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Decisive neutrality, restricted decisive neutrality, and split decisive neutrality on median semilattices and median graphs.

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DECISIVE NEUTRALITY, RESTRICTED DECISIVE NEUTRALITY, AND SPLIT DECISIVE NEUTRALITY ON MEDIAN SEMILATTICES AND MEDIAN GRAPHS

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DEDICATION

For my wife Robin and my son Parker.

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ABSTRACT

DECISIVE NEUTRALITY, RESTRICTED DECISIVE NEUTRALITY, AND SPLIT DECISIVE NEUTRALITY ON MEDIAN SEMILATTICES AND MEDIAN GRAPHS

Ulf Högnäs

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Consensus functions on finite median semilattices and finite median graphs are studied from an axiomatic point of view. We start with a new axiomatic characterization of majority rule on a large class of median semilattices we call *sufficient*. A key axiom in this result is the restricted decisive neutrality condition. This condition is a restricted version of the more well-known axiom of decisive neutrality given in [4]. Our theorem is an extension of the main result given in [7].

Another main result is a complete characterization of the class of consensus on a finite median semilattice that satisfies the axioms of decisive neutrality, bi-idempotence, and symmetry. This result extends the work of Monjardet [9]. Moreover, by adding monotonicity as a fourth axiom, we are able to correct a mistake from the Monjardet paper.

An attempt at extending the results on median semilattices to median graphs is given, based on a new axiom called split decisive neutrality. We are able to show that majority rule is the only consensus function defined on a path with three vertices that satisfies split decisive neutrality and symmetry.

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CHAPTER 1
INTRODUCTION

The results proved in this thesis are a contribution to the growing area of mathematics known as consensus theory. Two early and seminal results in this area are Arrow's impossibility theorem [1] and May's characterization of simple majority rule [6]. These two results first appeared in print over sixty years ago and, since then, many authors have extended these results to other areas of mathematics. See Boris Mirkin's theorem [8], which deals with consensus functions on a set of equivalence relations, as just one example of an extension of Arrow's theorem. Next, see [12] for a recent example of an extension of May's theorem.

Our starting point is an axiomatic characterization of majority rule for consensus functions defined on the set of hierarchies due to McMorris and Powers [7]. The McMorris and Powers characterization of majority rule on hierarchies is similar in spirit to May's characterization of simple majority rule. It is known that the set of hierarchies, with set inclusion as a partial order, is an example of an important mathematical object called a median semilattice. The main result given in the second chapter of this thesis is an extension of the McMorris and Powers theorem to a large class of median semilattices which we call sufficient. One of the main axioms of our new result is called restricted decisive neutrality since it is a restrictive version of the decisive neutrality condition due to Monjardet [9].

In Chapter 3, we look carefully at the Mondjardet model of consensus given in [9] and classify the class of consensus functions defined on a finite median semilattice satisfying the following axioms: decisive neutrality, bi-idempotence, and symmetry. Moreover, by adding mononotonicity as a fourth axiom, the final theorem of Chapter 3 corrects a mistake from the Mondjardet paper.

Chapter 4 is exploratory in nature. The hope is that the results proved in Chapter 3 in the context of median semilattices can be extended to median graphs. Many questions have to be answered. For example: What is the definition of majority rule on a median graph? Also: How does one extend the decisive neutrality condition from median semilattices to median graphs? We formulate a new condition called spilt decisive neutrality which, in our opinion, is a natural extension of decisive neutrality. At the end of this chapter, we prove a modest result. Namely, majority rule is the only consensus function defined on a path with three vertices that satisfies split decisive neutrality and symmetry. In the next section, some basic notation and terminology from the area of ordered sets is given.

1.1 Meet Semilattices and Median Semilattices

We begin by introducing some theory of ordered sets. Some of the definitions and terminology given in this section can be found in Davey and Priestley (2002) [3].

Let X be a nonempty set. An **order**, or **partial order**, on X is a binary relation $\leq$ on X such that, for all $x, y, z \in X$,

(i) $x \leq x$

(ii) $x \leq y$ and $y \leq x$ imply $x = y$

(iii) $x \leq y$ and $y \leq z$ imply $x \leq z$.

