\pi) = w(\sigma[x],\pi')$ for all $x \in X$. □

Proposition 2.27. *The rule F_w satisfies neutrality.*

Proof. Let π and π' be two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. Then, by Lemma 2.5, it follows that $v(x,\pi) = v(\sigma[x],\pi')$ for all $x \in X$. Similarly it follows from Proposition 2.26 that $w(x,\pi) = w(\sigma[x],\pi')$ for all $x \in X$. Also note that $F_w(\pi) \subseteq F_A(\pi)$ for all π. Since $y \in F_A(\pi)$ if and only if $\sigma[y] \in F_A(\pi')$, we have

$$\begin{aligned}
x \in F_w(\pi) &\iff w(x,\pi) \le w(y,\pi)\ \forall\ y \in F_A(\pi) \\
&\iff w(\sigma[x],\pi') \le w(\sigma[y],\pi')\ \forall\ \sigma[y] \in F_A(\pi') \text{ by Proposition 2.26} \\
&\iff \sigma(x) \in F_w(\pi').
\end{aligned}$$

Thus the rule F_w satisfies neutrality. □

Proposition 2.28. *The rule F_w satisfies consistency.*

Proof. Let $F_w(\pi) \cap F_w(\rho) \neq \emptyset$. To show $F_w(\pi) \cap F_w(\rho) \subseteq F_w(\pi+\rho)$, let $x \in F_w(\pi) \cap F_w(\rho)$.

Since $F_w(\pi) \subseteq F_A(\pi)$ for all π, and since F_A satisfies consistency, $x \in F_A(\pi) \cap F_A(\rho) = F_A(\pi+\rho)$. So it is left to show that $w(x,\pi+\rho) \le w(y,\pi+\rho)$ for all

$y \in F_A(\pi + \rho)$.

$$\begin{aligned} w(x, \pi + \rho) &= w(x, \pi) + w(x, \rho) \\ &\leq w(y, \pi) + w(y, \rho) \\ &= w(y, \pi + \rho). \end{aligned}$$

Hence it follows by definition of F_w, that $x \in F_w(\pi + \rho)$.

Now to show $F_w(\pi) \cap F_w(\rho) \supseteq F_w(\pi + \rho)$, let $z \in F_w(\pi + \rho)$. If $z \in F_w(\pi) \cap F_w(\rho)$ we are done. Then we may assume that $z \notin F_w(\pi)$. Choose $y \in F_w(\pi) \cap F_w(\rho)$. Then $w(z, \pi) > w(y, \pi)$ and so $w(z, \pi) - w(y, \pi) > 0$. Given $w(z, \pi + \rho) \leq w(y, \pi + \rho)$ for all $y \in X$ it follows that:

$$\begin{aligned} w(z, \pi) + w(z, \rho) &\leq w(y, \pi) + w(y, \rho) \\ w(z, \pi) - w(y, \pi) + w(z, \rho) &\leq w(y, \rho) \\ w(z, \rho) &< w(y, \rho) \ \text{ since } w(z, \pi) - w(y, \pi) > 0 \end{aligned}$$

contrary to $y \in F_w(\rho)$. Hence it follows that $F_w(\pi) \cap F_w(\rho) \supseteq F_w(\pi + \rho)$. Thus $F_w(\pi) \cap F_w(\rho) = F_w(\pi + \rho)$ and so F_w satisfies consistency. $\square$

Proposition 2.29. *The rule F_w is discerning.*

Proof. Let $\pi \in {\mathbb{N}_0}^{\mathbb{P}(X)}$ be a profile such that $v(x, \pi) > v(y, \pi)$ for all $y \in X \setminus \{x\}$. By definition of F_w, $F_w(\pi) = \{x\}$ hence $x \in F_w(\pi)$.

Now suppose that $v(x, \pi) < v(y, \pi)$ for all $y \in X \setminus \{x\}$. Then by definition of approval voting, it follows that $x \notin F_A(\pi)$ and thus $x \notin F_w(\pi)$. Hence the rule F_w is discerning. $\square$

The rules g, f_X, F_{mean}, and F_w allow us to make the following statement.

Corollary 2.30. *If $|X| \geq 3$, then the conditions of neutrality, discerning, consistency, and cancellation are necessary and sufficient in the characterization of approval voting rule stated in Theorem 2.15.*

2.4 APPROVAL VOTING WITH TWO ALTERNATIVES

By Corollary 2.30, we know that if $|X| \geq 3$, then the conditions of neutrality, consistency, cancellation, and discerning are necessary and sufficient. Observe that since $f(\pi) \neq \emptyset$, if $|X| = 1$, then no conditions are required to characterize approval voting, since it is the only function. This raises the question: under what minimal set of conditions can we characterize approval voting when $|X| = 2$?

Theorem 2.31. *For $|X| = 2$, a rule $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies cancellation and is discerning if and only if f is the approval voting rule. Furthermore, the conditions of cancellation and discerning are sufficient to characterize the approval voting rule.*

Proof. We have shown that the approval voting rule satisfies the conditions of cancellation and is discerning. We will now show it is the only rule satisfying these two conditions.

Let $X = \{x_1, x_2\}$, and consider that for any profile $\pi \in {\mathbb{N}_0}^{\mathbb{P}(X)}$, we have three cases. The first case is that $v(x_1, \pi) = v(x_2, \pi)$, in this case $f = X$ by cancellation, and thus agrees with the approval voting rule. The next case is that $v(x_1, \pi) > v(x_2, \pi)$, hence by discerning we have that $x_1 \in f(\pi)$ and $x_2 \notin f(\pi)$, and again the rule f agrees with the approval voting rule. The last case is $v(x_1, \pi) < v(x_2, \pi)$, we have that f agrees with approval voting by a similar argument to the previous case. Therefore, it follows that a rule $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies cancellation and is discerning if and only if f is the approval voting rule.

To show this characterization is sufficient, recall the rule $g : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ defined in section 2.3 by

$$g(\pi) = \begin{cases} X & \text{if } F_A(\pi) = X \\ \{min\,(F_A(\pi))\} & \text{otherwise.} \end{cases}$$

We showed that this rule is discerning but violated the condition of cancellation. Also recall the rule f_X defined in section 2.3 by $f_X(\pi) = X$ for all profiles π. We showed that this rule satisfies the condition of cancellation but is not discerning. Hence the conditions of cancellation and discerning are sufficient to characterize the approval voting rule. $\square$

We have now shown that there are two necessary and sufficient conditions to characterize approval voting. However, there is no other set of two conditions that can characterize the approval voting rule. To see this, consider the following figure that demonstrates a function that satisfies each possible set of two.

Table 2.1: Characterizing approval voting with two alternatives.

	Neutrality	Consistency	Discerning	Non-Deviating	Cancellation
Neutrality	F_X	F_X	g	F_X	F_X
Consistency	F_X	g	g	F_X	F_X
Discerning	g	g	g	g	Theorem 2.31
Non-Deviating	F_X	F_X	g	F_X	F_X
Cancellation	F_X	F_X	Theorem 2.31	F_X	F_X

2.5 ALLOWING THE EMPTY SET AS A SOCIAL OUTCOME

By Theorem 2.15, we know that under the conditions of neutrality, consistency, cancellation, and discerning that if $f(\pi) \neq \emptyset$ for all profiles π then f is the approval voting rule. Consider the following rule $g : {\mathbb{N}_0}^{\mathbb{P}(X)} \to \mathbb{P}(X)$ defined by:

$$g(\pi) = \begin{cases} F_A(\pi) & \text{if } |F_A(\pi)| \in \{1, |X|\} \\ \emptyset & \text{otherwise.} \end{cases}$$

Lemma 2.32. *The rule g satisfies consistency.*

Proof. Suppose π and π' are profiles such that $g(\pi) \cap g(\pi') \neq \emptyset$. If $g(\pi) = g(\pi') = X$, then $g(\pi + \pi') = X$ and thus $g(\pi) \cap g(\pi') = g(\pi + \pi')$.

Now suppose $g(\pi) \cap g(\pi') \neq X$. Then, without loss of generality, we may assume $g(\pi) \neq X$. Thus $g(\pi) = \{x\}$ where $v(x, \pi) > v(y, \pi)$ for all $y \in X \setminus \{x\}$ and $v(x, \pi') \geq v(y, \pi')$ for all y. Hence $g(\pi) \cap g(\pi') = \{x\}$. Observe that

$$\begin{aligned} v(x, \pi + \pi') &= v(x, \pi) + v(x, \pi') \\ &> v(y, \pi) + v(y, \pi') \text{ for all } y \in X \setminus \{x\} \\ &= v(y, \pi + \pi') \end{aligned}$$

Thus $F_A(\pi + \pi') = \{x\}$ which implies $g(\pi + \pi') = \{x\} = g(\pi) \cap g(\pi')$, and hence the g rule satisfies consistency. □

Lemma 2.33. *The rule g satisfies cancellation.*

Proof. Suppose π is a profile such that $v(x, \pi) = v(y, \pi)$ for all x, y. Then by definition of the approval voting rule, $|F_A(\pi)| = |X|$, thus $g(\pi) = F_A(\pi) = X$. And thus the rule g satisfies cancellation. □

Lemma 2.34. *The rule g is discerning.*

Proof. Let $\pi \in {\mathbb{N}_0}^{\mathbb{P}(X)}$ be a profile. Suppose $v(x, \pi) > v(y, \pi)$ for all $y \in X \setminus \{x\}$. Then, by definition of the approval voting rule, $|F_A(\pi)| = |\{x\}| = 1$. Thus $g(\pi) = \{x\}$ and so $x \in g(\pi)$.

Now suppose that $v(x, \pi) < v(y, \pi)$ for all $y \in X \setminus \{x\}$. Then by definition of the approval voting rule, $x \notin F_A(\pi)$ it follows that $x \notin g(\pi)$. Hence g is discerning.

□

Lemma 2.35. *The g rule satisfies neutrality.*

Proof. Let π and π' be two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathbb{P}(X)$. Then by Lemma 2.5 it follows that $v(x, \pi) = v(\sigma[x], \pi')$ for all $x \in X$. Also note that $g(\pi) \subseteq F_A(\pi)$ for all π.

Using the definition of g and Lemma 2.5, we get the following:

$$\begin{aligned} x \in g(\pi) &\iff v(x,\pi) \geq v(y,\pi) \\ &\iff v(\sigma[x],\pi') \geq v(\sigma[y],\pi') \\ &\iff \sigma(x) \in g(\pi') \end{aligned}$$

Thus g satisfies neutrality. □

Observe we have found a rule satisfying each of our 4 axioms, and thus we cannot characterize Approval Voting with these four axioms in this model. However we suspect the following characterization holds in this model.

Conjecture 2.1. *The aggregation function $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to \mathbb{P}(X)$ satisfies cancellation, consistency, and discerning if and only if $f = F_A$ or $f = g$.*

CHAPTER 3

BALLOT AGGREGATION RULES

Under the framework of collective approval rules in Chapter 2, our rules aggregated every voter's approval set of alternatives into a nonempty subset of X. In this chapter we will study social choice functions where voters may not be able to choose the set of alternatives they approve of as a ballot. Instead they must approve of a ballot among a set of admissible ballots.

In some cases a voter may be either more or less sincere about the alternatives they are approving of. The ballot space consisting of each alternative as a single ballot is one example, in this scenario voters are forced to pick a more sincere preference from their approval set of alternatives. Another scenario we can consider is where each voter must select a ballot of at least three alternatives. In this case some voters must choose the ballot with the largest amount of candidates they approve of along with some candidates they may not approve of.

As in Chapter 2, the finite set of alternatives is $X = \{x_1, \ldots, x_m\}$ with $m \geq 2$. Let

$$\mathfrak{B} \subseteq \mathbb{P}(X),$$

we define $\mathfrak{B}$ as a **ballot space** and $B \in \mathfrak{B}$ as an **Admissible Ballot**. In this model our ballot response profiles are functions

$$\pi : \mathfrak{B} \to \mathbb{N}_0$$

with the interpretation that $\pi(B)$ represents the number of voters that chose the ballot $B \in \mathfrak{B}$. The set of all profiles on the ballot space $\mathfrak{B}$ is denoted by ${\mathbb{N}_0}^{\mathfrak{B}}$. The type of functions we are considering will be of the form

$$f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X).$$

We will refer to this class of functions as **ballot aggregation rules**.

3.1 NON-PREFERENTIAL BALLOTS

In the model of collective approval rules, we allow any voter to choose the set of alternatives they approve of as their vote; even if that happens to be the empty set or the entire set of alternatives. Although allowing the empty set of alternatives or the entire set of alternatives as admissible ballots shows no preference among the alternatives, they are needed to satisfy the inherent purpose of collective approval rules, which is to allow a voter to cast their sincere vote. If one voter approves of the entire set of alternatives, but can not cast that as a ballot, then we can no longer consider it collective approval voting since the function isn't aggregating that voter's true approval set.

In this section we will be discussing ballot spaces $\mathfrak{B}$ such that

$$\{B \in \mathbb{P}(X) \ : \ B \neq \emptyset, \ B \neq X\} \subseteq \mathfrak{B}.$$

The results of Fishburn [9], Alós-Ferrer [2], Duddy and Piggins [6], and Ninjbat [14] vary slightly from each other depending on whether the empty set and/or the entire set of alternatives belong to the ballot space $\mathfrak{B}$.

We now will study other known characterizations of Approval Voting and whether the results still hold if you remove or add either of these sets as admissible ballots.

> "The Characterization remains unchanged if the full ballot is allowed for. Allowing for the empty ballot requires to specify that aggregation of a single, empty ballot results in the full set of candidates."

Alós-Ferrer is correct when he states that the characterization remains unchanged if the full ballot is allowed for. However, by Lemma 2.9, we have shown that we do not need to modify the definition of approval voting to require the aggregation of a single empty ballot to result in the full set of candidates if our rule satisfies non-deviating, faithfulness, and consistency. In the characterizations of Theorems 2.12 and 2.15 we do not explicitly assume the conditions of non-deviating, faithfulness, and consistency. However, by Lemmas 2.13 and 2.14 we have that the function still satisfies those three conditions, and thus we still do not need to modify the definition of approval voting to require the aggregation of a single empty ballot to result in the full set of candidates.

We now consider the role of neutrality and consistency in the matter of allowing the empty set of alternatives or the entire set of alternatives as admissible ballots. Suppose $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies neutrality and consistency. Now suppose π is a profile such that $\pi(B) = 0$ for all $B \in \mathbb{P}(X) \setminus \{\emptyset, X\}$. If σ is an arbitrary permutation of X and π' is the profile such that,

$$\pi'(\sigma[B]) = \pi(B) \text{ for all } B \in \mathbb{P}(X).$$

Then $\pi = \pi'$, thus by neutrality $f(\pi) = \sigma[f(\pi)]$. Since $f(\pi) \neq \emptyset$ and σ was arbitrary, it follows that $f(\pi) = X$.

For any profile $\rho \in {\mathbb{N}_0}^{\mathbb{P}(X)}$, define profiles $\hat{\rho}$ and $\tilde{\rho}$ By

$$\hat{\rho}(B) = \begin{cases} \rho(B) & \text{if } B \notin \{\emptyset, X\} \\ 0 & \text{otherwise,} \end{cases}$$

and

$$\tilde{\rho}(B) = \begin{cases} \rho(B) & \text{if } B \in \{\emptyset, X\} \\ 0 & \text{otherwise.} \end{cases}$$

Notice that $\rho = \hat{\rho} + \tilde{\rho}$. From above, $f(\tilde{\rho}) = X$; by consistency we have $f(\rho) = f(\hat{\rho})$. Therefore given neutrality and consistency it does not matter if the empty set of alternatives or the entire set of alternatives are allowed as admissible ballots or not.

3.2 LEXICOGRAPHICAL SCORING RULES

In this section we will be studying a generalization of rules introduced by Fishburn [9]. The ballot aggregation rules we will be considering are threshold rules that use score functions as a threshold. Let $\mathfrak{A}$ be the set containing all possible ballot spaces,

$$\mathcal{C} : \mathfrak{A} \to \mathbb{P}(\{1, 2, \dots, |X|\})$$

is a function with the interpretation that for a ballot space $\mathfrak{B}$, $\mathcal{C}(\mathfrak{B})$ is the set containing the natural numbers corresponding to the size of the ballots in $\mathfrak{B}$. That is to say that

$$B \in \mathfrak{B} \Rightarrow |B| \in \mathcal{C}(\mathfrak{B}).$$

For a ballot space $\mathfrak{B}$, a **score function on $\mathfrak{B}$** is a function

$$s : \mathcal{C}(\mathfrak{B}) \to \mathbb{R},$$

which assigns a real number to each ballot based on its cardinality. Next, for each alternative $x \in X$ and profile π, let

$$p(x, \pi, s) = \sum_{B \in \mathfrak{B}} s(|B|)\pi(B)\chi_x(B).$$

Using this notation, a social choice function f on $\mathfrak{B}$ is said to be a **simple scoring function on $\mathfrak{B}$** if there is a score function s such that

$$f(\pi) = \{x \in X \ : \ p(x, \pi, s) \geq p(y, \pi, s) \ \forall \ y \in X\} \text{ for all } \pi \in {\mathbb{N}_0}^{\mathfrak{B}}.$$

We will denote a simple scoring function on $\mathfrak{B}$ with score function s by f_s. When $s(|B|) = 1$ for all $B \in \mathfrak{B}$, we will say f is the **majority rule** and denote it F_M. In the case where f is a simple scoring function with score function s such that $s(|B|) = -1$ for all $B \in \mathfrak{B}$, f is said to be the **inverse majority rule** and we will denote it $F_{M^{-1}}$. Last, when f is a simple scoring function with score function s such that $s(|B|) = 0$ for all $B \in \mathfrak{B}$, f is said to be the **trivial rule** which always selects the whole set of alternatives, and we will denote it f_X. So $f(\pi) = X$ for all profiles $\pi \in \mathbb{N}_0{}^{\mathfrak{B}}$.

Now we will characterize the set of all simple scoring functions based on constant score functions.

Proposition 3.1. *If $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ is a simple scoring function determined by score function $s : \mathcal{C}(\mathfrak{B}) \to \mathbb{R}$ such that for all $B \in \mathfrak{B}$, $s(|B|) = a$ for some $a \in \mathbb{R}$. Then $p(x, \pi, s) = a \cdot v(x, \pi)$. Furthermore,*

1. *If $a > 0$, then $f = F_M$.*
2. *If $a < 0$, then $f = F_{M^{-1}}$.*
3. *If $a = 0$, then $f = f_X$.*

Proof. Let f be a simple scoring function determined by the score function such that for all $B \in \mathfrak{B}$, $s(|B|) = a$ for some $a \in \mathbb{R}$. Observe that;

$$\begin{aligned}
p(x, \pi, s) &= \sum_{B \in \mathfrak{B}} s(|B|)\pi(B)\chi_x(B) \\
&= \sum_{B \in \mathfrak{B}} a \cdot \pi(B)\chi_x(B) \\
&= a \cdot \sum_{B \in \mathfrak{B}} \cdot\pi(B)\chi_x(B) \\
&= a \cdot v(x, \pi).
\end{aligned}$$

First suppose that $a \neq 0$. Since $v(x_i, \pi) \geq 0$ for all $x_i \in X$, then for $a \geq 0$, $a \cdot v(x, \pi) \geq a \cdot v(y, \pi)$ if and only if $v(x, \pi) \geq v(y, \pi)$. By the definition of simple scoring function,

$$\begin{aligned}
f(\pi) &= \{x \in X \ : \ p(x, \pi, s) \geq p(y, \pi, s) \ \forall \ y \in X\} \\
&= \{x \in X \ : \ a \cdot v(x, \pi) \geq a \cdot v(y, \pi) \ \forall \ y \in X\} \\
&= \{x \in X \ : \ 1 \cdot v(x, \pi) \geq 1 \cdot v(y, \pi) \ \forall \ y \in X\} \\
&= F_M(\pi).
\end{aligned}$$

Now suppose $a < 0$. Then it follows that $(-a) > 0$ and hence,

$$\begin{aligned}
f(\pi) &= \{x \in X \ : \ p(x, \pi, s) \geq p(y, \pi, s) \ \forall \ y \in X\} \\
&= \{x \in X \ : \ a \cdot v(x, \pi) \geq a \cdot v(y, \pi) \ \forall \ y \in X\} \\
&= \{x \in X \ : \ (-a) \cdot (-1) \cdot v(x, \pi) \geq (-a) \cdot (-1) \cdot v(y, \pi) \ \forall \ y \in X\} \\
&= \{x \in X \ : \ (-1) \cdot v(x, \pi) \geq (-1) \cdot v(y, \pi) \ \forall \ y \in X\} \\
&= F_{M^{-1}}(\pi).
\end{aligned}$$

And finally if $a = 0$, then by $f = f_X$ by definition. $\square$

From now on we will use the fact that $F_M(\pi) = f_s(\pi)$ where s is the score function $s(B) = 1$ for all $B \in \mathfrak{B}$. Also notice in the proof above we have that

$$\begin{aligned}
F_M(\pi) &= \{x \in X \ : \ 1 \cdot v(x, \pi) \geq 1 \cdot v(y, \pi) \ \forall \ y \in X\} \\
&= \{x \in X \ : \ v(x, \pi) \geq v(y, \pi) \ \forall \ y \in X\} \\
&= \{x \in X \ : \ v(x, \pi) = \max v(\pi)\}
\end{aligned}$$

which aligns with the idea of Approval Voting, but since voters may not choose any set of alternatives they approve of as a ballot, we will only refer to it as Approval Voting in the case that $\mathfrak{B} = \mathbb{P}(X)$.

We will now consider a larger class of scoring rules introduced by Fishburn (1979). For score functions

$$s_1, \dots, s_T : \mathcal{C}(\mathfrak{B}) \to \mathbb{R},$$

an alternative $x \in X$ is said to **lexicographically dominate** the alternative $y \in X \setminus \{x\}$ for a profile π if either $p(x, \pi, s_1) > p(y, \pi, s_1)$ or there is a $j \in \{2, \dots, T\}$ such that $p(x, \pi, s_i) \geq p(y, \pi, s_i)$ for all $s_i \in \{1, \dots, j-1\}$ and $p(x, \pi, s_j) > p(y, \pi, s_j)$. We will denote "$x$ lexicographically dominates y" by $x \succ_{\text{LD}} y$.

A social choice function f is said to be a **Lexicographical Scoring Rule** if there exists score functions $s_1, \dots, s_T$ such that

$$f(\pi) = \{x \in X \ : \ \nexists\, y \in X \ s.t. \ y \succ_{\text{LD}} x\}.$$

We let $f_{s_1,\dots,s_T}$ denote a Lexicographical Scoring Rule determined by score functions $s_1, \dots, s_T$.

Lemma 3.2. *The relation $\succ_{LD}$ on the set X is irreflexive and transitive.*

Proof. We will first show that $(X, \succ_{\text{LD}})$ is irreflexive. Consider that for any profile π and $x \in X$, $p(x, \pi, s_i) \not< p(x, \pi, s_i)$ for all $i \in \{2, \dots, T\}$ and hence $x \not\succ_{\text{LD}} x$. Therefore $(X, \succ_{\text{LD}})$ is irreflexive.

We now show that $(X, \succ_{\text{LD}})$ is transitive. Let $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$ and $x, y, z \in X$ such that $x \succ_{\text{LD}} y$ and $y \succ_{\text{LD}} z$. Since $x \succ_{\text{LD}} y$ we have that $p(x, \pi, s_1) > p(y, \pi, s_1)$ or there exists $j \in \{2, \dots, T\}$ such that $p(x, \pi, s_i) \geq p(y, \pi, s_i)$ for all $s_i \in \{1, \dots, j-1\}$ and $p(x, \pi, s_j) > p(y, \pi, s_j)$. Similarly for $y \succ_{\text{LD}} z$.

We first consider that $p(x, \pi, s_1) > p(y, \pi, s_1)$, hence

$$p(x, \pi, s_1) > p(y, \pi, s_1) \geq p(z, \pi, s_1).$$

Thus, $p(x, \pi, s_1) > p(z, \pi, s_1)$ and we have that $x \succ_{\text{LD}} z$. Similarly if $p(y, \pi, s_1) > p(z, \pi, s_1)$ we have

$$p(x, \pi, s_1) \geq p(y, \pi, s_1) > p(z, \pi, s_1)$$

with $p(x,\pi,s_1) > p(z,\pi,s_1)$ and again we have that $x \succ_{\mathrm{LD}} z$.

Now consider that $p(x,\pi,s_1) \not> p(y,\pi,s_1)$ and $p(y,\pi,s_1) \not> p(z,\pi,s_1)$. Thus we have that there exists $j_1, j_2 \in \{2,\ldots,T\}$ such that

$$p(x,\pi,s_i) \geq p(y,\pi,s_i) \text{ for all } i \in \{2,\ldots,j_1-1\} \ \& \ p(x,\pi,s_{j_1}) > p(y,\pi,s_{j_1})$$

$$\text{and } p(y,\pi,s_{i'}) \geq p(z,\pi,s_{i'}) \text{ for all } i' \in \{2,\ldots,j_2-1\} \ \& \ p(y,\pi,s_{j_2}) > p(z,\pi,s_{j_2}).$$

We first consider that $j_1 \leq j_2$. Thus we have

$$p(x,\pi,s_{j_1}) > p(y,\pi,s_{j_1}) \geq p(z,\pi,s_{j_1}),$$

and for $i \in \{2,\ldots,j_1-1\}$ we have

$$p(x,\pi,s_i) \geq p(y,\pi,s_i) \geq p(z,\pi,s_i).$$

Therefore it follows that $x \succ_{\mathrm{LD}} z$. Similarly if $j_1 > j_2$ we have

$$p(x,\pi,s_{j_2}) \geq p(y,\pi,s_{j_2}) > p(z,\pi,s_{j_2}),$$

and for $i \in \{2,\ldots,j_2-1\}$ we have

$$p(x,\pi,s_i) \geq p(y,\pi,s_i) \geq p(z,\pi,s_i).$$

Therefore it follows that $x \succ_{\mathrm{LD}} z$ and thus $(X, \succ_{\mathrm{LD}})$ is transitive. We have shown that $(X, \succ_{\mathrm{LD}})$ is a irreflexive transitive relation. $\square$

To give an example of a lexicographical scoring rule that is not a simple scoring rule, consider the collective approval rule defined in Section 2.3 $f_w : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ defined by

$$f_w(\pi) = \{x \in F_M(\pi) \ : \ w(x,\pi) \leq w(y,\pi) \ \forall \ y \in F_M(\pi)\}$$

where

$$w(x,\pi) = \sum_{B \in \mathbb{P}(X)} |B| \, \pi(B) \, \chi_x(B).$$

To phrase this function in terms of lexicographical scoring rules consider the functions

$$s_1, s_2 : \mathcal{C}(\mathfrak{B}) \to \mathbb{R}$$

defined by for all ballots $B \in \mathfrak{B}$, $s_1(|B|) = 1$ and $s_2(|B|) = -|B|$. Observe that any alternative not in the Majority output gets lexicographically dominated in the score function s_1. In Chapter 2 we defined this as a refinement of Approval Voting. Alternatives that are lexicographically dominated in the score function s_2 are still in the Majority Outcome, but with less decisive voters who approve of them.

Lemma 3.3. *For all profiles $\pi \in \mathbb{N}_0{}^{\mathfrak{B}}$, $f_w(\pi) = f_{s_1,s_2}(\pi)$.*

Proof. We first consider that for $x \in X$ we have that

$$\begin{aligned} p(x, \pi, s_1) &= \sum_{B \in \mathfrak{B}} s_1(|B|)\pi(B)\chi_x(B) \\ &= \sum_{B \in \mathfrak{B}} \pi(B)\chi_x(B) \\ &= v(x, \pi). \end{aligned}$$

Similarly for $x \in X$ we have that,

$$\begin{aligned} p(x, \pi, s_2) &= \sum_{B \in \mathfrak{B}} s_2(|B|)\pi(B)\chi_x(B) \\ &= \sum_{B \in \mathfrak{B}} -|B|\pi(B)\chi_x(B) \\ &= -\sum_{B \in \mathfrak{B}} |B|\pi(B)\chi_x(B) \\ &= -w(x, \pi). \end{aligned}$$

Let $\pi \in \mathbb{N}_0{}^{\mathfrak{B}}$ be an arbitrary profile and let $x \in f_w(\pi)$. By way of contradiction suppose that $x \notin f_{s_1,s_2}(\pi)$, hence we have that there exists $z \in X$ such that $z \succ_{\mathrm{LD}} x$. Since $z \succ_{\mathrm{LD}} x$ we have either $p(z, \pi, s_1) > p(x, \pi, s_1)$ or we have that $p(z, \pi, s_1) \geq p(x, \pi, s_1)$ and $p(z, \pi, s_2) > p(x, \pi, s_2)$. Since $x \in f_w(\pi)$ we have that

$v(x, \pi) \geq v(y, \pi)$ for all $y \in X$; this fact along with the fact that $p(x, \pi, s_1) = v(x, \pi)$ gives us that $p(x, \pi, s_1) \geq p(y, \pi, s_1)$ for all $y \in X$ and hence $p(x, \pi, s_1) \geq p(z, \pi, s_1)$. It follows we are in the second case with $p(x, \pi, s_1) = p(z, \pi, s_1)$ and $p(z, \pi, s_2) > p(x, \pi, s_2)$. Since $p(x, \pi, s_1) = p(z, \pi, s_1)$ we have that $x, z \in F_M(\pi)$. Observe that $p(x, \pi, s_2) = -w(x, \pi)$ for all $x \in X$ and $p(z, \pi, s_2) > p(x, \pi, s_2)$ implies that $w(x, \pi) > w(z, \pi)$, contrary to $x \in \{x \in F_M(\pi) \ : \ w(x, \pi) \leq w(y, \pi) \ \forall \ y \in F_A(\pi)\}$. Thus we have that $x \in f_{s_1,s_2}(\pi)$ and thus $f_w(\pi) \subseteq f_{s_1,s_2}(\pi)$.

Now let $x \in f_{s_1,s_2}(\pi)$ and by way of contradiction suppose that $x \notin f_w(\pi)$. It follows that either $x \notin F_M(\pi)$ or $x \in F_M(\pi)$ with $w(x, \pi) > w(z, \pi)$ for some $z \in F_M(\pi)$. First lets consider that since $x \in f_{s_1,s_2}(\pi)$ along with the fact that $v(y, \pi) = p(y, \pi, s_1)$ for all $y \in X$, gives us that $v(x, \pi) \geq v(y, \pi)$ for all $y \in X$. But since $x \notin f_w(\pi)$ it follows that there exists a $z \in F_M(\pi)$ such that $v(x, \pi) = v(z, \pi)$ and $w(x, \pi) > w(z, \pi)$. By the fact $p(y, \pi, s_2) = -w(y, \pi)$ for all $y \in X$ we have that $p(z, \pi, s_2) > p(x, \pi, s_2)$ with $p(z, \pi, s_1) \geq p(x, \pi, s_1)$ and thus $z \succ_{\text{LD}} x$, contrary to $x \in f_{s_1,s_2}(\pi)$. It follows that $x \in f_w(\pi)$ and thus $f_{s_1,s_2}(\pi) \subseteq f_w(\pi)$.

Since π was chosen arbitrarily with $f_w(\pi) \subseteq f_{s_1,s_2}(\pi)$ and $f_{s_1,s_2}(\pi) \subseteq f_w(\pi)$, it follows that for all profiles $\pi \in \mathbb{N}_0{}^{\mathfrak{B}}$ we have $f_w(\pi) = f_{s_1,s_2}(\pi)$. $\square$

We now give some defining characteristics of ballot aggregation functions. We will say that a ballot space $\mathfrak{B}$ is **permutation closed** if, for any permutation σ on X and for any $B \in \mathfrak{B}$, $\sigma(B) \in \mathfrak{B}$. A social choice function f on a permutation closed ballot space $\mathfrak{B}$ satisfies **neutrality** if, for any profiles π and π' and for any permutation σ of X,

$$\pi'(\sigma(B)) = \pi(B) \text{ for all } B \in \mathfrak{B} \ \Rightarrow \ \sigma\left(f(\pi)\right) = f(\pi').$$

Lemma 3.4. *If $\mathfrak{B}$ is a permutation closed ballot space, $s : \mathcal{C}(\mathfrak{B}) \to \mathbb{R}$ is a score function and π, π' are two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathfrak{B}$ then $p(x, \pi, s) = p(\sigma[x], \pi', s)$ for all $x \in X$.*

Proof. Let π and π' be two profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathfrak{B}$. For any $x \in X$ notice that

$$\begin{aligned} p(x, \pi, s) &= \sum_{B \in \mathfrak{B}} s(|B|)\pi(B)\chi_x(B) \\ &= \sum_{B \in \mathfrak{B}} s(|\sigma[B]|)\pi'(\sigma[B])\chi_x(B) \\ &= \sum_{\sigma[B] \in \mathfrak{B}} s(|\sigma[B]|)\pi'(\sigma[B])\chi_{\sigma[x]}(\sigma[B]) \\ &= p(\sigma[x], \pi', s). \end{aligned}$$

Hence $p(x, \pi, s) = p(\sigma[x], \pi', s)$ for all $x \in X$. $\square$

Proposition 3.5. *If $\mathfrak{B}$ is a permutation closed ballot space, then majority rule, inverse majority rule, and the trivial rule satisfy neutrality.*

Proof. For each of these functions we will consider π, π' as profiles such that $\pi'(\sigma[B]) = \pi(B)$ for all $B \in \mathfrak{B}$. By proposition 3.1, F_M is determined by a positive constant score function s. Therefore for any $x \in X$ and $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$,

$$\begin{aligned} x \in F_M(\pi) &\iff p(x, \pi, s) \geq p(x, \pi, s) \; \forall \; y \in X \\ &\iff p(\sigma[x], \pi', s) \geq p(\sigma[y], \pi', s) \text{ by Lemma 3.4} \\ &\iff \sigma(x) \in F_M(\pi'). \end{aligned}$$

Hence $F_M(\pi') = \sigma(F_M(\pi))$ and so F_M satisfies neutrality.

To show Inverse Majority is neutral, consider that by proposition 3.1, $F_{M^{-1}}$ is determined by a negative constant score function s. Therefore for any $x \in X$ and $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$,

$$\begin{aligned} x \in F_{M^{-1}}(\pi) &\iff p(x, \pi, s) \leq p(y, \pi, s) \; \forall \; y \in X \\ &\iff p(\sigma[x], \pi', s) \leq p(\sigma[y], \pi', s) \text{ by Lemma 3.4} \\ &\iff \sigma(x) \in F_{M^{-1}}(\pi') \end{aligned}$$

Hence $F_{M^{-1}}(\pi') = \sigma(F_{M^{-1}}(\pi))$ and so $F_{M^{-1}}$ satisfies neutrality. To show f_X is neutral, consider that for every $x \in X$, $x \in f_X(\pi)$, hence every $\sigma(x) \in f_X(\pi')$. Hence the rules Majority, Inverse Majority, and f_X satisfies neutrality as was desired. $\square$

We define the remaining axioms of non-deviating, consistency, faithfulness, and cancellation the same way as is in Chapter 2 without modification.

Proposition 3.6. *Majority rule, inverse majority rule, and the trivial rule satisfy consistency.*

Proof. Let π and ρ be profiles and assume that $F_M(\pi) \cap F_M(\rho) \neq \emptyset$. To show $F_M(\pi) \cap F_M(\rho) \subseteq F_M(\pi+\rho)$, let $x \in F_M(\pi) \cap F_M(\rho)$ and recall that F_M is determined by a positive constant score function s. Thus $p(x, \pi, s) \geq p(y, \pi, s)$ and $p(x, \rho, s) \geq p(y, \rho, s)$ for all $y \in X$. Observe that:

$$\begin{aligned} p(x, \pi + \rho, s) &= p(x, \pi, s) + p(x, \rho, s) \\ &\geq p(y, \pi, s) + p(y, \rho, s) \ \forall \ y \in X \\ &= p(y, \pi + \rho, s) \end{aligned}$$

Thus $x \in F_M(\pi + \rho)$

Now to show $F_M(\pi) \cap F_M(\rho) \supseteq F_M(\pi + \rho)$, let $z \in F_M(\pi + \rho)$. Thus $p(z, \pi+\rho, s) = p(z, \pi, s) + p(z, \rho, s)$. If $z \in F_M(\pi) \cap F_M(\rho)$ we are done. Then we may assume that $z \notin F_M(\pi)$. Choose $y \in F_M(\pi) \cap F_M(\rho)$ such that $p(z, \pi, s) < p(y, \pi, s)$ and so, $p(y, \pi, s) - p(z, \pi, s) > 0$. Given $p(z, \pi + \rho, s) \geq p(y, \pi + \rho, s)$ for all $y \in X$ it follows that:

$$\begin{aligned} p(z, \pi, s) + p(z, \rho, s) &\geq p(y, \pi, s) + p(y, \rho, s) \\ p(z, \rho, s) &\geq (p(y, \pi, s) - p(z, \pi, s)) + p(y, \rho, s) \\ p(z, \rho, s) &> p(y, \rho, s) \ \text{ since } p(y, \pi, s) - p(z, \pi, s) > 0 \end{aligned}$$

contrary to $y \in F_M(\rho)$. Hence it follows that $F_M(\pi) \cap F_M(\rho) \supseteq F_M(\pi + \rho)$. Thus $F_M(\pi) \cap F_M(\rho) = F_M(\pi + \rho)$ and hence the F_M satisfies consistency.

A similar argument works to show that Inverse Majority is consistent. It is trivial to show that f_X is consistent. Hence the rules Majority, Inverse Majority, and f_X satisfy consistency. $\square$

Proposition 3.7. *Majority rule, inverse majority rule, and the trivial rule satisfy cancellation.*

Proof. For each of these functions we will consider a profile π such that $v(x, \pi) = v(y, \pi)$ for all $x, y \in X$. Observe that since each of these three functions are simple scoring functions, $f(\pi) = \{x \in X \ : \ p(x, \pi, s) \geq p(y, \pi, s)\} = X$ since for each of the three scoring functions $p(x, \pi, s) = p(y, \pi, s)$ for all $x, y \in X$. Hence the rules Majority, Inverse Majority, and f_X satisfies cancellation as was desired. $\square$

We now discuss a class of ballot spaces introduced by Fishburn (1979). Suppose a ballot space $\mathfrak{B} \subseteq \mathbb{P}(X)$ satisfies

1. For all $B \in \mathfrak{B}$, $\lambda(B) \in \mathfrak{B}$ for all permutations $\lambda : X \to X$.

2. For all $x, y \in X$, there is a $B \in \mathfrak{B}$ such that $x \in B$ and $y \notin B$.

Then we will say $\mathfrak{B}$ satisfies **basic richness**, or $\mathfrak{B}$ is a **rich ballot space**. We can describe such spaces as permutation closed ballot where $\mathfrak{B} \notin \{\{\emptyset\}, \{X\}, \{\emptyset, X\}\}$. Furthermore, if $\mathbb{P}(X) \backslash \{\emptyset, X\} \subseteq \mathfrak{B}$, we will say $\mathfrak{B}$ is an **unrestricted ballot space**.

We now state a result by Fishburn [8], about ballot spaces satisfying basic richness:

Theorem 3.8. *(Fishburn 1979) For any ballot space satisfying basic richness: a social choice function $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies Neutrality and Consistency if and only if it is a Lexicographical Scoring Rule. Moreover, f is determined by score functions $s_1, \ldots, s_T$ such that $0 < T \leq |X|$.*

We have shown that Majority, Inverse Majority, and f_X are three such functions. In addition these three functions also satisfy cancellation. This raises the question: what is the class of lexicographical scoring rules satisfying cancellation? For the unrestricted ballot spaces Ninjbat [14] provides an answer to this question.

Theorem 3.9. *(Ninjbat 2013) For $\mathfrak{B} = P_{ne}(X)$, a social choice function $f : {\mathbb{N}_0}^{P_{ne}(X)} \to P_{ne}(X)$ satisfies neutrality, consistency, and cancellation if and only if $f = F_M$, $f = F_{M^{-1}}$, or $f = f_X$.*

The question now is: Does this result extend to any larger class of ballot spaces? To answer this question we will return to the axiom we introduced in Chapter 2 of non-deviating. Our goal is to show that if f is cancellative and consistent, then f is non-deviating. We first state and prove two lemmas and a corollary.