These conditions are called **reflexivity, antisymmetry** and **transitivity**, respectively. If $\leq$ is a partial order on a set X, then the pair $(X, \leq)$ is called a **partially ordered set.** If there is an element z belonging to X such that $z \leq x$ for all $x \in X$, then z is called **the zero element** of the partially ordered set $(X, \leq)$.

Let $(X,\leq)$ be a partially ordered set and let u,v be elements of X. An element w is the **meet** (*greatest lower bound*) of u and v if the following conditions are met

- $w \leq u$ and $w \leq v$
- For any $x \in X$ such that $x \leq u$, $x \leq v$, we have $x \leq w$.

The meet is denoted by $u \wedge v$. A **meet semilattice** is a partially ordered set $(X,\leq)$ in which any two elements u,v have a meet.

An element w is the **join** (*least upper bound*) of u and v if the following conditions are met

- $u \leq w$ and $v \leq w$
- For any $x \in X$ such that $u \leq x$, $v \leq x$, we have $w \leq x$.

The join is denoted by $u \vee v$. A **join semilattice** is a partially ordered set $(X,\leq)$ in which any two elements u,v have a join.

Let x,y be elements of X. We say that y **covers** x if $x < y$ and $x \leq z < y$ implies $x = z$. A **maximal element** $x \in X$ is an element such that $x \leq y$ implies $x = y$.

We define **height** of a finite partially ordered set X, denoted by $h(X)$, as follows. Let $\mathscr{A} = \{A \subseteq X : \forall x,y \in A, x \leq y \text{ or } y < x\}$, then

$$h(X) = \max_{A \in \mathscr{A}}(|A|) - 1.$$

A partially ordered set $(X,\leq)$ is a **lattice** if it is both a join semilattice and a meet semilattice. A lattice $(X,\leq)$ is **distributive** if, for all elements u, v, and w belonging to X,

$$u \vee (v \wedge w) = (u \vee v) \wedge (u \vee w) \text{ and } u \wedge (v \vee w) = (u \wedge v) \vee (u \wedge w).$$

For the rest of this chapter, X will be a **fininte meet semilattice** and we allow for the possibility that X is a lattice. The semilattices studied in this thesis belong to the class of semilattices called *median semilattices*, which we will define shortly.

A subset I of a partially ordered set $(X, \leq)$ is an (order) **ideal** if for all $x, y \in X$ $x \in X$, $y \in I$, and $x \leq y$ implies $x \in I$. It can be shown that the smallest ideal with respect to set inclusion containing an element a is equal to $\{x \in X : x \leq a\}$. This is called the **principal ideal** of a.

A meet semilattice $(X, \leq)$ satisfies the **Join-Helly Property** if for all $x, y, z \in X$ such that $x \vee y, x \vee z, y \vee z$ all exist, then $x \vee y \vee z$ exists.

A meet semilattice $(X, \leq)$ is **distributive** if , for every $x \in X$,

$$\{y \in X : y \leq x\}$$

is a distributive lattice. In other words, every principal ideal is a distributive lattice. The next definition is one of the most important definitions of the chapter. A **median semilattice** is a finite meet semilattice $(X, \leq)$ such that

- $(X, \leq)$ is a distributive semilattice
- $(X, \leq)$ satisfies the Join-Helly Property.

Any distributive lattice is a median semilattice. Figure 1.1 is a simple example of a distributive meet lattice which is not a lattice.

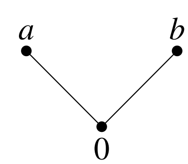


Figure 1.1: A median semilattice

Figure 1.2 is an example of a distributive meet semilattice X, which is not a median semilattice.

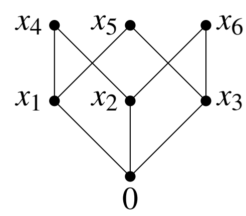


Figure 1.2: A semilattice that does not satisfy the Join-Helly Property.

Note that $\{y \in X : y \leq x\}$ is a distributive lattice for all $x \in X$. However, X does not satisfy the Join-Helly Property. To see this, notice that each of $x_1 \vee x_2$, $x_1 \vee x_3$, and $x_2 \vee x_3$ exist, while $x_1 \vee x_2 \vee x_3$ does not.