Lemma 3.10. *For every $B \in \mathfrak{B}$, there exists $\rho \in {\mathbb{N}_0}^{\mathfrak{B}}$ such that for all $x \in X$, $v(x, \pi_B + \rho) = k$ for some $k \in \mathbb{N}_0$.*

Proof. Let $X = \{x_1, \ldots, x_m\}$, and let $B \in \mathfrak{B}$ with $|B| = c \in \mathcal{C}(\mathfrak{B})$. We introduce the profile

$$\rho = \sum_{C \in \mathfrak{B}_c, C \neq B} \pi_C,$$

where

$$\mathfrak{B}_c = \{C \subseteq X \;:\; |C| = c\}.$$

For each $x_i \in X$, observe that

$$v(x_i, \rho) = \begin{cases} \binom{m-1}{c-1} & \text{if } x_i \notin B. \\ \binom{m-1}{c-1} - 1 & \text{if } x_i \in B \end{cases}$$

Consider the profile $\pi_B + \rho$, and observe that

$$v(x, \pi_B + \rho) = v(x, \pi_B) + v(x, \rho) \text{ for each } x \in X.$$

Since $v(x, \pi_B) = 1$ if $x \in B$ and $v(x, \pi_B) = 0$ if $x \notin B$ it follows that,

- If $x \in B$,

$$\begin{aligned} v(x, \pi_B + \rho) &= v(x, \pi_B) + v(x, \rho) \\ &= 1 + \binom{m-1}{c-1} - 1 \\ &= \binom{m-1}{c-1} \end{aligned}$$

- If $x \notin B$,

$$\begin{aligned} v(x, \pi_B + \rho) &= v(x, \pi_B) + v(x, \rho) \\ &= 0 + \binom{m-1}{c-1} \\ &= \binom{m-1}{c-1} \end{aligned}$$

Hence it follows that $v(x, \pi_B + \rho) = \binom{m-1}{c-1} \in \mathbb{N}_0$ for every $x \in X$. □

Lemma 3.11. *For every profile $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$, there exists a profile $\rho \in {\mathbb{N}_0}^{\mathfrak{B}}$ such that for all $x \in X$, $v(x, \pi + \rho) = k$ for some $k \in \mathbb{N}_0$.*

Proof. Let $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$ be a fixed profile. Observe that

$$\pi = \sum_{B_i \in \mathfrak{B}} \pi(B_i) \cdot \pi_{B_i}.$$

By Lemma 3.10, for each π_{B_i}, there exists a $\rho_i \in {\mathbb{N}_0}^{\mathfrak{B}}$ such that for every $x \in X$, $v\left(x, \pi_{B_i, +\rho_i}\right) = k_i$ for some $k_i \in \mathbb{N}_0$. Consider the profile

$$\rho = \sum_{B_i \in \mathfrak{B}} \pi(B_i) \cdot \pi_{\rho_i}.$$

Thus

$$\begin{aligned} \pi + \rho &= \sum_{B_i \in \mathfrak{B}} \pi(B_i) \cdot \pi_{B_i} + \sum_{B_i \in \mathfrak{B}} \pi(B_i) \cdot \pi_{\rho_i} \\ &= \sum_{B_i \in \mathfrak{B}} \pi(B_i) \cdot \left[\pi_{B_i} + \pi_{\rho_i}\right]. \end{aligned}$$

It follows that for each $x \in X$,

$$\begin{aligned} v(x, \pi + \rho) &= v(x, \pi_{B_1} + \pi_{\rho_1}) + \cdots + v(x, \pi_{B_n} + \pi_{\rho_n}) \\ &= k_1 + \cdots + k_n \end{aligned}$$

Hence for all $x \in X$, $v(x, \pi + \rho) = k$ for $k = k_1 + \cdots k_n \in \mathbb{N}_0$. $\square$

Corollary 3.12. *If $\mathfrak{B}$ is a permutation closed ballot space and $f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies cancellation, then for every profile $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$, there exists a profile $\rho \in {\mathbb{N}_0}^{\mathfrak{B}}$ such that*

$$f(\pi + \rho) = X.$$

Proof. By Lemma 3.11, there exists a profile $\rho \in {\mathbb{N}_0}^{\mathfrak{B}}$ such that for all $x \in X$, $v(x, \pi + \rho) = k$ for some $k \in \mathbb{N}_0$. It follows by cancellation that $f(\pi + \rho) = X$ as was desired. $\square$

Proposition 3.13. *If $f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies cancellation and consistency then f is non-deviating.*

Proof. Let π and π' be profiles such that $v(x, \pi) = v(x, \pi')$ for every alternative $x \in X$. By Lemma 3.11, let ρ be a profile such that for every $x \in X$, $v(x, \pi + \rho) = k$ for some $k \in \mathbb{N}_0$. By Corollary 3.12, we have that $f(\pi + \rho) = X$.

Observe that since for each $x \in X$ $v(x, \pi + \rho) = k$, we have $v(x, \pi + \rho) = v(x, \pi) + v(x, \rho) = k$. Thus,

$$v(x, \rho) = k - v(x, \pi). \tag{3.1}$$

Using Equation 3.1 and the hypothesis, for any $x \in X$,

$$\begin{aligned} v(x, \pi' + \rho) &= v(x, \pi') + v(x, \rho) \text{ for } x \in X \\ &= v(x, \pi') + k - v(x, \pi) \\ &= v(x, \pi') + k - v(x, \pi') \\ &= k. \end{aligned}$$

Thus, by cancellation, $f(\pi' + \rho) = X$. Notice that by consistency we have that

$$f(\pi + \pi' + \rho) = f(\pi) \cap X = f(\pi)$$

since $f(\pi' + \rho) = X$. Similarly since $f(\pi + \rho) = X$ we have

$$f(\pi' + \pi + \rho) = f(\pi') \cap X = f(\pi').$$

We have $f(\pi) = f(\pi + \pi' + \rho) = f(\pi')$ and thus f is non-deviating. $\square$

We now will use the axiom of non-deviating to show that every score function constituting a cancellative lexicographical scoring rule must be a constant function.

Lemma 3.14. *If $s_1, \ldots, s_T : \mathcal{C}(\mathfrak{B}) \to \mathbb{R}$ are score functions determining a lexicographical scoring rule f satisfying cancellation, then for each $i \in \{1, \ldots, T\}$, s_i is a constant function. That is for all ballots $B \in \mathfrak{B}$, $s_i(|B|) = k_i$ for some $k_i \in \mathbb{R}$.*

Proof. Let s_t be a score function determining the lexicographical scoring rule f satisfying cancellation. First consider the case that $\mathcal{C}(\mathfrak{B}) = \{c\}$. For $B, B' \in \mathfrak{B}$ it follows that

$$s_t(|B|) = s_t(c) = s_t(|B'|).$$

We will now assume that $|\mathcal{C}(\mathfrak{B})| \geq 2$ and $s_t(c_j) = k_t \in \mathbb{R}$ where $c_j = \min\ \mathcal{C}(\mathfrak{B})$. We will show that $s_t(c_k) = s_t(c_j) = k_t$ for any $c_k \in \mathcal{C}(\mathfrak{B})$. Let $c_k \in \mathcal{C}(\mathfrak{B})$, if $c_k = c_j$ we are done, so by minimality of c_j we may assume $c_j < c_k$. Choose $B \in \mathfrak{B}$ such that $|B| = c_k$. Consider the profiles

$$\pi = \sum_{|D| = c_j} \pi_D$$

and

$$\rho = \binom{|B| - 1}{c_j - 1} \pi_B + \sum_{D \not\subseteq B, |D| = c_j} \pi_D.$$

Observe that

$$
\begin{aligned}
f(\pi) &= f\left(\sum_{|D|=c_j} \pi_D\right) \\
&= f\left(\sum_{D\subset B, |D|=c_j} \pi_D + \sum_{D\not\subset B, |D|=c_j} \pi_D\right) \\
&= f\left(\binom{|B|-1}{c_j-1}\pi_B + \sum_{D\not\subset B, |D|=c_j} \pi_D\right) \text{ by Proposition 3.13} \\
&= f(\rho).
\end{aligned}
$$

By construction $v(x,\pi) = v(y,\pi)$ for all $x, y \in X$ and thus by cancellation it follows that $f(\pi) = f(\rho) = X$. Since f is a lexicographical scoring rule with $f(\rho) = X$ it follows that $p(x,\rho,s_t) = p(y,\rho,s_t)$ for all $x, y \in X$. Choose $x' \in B$ and $y' \notin B$. We have that

$$p(x',\rho,s_t) = s_t(|B|)\cdot\binom{|B|-1}{c_j-1} + s_t(|D|)\cdot\left[\binom{|X|-1}{c_j-1} - \binom{|B|-1}{c_j-1}\right]$$

and

$$p(y',\rho,s_t) = s_t(|D|)\cdot\binom{|X|-1}{c_j-1}.$$

Since $p(x',\rho,s_t) = p(y',\rho,s_t)$ with $|B| = c_k$ and $|D| = c_j$ we have that

$$s_t(c_k)\cdot\binom{|B|-1}{c_j-1} + s_t(c_j)\cdot\binom{|X|-1}{c_j-1} - s_t(c_j)\binom{|B|-1}{c_j-1} = s_t(c_j)\cdot\binom{|X|-1}{c_j-1}.$$

Adding $s_t(c_j)\binom{|B|-1}{c_j-1} - s_t(c_j)\cdot\binom{|X|-1}{c_j-1}$ to both sides gives us

$$s_t(c_k)\cdot\binom{|B|-1}{c_j-1} = s_t(c_j)\cdot\binom{|B|-1}{c_j-1}.$$

By dividing both sides by $\binom{|B|-1}{c_j-1}$ we get that $s_t(c_k) = s_t(c_j)$. It follows for each $c_k \in \mathfrak{C}(\mathfrak{B})$ that $s_t(c_k) = s_t(c_j) = k_t$. Hence each score function s_t must be a constant function. □

Theorem 3.15. *If $\mathfrak{B}$ is a rich ballot space, then a lexicographical scoring function $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies cancellation if and only if f is majority rule, inverse majority rule, or the trivial rule.*

Proof. We have shown that the rules F_M, $F_{M^{-1}}$, and f_X each satisfy cancellation. Let $f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ be a fixed social choice function which satisfies cancellation. So let $s_1, ..., s_T$ be the score functions determining f. By Lemma 3.14, we know that each s_i is a constant function, so let $s_i(|B|) = k_i$ for all $B \in \mathfrak{B}$.

Choose a ballot $B \in \mathfrak{B}$ and consider $f(\pi_B)$. Since $f(\pi_B) \neq \emptyset$, there exists an $x \in X$ such that $x \in f(\pi_B)$. There are two possibilities, either $x \in B$ or $x \notin B$. First consider that $x \in B$. Consider $x' \in X$ such that $x' \in B$, then for the permutation $\sigma = (x\, x')$, consider the profile π' defined by $\pi'(\sigma[C]) = \pi_B(C)$ for all $C \in \mathfrak{B}$. Observe that $\pi_B = \pi'$ and thus $f(\pi_B) = f(\pi')$. By neutrality we have that;

$$\sigma[f(\pi_B)] = f(\pi') = f(\pi_B).$$

Thus it follows that $x' = \sigma(x) \in f(\pi_B)$. Since x' was chosen arbitrarily, it follows that $B \subseteq f(\pi_B)$. Conversely if $x \notin f(\pi_B)$ then $B \not\subseteq f(\pi_B)$.

Now we will consider another element $y \notin B$. If $y \in f(\pi_B)$ then a similar argument shows that $B^c \subseteq f(\pi_B)$; and conversely if $y \notin f(\pi_B)$ then $B^c \not\subseteq f(\pi_B)$. Hence it follows that we either have $f(\pi_B) = B$, or $f(\pi_B) = X \setminus B$, or $f(\pi_B) = X$. We will consider each of these three cases separately.

<u>Case 1</u>: Lets consider that $f(\pi_B) = B$. Observe that by definition of a lexicographical score that there is a score function s_i such that for $x \in B$ and $y \in B^c$ such that $x \succ_{\text{LD}} y$. Hence $p(x, \pi_B, s_i) > p(y, \pi_B, s_i)$. Since $s_i(B) = k_i$, and $p(y, \pi_B, s_i) = 0$ it follows that $k_i > 0$.

If f is a simple scoring rule, then $f = F_M$ by Lemma 3.10 and we are done. Otherwise, suppose that f is not a simple scoring rule, thus there is a $j > i$ such that $x \succ_{\text{LD}} y$ for s_j but $x \not\succ_{\text{LD}} y$ for s_i. Since $x \not\succ_{\text{LD}} y$ for s_i, it follows that $p(x, \pi, s_t) = p(y, \pi, s_t)$ for $t \in \{1, .., i\}$. Thus $k_1 \cdot v(x, \pi) = k_1 \cdot v(y, \pi)$, and since $k_1 > 0$ it follows that $v(x, \pi) = v(y, \pi)$. By hypothesis $p(x, \pi, s_j) > p(y, \pi, s_j)$, since $s_j(|B|) = k_j$ it follows that $k_j \cdot v(x, \pi) > k_j \cdot v(y, \pi)$, but $v(x, \pi) = v(y, \pi)$ implies that $k_j \cdot v(x, \pi) = k_j \cdot v(y, \pi)$

a contradiction. Thus f is a simple scoring rule and thus $f = F_M$.

Case 2: Consider that $f(\pi_B) = X \setminus B$. Observe that by definition of a lexicographical score that there is a score function s_i such that for $x \in B$ and $y \in B^c$ such that $y \succ_{\text{LD}} x$. Hence $p(y, \pi_B, s_i) > p(x, \pi_B, s_i)$. Since $s_i(B) = k_i$, and $p(y, \pi_B, s_i) = 0$ it follows that $k_i < 0$.

If f is a simple scoring rule, then $f = F_{M^{-1}}$ by lemma [3.10] and we are done. Now suppose that f is not a simple scoring rule, thus there is a $j > i$ such that $x \succ_{\text{LD}} y$ for s_j but $x \not\succ_{\text{LD}} y$ for s_i. Since $x \not\succ_{\text{LD}} y$ for s_i, it follows that $p(x, \pi, s_t) = p(y, \pi, s_t)$ for $t \in \{1, \ldots, i\}$. Thus $k_1 \cdot v(x, \pi) = k_1 \cdot v(y, \pi)$, and since $k_1 > 0$ it follows that $v(x, \pi) = v(y, \pi)$. By hypothesis $p(x, \pi, s_j) > p(y, \pi, s_j)$, since $s_j(|B|) = k_j$ it follows that $k_j \cdot v(x, \pi) > k_j \cdot v(y, \pi)$, but $v(x, \pi) = v(y, \pi)$ implies that $k_j \cdot v(x, \pi) = k_j \cdot v(y, \pi)$ a contradiction. Thus f is a simple scoring rule and thus $f = F_{M^{-1}}$.

Case 3: The last case to consider is that $f(\pi_B) = X$. Since we chose B arbitrarily in the first step, we may assume that $|B|$ is minimal in $\mathcal{C}(\mathfrak{B})$, we will first show that $f(\pi_{B'}) = X$ for all $B' \in \mathfrak{B}$.

Let $D \in \mathfrak{B}$ such that $B \subset D$. Consider that by Non-Deviating,

$$f\left(\sum_{C \subset D, |C| = |B|} \pi_C\right) = f\left(\binom{|D| - 1}{|B| - 1} \pi_D\right) \tag{3.2}$$

By consistency the left hand side of equation (3.2) is X. Observe that by consistency that $f(k \cdot \pi_D) = f(\pi_D)$ for all $k \in \mathbb{N}$. Hence by transitivity $f(\pi_B) = X$.

Now let n be the number of voters determining the profile π, that is

$$n = \sum_{B \in \mathfrak{B}} \pi(B).$$

Hence $\pi = \pi_{B_1} + \cdots + \pi_{B_n}$. But since $f(\pi_{B_i}) = X$ for each B_i. It follows by consistency that

$$f(\pi) = f(\pi_{B_1} + \cdots + \pi_{B_n}) = f(\pi_{B_1}) \cap \cdots \cap f(B_n) = X \cap \cdots \cap X = X$$

Hence it follows that $f = f_X$. $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ is a lexicographical scoring function with $\mathfrak{B}$ a rich ballot space satisfies cancellation if and only if $f = F_M$, $f = F_{M^{-1}}$, or $f = f_X$ as was desired. □

Using the fact that lexicographical scoring function are implicitly neutral and consistent, we get a nice extension of Theorem 3.9 to include all rich ballot spaces.

Theorem 3.16. *A rule $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ defined on any rich ballot space satisfying neutrality, consistency, and cancellation if and only if $f = F_M$, $f = F_{M^{-1}}$ or $f = f_X$.*

Another natural extension we can discuss is the extension of Theorem 2.15 in Chapter 2 to include rich ballot spaces.

Lemma 3.17. *The social choice rules inverse approval voting and the trivial rule violate the axiom of discerning.*

Proof. Choose $B \in \mathfrak{B}$, and let $x \notin B$. Consider the profile ρ generated by all ballots of size $|B|$ where x is not an alternative of the ballot.

$$\rho = \sum_{C \subseteq X \setminus \{x\}, |C| = |B|} \pi_C$$

Observe that it follows that $v(x, \rho) = 0$ whereas $v(y, \rho) > 0$ for all $y \in X \setminus \{x\}$. By definition of Inverse Majority, $F_{M^{-1}}(\rho) = \{x\}$. By definition of f_X, $f_X(\rho) = X$. Hence both functions violate the axiom of discerning. □

We now can extend Theorem 2.15 to include all rich ballot spaces as a result of Theorem 3.15 and the previous Lemma.

Theorem 3.18. *For $\mathfrak{B}$ satisfying basic richness, a social choice function $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies neutrality, discerning, consistency, and cancellation if and only if f is majority rule.*

3.3 CHARACTERIZING MAJORITY ON j-RICH BALLOT SPACES

In this section we will assume $m \geq 4$ and so $X = \{x_1, \ldots, x_m\}$. For any integer k belonging to the interval $[1, m]$, the set

$$\mathfrak{B}_k = \{B \subseteq X : |B| = k\}$$

is the ballot space consisting of all ballots of size k. For example, $\mathfrak{B}_1$ is the ballot space consisting of each alternative on an individual ballot. That is,

$$\mathfrak{B}_1 = \{\{x\} \, : \, x \in X\}.$$

These ballot spaces satisfy basic richness and in fact if a ballot space $\mathfrak{B}$ satisfies basic richness then

$$\mathfrak{B} = \bigcup_{k \in I} \mathfrak{B}_k$$

for some nonempty subset I of $\{0, \ldots, m\}$ where $\mathfrak{B}_0 = \{\emptyset\}$.

For some $j \in \{1, \ldots, m-1\}$, we will say that a ballot space $\mathfrak{B}$ is j**-rich** if $\mathfrak{B}_j \subseteq \mathfrak{B}$. It should be noted that although ballot spaces satisfying basic richness are j-rich, a j-rich ballot space need not satisfy basic richness. For example $\mathfrak{B}_1 \cup \{\{x_1, x_2\}\}$ is a j rich ballot space, but does not satisfy basic richness since it is not permutation closed. Hence j-richness is more general than the basic richness conditions. However, both types of richness incorporate some interesting classes of ballot spaces.

We now give two simple characterizations on $(m-1)$-rich ballot spaces and 3-rich ballot spaces.

Theorem 3.19. *The function $f : {\mathbb{N}_0}^{\mathfrak{B}_{m-1}} \rightarrow P_{ne}(X)$ satisfies faithfulness, consistency, and cancellation if and only if $f = F_M$.*

Proof. We have shown that F_M satisfies consistency, faithfulness, and cancellation for all ballot spaces. It is left to show that if $f : {\mathbb{N}_0}^{\mathfrak{B}_{m-1}} \rightarrow P_{ne}(X)$ satisfies consistency, faithfulness, and cancellation, then $f = F_M$.

Let $f : {\mathbb{N}_0}^{\mathfrak{B}_{m-1}} \to P_{ne}(X)$ be an aggregation function that is faithful, consistent, and cancellative. By faithfulness,

$$f\left(\pi_{X\setminus\{x_i\}}\right) = X \setminus \{x_i\}$$

for $i = 1, \ldots, m$. By cancellation,

$$f\left(\sum_{i=1}^{m} \pi_{X\setminus\{x_i\}}\right) = X.$$

Let π be an arbitrary profile, and let

$$j = \min\{\pi(B) \,:\, B \in \mathfrak{B}\}.$$

If $j = 0$, then there exists a ballot $X \setminus \{x'\} \in \mathfrak{B}$ that gets no votes. Notice that $x' \in B$ for all $B \in \mathfrak{B}$ such that $\pi(B) > 0$. Now by faithfulness $x' \in f(\pi_B)$ for each B such that $\pi(B) > 0$. Observe that

$$\bigcap_{\pi(B)>0} f\left(\pi(B) \cdot \pi_B\right) \neq \emptyset.$$

By consistency,

$$\begin{aligned}
f(\pi) &= \bigcap_{\pi(B)>0} f\left(\pi(B) \cdot \pi_B\right) \\
&= \bigcap_{i\in I} X \setminus \{x_i\} \qquad \text{where } I = \{i : \pi(X \setminus \{x_i\}) > 0\} \\
&= F_M(\pi)
\end{aligned}$$

If $j > 0$,

$$\begin{aligned} f(\pi) &= f\left(\sum_{B\in\mathfrak{B}} \pi(B)\cdot \pi_B\right) \\ &= f\left(\left[\sum_{B\in\mathfrak{B}} (\pi(B)-j)\cdot \pi_B\right] + j\cdot\left[\sum_{B\in\mathfrak{B}} \pi_B\right]\right) \\ &= f\left(\sum_{B\in\mathfrak{B}} (\pi(B)-j)\cdot \pi_B\right) \bigcap f\left(j\cdot\left[\sum_{B\in\mathfrak{B}} \pi_B\right]\right) \\ &= f\left(\sum_{B\in\mathfrak{B}} (\pi(B)-j)\cdot \pi_B\right) \bigcap X \text{ by cancellation} \\ &= f\left(\sum_{B\in\mathfrak{B}} (\pi(B)-j)\cdot \pi_B\right) \text{ by consistency.} \end{aligned}$$

Observe that there exists a $B \in \mathfrak{B}$ such that $\pi(B) - j = 0$, hence by the previous case we have that

$$f(\pi) = f\left(\sum_{B\in\mathfrak{B}} (\pi(B)-j)\cdot \pi_B\right) = F_M(\pi)$$

as was desired. □

Theorem 3.20. *Let $\mathfrak{B}$ be a ballot space such that $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3 \subseteq \mathfrak{B}$. The aggregation function $f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies faithfulness, consistency, and cancellation if and only if $f = F_M$.*

Proof. Let $\mathfrak{B}$ be a ballot space such that $\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_3 \subseteq \mathfrak{B}$. We have shown that F_M satisfies consistency, faithfulness, and cancellation for all ballot spaces. It is left to show that if $f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies consistency, faithfulness, and cancellation, then $f = F_M$.

Let $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$, if $F_M(\pi) = X$ then we have that $v(x, \pi) = v(y, \pi)$ for all alternatives $x, y \in X$. Hence by cancellation we have that $f(\pi) = X$ which coincides with Majority Rule.

Now we will suppose that $|F_M(\pi)| = m - 1$. Without loss of generality we may assume that $F_M(\pi) = \{x_1, \ldots, x_{m-1}\}$. By way of contradiction we will assume that $x_m \in f(\pi)$ and let $l = v(x_1, \pi) - v(x_m, \pi) > 0$. Observe that by faithfulness and consistency we have that

$$\begin{aligned} f\left(\pi + l \cdot \pi_{x_m}\right) &= f(\pi) \cap f\left(\pi_{x_m}\right) \\ &= \{x_m\} \end{aligned}$$

By cancellation, $f(\pi + l \cdot \pi_{x_m}) = X$. Hence $x_m \notin f(\pi)$, that is $f(\pi) \subseteq F_M(\pi)$. Now we will suppose that $f(\pi) \neq F_M(\pi)$, hence that there exists an $x_j \in F_M(\pi) \setminus f(\pi)$, and we will let $x_i \in f(\pi)$. Since f satisfies consistency and cancellation, we have that f satisfies non-deviating by Proposition 3.13. Thus,

$$\begin{aligned} f\left(\pi + l \cdot \pi_{\{x_i, x_j, x_m\}}\right) &= f\left(\sum_{i=1}^{m} v(x_i, \pi) \cdot \pi_{\{x_i\}} + l \cdot \pi_{\{x_i, x_j\}}\right) \\ &= X \cap \{x_i, x_j\} \\ &= \{x_i, x_j\}. \end{aligned}$$

But,

$$x_i \in f(\pi) \cap f\left(l \cdot \pi_{\{x_i, x_j, x_m\}}\right)$$

and

$$x_j \notin f(\pi) \cap f\left(l \cdot \pi_{\{x_i, x_j, x_m\}}\right) \Rightarrow x_j \notin f\left(\pi + l \cdot \pi_{\{x_i, x_j, x_m\}}\right)$$

which is a contradiction. Hence we know that if $|F_M(\pi)| = m - 1$, then we have that $f(\pi) = F_M(\pi)$.

Now we will let let $\pi \in \mathbb{N}_0{}^{\mathfrak{B}}$ be a profile such that $|F_M(\pi)| = n$, for some $n \in \{1, ..., m - 2\}$. Without loss of generality, we will let $x \in F_M(\pi)$ and hence $v(x, \pi) = \max v(\pi)$. By way of induction we will suppose that if $|F_M(\pi)| = n + 1$ then $f(\pi) = F_M(\pi)$.

We will first show that, $f(\pi) \subseteq F_M(\pi)$. Consider $z \notin F_M(\pi)$ and let $l = \max v(\pi) - v(z, \pi) > 0$. By consistency, observe that if $z \in f(\pi)$ then we have that,

$$f\left(\pi + l \cdot \pi_{\{z\}}\right) = \{z\}$$

But observe that $F_M(\pi) = F_M(\pi) \cup \{z\}$ and thus $\left|F_M\left(\pi + l \cdot \pi_{\{z\}}\right)\right| = n+1$. Hence by our induction hypothesis we have that $f\left(\pi + l \cdot \pi_{\{z\}}\right) = F_M\left(\pi + l \cdot \pi_{\{z\}}\right) \neq \{z\}$ since $|\{z\}| \neq n+1$. Thus $z \notin f(\pi)$ and hence $f(\pi) \subseteq F_M(\pi)$ as was desired.

Now we will show that $F_M(\pi) \subseteq f(\pi)$. By way of contradiction, we will suppose that $F_M(\pi) \nsubseteq f(\pi)$. Thus, there exists $y \in F_M(\pi) \setminus f(\pi)$. First consider that since $z \notin F_M(\pi)$,

$$\left|F_M\left(\sum_{x_k \in X \setminus \{z\}} v(x_k, \pi)\pi_{\{x_k\}} + v(x_1, \pi) \cdot \pi_{\{z\}}\right)\right| = n+1$$

and hence we have by the induction hypothesis that,

$$f\left(\sum_{x_k \in X \setminus \{z\}} v(x_k, \pi)\pi_{\{x_k\}} + v(x_1, \pi)\pi_{\{z\}}\right) = F_M(\pi) \cup \{z\}.$$

Now observe that by non-deviating we have that

$$\begin{aligned} f\left(\pi + l \cdot \pi_{\{x,y,z\}}\right) &= f\left(\sum_{x_k \in X \setminus \{z\}} v(x_k, \pi)\pi_{\{x_k\}} + v(x_1, \pi)\pi_{\{z\}} + l \cdot \pi_{\{x,y\}}\right) \\ &= \{F_M(\pi) \cup \{z\}\} \cap \{x, y\} \\ &= \{x, y\} \end{aligned}$$

But by consistency, we have that

$$f\left(\pi + l \cdot \pi_{\{x,y,z\}}\right) = f(\pi) \cap f\left(\pi_{\{x,y,z\}}\right) = \{x\}$$

a contradiction, thus we have that $f(\pi) = F_M(\pi)$ when $|F_M(\pi)| = n$. Hence by Induction we have that $f(\pi) = F_M(\pi)$ for all $n \in \{1, \ldots, m-2\}$.

We have now shown that $f(\pi) = F_M(\pi)$ when $|F_M(\pi)| = n$ for all $n \in \{1, \ldots, m\}$, and hence $f(\pi) = F_M(\pi)$. $\square$

Using the ideas from the previous two theorems we can characterize Majority Rule on a much larger class of j-rich ballot spaces. First we state three lemmas.

Lemma 3.21. *Let $j \in \{3, \ldots, m-1\}$ and let π be any profile such that $F_M(\pi) \in \mathfrak{B}_j$ and $v(x,\pi) = v(y,\pi)$ for all $x, y \in X \setminus F_M(\pi)$. Then $f(\pi) = F_M(\pi)$.*

Proof. Let $\alpha = \max v(\pi)$ and $\beta = v(x,\pi)$ for some $x \in X \setminus F_M(\pi)$. Then $\alpha > \beta$ and so $(\alpha - \beta) > 0$. Compare the profiles

$$\rho = \binom{m-1}{j-1} \cdot \pi$$

and

$$\widehat{\rho} = \beta \cdot \sum_{B \in \mathfrak{B}_j} B + (\alpha - \beta)\binom{m-1}{j-1} F_M(\pi).$$

We get

$$v(y,\rho) = v(y,\widehat{\rho}) = \alpha \cdot \binom{m-1}{j-1}$$

for all $y \in F_M(\pi)$ and

$$v(z,\rho) = v(z,\widehat{\rho}) = \beta \cdot \binom{m-1}{j-1}$$

for all $z \in X \setminus F_M(\pi)$. Since f satisfies consistincy and cancellation, then by Lemma 3.13 f is non-deviating and thus $f(\rho) = f\left(\widehat{\rho}\right)$.

Let μ be the profile defined by

$$\mu = (\alpha - \beta) \cdot \sum_{B' \in \mathfrak{B}_j, B' \neq F_M(\pi)} \pi_{B'}$$

and note that

$$\mu + \widehat{\rho} = \alpha \cdot \sum_{B \in B_j} \pi_B.$$

Thus,

$$v(x, \rho + \mu) = v(x, \mu + \widehat{\rho}) = \alpha \cdot \binom{m-1}{j-1}$$

for all $x \in X$. By cancellation,

$$f(\rho + \mu) = f(\mu + \widehat{\rho}) = X.$$

Using consistency we get

$$f(\rho) = f(\rho + (\mu + \widehat{\rho})) = f((\rho + \mu) + \widehat{\rho}) = f(\widehat{\rho}).$$

Recall that $\rho = \binom{m-1}{j-1} \cdot \pi$ and so, by consistency, $f(\pi) = f(\rho)$. Therefore,

$$\begin{aligned}
f(\pi) &= f(\widehat{\rho}) \\
&= f\left(\beta \cdot \sum_{B \in \mathfrak{B}_j} \pi_B + (\alpha - \beta)\binom{m-1}{j-1} F_M(\pi)\right) \\
&= f\left(\beta \cdot \sum_{B \in \mathfrak{B}_j} \pi_B\right) \cap f\left(F_M(\pi)\right) \text{ by consistincy} \\
&= X \cap f\left(F_M(\pi)\right) \text{ by cancellation} \\
&= F_M(\pi) \text{ by faithfulness.}
\end{aligned}$$

Hence $f(\pi) = F_M(\pi)$ and we're done. □

For each alternative $x_i \in X$, let the profile π_{x_i} be the profile that consists of each of the j sized ballots containing x_i. That is,

$$\pi_{x_i} = \sum_{B \in \mathfrak{B}_j, x_i \in B} \pi_B.$$

Then

$$v(x_i, \pi_{x_i}) = \binom{m-1}{j-1} \text{ and } v(x_t, \pi_{x_i}) = \binom{m-2}{j-2}$$

for all $t \neq i$. It follows from consistency and faithfulness that $f(\pi_{x_i}) = \{x_i\}$.

Lemma 3.22. *Let $j \in \{3, \ldots, m-1\}$. If I is a j-element subset of $\{1, \ldots, m\}$ and*

$$\rho = \sum_{i \in I} \pi_{x_i}$$

then $f(\rho) = \{x_i : i \in I\}$.

Proof. Observe that

$$v(x_i, \rho) = \binom{m-1}{j-1} + (j-1) \cdot \binom{m-2}{j-2}$$

for all $i \in I$ and

$$v(x_t, \rho) = j \cdot \binom{m-2}{j-2}$$

for all $t \in \{1, \ldots, m\} \setminus I$. Since $F_M(\rho) = \{x_1 \ : \ i \in I\} \in \mathfrak{B}_j$, it follows from Lemma 3.21 that $f(\rho) = \{x_i : i \in I\}$. □

Lemma 3.23. *Let $j \in \{3, ..., m-1\}$. If I is a $(j-1)$-element subset of $\{1, \ldots, m\}$ and*

$$\rho = \sum_{i \in I} \pi_{x_i},$$

then $f(\rho) = \{x_i : i \in I\}$.

Proof. For each $t \in \{1, \ldots, m\} \setminus I$,

$$F_M\left(\rho + \pi_{x_t}\right) = \{x_i \ : \ i \in I\} \cup \{x_t\}.$$

Using consistency and Lemma 3.22 we get

$$f\left(\sum_{t \in \{1,\ldots,m\} \setminus I} [\rho + \pi_{x_t}]\right) = \{x_i : i \in I\}.$$

Note that

$$\sum_{t \in \{1,\ldots,m\} \setminus I} [\rho + \pi_{x_t}] = \sum_{i=1}^{m} \pi_{x_i} + (m-j)\rho.$$

By consistency and cancellation,

$$f\left(\sum_{i=1}^{m} \pi_{x_i} + (m-j)\rho\right) = f(\rho).$$

□

We now state and prove one of the main theorems of this thesis.

Theorem 3.24. *If $m \geq 4$ and $2 < j < m$, then majority rule is the only social choice function on a j-rich ballot space satisfying faithfulness, consistency, and cancellation.*

Proof. Assume that the set

$$D = \{\pi \in {\mathbb{N}_0}^{\mathfrak{B}} : f(\pi) \neq F_M(\pi)\}$$

is nonempty. So D is the set of profiles where the functions f and F_M disagree. Choose $\rho \in D$ such that $|F_M(\rho)|$ is maximal. This means that if π is a profile such that $|F_M(\pi)| > |F_M(\rho)|$, then $f(\pi) = F_M(\pi)$. Since f is cancellative and $\rho \in D$ it follows that $F_M(\rho) \neq X$. So

$$|F_M(\rho)| \leq m - 1.$$

Assume that there exists $x \in f(\rho)$ such that $x \notin F_M(\rho)$. We may assume that $x = x_1$. Let

$$\ell = \max v(\rho) - v(x_1, \rho)$$

and note that $\ell > 0$. Next, let

$$\widehat{\rho} = \alpha\rho + \ell\pi_{x_1}$$

where

$$\alpha = \binom{m-1}{j-1} - \binom{m-2}{j-2}.$$

Then

$$v(x_1, \widehat{\rho}) = \alpha \cdot v(x_1, \rho) + \ell \cdot \binom{m-1}{j-1}$$

and

$$v(x_i, \widehat{\rho}) = \alpha \cdot v(x_i, \rho) + \ell \cdot \binom{m-2}{j-2}$$

for all $i \neq 1$. If $v(x_i, \rho) = \max v(\rho) = [\ell + v(x_1, \rho)]$, then

$$\begin{aligned} v(x_i, \widehat{\rho}) &= \alpha \cdot [\ell + v(x_1, \rho)] + \ell \cdot \binom{m-2}{j-2} \\ &= \alpha \cdot v(x_1, \rho) + \ell \cdot \binom{m-1}{j-1} \\ &= v(x_1, \widehat{\rho}). \end{aligned}$$

It follows that

$$F_M(\widehat{\rho}) = F_M(\rho) \ \cup \{x_1\}.$$

By our choice of ρ and the fact that $|F_M(\widehat{\rho})| > |F_M(\rho)|$ it follows that

$$f(\widehat{\rho}) = F_M(\widehat{\rho}) = F_M(\rho) \ \cup \{x_1\}.$$

On the other hand, by consistency,

$$f(\widehat{\rho}) = f(\rho) \cap f(\pi_{x_1}) = \{x_1\}.$$

Since $F_M(\rho) \ \cup \{x_1\} \neq \{x_1\}$ we get a contradiction. It now follows that $f(\rho) \subset F_M(\rho)$.

Since $f(\rho) \subset F_M(\rho)$ and $f(\rho) \neq F_M(\rho)$, there exists $y \in F_M(\rho) \setminus f(\rho)$. Let $x \in X \setminus F_M(\rho)$ and $z \in f(\rho)$. We may assume that $x = x_1$, $y = x_2$, and $z = x_3$. As above, let

$$\ell = \max\, v(\rho) - v(x_1, \rho)$$

and note that $\ell > 0$. We now introduce the profile

$$\mu = \alpha\rho + \ell[\pi_{x_1} + \pi_{x_2} + \cdots + \pi_{x_j}].$$

By our choice of ρ we know that

$$f(\alpha\rho + \ell\pi_{x_1}) = F_M(\alpha\rho + \ell\pi_{x_1}) = F_M(\rho) \cup \{x_1\}.$$

Using consistency and Lemma 3.23,

$$f\left(\ell[\pi_{x_2} + \cdots + \pi_{x_j}]\right) = \{x_2, \ldots, x_j\}.$$

Using consistency and the fact that $x_2, x_3 \in F_M(\rho)$ we get

$$f(\mu) = f(\alpha\rho + \ell\pi_{x_1}) \cap f\left(\ell[\pi_{x_2} + \cdots + \pi_{x_j}]\right) \supseteq \{x_2, x_3\}.$$

Next, using consistency and Lemma 2, we get

$$f\left(\ell[\pi_{x_1} + \cdots + \pi_{x_j}]\right) = \{x_1, \ldots, x_j\}.$$

Since $x_3 \in f(\rho) = f(\alpha\rho)$ and $x_3 \in f\left(\ell[\pi_{x_1} + \cdots + \pi_{x_j}]\right)$ it follows that

$$f(\mu) = f(\alpha\rho) \cap f\left(\ell[\pi_{x_1} + \cdots + \pi_{x_j}]\right).$$

Since $x_2 \notin f(\rho) = f(\alpha\rho)$ it follows from the previous equation that $x_2 \notin f(\mu)$. But this contradicts the fact that $\{x_2, x_3\} \subseteq f(\mu)$. This final contradiction shows that the set $D = \{\pi \in \mathbb{N}_0 : f(\pi) \neq F_M(\pi)\}$ must be the empty set. Hence $f = F_M$ and we're done. $\square$

3.4 CHARACTERIZING MAJORITY ON RICH BALLOT SPACES

Let $|X| \geq 3$. We now introduce a ballot aggregation function on $\mathfrak{B}_1$ similar to a function we studied Section 2.3. Suppose $\leq$ is a linear order on X. For any subset $Y \subseteq X$, let $\min(Y)$ be the unique element belonging to Y such that $\min(Y) \leq y$ for all $y \in Y$. Consider the function $g' : {\mathbb{N}_0}^{\mathfrak{B}_1} \to P_{ne}(X)$ defined by

$$g'(\pi) = \begin{cases} X & \text{if } F_M(\pi) = X \\ \{\min(F_M(\pi))\} & \text{otherwise.} \end{cases}$$

Recall in Section 2.3, we showed that the rule $g : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies consistency and cancellation, it is similar to show that the rule g' satisfies consistency and cancellation. Since the ballot space is $\mathfrak{B}_1$, g' satisfies faithfulness. Hence there is no extension of Theorem 2.12 that includes all possible rich ballot spaces. Using Theorem 3.24, however, we can extend it to a large class of rich ballot spaces.

By Theorem 3.24, the last rich ballot spaces we have to consider are the 2-rich ballot spaces $\mathfrak{B}_2$ and $\mathfrak{B}_1 \cup \mathfrak{B}_2$. We first will introduce some notation and terminology. Let $\mathfrak{B}$ be a ballot space. For any profile $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$, the **non-minimal support** of π is defined by

$$supp_{nm}(\pi) = \{x \in X : v(x, \pi) > \min\, v(\pi)\}.$$

We will say that a profile π is **triangular** if $|supp_{nm}(\pi)| = 3$ and

$$\max v(\pi) + \min v(\pi) < v(y, \pi) + v(z, \pi)$$

for all $y \neq z$ belonging to $supp_{nm}(\pi)$. For example, if $m \geq 4$ and $\pi = \{x_1\} + \{x_2\} + \{x_3\}$, then π is triangular since $supp_{nm}(\pi) = \{x_1, x_2, x_3\}$, $\max v(\pi) + \min v(\pi) = 1$, and $v(x_j, \pi) + v(x_k, \pi) = 2$ for all $x_j \neq x_k$ in $supp_{nm}(\pi)$.

Let $X = \{x_1, x_2, x_3, x_4\}$, $\mathfrak{B} \in \{\mathfrak{B}_2, \mathfrak{B}_1 \cup \mathfrak{B}_2\}$, and ϕ be the permutation on X defined by $\phi(x_1) = x_2, \phi(x_2) = x_3, \phi(x_3) = x_4$, and $\phi(x_4) = x_1$. Define the rule $f_\triangle : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ by

$$f_\triangle(\pi) = \begin{cases} \{\phi(x_\ell)\} & \text{if } \pi \text{ is triangular and } \{x_\ell\} = X \setminus supp_{nm}(\pi) \\ F_M(\pi) & \text{otherwise.} \end{cases}$$

Example 3.1. *Consider the profile $\pi = \pi_{\{x_1,x_2\}} + \pi_{\{x_2,x_3\}} + \pi_{\{x_3,x_1\}}$, then π is triangular and $supp_{nm}(\pi) = \{x_1, x_2, x_3\}$. In this case, $\{x_4\} = X \setminus supp_{nm}(\pi)$ and so*

$$f_\triangle(\pi) = \{\phi(x_4)\} = \{x_1\}.$$

Observe that $F_M(\pi) = \{x_1, x_2, x_3\}$ and so $g(\pi) \neq F_M(\pi)$.

Since F_M satisfies faithfulness and cancellation it follows that the rule $f_\triangle$ satisfies faithfulness and cancellation. The difficulty arises when showing why the rule $f_\triangle$ satisfies consistency.

Lemma 3.25. *The rule $f_\triangle$ satisfies the condition of consistency.*

Proof. Let π and ρ be profiles such that $f_\triangle(\pi) \cap f_\triangle(\rho) \neq \emptyset$. We may assume without loss of generality that

$$x_1 \in f_\triangle(\pi) \cap f_\triangle(\rho).$$

We will consider three case.