The following definition will be important for this thesis. Let $(X, \leq)$ be a median semilattice. A non-zero element $j \in X$ is **join irreducible** if for all $x, y \in X$

$$j = x \vee y \implies j = x \text{ or } j = y.$$

We denote by $J(X)$ the set of all join irreducibles of $(X, \leq)$. An **atom** is a join irreducible that covers the zero element.

In the next section, we introduce another example of a median semilattice motivated by the area of mathematical classification [8].

1.2 Hierarchies

The main reference for this section is the book authored by Day and McMorris (2003) [4]. Let S be a finite set with $n \geq 3$ elements. A **hierarchy** on S is a collection H of nonempty subsets of S such that

(i) $S \in H$

(ii) $\{x\} \in H$ for all $x \in S$

(iii) $A \cap B \in \{A, B, \emptyset\}$ for all $A, B \in H$.

If $S = \{x,y,z\}$ then $H = \{\{x,y,z\},\{x\},\{y\},\{z\},\{x,y\}\}$ is a hierarchy on S. Figure 1.3 is a visual representation of H.

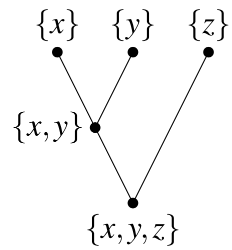


Figure 1.3: A hierarchy on $S = \{x,y,z\}$.

Let S be a set with at least three element and let $\mathscr{H}(S)$ be the set of all hierarchies on S. We define $\leq$ on $\mathscr{H}(S)$ as follows: for all $H_1, H_2 \in \mathscr{H}(S)$,

$$H_1 \leq H_2 \quad \text{if} \quad H_1 \subseteq H_2.$$

By the properties of set containment, it is easily shown that $(\mathscr{H}(S), \leq)$ is a partially ordered set. Moreover, it is not hard to verify that for any $H, H' \in \mathscr{H}(S)$, $H \cap H' \in \mathscr{H}(S)$. We now show that

$$H \wedge H' = H \cap H'.$$

Note that since $H \cap H' \leq H$ and $H \cap H' \leq H'$, $H \cap H'$ is a lower bound of $\{H, H'\}$. We claim that it is the greatest lower bound. Suppose that $J \leq H$ and $J \leq H'$. This implies that $J \subseteq H$ and $J \subseteq H'$, so $J \in H \cap H'$. It follows that $J \leq H \cap H'$ and therefore, $H \cap H'$ is the greatest lower bound of $\{H, H'\}$. Thus, the partially ordered set $(\mathscr{H}(S), \leq)$ is a meet semilattice with set intersection as the meet operation.

We claim that for any $H, H' \in \mathscr{H}(S)$, if $H \cup H' \in \mathscr{H}(S)$ then

$$H \vee H' = H \cup H'.$$

Note that since $H \leq H \cup H'$ and $H' \leq H \cup H'$, $H \cup H'$ is an upper bound of $\{H, H'\}$. We claim that it is the least upper bound. Suppose that $H \leq K$ and $H' \leq K$. Then $H \cup H' \leq K$. This proves that $H \cup H'$ is the least upper bound of $\{H, H'\}$.

We claim that for any $H \in \mathscr{H}(S)$,

$$X = \{J \in \mathscr{H}(S) : J \leq H\} \tag{1.1}$$

is a distributive lattice. Note that for any $J, J' \in X$,

$$\begin{aligned} J \vee J' = J \cup J' &\subseteq H \\ J \wedge J' = J \cap J' &\subseteq H. \end{aligned}$$

Since intersections distribute over unions and vice versa, X is a distributive semilattice. Therefore, since X is an arbitrary principal ideal of the meet semilattice $(\mathscr{H}(S), \leq)$, we can conclude that $(\mathscr{H}(S), \leq)$ is a distributive semilattice.

We now claim that the distributive semilattice $(\mathscr{H}(S), \leq)$ satisfies the Join Helly property. To see this, suppose that $J \vee K, J \vee L$, and $K \vee L$ all exist. for some $J, K, L \in \mathscr{H}(S)$. We claim that $J \vee K \vee L$ exists. This is equivalent to showing that $J \cup K \cup L$ is a hierarchy. Hierarchy conditions (i) and (ii) are obviously fulfilled. It remains to verify (iii). Let $A, B \in J \cup K \cup L$ and assume without loss of generality that $A \in J$.