Case 1. Assume that π and ρ are triangular. Since π is triangular and $x_1 \in f_\triangle(\pi)$ it follows that $f_\triangle(\pi) = \{x_1\}$ and $supp_{nm}(\pi) = \{x_1, x_2, x_3\}$. Therefore, $\min\, v(\pi) = v(x_4, \pi)$ and

$$v(x_i, \pi) + v(x_4, \pi) \leq \max\, v(\pi) + \min\, v(\pi) < v(x_j, \pi) + v(x_k, \pi)$$

whenever $\{i, j, k\} = \{1, 2, 3\}$. Similarly, $f_\triangle(\rho) = \{x_1\}$, $\min\, v(\rho) = v(x_4, \rho)$, and

$$v(x_i, \rho) + v(x_4, \rho) < v(x_j, \rho) + v(x_k, \rho)$$

whenever $\{i, j, k\} = \{1, 2, 3\}$. Notice that $\min\, v(\pi+\rho) = v(x_4, \pi+\rho)$ and $supp_{nm}(\pi+\rho) = \{x_1, x_2, x_3\}$. Choose $i \in \{1, 2, 3\}$ such that $\max\, v(\pi + \rho) = v(x_i, \pi + \rho)$. Then

$$\begin{aligned}
\max\, v(\pi + \rho) + \min\, v(\pi + \rho) &= v(x_i, \pi + \rho) + v(x_4, \pi + \rho) \\
&= (v(x_i, \pi) + v(x_4, \pi)) + (v(x_i, \rho) + v(x_4, \rho)) \\
&< (v(x_j, \pi) + v(x_k, \pi)) + (v(x_j, \rho) + v(x_k, \rho)) \\
&= v(x_j, \pi + \rho) + v(x_k, \pi + \rho)
\end{aligned}$$

for all $j \neq k$ in $\{1, 2, 3\}$. Hence $\pi + \rho$ is triangular and $f_\triangle(\pi + \rho) = \{x_1\} = f_\triangle(\pi) \cap f_\triangle(\rho)$.

Case 2. Assume that both π and ρ are not triangular. Then $f_\triangle(\pi) = F_M(\pi)$ and $f_\triangle(\rho) = F_M(\rho)$. Now $x_1 \in f_\triangle(\pi) \cap f_\triangle(\rho)$ implies that

$$v(x_1, \pi) = \max\, v(\pi) \text{ and } v(x_1, \rho) = \max\, v(\rho).$$

Thus, $\max\, v(\pi + \rho) = v(x_1, \pi + \rho)$.

Assume that there exist $j \neq k$ in $\{2, 3, 4\}$ such that

$$v(x_1, \pi) + \min\, v(\pi) < v(x_j, \pi) + v(x_k, \pi)$$

and keep in mind that $v(x_1, \pi) = \max\, v(\pi)$. Since $v(x_1, \pi) \geq v(x_j, \pi)$ and $v(x_1, \pi) \geq v(x_k, \pi)$ it follows that $v(x_j, \pi) > \min\, v(\pi)$ and $v(x_k, \pi) > \min\, v(\pi)$. Therefore,

$supp_{nm} = \{x_1, x_j, x_k\}$. Moreover, max $v(\pi) +$ min $v(\pi) < v(x_1, \pi) + v(x_j, \pi)$ and max $v(\pi) +$ min $v(\pi) < v(x_1, \pi) + v(x_k, \pi)$. Thus, by assumption,

$$\max\ v(\pi) + \min\ v(\pi) < v(x_r, \pi) + v(x_s, \pi)$$

for all $x_r \neq x_s$ in $supp_{nm}(\pi)$. This means that π is triangular contrary to the assumption that π is not triangular. Therefore,

$$v(x_1, \pi) + \min\ v(\pi) \geq v(x_j, \pi) + v(x_k, \pi)$$

for all $j \neq k$ in $\{2, 3, 4\}$. In a similar way, $v(x_1, \rho) =$ max $v(\rho)$ along with ρ not triangular implies that

$$v(x_1, \rho) + \min\ v(\rho) \geq v(x_j, \rho) + v(x_k, \rho)$$

for all $j \neq k$ in $\{2, 3, 4\}$. Since min $v(\pi + \rho) \geq$ min $v(\pi) +$ min $v(\rho)$ we have,

$$\begin{aligned} \max\ v(\pi + \rho) + \min\ v(\pi + \rho) &= v(x_1, \pi + \rho) + \min\ v(\pi + \rho) \\ &\geq [v(x_1, \pi) + \min\ v(\pi)] + [v(x_1, \rho) + \min\ v(\rho)] \\ &\geq [v(x_j, \pi) + v(x_k, \pi)] + [v(x_j, \rho) + v(x_k, \rho)] \\ &= v(x_j, \pi + \rho) + v(x_k, \pi + \rho) \end{aligned}$$

for all $j \neq k$ in $\{2, 3, 4\}$. It now follows that $\pi + \rho$ is not triangular and so

$$f_\triangle(\pi + \rho) = F_M(\pi + \rho) = F_M(\pi) \cap F_M(\rho) = f_\triangle(\pi) \cap f_\triangle(\rho).$$

Case 3. Assume π is triangular and ρ is not triangular. Since π is triangular and $x_1 \in f_\triangle(\pi)$ it follows that $f_\triangle(\pi) = \{x_1\}$, $supp_{nm}(\pi) = \{x_1, x_2, x_3\}$, and min $v(\pi) = v(x_4, \pi)$. Moreover,

$$\max\ v(\pi) + \min\ v(\pi) < v(x_j, \pi) + v(x_k, \pi)$$

for all $j \neq k$ in $\{1, 2, 3\}$. Therefore,

$$v(x_2, \pi) + v(x_4, \pi) < v(x_1, \pi) + v(x_3, \pi) \tag{3.3}$$

and

$$v(x_3, \pi) + v(x_4, \pi) < v(x_1, \pi) + v(x_2, \pi). \tag{3.4}$$

Recall from the argument given in Case 2 that $v(x_1, \rho) = \max v(\rho)$ along with ρ not triangular implies that

$$v(x_1, \rho) + \min v(\rho) \geq v(x_j, \rho) + v(x_k, \rho)$$

for all $j \neq k$ in $\{2, 3, 4\}$. Therefore,

$$v(x_1, \rho) + v(x_3, \rho) \geq v(x_2, \rho) + v(x_4, \rho) \tag{3.5}$$

and

$$v(x_1, \rho) + v(x_2, \rho) \geq v(x_3, \rho) + v(x_4, \rho). \tag{3.6}$$

Adding inequalities (3.3) and (3.5) leads to

$$v(x_2, \pi + \rho) + v(x_4, \pi + \rho) < v(x_1, \pi + \rho) + v(x_3, \pi + \rho).$$

Also, adding inequalities (3.4) and (3.6) gives

$$v(x_3, \pi + \rho) + v(x_4, \pi + \rho) < v(x_1, \pi + \rho) + v(x_2, \pi + \rho).$$

If $v(x_1, \pi + \rho) + v(x_4, \pi + \rho) < v(x_2, \pi + \rho) + v(x_3, \pi + \rho)$, then $\pi + \rho$ is triangular and $f_\triangle(\pi + \rho) = \{x_1\} = f_\triangle(\pi) \cap f_\triangle(\rho)$. Finally, if

$$v(x_1, \pi + \rho) + v(x_4, \pi + \rho) \geq v(x_2, \pi + \rho) + v(x_3, \pi + \rho),$$

then $\pi + \rho$ is not triangular and $x_1 \in f_\triangle(\pi + \rho) = F_M(\pi + \rho)$. In fact, looking at the last three inequalities we can see that $f_\triangle(\pi + \rho) = \{x_1\}$ and we're done. Thus the rule $f_\triangle$ satisfies the condition of consistency. $\square$

The rule $f_\triangle$ satisfies the conditions of faithfulness, consistency, and cancellation. This example shows why neutrality is needed in Fishburn's theorem when the ballot space is either $\mathfrak{B}_2$ or $\mathfrak{B}_1 \cup \mathfrak{B}_2$ when $m = 4$. We now consider the case of when $m \geq 4$.

Let $X = \{x_1, \ldots, x_m\}$ with $m \geq 4$ and $\mathfrak{B} \in \{\mathfrak{B}_2, \mathfrak{B}_1 \cup \mathfrak{B}_2\}$. Let $\sigma : X \to X$ be a bijective function such that $\sigma(x) \neq x$ for any $x \in X$. For any profile $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$, let

$$\widehat{\pi} = \pi + \sum_{x \in X} [v(x, \pi) + v(\sigma(x), \pi)] \cdot \pi_{\{x, \sigma(x)\}}$$

and define $f_\sigma : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ by $f_\sigma(\pi) = F_M(\widehat{\pi})$ for all profiles $\pi \in {\mathbb{N}_0}^{\mathfrak{B}}$.

Example 3.2. *Let $\sigma : X \to X$ be the permutation such that $\sigma(x_i) = x_{i+1}$ for all $i \leq m$ with the convention that $x_{m+1} = x_1$ and consider the profile*

$$\pi = \pi_{\{x_1, x_2\}} + \pi_{\{x_2, x_3\}} + \pi_{\{x_1, x_3\}}.$$

Since $v(x, \pi) = 2$ for $x \in \{x_1, x_2, x_3\}$ and $v(x, \pi) = 0$ for $x \in X \setminus \{x_1, x_2, x_3\}$, it follows that $F_M(\pi) = \{x_1, x_2, x_3\}$. Now observe that

$$\begin{aligned} f_\sigma(\pi) &= F_M(\widehat{\pi}) \\ &= F_M\left(\pi + 4 \cdot \pi_{\{x_1, x_2\}} + 4 \cdot \pi_{\{x_2, x_3\}} + 2 \cdot \pi_{\{x_3, x_4\}} + 2 \cdot \pi_{\{x_m, x_1\}}\right) \\ &= F_M\left(5 \cdot \pi_{\{x_1, x_2\}} + 5 \cdot \pi_{\{x_2, x_3\}} + \pi_{\{x_1, x_3\}} + 2 \cdot \pi_{\{x_3, x_4\}} + 2 \cdot \pi_{\{x_m, x_1\}}\right), \end{aligned}$$

and so

$$v(x, \widehat{\pi}) = \begin{cases} 8 & \text{if } x = x_1, x_3 \\ 10 & \text{if } x = x_2 \\ 2 & \text{if } x = x_4, x_m \text{ and } x_4 \neq x_m \\ 4 & \text{if } x = x_m \text{ and } x_4 = x_m \\ 0 & \text{otherwise.} \end{cases}$$

It follows that $f_\sigma(\pi) = \{x_2\} \neq F_M(\pi)$. However, we will show that f_σ satisfies cancellation, faithfulness, and consistency.

Lemma 3.26. *The rule f_σ satisfies the condition of cancellation.*

Proof. Let $\pi \in \mathbb{N}_0{}^{\mathfrak{B}}$ be a profile such that $v(x,\pi) = v(y,\pi)$ for all $x, y \in X$. By definition of $\widehat{\pi}$ we have that for any $x \in X$ that

$$\begin{aligned} v(x,\widehat{\pi}) &= v(x,\pi) + [v(x,\pi) + v(\sigma(x),\pi)] + [v(\sigma^{-1}(x),\pi) + v(x,\pi)] \\ &= 3 \cdot v(x,\pi) + v(\sigma(x),\pi) + v(\sigma^{-1}(x),\pi) \\ &= 5 \cdot v(x,\pi) \text{ by hypothesis.} \end{aligned}$$

Hence $f_\sigma(\pi) = F_M(\widehat{\pi}) = X$ and thus f_σ satisfies the condition of cancellation. $\square$

Lemma 3.27. *The rule f_σ satisfies the condition of faithfulness.*

Proof. We first consider the ballot $B = \{x\}$. We have for $x \in X$ and $\pi \in \mathbb{N}_0{}^{\mathfrak{B}}$ that,

$$\begin{aligned} v(x,\widehat{\pi}) &= v(x,\pi) + [v(x,\pi) + v(\sigma(x),\pi)] + \left[v(\sigma^{-1}(x),\pi) + v(x,\pi)\right] \\ &= 3 \cdot v(x,\pi) + v(\sigma(x),\pi) + v(\sigma^{-1}(x),\pi). \end{aligned}$$

It follows that $v\left(x, \widehat{\pi_{\{x\}}}\right) = 3$ and $v\left(y, \widehat{\pi_{\{x\}}}\right) \leq 2$ for $y \neq x$. Hence we have that $f_\sigma\left(\pi_{\{x\}}\right) = F_M\left(\widehat{\pi_{\{x\}}}\right) = \{x\}$.

We now consider $|B| = 2$. Without loss of generality, suppose that $B = \{x_1, x_i\}$ with $i > 1$. We first consider the case that $x_i \in \{x_2, x_m\}$, it follows that if $x_i = x_2$ then $\sigma(x_i) = x_3$ and $\sigma^{-1}(x_i) = x_1$ and if $x_i = x_m$ then $\sigma(x_i) = x_1$ and $\sigma^{-1}(x_i) = x_{m-1} \notin \{x_1, x_2,\}$ since $m \geq 3$. In either case we have that

$$\begin{aligned} &v\left(x_1, \widehat{\pi_{\{x_1,x_i\}}}\right) = 3 \cdot v\left(x_1, \pi_{\{x_1,x_i\}}\right) + v\left(x_2, \pi_{\{x_1,x_i\}}\right) + v\left(x_m, \pi_{\{x_1,x_i\}}\right) = 4, \\ &v\left(x_i, \widehat{\pi_{\{x_1,x_i\}}}\right) = 3 \cdot v\left(x_i, \pi_{x_1,x_i}\right) + v\left(\sigma(x_i), \pi_{\{x_1,x_i\}}\right) + v\left(\sigma^{-1}(x_i), \pi_{\{x_1,x_i\}}\right) = 4, \text{ and} \\ &v\left(x_j, \widehat{\pi_{\{x_1,x_i\}}}\right) = 3 \cdot v\left(x_j, \pi_{\{x_1,x_i\}}\right) + v\left(\sigma(x_j), \pi_{\{x_1,x_i\}}\right) + v\left(\sigma^{-1}(x_j), \pi_{\{x_1,x_i\}}\right) \leq 2 \\ &\quad \text{for } j \neq 1, 2. \end{aligned}$$

It follows that $f_\sigma\left(\pi_{\{x_1,x_i\}}\right) = F_M\left(\widehat{\pi_{\{x_1,x_i\}}}\right) = \{x_1, x_i\}$.

Now consider the case that $x_i \notin \{x_2, x_m\}$. It follows that $x_{i-1} \neq x_1, x_{m-1}$ and $x_{i+1} \neq x_1, x_3$ and observe that

$$v\left(x_1, \widehat{\pi_{\{x_1,x_i\}}}\right) = 3 \cdot v\left(x_1, \pi_{\{x_1,x_i\}}\right) + v\left(x_2, \pi_{\{x_1,x_i\}}\right) + v\left(x_m, \pi_{\{x_1,x_i\}}\right) = 3,$$
$$v\left(x_i, \widehat{\pi_{\{x_1,x_i\}}}\right) = 3 \cdot v\left(x_i, \pi_{x_1,x_i}\right) + v\left(\sigma(x_i), \pi_{\{x_1,x_i\}}\right) + v\left(\sigma^{-1}(x_i), \pi_{\{x_1,x_i\}}\right) = 3, \text{ and}$$
$$v\left(x_j, \widehat{\pi_{\{x_1,x_i\}}}\right) = 3 \cdot v\left(x_j, \pi_{\{x_1,x_i\}}\right) + v\left(\sigma(x_j), \pi_{\{x_1,x_i\}}\right) + v\left(\sigma^{-1}(x_j), \pi_{\{x_1,x_i\}}\right) \leq 2$$

for $j \neq 1, 2$.

Again it follows that It follows $f_\sigma\left(\pi_{\{x_1,x_i\}}\right) = F_M\left(\widehat{\pi_{\{x_1,x_i\}}}\right) = \{x_1, x_i\}$.

□

Lemma 3.28. *The rule f_σ satisfies the condition of consistency.*

Proof. To prove the consistency of f_σ suppose that π and ρ are profiles such that $f_\sigma(\pi) \cap f_\sigma(\rho) \neq \emptyset$. We will first observe that

$$\begin{aligned}
\widehat{\pi + \rho} &= (\pi + \rho) + \sum_{x \in X} \left[v(x, \pi + \rho) + v(\sigma(x), \pi + \rho)\right] \pi_{\{x,\sigma(x)\}} \\
&= (\pi + \rho) + \sum_{x \in X} \left[(v(x, \pi) + v(x, \rho)) + (v(\sigma(x), \pi) + v(\sigma(x) + \rho))\right] \pi_{\{x,\sigma(x)\}} \\
&= \pi + \rho + \sum_{x \in X} \left[(v(x, \pi) + v(\sigma(x), \pi)) + (v(x, \rho) + v(\sigma(x) + \rho))\right] \pi_{\{x,\sigma(x)\}} \\
&= \pi + \sum_{x \in X} \left[v(x, \pi) + v(\sigma(x), \pi)\right] \pi_{\{x,\sigma(x)\}} + \rho + \sum_{x \in X} \left[v(x, \rho) + v(\sigma(x), \rho)\right] \pi_{\{x,\sigma(x)\}} \\
&= \widehat{\pi} + \widehat{\rho}
\end{aligned}$$

Using this fact along with consistency of F_M we have that

$$\begin{aligned}
f_\sigma(\pi + \rho) &= F_M\left(\widehat{\pi + \rho}\right) \\
&= F_M\left(\widehat{\pi} + \widehat{\rho}\right) \\
&= F_M\left(\widehat{\pi}\right) \cap F_M\left(\widehat{\rho}\right) \\
&= f_\sigma(\pi) \cap f_\sigma(\rho).
\end{aligned}$$

Hence f_σ satisfies the condition of consistency. □

Using the three previous lemmas along with Example 3.2 we see that neutrality is necessary to characterize F_M on the rich ballot spaces $\mathfrak{B}_1$,$\mathfrak{B}_2$, and $\mathfrak{B}_1 \cup \mathfrak{B}_2$ when $m \geq 4$.

The remaining ballot spaces to consider occur when $m = 2$ and $m = 3$.

Lemma 3.29. *For $m = 2$ and $\mathfrak{B}$ a rich ballot space, $f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies faithfulness, consistency, and cancellation if and only if $f = F_M$.*

Proof. Since $m = 2$, $X = \{x, y\}$. Consider the case where π is a profile such that that $v(x, \pi) = v(y, \pi)$. By cancellation, $f(\pi) = X$ which agrees with $F_M(\pi)$. Now suppose π is a profile such that $v(x, \pi) \neq v(y, \pi)$, without loss of generality we will assume that $v(x, \pi) > v(y, \pi)$. Since $\mathfrak{B}$ is a rich ballot space with $m = 2$ it follows by the separation condition that $\mathfrak{B}_1 \subseteq \mathfrak{B}$. Let $l = v(x, \pi) - v(y, \pi) > 0$. By way of contradiction suppose $y \in f(\pi)$. By faithfulness and consistency, we have that

$$f(\pi + l \cdot \pi_{\{y\}}) = \{y\}.$$

However, observe that $v(x, \pi + l \cdot \pi_{\{y\}}) = v(y, \pi + l \cdot \pi_{\{y\}})$ and thus by cancellation $f(\pi + l \cdot \pi_{\{y\}}) = \{x, y\}$ contrary to $f(\pi + l \cdot \pi_{\{y\}}) = \{y\}$. Since $f(\pi) \neq \emptyset$ we have that $f(\pi) = \{x\}$ which agrees with $F_M(\pi)$. Thus $f(\pi) = F_M(\pi)$ for all profiles π. $\square$

When $m = 3$ the remaining ballot spaces to consider are $\mathfrak{B}_2$ and $\mathfrak{B}_1 \cup \mathfrak{B}_2$. Observe that when $m = 3$, $m - 1 = 2$ and thus by Theorem 3.19 we have that neutrality is not needed to characterize majority rule on $\mathfrak{B}_2$. We now consider the final rich ballot space.

While the following lemma is a consequence of Alós-Ferrer [2], we include a proof for completeness.