1. Suppose that $B \in J$. Then $A, B \in J$ and since J is a hierarchy, it follows that $A \cap B \in \{A, B, \emptyset\}$.

2. Suppose that $B \notin J$ and without loss of generality that $B \in K$. Notice that $J \cup K \in \mathscr{H}(S)$ and hence $A \cap B \in \{A, B, \emptyset\}$.

This proves that X is a median semilattice.

The previous comments lead to the following result.

Theorem 1.2.1. *For any finite set S consisting of $n \geq 3$ elements, the pair $(\mathscr{H}(S), \leq)$ is a median semilattice.*

If H is a hierarchy on S, with $|S| = n$, and $X \in H$, then X is called a *cluster*. If $|X| = n$ or $|X| = 1$, then X is a *trivial cluster*. If $1 < |X| < n$, then X is a *non-trivial cluster*. We call $H_\emptyset = \{S, \{x\} : x \in S\}$ the trivial hierarchy and note that $H_\emptyset$ is the zero element of the median semilattice $(\mathscr{H}(S), \leq)$. For any nontrivial subset A of S, let $H_A = H_\emptyset \cup \{A\}$ and note that $H_A \in \mathscr{H}(S)$.

We claim that $J(\mathscr{H}(S)) = \{H_A : 1 < |A| < |S|\}$. Let $H_A \in \{H_A : 1 < |A| < |S|\}$. We will show that H_A is join irreducible. Suppose that $H_A = J \vee J'$. This implies that $J \leq H_A$. Since $H_A = H_\emptyset \cup A$, where A is a single set, we have that $J \in \{H_A, H_\emptyset\}$. We cannot have that both of J, J' are the trivial hierarchy since $H_A \neq H_\emptyset$. Hence H_A is join irreducible.

Consider $H \in \mathscr{H}(S)$ such that H contains at least two non-trivial clusters A and B. We note that $H \backslash \{A\} \in \mathscr{H}(S)$, and that $H_A \cup H \backslash \{A\} = H$. Therefore, $H \notin J(\mathscr{H}(S))$. We conclude that the hierarchies in $\{H_A : 1 < |A| < |S|\}$ are the only join irreducibles.

1.3 Consensus

The main reference for this section is the book authored by Day and McMorris (2003) [4], which provides a more complete discussion with many examples of applications in bioinformatics.

Given a generic set of objects X, a **consensus function**, or **consensus rule**, is a function that maps from a tuple of choices from this set $(x_1, x_2, ..., x_k)$ to X itself, i.e. it is a function of the form

$$f : X^k \longrightarrow X.$$

The tuple $(x_1, x_2, ..., x_k)$ is called a **profile** and can be thought of as the preferences of k individuals. We will focus on consensus functions defined on finite median semilattices.

Let $(X, \leq)$ be a finite median semilattice and $k \geq 2$ an integer. The consensus function $f_u : X^k \longrightarrow X$ defined as follows: for any profile $P = (x_1, ..., x_k)$,

$$f_u(P) = \bigwedge_{i=1}^{k} x_i$$

is sometimes called **unanimity rule** or the **strict consensus rule**. There is an equivalent way to define unanimity rule. For any profile $P = (x_1, ..., x_k)$,

$$f_u(P) = \bigvee\{j \in J(X) : j \le x_i \text{ for } i = 1, ..., n\}$$

More examples of consensus functions will be given in subsequent chapters of this thesis.

Let $P \in X^k$ and $j \in J(X)$. Define

$$K_j(P) = \{i : j \le x_i\}. \tag{1.2}$$

With that, we are ready to define **majority rule** on median semilattices.

Definition 1.3.1. Let X be a median semilattice and $k \ge 3$.We define $\mathrm{Maj} : X^k \longrightarrow X$ by

$$\mathrm{Maj}(P) = \bigvee\{j \in J(X) : |K_j(P)| > k/2\}.$$

We will use mathematical induction to show that the join $\bigvee\{j \in J(X) : |K_j(P)| > k/2\}$ exists so that Maj is well defined. Let $j_1, j_2 \in J(X)$ such that $|K_{j_1}(P)| > \frac{k}{2}$ and $|K_{j_2}(P)| > \frac{k}{2}$. It follows from the pigeon hole principle that there must exist at least one element $x_i \in P$ such that $j_1 \le x_i$ and $j_2 \le x_i$, and hence $j_1 \vee j_2$ exists.