Lemma 3.30. *For $m = 3$ and $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2$, $f : {\mathbb{N}_0}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies faithfulness, consistency, and cancellation if and only if $f = F_M$.*

Proof. Let $\pi \in {\mathbb{N}_0}^{\mathfrak{B}_1 \cup \mathfrak{B}_2}$ be a fixed profile and without loss of generality let

$$v(x_1, \pi) \geq v(x_2, \pi) \geq v(x_3, \pi).$$

Define the profile $\hat{\pi} = \pi_{\{x_1\}} + \pi_{\{x_2\}} + \pi_{\{x_3\}}$ and observe that $v(x, \pi) = 1$ for all $x \in X$ and thus by cancellation we have $f(\hat{\pi}) = X$. Let $\alpha = v(x_1, \pi) - v(x_2, \pi) \geq 0$ and $\beta = v(x_2, \pi) - v(x_3, \pi) \geq 0$. Consider the new profile

$$\rho = v(x_3, \pi) \cdot \hat{\pi} + \beta \cdot \pi_{\{x_1, x_2\}} + \alpha \cdot \pi_{\{x_1\}},$$

and observe that

$$\begin{aligned}
v(x_1, \rho) &= v(x_3, \pi) + \beta + \alpha = v(x_3, \pi) + v(x_2, \pi) - v(x_3, \pi) + v(x_1, \pi) - v(x_2, \pi) \\
&= v(x_1, \pi), \\
v(x_2, \rho) &= v(x_3, \pi) + \beta = v(x_3, \pi) + v(x_2, \pi) - v(x_3, \pi) = v(x_2, \pi), \text{ and} \\
v(x_3, \rho) &= v(x_3, \pi).
\end{aligned}$$

Since f satisfies consistency and cancellation it follows from Proposition 3.13 that f is non-deviating and thus $f(\pi) = f(\rho)$.

By our hypothesis $F_M(\pi) \in \{\{x_1\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}\}$. We first consider that $F_M(\pi) = \{x_1, x_2, x_3\}$ and hence $v(x_1, \pi) = v(x_2, \pi) = v(x_3, \pi)$ and thus $\alpha = \beta = 0$ and $\rho = v(x_3, \pi) \cdot \hat{\pi}$. By consistency of f and the fact $f(\hat{\pi}) = X$, we have

$$f(\pi) = f(\rho) = f(v(x_3, \pi) \cdot \hat{\pi}) = f(\hat{\pi}) = X = F_M(\pi).$$

If $F_M(\pi) = \{x_1, x_2\}$, then $v(x_1, \pi) = v(x_2, \pi) > v(x_3, \pi)$ and hence $\alpha = 0$

and $\beta > 0$. By faithfulness, consistency, and the fact $f(\hat{\pi}) = X$, we have that

$$\begin{aligned} f(\pi) = f(\rho) &= f\left(v(x_3,\pi)\cdot\hat{\pi} + \beta\cdot\pi_{\{x_1,x_2\}}\right) \\ &= f\left(v(x_3,\pi)\cdot\hat{\pi}\right) \cap f\left(\beta\cdot\pi_{\{x_1,x_2\}}\right) \\ &= X \cap \{x_1,x_2\} \\ &= \{x_1,x_2\} \\ &= F_M(\pi). \end{aligned}$$

Finally we consider the case that $F_M(\pi) = \{x_1\}$. We now have $v(x_1,\pi) > v(x_2,\pi) \geq v(x_3,\pi)$ and thus $\alpha > 0$. We will suppose $\beta > 0$, then by faithfulness, consistency, and the fact $f(\hat{\pi}) = X$, we have

$$\begin{aligned} f(\pi) = f(\rho) &= f\left(v(x_3,\pi)\cdot\hat{\pi} + \beta\cdot\pi_{\{x_1,x_2\}} + \alpha\cdot\pi_{\{x_1\}}\right) \\ &= f\left(v(x_3,\pi)\cdot\hat{\pi}\right) \cap f\left(\beta\cdot\pi_{\{x_1,x_2\}}\right) \cap f\left(\alpha\cdot\pi_{\{x_1\}}\right) \\ &= X \cap \{x_1,x_2\} \cap \{x_1\} \\ &= \{x_1\} \\ &= F_M(\pi). \end{aligned}$$

A similar argument takes care of the case where $\beta = 0$. Hence for $m = 3$ and $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2$, $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ satisfies faithfulness, consistency, and cancellation if and only if $f = F_M$. $\square$

As a result of the previous propositions, lemmas, and theorems we can now state the main result of this dissertation. This Theorem is a complete characterization of all rich ballot spaces in which the condition of neutrality can be dropped and still characterize majority rule; thus this result is a bridge between the results of Fishburn and Alós-Ferrer.

Theorem 3.31.

1. *The majority rule is the only rule fulfilling faithfulness, cancellation, and consistency in any rich ballot space* $\mathfrak{B}$ *of 2 alternatives.*

2. *The majority rule is the only rule fulfilling faithfulness, cancellation, and consistency in any rich ballot space* $\mathfrak{B}$ *of 3 alternatives if and only if* $\mathfrak{B} \neq \mathfrak{B}_1$.

3. *The majority rule is the only rule fulfilling faithfulness, cancellation, and consistency in any rich ballot space* $\mathfrak{B}$ *of at least 4 alternatives if and only if* $\mathfrak{B} \notin \{\mathfrak{B}_1, \mathfrak{B}_2, \mathfrak{B}_1 \cup \mathfrak{B}_2\}$.

Now that we have a complete characterization of all rich ballot spaces in which the condition of neutrality can be dropped and still characterize majority rule; the question still remains for which j-rich ballot spaces can we characterize majority rule without the condition of neutrality.

For the remainder of this chapter we will assume $m \geq 6$ and m is even. Given Theorem 3.24, we may restrict our search to 1-rich and 2-rich ballot spaces which are not j-rich for $j \geq 3$. Two examples of ballot spaces we are considering are the 1-rich and 2-rich ballot spaces $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \{x_1, x_3, x_5\}$ and $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3 \setminus \{x_1, x_3, x_5\}$. We will show we cannot characterize majority rule without neutrality on the first ballot space. Surprisingly, we have the following conjecture:

Conjecture 3.1. *There is a function defined on the ballot space* $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_3 \setminus \{x_1, x_3, x_5\}$ *satisfying the conditions of faithfulness, cancellation, and consistency other than majority rule.*

To motivate why we believe this conjecture to be true, we will introduce the following two non-permutation closed ballot spaces

$$\mathfrak{B}_{odd} = \{B \subseteq X \ : \ i \text{ odd} \ \forall \, x_i \in B\} \text{ and}$$

$$\mathfrak{B}_{even} = \{B \subseteq X \ : \ i \text{ even} \ \forall \, x_i \in B\}.$$

Recall the function f_σ from Example 3.2, we will be considering $f_\sigma : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ where $\mathfrak{B} = \mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_{odd} \cup \mathfrak{B}_{even}$. In Lemma 3.26 we showed that f_σ satisfies the condition of cancellation for any ballot space $\mathfrak{B}$, similarly in Lemma 3.28 we showed that f_σ satisfies the condition of consistency for any ballot space $\mathfrak{B}$. It is left to show that f_σ is faithful. We will first motivate this by an example.

Example 3.3. *We will suppose that $m = 6$ and consider the profile $\pi_{\{x_1,x_3,x_5\}}$ and so*

$$\widehat{\pi_{\{x_1,x_3,x_5\}}} = \pi_{\{x_1,x_3,x_5\}} + \pi_{\{x_1,x_2\}} + \pi_{\{x_2,x_3\}} + \pi_{\{x_3,x_4\}} + \pi_{\{x_4,x_5\}} + \pi_{\{x_5,x_6\}} + \pi_{\{x_6,x_1\}}.$$

It follows that

$$v(x, \widehat{\pi_{\{x_1,x_3,x_5\}}}) = \begin{cases} 3 & \text{if } x = x_1, x_3, x_5 \\ 2 & \text{if } x = x_2, x_4, x_6. \end{cases}$$

Thus by definition of f_σ,

$$f_\sigma\left(\pi_{\{x_1,x_3,x_5\}}\right) = F_M\left(\widehat{\pi_{\{x_1,x_3,x_5\}}}\right) = \{x_1, x_3, x_5\}.$$

Lemma 3.32. *The rule f_σ defined on the ballot space $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_{odd} \cup \mathfrak{B}_{even}$ satisfies the condition of faithfulness.*

Proof. By Lemma 3.27 we have that $f_\sigma(B) = B$ for all $B \in \mathfrak{B}_1 \cup \mathfrak{B}_2$. It is left to show that $f_\sigma(B) = B$ for all $B \in \mathfrak{B}_{odd} \cup \mathfrak{B}_{even}$. To do so we will use the fact that

$$v\left(x, \widehat{\pi_B}\right) = 3 \cdot v\left(x, \pi_B\right) + v\left(\sigma(x), \pi_B\right) + v\left(\sigma^{-1}(x), \pi_B\right),$$

along with the convention that $\sigma(x_i)$ is even (odd) means that $\sigma(x_i) = x_{i+1}$ where $i+1$ is even (odd). Moreover, $\sigma(x_m) = x_1$ is odd with m even. Since m is even we have that if i is even (odd), then $\sigma(x_i)$ and $\sigma^{-1}(x_i)$ is odd (even).

Let $B \in B_{odd}$, we will first consider $v\left(x_i, \widehat{\pi_B}\right)$ with i even. Since i is even $x_i \notin B$ and thus $v(x_i, \pi_B) = 0$. Hence from the equation above we have that

$$v\left(x_i, \widehat{\pi_B}\right) = v\left(\sigma(x_i), \pi_B\right) + v\left(\sigma^{-1}(x_i), \pi_B\right) \le 2.$$

Now consider $x_i \in B$. Since i is odd, we have $\sigma(x_i)$ and $\sigma^{-1}(x_i)$ are even and hence $v\left(\sigma(x_i), \pi_B\right) + v\left(\sigma^{-1}(x_i), \pi_B\right) = 0$. From the equation above we have that

$$v\left(x_i, \widehat{\pi_B}\right) = 3 \cdot v\left(x_i, \pi_B\right) = 3.$$

Finally consider $x_i \notin B$ with i odd. Similar to above we have that $v\left(\sigma(x_i), \pi_B\right) + v\left(\sigma^{-1}(x_i), \pi_B\right) = 0$. Thus $v\left(x_i, \widehat{\pi_B}\right) = 0$.

In total we have that $v\left(x, \widehat{\pi_B}\right) = 3$ if $x \in B$, and $v\left(x, \widehat{\pi_B}\right) \leq 2$ otherwise. Hence

$$f_\sigma\left(\pi_B\right) = F_M\left(\widehat{\pi_B}\right) = B.$$

A similar argument works to show that $f_\sigma\left(\pi_B\right) = B$ for $B \in \mathfrak{B}_{even}$. Hence the rule f_σ defined on the ballot space $\mathfrak{B}_1 \cup \mathfrak{B}_2 \cup \mathfrak{B}_{odd} \cup \mathfrak{B}_{even}$ satisfies the condition of faithfulness. $\square$

CHAPTER 4
RESTRICTED BALLOT AGGREGATION RULES

In Chapter 3, we considered the model where we restrict voters to approve of ballots from a set of admissible ballots. We now consider the case that the set of admissible ballots are the possible committees to be formed.

In this model any voter of the electorate may choose any committee they approve of from the collection of possible committees, the function will collectively aggregate every voters approval committee into one of the possible committees. Thus the class of aggregation rules we will be modeling in this chapter are of the form:

$$f : \mathbb{N}_0{}^{\mathfrak{B}} \to \mathfrak{B} \setminus \{\emptyset\}.$$

Since these functions are ballot aggregation rules with a restriction on the social outcome, we refer to this class of rules as **restricted ballot aggregation rules**. The following example shows why majority rule may not be well defined.

Example 4.1. *Let* $X = \{x_1, x_2, x_3, x_4\}$ *and consider the ballot space* $\mathfrak{B} = \mathfrak{B}_1 \cup \{\emptyset, X\}$*. Observe that* $F_M : \mathbb{N}_0{}^{\mathfrak{B}} \to \mathfrak{B} \setminus \{\emptyset\}$ *is not well defined. To see this, consider for the profile* $\pi = \pi_{\{x_1\}} + \pi_{\{x_2\}}$ *we have that* $\{x \ : \ v(x, \pi) \geq v(y, \pi) \ \forall y \in X\} = \{x_1, x_2\} \notin \mathfrak{B}$*.*

Since Majority Rule is not well-defined on this space we will adjust the definitions and axioms as needed for this new model of aggregation functions. To do so we will first introduce a new set of tools to help us.

For any two ballots B_1, B_2, we say

$$B_1 \leq B_2 \text{ if } B_1 \subseteq B_2$$

Thus we have a partial order, on our ballot space. Hence when considering working with such functions we will choose to view the ballot space as a **partially ordered set** (or **poset**).

In 1990 Monjardet [4] began the first construction of an abstract axiomatic theory of consensus functions in order to account for several classes of similar concrete results in the domain of the axiomatic approach to consensus. In this chapter we will discuss characterizing the approval voting rule on lattices under the latticial framework of Monjardet. Before we begin we will introduce some preliminaries from the theory of partially ordered sets.

Once we have developed the framework to introduce our model, it will be our goal to see when we can generalize the three main characterizations of approval voting from Chapter 2. The main characterizations of approval voting are:

1. **Theorem 2.10.** A rule $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies faithfulness, consistency, and is non-deviating if and only if f is the approval voting rule.

2. **Theorem 2.12.** A rule $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies faithfulness, consistency, and cancellation if and only if f is the approval voting rule.

3. **Theorem 2.15.** A rule $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies neutrality, consistency, cancellation, and is discerning if and only if f is the approval voting rule.

In Section 4.3 we will present a generalization of Theorem 2.10 to offer a characterization using the conditions of faithfulness, consistency, and non-deviating to include all distributive lattices. In this section we will also show that we cannot extend Theorem 2.15 to include all distributive lattices.

Next in Section 4.4 we first show we can extend our result to include analogues of Theorem 2.12 and Theorem 2.15 to include boolean lattices. We then conclude the Chapter with a conjecture about a class of distributive classes.

4.1 PRELIMINARIES

Before we begin working with aggregation functions on lattices, we start by providing the definitions of some necessary terms as well as some well-known results. Much of this can be found in reference [5] or other introductory books on order. For the purposes of this dissertation, we will assume the lattices we refer to are finite lattices unless otherwise stated.

For a non-empty set P, a partial order on P is a binary relation $\leq$ on P that is reflexive, antisymmetric, and transitive. That is, for all x, y, z in P,

(i) $x \leq x$,

(ii) $x \leq y$ and $y \leq x$ imply $x = y$,

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$.

A set P equipped with a partial order is called a **partially ordered set**, or **poset** for short and is denoted by $(P, \leq)$. Furthermore, P is said to have a **bottom** element 0 if for all $x \in P$, $0 \leq x$ and a **top** element 1 if for all $x \in P$, $x \leq 1$. Let $x < y$ denote the case that $x \leq y$ and $y \nleq x$. We say that **x covers y** in the poset if $y < x$ and there is no z with $y < z < x$.

We use the language *partial* to imply that not all elements are related by this relationship. If $x \nleq y$ and $y \nleq x$, then we say that x and y are **non-comparable** and denote it $x||y$. A poset in which every two elements are related is called a **totally ordered set**. For example, any non-empty subset of real numbers $\mathbb{R}$ with the standard order, is a totally ordered set.

Given any subset S of a poset P, the **join of S** is the supremum (or least upper bound) and denoted, $\bigvee S$, if it exists. Similarly the **meet of S** is the infimum (or greatesest lower bound) and denoted, $\bigwedge S$. If $S = \{x_1, \dots, x_n\}$ we denote $\bigvee S = x_1 \vee \cdots \vee x_n$ and $\bigwedge S = x_1 \wedge \cdots \wedge x_n$. Although the meet and join need not exist, for our purposes we want to consider the case in which it does. A partially ordered set P is called a **lattice** if for all $x, y \in P$ $x \vee y \in P$ and $x \wedge y \in P$ and in this case we denote $P = L$. Note, in the case we have that L is a finite lattice, we have that $\bigvee \emptyset = 0$ and $\bigwedge \emptyset = 1$.

Let L be a lattice. An element $x \in L$ is **meet-irreducible** if $x \neq 1$ and $x = a \wedge b$ implies $x = a$ or $x = b$ for all $a, b \in L$. Similarly $x \in L$ is **join-irreducible** if $x \neq 0$ and $x = a \vee b$ implies $x = a$ or $x = b$ for all $a, b \in L$. We let $J(L)$ be the set of all join-irreducible elements of L and $M(L)$ be the set of all meet-irreducible elements L.

The notion of join-irreducibles give us a nice lemma involving equivalence of elements in a lattice. It is well known that every element in a finite lattice L is the join of all the join-irreducibles less than or equal to it. This leads us to the following proposition, a proof of which can be found in any introductory text on partially ordered sets.

Lemma 4.1. *For any finite lattice L and $x, y \in L$,*

$$x = y \iff [j \leq x \iff j \leq y \text{ for all } j \in J(L)].$$

We now will consider a special class of lattices. A lattice L is said to be **distributive** if it satisfies the distributive law, that is, for all $a, b, c \in L$, we have $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$ and $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$. The topic of distributivity of lattice leads us to two very important well known facts about distributive lattices.

Lemma 4.2. *For any distributive lattice L, if $j \leq x_1 \vee \cdots \vee x_n$ with $j \in J(L)$. Then $j \leq x_i$ for some $i \in \{1, \dots, n\}$.*

Proof. Suppose that $j \leq x_1 \vee \cdots \vee x_n$, and observe that since $j \leq j$ we have by the definition of meet that

$$\begin{aligned} j &= j \wedge (x_1 \vee \cdots \vee x_n) \\ &= (j \wedge x_1) \vee \cdots \vee (j \wedge x_n) \text{ by distributivity.} \end{aligned}$$

By the definition of join irreducible we have that that $j = j \wedge x_i$ for some x_i. Thus by the definition of meet we have that $j \leq x_i$. □

To introduce the next important fact about distributive lattices we first establish definitions of two classes of lattices. The lattice M_n has a bottom element 0, a top element 1, and n elements $a_1, \dots a_n$ such that $a_i || a_j$ for all i, j where $i \neq j$. The lattice N_k has a bottom element 0, a top element 1, $k - 3$ elements such that $b_1 \leq b_2 \leq \cdots \leq b_{k3}$, and then a single element denoted a that is non-comparable to any b_i.

In this dissertation we will sometimes define a lattice by its Hasse Diagram. A **Hasse Diagram** is a graphical representation of the relation of elements of a partially ordered set with an implied upward orientation. A point is drawn for each element of the partially ordered set and joined with the line segment according to the following rules:

1. If $p < q$ in the poset, then the point corresponding to p appears lower in the drawing then the point corresponding to q.

2. The two points p and q will be joined by a line segment if and only if p covers q or q covers p.

Hasse Diagrams of M_n and N_k can be found in Figure 4.2 and Figure 4.3, respectively. It is well known that the lattices M_n and N_k are not distributive.

We now introduce an important class of distributive lattices. In a lattice L with 0 and 1, y is a **complement** of x if $x \wedge y = 0$ and $x \vee y = 1$. A **boolean**

lattice is a distributive lattice with 0 and 1 with $0 \neq 1$ in which every element has a complement.

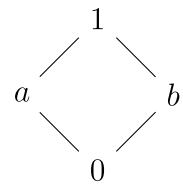

Figure 4.1: M_2

Example 4.2. *For the boolean lattice $M_2 = \{0, a, b, 1\}$, we have $0 < a < 1$ and $0 < b < 1$ with $a||b$. We also have the following tables of joins of any 2 elements of M_2, and the meets of any 2 elements of M_2.*

Table 4.1: Joins of M_2

	0	a	b	1
0	0	a	b	1
a	a	a	1	1
b	b	1	b	1
1	1	1	1	1

Table 4.2: Meets of M_2

	0	a	b	1
0	0	0	0	0
a	0	a	0	a
b	0	0	b	b
1	0	a	b	1

4.2 APPROVAL VOTING ON LATTICES

Let L be a finite lattice, and let $J(L)$ be the set of join-irreducibles of L. Moreover, we will require $|J(L)| \geq 2$. Let L^k be the set of all k-tuples of L. A k-tuple $(x_1, \ldots, x_k) \in L^k$ is called a **profile** on L of length k and we denote the length by $|\pi|$. Let L^* be the set of all profiles of positive finite length, that is

$$L^* = \bigcup_{k \geq 1} L^k.$$

The type of functions we are considering will be of the form

$$f : L^* \to L.$$

We will refer to this class of functions as **aggregation functions on L** or **aggregation rules on L**.

For a profile $\pi = (x_1, \ldots, x_k) \in L^*$ and join irreducible $s \in J(L)$ we let,

$$K_s(\pi) = \{i \,:\, s \leq x_i\}.$$

Note that if π is a profile such that $K_s(\pi) = \emptyset$ for all join irreducibles $s \in J(L)$, then $x_1 = \cdots = x_k = 0$. In this case we will call π a **zero profile**. An example is the profile consisting of a single voter casting the ballot 0.