Assume that $\bigvee_{i=1}^{k} j_i$ exists for every integer $k \in \{1, ..., n\}$ if every pairwise join from $\{j_1, j_2, ..., j_n\} \subseteq J(X)$ exists. This is the inductive hypothesis. Assume that every pairwise join from $\{j_1, j_2, ..., j_{n+1}\} \subseteq J(X)$ exists. Now note that $j_{n+1} \vee \bigvee_{i=1}^{n-1} j_i$ and $j_n \vee \bigvee_{i=1}^{n-1} j_i$ both exists by the inductive hypothesis. Since $j_{n+1} \vee j_n$ exists, it follows from the join helly property that $j_n \vee j_{n+1} \vee \bigvee_{i=1}^{n-1} j_i$ exists, which proves the claim and hence Maj is well defined.

We will return to this version of majority rule in the next chapter. I

1.4 Majority Rule on Hierarchies

For this section, recall, by Theorem 1.2.1, that for any set S with $|S| \ge 3$, the pair $(\mathscr{H}(S), \le)$ is a median semilattice with subset containment as the partial order. Also recall that $\{H_A : 1 < |H| < n\}$ is the set of join irreducibles of $\mathscr{H}(S)$ where $H_A = H_\emptyset \cup \{A\}$. Let $K = \{1, ..., k\}$ with $k \geq k$. For any profile $P = (H_1, ..., H_k)$ belonging to $\mathscr{H}(S)^k$, where $k \geq 3$, and for any nontrivial subset X of S,

$$K_X(P) = \{i \in K : H_X \subseteq H_i\} = \{i \in K : X \in H_i\}.$$

By Definition 1.3.1, Maj : $\mathscr{H}(S)^k \longrightarrow \mathscr{H}(S)$ is defined by

$$\text{Maj}(P) = \left\{ X : |K_X(P)| > \frac{k}{2} \right\}.$$

In this context, Maj is called **majority rule on hierarchies**. Notice that Maj(P) consists of all the clusters that appear in more than half of the input profile P. Majority rule on hierarchies is well known and it is a popular method of consensus [4]. Part of its appeal is based on the fact that Maj satisfies some desirable properties.

In their paper, McMorris and Powers gave two axiomatic characterizations of majority rule in term of six axioms. To state the McMorris and Powers theorem, we need to define these six axioms.

Definition 1.4.1. A profile P is called a **biprofile** if there exist proper subsets X and Y of S such that $H_i \in \{H_X, H_Y\}$ for all $i \in K$ and $|K_X(P)| > |K_Y(P)| > 0$. We say that a consensus function $f : \mathscr{H}(S)^k \longrightarrow \mathscr{H}(S)$ is **biprofile nontrivial (BNT)** if $f(P) \neq H_\emptyset$ for all biprofiles P.

Notice that a biprofile consists of two different hierarchies, each of which is the join of the trivial hierarchy and some subset of S, and where one is more frequent than the other. A consensus function satisfies this axiom if no such profile maps to the trivial hierarchy.

Definition 1.4.2. We say that a consensus function is **bi-idempotent (BI)** if for any profile $P = (H_1, ..., H_k)$ with $H_i \in \{H_X, H_Y\}$ for all i and $|K_X(P)| \neq k/2$, it follows that $f(P) \in \{H_X, H_Y\}$.

Note that the profile P given in Definition (1.4.2) is a biprofile. Therefore, (BI) implies (BNT).

For the next definition, we let

$$\mathscr{W} = \{P \in \mathscr{H}^k : P \text{ is not a biprofile}\}.$$

Definition 1.4.3. A consensus function f satisfies **restricted decisive neutrality (RDN)** if for all $X, Y \subset S$ and for all $P, P' \in \mathscr{W}$,

$$K_X(P) = K_Y(P') \implies [X \in f(P) \iff Y \in f(P')]$$

If we replace $\mathscr{W}$ in Definition (1.4.3) with $\mathscr{H}(S)^k$, then the resulting condition is a well-known axiom called **decisive neutrality** [4]. We will consider a more general version of decisive neutrality in the next chapter.

Definition 1.4.4. A consensus function f satisfies **monotonicity (M)**, if for any profiles $P = (H_1, ..., H_k)$ and $P' = (H_1', ..., H_k')$,

$$P \leq P' \implies f(P) \leq f(P')$$

Here $P \leq P'$ means that $H_i \subseteq H_i'$ for all $i \in K$.

The monotonicity axiom implies that if for every slot in P, the hierarchies are subsets of the hierarchies in the corresponding slots in P', then all the clusters in the output of P must also belong to the output of P'.

Definition 1.4.5. A consensus function satisfies **monotone neutrality (MN)**, if for any profiles P and P' and clusters X and Y,

$$K_X(P) \subseteq K_Y(P') \implies [X \in f(P) \implies Y \in f(P')].$$

This axiom implies that if the set of slots with a hierarchy that contain X in P is a subset of the slots in P' with a hierarchy that contains Y, then if X belongs to the output of P, then Y must also belong to the output of P'. Notice that if $P \leq P'$, then $K_X(P) \subseteq K_Y(P')$, so (MN) implies (M).

Definition 1.4.6. A consensus function f satisfies **symmetry (S)** if for any profile $P = (H_1,...,H_k)$ and any permutation σ of K, $f(P) = f(H_{\sigma(1)},...,H_{\sigma(k)})$.

Theorem 1.4.7. *Let $f : \mathscr{H}^k \longrightarrow \mathscr{H}$ be a consensus function on hierarchies. Then the following are equivalent:*

1. f = Maj
2. *f satisfies symmetry, (MN), and is bi-idempotent*
3. *f satisfies (RDN), symmetry, monotonicity, and is biprofile nontrivial.*

Our goal for the next chapter is to prove a generalized version of Theorem 1.4.7 for a broader class of semilattices.

CHAPTER 2
RESTRICTIVE DECISIVE NEUTRALITY AND MAJORITY RULE

In this chapter we introduce a class of median semilattices that we call sufficient, which will help us to generalize Theorem 1.4.7. As the name suggests, this condition is not necessary, which we illustrate with some examples in section 2.2.

2.1 Majority Rule on Semilattices

For the following definitions, $f : X^k \longrightarrow X$ is a consensus function with X being an arbitrary finite median semilattice. As stated in the previous chapter, our goal is to characterize Majority Rule on median semilattices. We repeat the definition here for convenience.

Definition 2.1.1. We define $\mathrm{Maj} : X^k \longrightarrow X$ by

$$\mathrm{Maj}(P) = \bigvee\{j \in J(X) : |K_j(P)| > k/2\}. \tag{2.1}$$

The axioms in Theorem 1.4.7 apply to hierarchies, so we need to formulate the axioms for our more general setting. The axioms that we have chosen are analogues to Definitions 1.4.1, 1.4.2, 1.4.3, 1.4.4, and 1.4.6.

Definition 2.1.2. Let $K = \{1,...,k\}$. We say that $f : X^k \longrightarrow X$ satisfies **Symmetry**, or (S), if for any permutation σ of K and any profile $P = (x_1,...,x_k)$,

$$f(P) = f(x_{\sigma(1)},...,x_{\sigma(k)}). \tag{2.2}$$

Definition 2.1.3. Let $P = (x_1,...,x_k)$ such that $x_i \in \{j, j'\}$ for all $i \in K$ with $j, j' \in J(X)$, and such that $0 < |K_j(P)| < |K_{j'}(P)|$. We call such a profile a **biprofile** and say that f satisfies **Biprofile Non-Trivial**, or (BNT), if for all such profiles P, $f(P) \neq 0$.

Definition 2.1.4. Let $P = (x_1, ..., x_k), P' = (y_1, ..., y_k)$. We write $P \leq P'$ when $x_i \leq y_i$ for all $i \in K$. We say that f satisfies **Monotonicity**, or (M), if

$$P \leq P' \implies [f(P) \leq f(P')]. \tag{2.3}$$

Definition 2.1.5. Let $\mathscr{W} = \{P \in X^k : P \text{ is not a biprofile}\}$. Let $P, P' \in \mathscr{W}$ and let $j, j' \in J(X)$. We say that f satisfies **Restricted Decisive Neutrality**, or (RDN), if

$$K_j(P) = K_{j'}(P') \implies [j \leq f(P) \iff j' \leq f(P')]. \tag{2.4}$$

From now on, let X be a median semilattice that is not a lattice.