Lemma 4.3. *For any profile $\pi \in L^*$ and join irreducibles $j, j' \in J(L)$,*

$$j \leq j' \Rightarrow |K_{j'}(\pi)| \leq |K_j(\pi)|.$$

Proof. Suppose $j \leq j'$, it follows that $K_j(\pi) \subseteq K_{j'}(\pi)$ and hence we have $|K_{j'}(\pi)| \leq |K_j(\pi)|$ as was desired. $\square$

Monjardet [13] defines **majority rule** as the consensus function $F_M : L^n \to L$ defined by

$$F_M(\pi) = \bigvee \{j \in J(L) \;:\; |K_j(\pi)| \geq n/2\}.$$

However this version of majority rule doesn't fit with the one we have in this dissertation, thus we will provide a new version that agrees with the model we have discussed. The **approval set with respect to the profile** π is defined by

$$A(\pi) = \{s \in J(L) \,:\, |K_s(\pi)| \geq |K_{s'}(\pi)| \;\forall\, s' \in J(L)\}.$$

Using this, the **approval voting rule** is the aggregation rule $F_A : L^* \to L$ defined by, for all profiles $\pi \in L^*$,

$$F_A(\pi) = \bigvee A(\pi).$$

Observe that for any profile π, $A(\pi) \neq \emptyset$ and thus $F_A(\pi) \neq 0$. Furthermore, in the case that π is a zero profile, $A(\pi) = J(L)$ and thus $F_A(\pi) = \bigvee J(L) = 1$.

Example 4.3. *For $L = M_2$ (Figure 4.1), we have $J(M_2) = \{a, b\}$ and consider the profile $\pi = (0, a, b, 1)$. We have $K_a(\pi) = \{2, 4\}$ and $K_b(\pi) = \{3, 4\}$, and it follows that $A(\pi) = \{a, b\}$ thus $F_A(\pi) = a \vee b = 1$.*

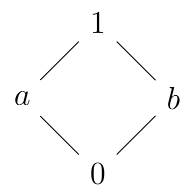

Figure 4.1: M_2

Proposition 4.4. *For L a distributive lattice and any $j \in J(L)$ the following holds,*

$$j \leq F_A(\pi) \iff j \in A(\pi).$$

Proof. Let L be a distributive lattice and suppose that $j \in A(\pi)$ for some $j \in J(L)$. By definition of F_A we have that $j \leq \bigvee A(\pi) = F_A(\pi)$.

Now suppose L is a distributive lattice and suppose $j \leq F_A(\pi)$ for some $j \in J(L) \setminus A(\pi)$. It follows that $|K_j(\pi)| < |K_{j'}(\pi)|$ for all $j' \in A(\pi) = \{s_1, \ldots, s_n\}$. Consider that $j \leq F_A(\pi) = s_1 \vee \cdots \vee s_n$. Since L is distributive, it follows by Lemma 4.2 that $j \leq s_i$ for some $s_i \in A(\pi)$. Observe that since $j \leq s_i$ it follows by Lemma 4.3 that $|K_j(\pi)| > |K_{s_i}(\pi)|$ contrary to assumption that $j \notin A(\pi)$. Therefore it follows that $j \leq F_A(\pi)$ if and only if $j \in A(\pi)$. $\square$

The functions we discussed in Chapters 2 and 3 were implicitly anonymous, these are not. A function $f : L^* \to L$ is said to be **anonymous** if for any profile $\pi = (x_1, \ldots, x_k)$ and every permutation $\sigma : N_k \to N_k$, where $N_k = \{1, \ldots, k\}$, $f(\pi^\sigma) = f(\pi)$. That is,

$$f(x_{\sigma(1)}, \ldots, x_{\sigma(k)}) = f(x_1, \ldots, x_k).$$

Proposition 4.5. *The approval voting rule satisfies the condition of anonymity.*

Proof. Suppose π is a zero profile, then given any permutation σ, π^σ is also a zero profile. Thus it follows by definition of F_A that $F_A(\pi) = 1 = F_A(\pi^\sigma)$.

Now suppose π is a nonzero profile and $\sigma : N_{|\pi|} \to N_{|\pi|}$ is a permutation. Consider that,

$$\begin{aligned} A(\pi) &= \{s \in J(L) \,:\, |K_s(\pi)| \geq |K_{s'}(\pi)| \,\forall\, s' \in J(L)\} \\ &= \{s \in J(L) \,:\, |K_s(\pi^\sigma)| \geq |K_{s'}(\pi^\sigma)| \,\forall\, s' \in J(L)\} \\ &= A(\pi^\sigma). \end{aligned}$$

Therefore

$$F_A(\pi^\sigma) = \bigvee A(\pi^\sigma) = \bigvee A(\pi) = F_A(\pi).$$

Hence the approval voting rule satisfies the condition of anonymity. $\square$

A function $f : L^* \to L$ is said to be **consistent** if for all profiles $\pi = (x_1, \ldots, x_k)$ and $\pi' = (y_1, \ldots, y_j)$,

$$f(\pi) \wedge f(\pi') \neq 0 \Rightarrow f(\pi\pi') = f(\pi) \wedge f(\pi'),$$

where $\pi\pi'$ is the profile where you start with the elements from the profile π and end with the elements from the profile π'. That is,

$$\pi\pi' = (x_1, \ldots, x_k, y_1, \ldots, y_j).$$

Proposition 4.6. *For any distributive lattice, the approval voting rule is consistent.*

Proof. Let L be a distributive lattice and let π and π' be profiles such that $F_A(\pi) \wedge F_A(\pi') \neq 0$. First observe that for any join irreducible $j \in J(L)$,

$$|K_j(\pi\pi')| = |K_j(\pi)| + |K_j(\pi')|.$$

We will first show that $j \leq F_A(\pi\pi') \Rightarrow j \leq F_A(\pi) \wedge F_A(\pi')$ for all $j \in J(L)$. Let $j \in J(L)$ such that $j \leq F_A(\pi\pi')$. Hence it follows by Proposition 4.4 that

$j \in A(\pi\pi')$. By way of contradiction suppose that $j \not\leq F_A(\pi)$. Since $F_A(\pi) \neq 0$ and $F_A(\pi) \wedge F_A(\pi') \neq 0$, there exists $s \in J(L)$ such that $s \leq F_A(\pi) \wedge F_A(\pi')$. Hence $s \leq F_A(\pi)$ and thus $|K_s(\pi)| > |K_j(\pi)|$. Given $|K_j(\pi\pi')| \geq |K_s(\pi\pi')|$ it follows that

$$\begin{aligned} |K_j(\pi)| + |K_j(\pi')| &\geq |K_s(\pi)| + |K_s(\pi')| \\ |K_j(\pi')| &\geq |K_s(\pi')| + |K_s(\pi)| - |K_j(\pi)| \\ |K_j(\pi')| &> |K_s(\pi')| \ \text{ since } |K_s(\pi)| - |K_j(\pi)| > 0, \end{aligned}$$

contrary to $s \leq F_A(\pi)$. Thus $j \leq F_A(\pi) \wedge F_A(\pi')$.

Now we will show that if $j \leq F_A(\pi) \wedge F_A(\pi')$ then $j \leq F_A(\pi\pi')$ for all $j \in J(L)$. Let $j \in J(L)$ such that $j \leq F_A(\pi) \wedge F_A(\pi')$. By definition of meet, $j \leq F_A(\pi)$ and $j \leq F_A(\pi')$. Hence it follows by Proposition 4.4 that $j \in A(\pi)$ and $j \in A(\pi')$.

Given $j \in A(\pi)$ and $j \in A(\pi')$, it follows that $|K_j(\pi)| \geq |K_{j'}(\pi)|$ for all $j' \in J(L)$. Similarly that $|K_j(\pi')| \geq |K_{j'}(\pi')|$ for all $j' \in J(L)$. Observe that,

$$\begin{aligned} |K_j(\pi\pi')| &= |K_j(\pi)| + |K_j(\pi')| \\ &\geq |K_{j'}(\pi)| + |K_{j'}(\pi')| \, \forall \, j' \in J(L) \\ &= |K_j(\pi\pi')|. \end{aligned}$$

Thus it follows that $j \in A(\pi\pi')$. Therefore by Proposition 4.4 $j \leq F_A(\pi\pi')$.

Hence it follows by Lemma 4.1 that $F_A(\pi\pi') = F_A(\pi) \wedge F_A(\pi')$ and thus F_A satisfies consistency. □

We will now show that Proposition 4.6 does not hold for the non-distributive lattices M_n for $n \geq 3$.

Example 4.4. *Let $n \geq 3$ and consider approval voting defined on M_n (Figure 4.2). Let $\pi = (a_1, a_2)$ and let $\rho = (a_1, a_3)$. Observe that $F_A(\pi) = \bigvee\{a_1, a_2\} = 1$, $F_A(\pi) = \bigvee\{a_1, a_3\} = 1$, and $F_A(\pi\rho) = \bigvee\{a_1\} = a_1$. Therefore we have that*

$$1 = F_A(\pi) \wedge F_A(\rho) \neq F_A(\pi\rho) = a_1.$$

And thus approval voting on M_n violates the condition of consistency.

Based on the previous proposition and example we offer the following conjecture:

Conjecture 4.1. *If F_A is consistent on a finite lattice L, then L is distributive.*

Here is a nice generalization of the discerning condition introduced by Duddy & Piggins [6]: a function $f : L^* \to L$ is said to be **discerning** if for all profiles π, and all $s \in J(L)$,

(i) If $|K_s(\pi)| > |K_{s'}(\pi)|$ for all $s' \in J(L) \setminus \{s\}$, then $s \leq f(\pi)$.

(ii) If $|K_s(\pi)| < |K_{s'}(\pi)|$ for all $s' \in J(L) \setminus \{s\}$, then $s \not\leq f(\pi)$.

Proposition 4.7. *For any lattice, the approval voting rule satisfies the first condition of discerning.*

Proof. To show that F_A satisfies the first condition of discerning, consider a profile π such that $|K_s(\pi)| > |K_{s'}(\pi)|$ for all $s' \in J(L) \setminus \{s\}$. Therefore, $s \in A(\pi)$ and so $s \leq \bigvee A(\pi) = F_A(\pi)$. In fact, $A(\pi) = \{s\}$ and $F_A(\pi) = s$. Therefore, F_A satisfies the first condition of discerning on any finite lattice L. □

Proposition 4.8. *For any distributive lattice, the approval voting rule satisfies the second condition of discerning.*

Proof. Let L be a distributive lattice. Suppose π is a profile such that $|K_s(\pi)| < |K_{s'}(\pi)|$ for all $s' \in J(L) \setminus \{s\}$. Then $s \notin A(\pi)$ and so by Proposition 4.4, $s \not\leq F_A(\pi)$. □

Consider the Non-Distributive Lattice M_n for $n \geq 3$.

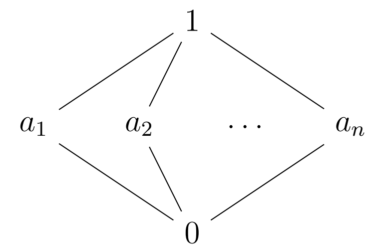

Figure 4.2: M_n

Non-Distributive for $n \geq 3$

Lemma 4.9. *The rule F_A defined on M_n for $n \geq 3$ violates condition (ii) of discerning.*

Proof. Given the profile $\pi = (a_2, \ldots, a_n)$, $|K_{a_1}(\pi)| = 0 < |K_{a_i}(\pi)| = 1$ for all $a_i \in \{a_2, \ldots, a_n\} = J(L) \backslash \{a_1\}$. By the definition of F_A, $F_A(\pi) = \bigvee\{a_2, \ldots, a_n\} = 1$ hence $a_1 \leq 1$ so F_A does not satisfy condition (ii) of Discerning. $\square$

Consider the non-distributive lattice N_k for $k \geq 5$.

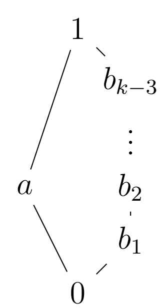

Figure 4.3: N_k

Nondistributive for $k \geq 5$

Lemma 4.10. *The rule F_A defined on N_k for $k \geq 5$ violates condition (ii) of Discerning.*

Proof. Given the profile $\pi = (a, b_{k-4})$, consider that $|K_{b_{k-3}}(\pi)| = 0 < |K_x(\pi)| = 1$ for all $x \in \{a, b_1, \ldots, b_{k-4}\} = J(L) \setminus \{b_{k-3}\}$. By the definition of F_A, $F_A(\pi) =$

$\bigvee\{a, b_1, \ldots, b_{k-4}\} = 1$ hence $b_{k-3} \leq 1$ and so F_A does not satisfy condition (ii) of discerning. $\square$

By Proposition 4.8 and the two previous lemmas we are left with the following conjecture:

Conjecture 4.2. *The approval voting rule on a lattice L is discerning if and only if L is distributive.*

A function $f : L^* \to L$ is said to be **non-deviating** if for any $\pi, \pi' \in L^*$ such that $|K_j(\pi)| = |K_j(\pi')|$ for all $j \in J(L)$ then $f(\pi) = f(\pi')$

Proposition 4.11. *The F_A rule satisfies the condition of non-deviating.*

Proof. Suppose π and π' are profiles such that $|K_j(\pi)| = |K_j(\pi')|$ for all $j \in J(L)$. Thus it follows that $A(\pi) = A(\pi')$, and thus $F_A(\pi) = \bigvee A(\pi) = \bigvee A(\pi') = F_A(\pi')$. Hence the F_A rule is non-deviating. $\square$

A function $f : L^* \to L$ is said to satisfy **cancellation** if for all profiles π, if $|K_s(\pi)| = |K_{s'}(\pi)|$ for all $s, s' \in J(L)$ then $f(\pi) = 1$.

Proposition 4.12. *The F_A rule satisfies the condition of cancellation.*

Proof. Let π be a profile such that $|K_s(\pi)| = |K_{s'}(\pi)|$ for all join irreducibles $s, s' \in J(L)$. Then $A(\pi) = J(L)$ and it follows from the definition of F_A that $F_A(\pi) = \bigvee A(\pi) = 1$, thus F_A satisfies cancellation. $\square$

Another nice generalization of our previous axioms is the extension of neutrality to latticial models of consensus. A function $f : L^* \to L$ is said to be **neutral** if for any profiles π and π' and for any permutation $\sigma : J(L) \to J(L)$ if $K_s(\pi) = K_{\sigma(s)}(\pi')$ for all join irreducibles $s \in J(L)$ then

$$j \leq f(\pi) \text{ for some } j \in J(L) \Rightarrow \sigma(j) \leq f(\pi').$$

Proposition 4.13. *For any distributive lattice, the approval voting rule satisfies neutrality.*

Proof. Suppose π and π' are profiles, and $\sigma : J(L) \to J(L)$ is a permutation such that $K_s(\pi) = K_{\sigma(s)}(\pi')$ for all join irreducibles $s \in J(L)$. Observe that $\sigma[A(\pi)] = A(\pi')$.

Suppose $j \leq F_A(\pi)$, given the rule is defined on a distributive lattice, then it follows by Proposition 4.4 that $j \in A(\pi)$. Therefore $\sigma(j) \in A(\pi')$ and thus $\sigma(j) \leq F_A(\pi')$. Thus the rule F_A defined on a distributive lattice satisfies Neutrality. $\square$

A function $f : L^* \to L$ is said to be **faithful**[1] if for all $x \in L\setminus\{0\}$, $f\left(\pi_{\{x\}}\right) = x$ where $\pi_{\{x\}} = (x)$.

Proposition 4.14. *The approval voting rule satisfies the condition of faithfulness.*

Proof. Consider the profile $\pi_{\{x\}}$. By definition of the approval set, $A\left(\pi_{\{x\}}\right) = \{s \in J(L) : s \leq x\}$. Recall that in any finite lattice, $x = \bigvee\{s \in J(L) : s \leq x\}$, thus it follows that $F_A\left(\pi_{\{x\}}\right) = x$. Hence the approval voting rule satisfies the condition of faithfulness. $\square$

4.3 CHARACTERIZING APPROVAL VOTING ON DISTRIBUTIVE LATTICES

To begin our first characterization of the approval voting rule on the latticial model we first introduce a key property the previous models of consensus had. A function $f : L^* \to L$ is said to be **productive** if for all profiles π, $f(\pi) \neq 0$. Observe that since $A(\pi) \neq \emptyset$ for any profile π, it follows that the approval voting rule is indeed productive.

[1]We define faithfulness to be consistent with our definition of faithful in the model of Chapter 2. However, this is different from the well establish definition of faithfulness on the latticial model which requires $f(0) = 0$.

We now state and prove the main result of Chapter 4. This is a generalization of Theorem 2.10.

Theorem 4.15. *The aggregation function $f : L^* \to L$ on a distributive lattice, satisfies productivity, consistency, faithfulness, and non-deviating if and only if $f = F_A$.*

Proof. We have shown that F_A defined on a distributive lattice satisfies productivity, consistency, faithfulness, and non-deviating. It is left to show that if $f : L^* \to L$ satisfies productivity, consistency, faithfulness, and non-deviating; then $f = F_A$.

Let π be a zero profile, by productiveness we have that $f(\pi) \neq 0$. By faithfulness we have that $f(1) = 1$. So observe that by consistency we have that

$$0 \neq f(\pi) = f(\pi) \wedge 1 = f(\pi) \wedge f(1) = f(\pi 1).$$

Since π is a zero profile, we have by non-deviating that $f(\pi 1) = f(1) = 1$. Hence $f(\pi) = 1$ as was desired.

Now let π be a fixed non-zero profile. Since π is non-zero, the integer $l = |K_j(\pi)|$ for $j \in A(\pi)$ is strictly greater than 0. For each integer t in the interval $[1, l]$ let

$$B_t = \{j \in J(X) : t \leq |K_j(\pi)|\} \text{ and } b_t = \bigvee B_t.$$

Observe that

$$b_1 = \bigvee \{j \in J(L) \ : \ K_j(\pi) \neq \emptyset\} \text{ and } b_l = F_A(\pi).$$

Also notice that

$$b_{j_1} \leq b_{j_2} \text{ if } j_1 \geq j_2.$$

Consider the profile

$$\pi^* = (b_1, \ldots, b_l).$$

Since f is faithful, it follows that $f(b_j) = b_j$ for $j = 1, \cdots, l$. Therefore,

$$\bigwedge \{f(b_j) \, : j \in [1, l]\} = \bigwedge \{b_j \, : j \in [1, l]\} = b_l \neq 0.$$

Since f satisfies consistency we get

$$f(\pi^*) = \bigwedge \{f(b_j) \, : j \in [1, l]\} = b_l = F_M(\pi).$$

Thus $f(\pi^*) = F_A(\pi)$.

For each $s \in J(X)$, $s \leq b_j$ if and only if $j = 1, 2, \cdots, |K_s(\pi)|$ and so $|K_s(\pi^*)| = |K_s(\pi)|$. Since f satisfies non-deviating it follows that $f(\pi^*) = f(\pi)$. Hence $f(\pi) = F_A(\pi)$ and we're done. $\square$

Theorem 2.15 in Chapter 2, we characterized approval voting with the axioms of productivity, neutrality, consistency, cancellation, and discerning. We will show that on the latticial model we cannot extend this characterization to include distributive lattices.

Let $n \geq 2$ and consider the following distributive lattice:

$$\begin{array}{c} 1 \\ | \\ a_{n-1} \\ | \\ \cdots \\ | \\ a_1 \\ | \\ 0 \end{array}$$

Figure 4.4: n-chain

Define $f : L^* \to L$ by,

$$f(\pi) = \begin{cases} a_{n-1} & \text{if } a_{n-1} \in \pi \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.16. *The rule f satisfies productivity.*

Proof. For any profile $\pi \in L^*$ we have that $f(\pi) \in \{a_{n-1}, 1\}$. Hence the rule f satisfies the condition of productivity. $\square$

Lemma 4.17. *The rule f satisfies neutrality.*

Proof. To show f satisfies Neutrality, suppose π and π' are profiles such that given $\sigma : J(X) \to J(X)$, $K_s(\pi) = K_{\sigma(s)}(\pi')$ for all join irreducibles $s \in J(X)$.
By way of contradiction suppose that

$$j \leq f(\pi) \text{ and } \sigma(j) \not\leq f(\pi')$$

Observe that $\sigma(j) = 1$ and $f(\pi') = a_{n-1}$. Since $f(\pi') = a_{n-1}$ it follows that

$$|K_1(\pi')| < |K_{a_{n-1}}(\pi')| \leq |K_{a_i}(\pi')| \text{ for all } i \leq n-1$$

But since $K_s(\pi) = K_{\sigma(s)}(\pi')$ it follows that

$$|K_{\sigma^{-1}(1)}(\pi')| < |K_{\sigma^{-1}(a_{n-1})}(\pi')| \leq |K_{\sigma^{-1}(a_i)}(\pi')| \text{ for all } i \leq n-1$$

Observe that since $a_1 \leq a_2 \leq \cdots \leq a_n \leq 1$ for all profiles π we have that

$$|K_1(\pi)| \leq |K_{a_n}(\pi)| \leq \ldots \leq |K_{a_2}(\pi)| \leq |K_{a_1}(\pi)|.$$

It follows that $\sigma(1) = 1$ and $\sigma(a_{n-1}) = a_{n-1}$, and thus $1 = j \not\leq f(\pi) = a_{n-1}$ a contradiction. Thus f satisfies neutrality. $\square$

Lemma 4.18. *The rule f satisfies consistency.*

Proof. Let $\pi, \pi' \in L^*$ such that $f(\pi) \wedge f(\pi') \neq 0$. First consider the case that $a_{n-1} \notin \pi$ and $a_{n-1} \notin \pi'$, it follows that $a_{n-1} \notin \pi\pi'$. It now follows from the definition of f that $f(\pi) = 1$, $f(\pi') = 1$, and $f(\pi\pi') = 1$, thus we have

$$f(\pi) \wedge f(\pi') = 1 \wedge 1 = 1 = f(\pi\pi').$$

Now consider the case that $a_{n-1} \in \pi$ or $a_{n-1} \in \pi'$, without loss of generality we will assume that $a_{n-1} \in \pi$ and it follows that $a_{n-1} \in \pi\pi'$. By definition of f we have that $f(\pi) = a_{n-1}$, $f(\pi') \in \{a_{n-1}, 1\}$, and $f(\pi\pi') = a_{n-1}$, thus we have

$$f(\pi) \wedge f(\pi') = a_{n-1} \wedge f(\pi') = a_{n-1} = f(\pi\pi').$$

Hence we have that the rule f satisfies consistency. $\square$

Lemma 4.19. *The rule f satisfies the condition of discerning.*

Proof. To show f satisfies discerning, we first consider a profile π where there exists $s \in J(X)$ such that $|K_s(\pi)| < |K_{s'}(\pi)| \,\forall\, s' \in J(X) \setminus \{s\}$. Observe that since $a_1 \leq a_2 \leq \cdots \leq a_n \leq 1$ for all profiles π we have that

$$|K_1(\pi)| \leq |K_{a_n}(\pi)| \leq \ldots \leq |K_{a_2}(\pi)| \leq |K_{a_1}(\pi)|.$$

It follows that $s = 1$ and $s' = a_n$, observe the only way this can happen is if $a_{n-1} \in \pi$ and thus $f(\pi) = a_{n-1}$ with $1 \not\leq f(\pi)$.

We now consider a profile π where there exists $s \in J(X)$ such that $|K_s(\pi)| > |K_{s'}(\pi)| \,\forall\, s' \in J(X) \setminus \{s\}$. From above we know that $s = a_1$ and thus $a_1 \leq f(\pi) \in \{a_{n-1}, 1\}$. It follows that the rule f is discerning. $\square$

Lemma 4.20. *The rule f satisfies cancellation.*

Proof. Let $\pi \in L^*$ such that

$$|K_1(\pi)| = |K_{a_n}(\pi)| = \ldots = |K_{a_2}(\pi)| = |K_{a_1}(\pi)|.$$

The only way that $|K_1(\pi)| = |K_{a_n}(\pi)|$ is if $a_{n-1} \notin \pi$. Hence by the definition of f we have that $f(\pi) = 1$ and thus f satisfies cancellation. $\square$

Based on the previous lemmas, we now know that we cannot characterize approval voting on the entire class of distributive lattices with the axioms of productivity, neutrality, consistency, cancellation, and discerning. Hence Theorem 2.15 cannot be generalized to include all distributive lattices.

4.4 CHARACTERIZING APPROVAL VOTING ON BOOLEAN LATTICES

In the previous section we showed that F_A cannot be characterized with the conditions of productivity, neutrality, consistency, cancellation and discerning. In this section we want to consider F_A defined on Boolean lattices to show that we can characterize F_A with these conditions.

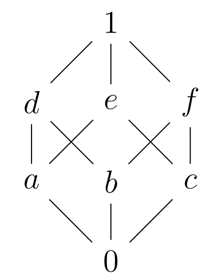

Figure 4.5: Boolean Lattice on 3 Elements

Proposition 4.21. *If a function $f : L^* \to L$ is defined on a boolean lattice satisfying cancellation and consistency, then f is non-deviating.*

Proof. Let $f : L^* \to L$ be a rule satisfying consistency and cancellation. Let π and π' be profiles such that $|K_j(\pi)| = |K_j(\pi')|$ for all $j \in J(L)$. For each $j_i \in J(L) = \{j_1, \ldots, j_m\}$, let

$$l_i = \alpha - |K_i(\pi)| \geq 0.$$

where

$$\alpha = \max\{|K_j(\pi)| \ : \ j \in J(L)\}.$$

Construct the profile ρ consisting of $\sum_{i=1}^{m} l_i$ voters such that there are l_i voters approving of j_i for each i. That is

$$\rho = \left(\underbrace{j_1, \ldots, j_1}_{l_1 \text{ times}}, \underbrace{j_2, \ldots, j_2}_{l_2 \text{ times}}, \ldots, \underbrace{j_{m-1}, \ldots, j_{m-1}}_{l_{m-1} \text{ times}}, \underbrace{j_m, \ldots, j_m}_{l_m \text{ times}} \right).$$

Since L is a boolean lattice, we have that $j||j'$ for all $j \neq j' \in J(L)$ and thus $|K_{j_i}(\rho)| = l_i$ for all $j_i \in J(L)$.

Now consider the profile $\pi\rho$. For each $j \in J(L)$,

$$\begin{aligned}|K_j(\pi\rho)| &= |K_j(\pi)| + |K_j(\rho)| \\ &= |K_j(\pi)| + \alpha - |K_j(\pi)| \\ &= \alpha.\end{aligned}$$

Thus it follows by cancellation that $f(\pi\rho) = 1$. Now consider the profile $\rho\pi'$. Since $|K_j(\pi)| = |K_j(\pi')|$ for all $j \in J(L)$, we have that

$$\begin{aligned}|K_j(\rho\pi')| &= |K_j(\rho)| + |K_j(\pi')| \\ &= \alpha - |K_j(\pi')| + |K_j(\pi')| \\ &= \alpha.\end{aligned}$$

Thus it follows by cancellation that $f(\rho\pi') = 1$.

Now notice by consistency we have that

$$f(\pi\rho\pi') = f(\pi) \wedge 1 = f(\pi)$$

since $f(\rho\pi') = 1$. Similarly since $f(\pi\rho) = 1$ we have

$$f(\pi\rho\pi') = 1 \wedge f(\pi') = f(\pi').$$

Thus by transitivity we have that

$$f(\pi) = f(\pi\rho\pi') = f(\pi')$$

and thus f is non-deviating. $\square$

We now state our characterization of the approval voting rule on boolean lattices as a consequence of Theorem 4.15 and Proposition 4.21.

Theorem 4.22. *The aggregation function $f : L^* \to L$ on a boolean lattice, satisfies productivity, consistency, faithfulness, and cancellation if and only if $f = F_A$.*

Notice that Theorem 4.22 is a lattice analogue of the Alós-Ferrer Theorem. Our goal now is to characterize F_A on all boolean lattices with the conditions of productivity, neutrality, consistency, cancellation, and discerning.

Proposition 4.23. *For a boolean lattice L, a function $f : L^* \to L$ satisfying neutrality and productivity, for any $x \in L$ there are three possibilities for $f(x)$.*

$$1)\ f(x) = \bigvee\{s \in J(L)\} = 1$$

$$2)\ f(x) = \bigvee\{s \in J(L)\ :\ s \not\leq x\} = x^c$$

$$3)\ f(x) = \bigvee\{s \in J(L)\ :\ s \leq x\} = x.$$

Proof. Consider the profile $\pi = (x)$. Since $f(x) \neq 0$, there exists an $s \in J(L)$ such that $s \leq f(x)$. There are two possibilities, either $s \leq x$ or $s \not\leq x$.

We first note that since L is boolean we have that $j||j'$ for all $j, j' \in J(L)$. Now consider that $s \leq x$. Consider $s' \in J(L)$ such that $s' \leq x$, then for the permutation $\sigma = (s\, s')$, $K_t(x) = K_{\sigma(t)}(x)$ for all $t \in J(L)$. Hence, by neutrality, $s' \leq f(x)$. Since s' was chosen arbitrarily, $\bigvee\{s \in J(L) : s \leq x\} = x \leq f(\pi)$.

Now consider $j \in J(L)$ such that $j \not\leq x$. A similar argument shows that for $j' \in J(L)$ such that $j' \not\leq x$, $j \leq f(x)$ if and only if $j' \leq f(x)$. Hence if $j \leq f(x)$ then $f(x) = 1$, and if $j \not\leq f(x)$ then $f(x) = x$.

For the second possibility suppose that $s \leq f(x)$ such that $s \not\leq x$. Notice that $x \neq 1$. Then it follows by a similar argument that $f(x) = \bigvee\{s \in J(L) : s \not\leq x\} \in \{x^c, 1\}$.

It follows that for any $x \in X \setminus \{0\}$ that

$$f(x) \in \{x, 1, x^c\}.$$

□

Proposition 4.24. *For a boolean lattice L, if $f : L^* \to L$ satisfies productivitity, neutrality, consistency, and is discerning then f is faithful.*

Proof. Let L be a boolean lattice and let $f : L^* \to L$ satisfy neutrality, consistency, and the condition of discerning. Let $x \in L$ and consider by the previous proposition we have

$$f(x) \in \{x, \bigvee\{s \in J(L) : s \not\leq x\} = x^c, 1\}$$

By way of contradiction suppose $f(x) \neq x$. Thus it follows that $\bigvee\{s \in J(L) : s \not\leq x\} \leq f(x)$.

If $|\{s \in J(L) : s \not\leq x\}| = 1$ then we have a contradiction to discerning, so we may assume $|\{s \in J(L) : s \not\leq x\}| \geq 2$. Suppose $\{s \in J(L) : s \not\leq x\} = \{y, j_1, j_2, \ldots, j_k\}$. Let $z \in J(L)$ such that $z \leq x$ and define a transposition on $J(L)$ by $\sigma_i = (z\ j_i)$ for $j_i \in \{j_1, \ldots, j_k\}$ and let $\sigma_i(x) = \bigvee\{\sigma_i(s) : s \leq x\}$. Since $K_s(x) = K_\sigma(s)(\sigma_i(x))$ for all join irreducibles $s \in J(L)$, by neutrality we have that $\sigma(y) = y \leq f(\sigma_i(x))$ for each i and thus by consistency we have

$$y \leq f(x) \wedge f(\sigma_1(x)) \wedge \cdots \wedge f(\sigma_k(x)) = f(\pi\sigma_1(x) \cdots \sigma_k(x)).$$

Let $\rho = \pi\sigma_1(x) \cdots \sigma_k(x)$, and consider that for $s \in J(L)$ we have

$$|K_s(\rho)| = \begin{cases} 0 & \text{if } s = y \\ 1 & \text{if } s \not\leq x \text{ and } s \neq y \\ k+1 & \text{otherwise.} \end{cases}$$

Hence by discerning $y \not\leq f(\rho)$ a contradiction. Hence

$$\bigvee\{s \in J(L) : s \not\leq x\} \not\leq f(\pi)$$

and thus $f(x) = x$ as was desired. □

We now state our final characterization of approval voting for boolean lattices as a result of Theorem 4.22 and Proposition 4.24.

Theorem 4.25. *The aggregation function $f : L^* \to L$ on a boolean lattice, satisfies productivity, neutrality, consistency, cancellation, and is discerning if and only if $f = F_A$.*

Now that we have shown the extensions of Theorem 2.12 and Theorem 2.15 to include Boolean Lattices, the question arises: for what classes of distributive lattices does the result of Theorem 2.12 and Theorem 2.15 extend to? Whereas Boolean lattices are very natural extensions, we also know that we cannot extend them to include chains.

We now introduce a new class of distributive lattices. For $n \geq 2$, consider the lattice $C_2 \times C_n$ defined by its Hasse Diagram below.

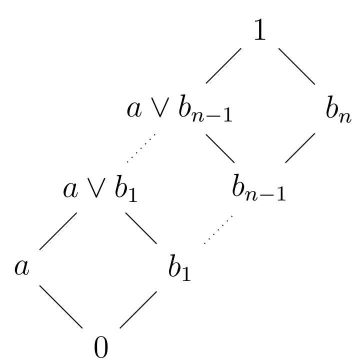

Figure 4.6: $C_2 \times C_n$

Conjecture 4.3. *The extensions of Theorem 2.12 and Theorem 2.15 hold for the lattices $C_2 \times C_n$ for all $n \geq 2$.*

This conjecture certainly holds for $n = 2$ since $C_2 \times C_2$ is a boolean lattice. The question about $n \geq 3$ is not known.

CHAPTER 5
CONCLUSIONS

In this dissertation we explored the problem of aggregating individual preferences into a collective preference and under what conditions we are required to select a collective majority. We studied majority rule on several models of consensus, and for each model, the conditions varied based on the model.

Starting with the least generalized model, we studied collective approval rules in which every member of the electorate can approve of whomever they deem best. The pinnacle result of this section was to characterize majority rule on this model with the conditions of neutrality, consistency, cancellation, and discerning.

Next we moved to the more general model working with ballot aggregation rules. In this section we presented the main theorem of this dissertation by characterizing majority rule with the necessary and sufficient conditions of faithfulness, consistency, and cancellation if and only if the domain is at least 3-rich.

We then concluded the results of this dissertation by working towards extending some results to a latticial model of consensus.

Whereas Arrow [3] and many others have provided examples to show that there is no unanimously fair voting rule. In this dissertation we provided many compelling reasons why one would would select a collective majority. We now conclude this dissertation by proposing some future work.

5.1 FUTURE WORK

Here we will discuss some of the current on going projects and potential future work related to the material in this dissertation. It would be useful to extend this research to vote aggregation processes other than majority rule as demonstrated by Gill & Gainous [10].

Throughout this dissertation we have stated several conjectures with some motivation as to why we believe they are true. In addition to these conjectures there are three noteworthy projects to be discussed.

5.1.1 PROJECT 1: EXTENDING DUDDY-PIGGINS CONSISTENCY

In 2013, Duddy & Piggins [6] presented a modified version of the consistency axiom to characterize the class of mean based rules. To not confuse this condition with the already well established axiom of consistency, we will say that a function $f : {\mathbb{N}_0}^{\mathbb{P}(X)} \to P_{ne}(X)$ satisfies the axiom of **Duddy-Piggins consistency** if for all profiles π, π',

$$f(\pi) \cap f(\pi') \subseteq f(\pi\pi') \subseteq f(\pi) \cup f(\pi').$$

Observe that this axiom has a very natural extension to the model of aggregation functions on a lattice L. Namely we say that $f : L^* \to L$ satisfies Duddy-Piggins Consistency if, for any $\pi, \pi' \in L^*$,

$$f(\pi) \wedge f(\pi') \leq f(\pi\pi') \leq f(\pi) \vee f(\pi').$$

We will refer to this axiom as the *lattice version of the Duddy-Piggins consistency axiom.*

Using this new consistency axiom, it is our hope to prove a lattice version of the main result given in [6].

5.1.2 PROJECT 2: A DIRECT PROOF OF THEOREM 3.15

Consider the following equivalent statement of Theorem 3.15:

Theorem 5.1. *A rule* $f : \mathbb{N}_0{}^{\mathfrak{B}} \to P_{ne}(X)$ *defined on any rich ballot space* $\mathfrak{B}$ *satisfies neutrality, consistency, and cancellation if and only if* $f = F_M$*,* $f = F_{M^{-1}}$ *or* $f = f_X$*.*

We want to find a direct proof of Theorem 3.15, instead of relying on the results of Fishburn.

5.1.3 PROJECT 3: A TRICHOTOMOUS EXTENSION

Following Alcantud [1], one can extend the set model used in Chapter 2 to allow a voter three possibilities. Namely, a voter can approve, disapprove, or be indifferent about any given alternative x_i belonging to the set X of all possible alternatives.

If $X = \{x_1, \ldots, x_m\}$ with $m \geq 2$, then let

$$\Pi = (\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0)^X.$$

For any $\pi \in (\mathbb{N}_0 \times \mathbb{N}_0 \times \mathbb{N}_0)^X$ and for any $x \in X$, $\pi(x) = (i, j, k)$ for some $i, j, k \in \mathbb{N}_0$. We write i as $\pi_+(x)$, which represents the number of voters who approve of an alternative $x \in X$. Similarly we will write j as $\pi_-(x)$, which represents the number of voters who disapprove of an alternative x. Lastly we will write k as $\pi_0(x)$, which represent as the number of voters who are indifferent about an alternative x.

An aggregation function is any function of the form:

$$f : \Pi \to P_{ne}(X).$$

Following [1], <u>Dis&Approval Voting</u> is the aggregation rule $W_D : \Pi \to P_{ne}(X)$ defined by:

$$W_D(\pi) = \{x \in X \ : \ \pi_+(x) - \pi_-(x) \geq \pi_+(y) - \pi_-(y)) \ \forall \, y \in X\},$$

The goal of this project is to characterize Dis&Approval Voting with axioms similar to the conditions used in Chapter 2.

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TEACHING EXPERIENCE

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- MATH 105: Contemporary Mathematics
  Made use of mathematical modeling to solve practical problems. Applications include management science, social choice, population growth, and personal finance.

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  Prepared future elementary educators for teaching problem solving skills and mathematical reasoning. This is a required lecture and lab for admittance to the Elementary Teacher Program which focuses on problem solving, number systems, number theory, and operations. Also utilized a basic skills assessment to ensure candidate fluency.
- MATH 152: Mathematics for Elementary Education II
  This is a required lecture and lab continuation of Mathematics for Elementary Education I and features elements of geometry, patterns, statistics, probability, and counting. This is a required preparation class for admittance to the Elementary Education program.
- MATH 190: Pre-Calculus
  Worked with students to recall and learn techniques from Algebra and Trigonometry which are necessary for success in the traditional Calculus sequence. Presented questions and problem-solving ideas in order to polish these skills and foster critical thinking.
- MATH 205: Calculus 1
  Unveiled the remarkable set of tools in differential and integral calculus of single variable functions to students seeking degrees in natural and applied sciences. This course is required for STEM field majors.
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Student Support Services **2013-2014**
Provided tutoring and instruction for first generation and low income undergraduates in mathematics and psychology.

See Blue Math Clinic **2013**
Creating fun/enriching lesson plans, which promote/encourage critical thinking, and mathematical learning for students that are considered "Struggling Math Students" who have various cognitive learning issues.

LEADERSHIP AND SERVICE

American Mathematical Society Student Chapter, University of Louisville

President & Founder **2017 -2019**
In 2017, I founded a University of Louisville chapter of the American Mathematical Society. Over the past 2 years, as the President for the chapter, I have helped generate almost $2,000 from various sources to help fund the chapter. In addition to weekly meetings, and other AMS events, I have organized trips for our members to attend regional math conferences at no cost to the members.

School of Interdisciplinary and Graduate Studies, University of Louisville

Graduate Student Ambassador **2017-2019**
In addition to providing both current and potential graduate students with additional networking and professional development skills, I also help during events such as, graduate student orientation, panelists for admissions programs, and doctoral hooding ceremonies.

Graduate Student Council, University of Louisville

Mathematics Department Representative **2017-2019**
While the purpose of the graduate student council is to serve as a governing body, to provide a voice to all graduate students, to promote academic research, and to facilitate leadership opportunities, my main responsibilities involve helping my constituents get access to research and travel funding as well as serving on multiple committees that benefit all graduate students.

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Graduate Student Peer Mentor **2016-2019**
In addition to mentoring the new mathematics graduate students, both as a I organize social events for the graduate students as well as an end of the year faculty appreciation picnic.

SELECT PRESENTATIONS AND PUBLICATIONS

Publications

Trevor Leach and Robert C Powers. "Majority rule on j-rich ballot spaces." Submitted August 2018.

Presentations

Dissertation Defense, University of Louisville — *Louisville, KY*
Characterizing majority rule on various discrete models of consensus. — **May 23, 2019**

Mathematics Association of America Sectional Meeting — *Danville, KY*
Aggregating preference on permutation closed j-rich domains. — **March 29, 2019**

2019 Joint Mathematics Meetings — *Baltimore, MD*
An axiomatic approach to aggregating individual preferences into a collective preference. — **January 18, 2019**

AMS Student Chapter, University of Louisville — *Louisville, KY*
Majority rule on j-rich ballot spaces. — **August 29, 2018**

Mathematics Association of America Sectional Meeting — *Bowling Green, KY*
Characterizing majority rule on rich ballot spaces. — **April 7, 2018**

37^{th} Mathematics Symposium, Western Kentucky University — *Bowling Green, KY*
Using 'discerning' to axiomatically characterize F_A. — **November 18, 2017**

Candidacy Exam, University of Louisville — *Louisville, KY*
Collective approval on lattices. — **November 7, 2017**

AMS Student Chapter, University of Louisville — *Louisville, KY*
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