The aim of the chapter is to characterize a class of semilattices such that a consensus function $f = \text{Maj}$ if and only if f satisfies (M), (S), (BTN), and (RDN).

Lemma 2.1.6. Suppose that f satisfies (RDN). Let $P_0 = (0, 0, ..., 0)$. Then $f(P_0) = 0$.

Proof. Suppose that $j \leq f(P_0)$ for some $j \in J(X)$. Since $K_j(P_0) = K_{j'}(P_0)$ for all $j' \in J(X)$, it follows from (RDN) that $j' \leq f(P_0)$ for all $j' \in J(X)$ i.e. $f(P_0)$ is an upper bound for all $j \in J(X)$. Since X is not a lattice, this is a contradiction. □

Lemma 2.1.7. Suppose that f satisfies (RDN). If a join irreducible j satisfies $j \leq f(R)$ with $R \in \mathscr{W}$, then $K_j(R) \neq \emptyset$.

Proof. Let $R \in \mathscr{W}$ and suppose that $j \leq f(R)$ for some $j \in J(X)$. Assume that $K_j(R) = \emptyset$. Note that $K_j(P_0) = \emptyset$, so by (RDN), it follows that $j \leq f(P_0)$, contradicting Lemma 2.1.6. Therefore, $K_j(R) \neq \emptyset$. □

Lemma 2.1.8. Suppose that f satisfies (S), (M), and (RDN). Let $P \in \mathscr{W}$ such that $|K_s(P)| \leq k/2$ for some $s \in J(X)$. Then $s \not\leq f(P)$.

Proof. Since X is a median semilattice that is not a lattice, there exist $j, j' \in J(X)$ such that $j \vee j'$ does not exist. Let $P' = (j, j, ..., j, 0, 0, ..., 0)$ such that $K_j(P') = \{1, ..., l\}$ with $l = |K_s(P)| \leq k/2$. Suppose that $j \leq f(P')$.

Let $Q = (0, 0, ..., 0, j', j', ..., j', 0, 0, ..., 0)$ such that $K_{j'}(Q) = \{l+1, ..., 2l\}$ and let $R = (j, j, ..., j, j', j', ..., j', 0, 0, ..., 0)$ such that $K_j(R) = \{1, ..., l\}$ and $K_{j'}(R) = \{l+1, ..., 2l\}$. Lastly, let $Q' = (j', j', ..., j', 0, 0, ..., 0)$ with $K_{j'}(Q') = \{1, ..., l\}$. By (S), $f(Q) = f(Q')$. Since $K_{j'}(Q') = K_j(P') = \{1, ..., l\}, (RDN)$ together with our assumption that $j \leq f(P')$ ensure that $j' \leq f(Q') = f(Q)$.

Note that $P' \leq R$ and $Q \leq R$. By (M) then, it follows that $j \leq f(P') \leq f(R)$ and $j' \leq f(Q) \leq f(R)$ and hence, $j \vee j' \leq f(R)$. This is a contradiction since j, j' were chosen so that $j \vee j'$ does not exist. Therefore, $j \not\leq f(P')$.

Now let $P'' \in \mathscr{W}$ with $K_s(P'') = \{1, ..., l\}$ and P'' a permutation of P. Since f satisfies (RDN), $|K_s(P'')| = |K_s(P')|$ and $j \not\leq f(P')$ imply $s \not\leq f(P'')$. By (S), $f(P) = f(P'')$ and therefore, $s \not\leq f(P)$. □

Lemma 2.1.9. Let X be a median semilattice that is not a lattice. Assume that X is of height at least two and assume that X has at least two atoms. Assume that f satisfies (S), (M), (BNT), and (RDN). If $k \geq 3$ and $P \in X^k$ such that $|K_j(P)| > k/2$ for some $j \in J(X)$, then $j \leq f(P)$.

Proof. For the first part of the argument, we will assume that $k = 3$. Based on our assumptions on X, there exists atoms $s \neq t$ and $x \in X$ such that $x > s$. Let

$$Q = (s, s, t) \text{ and } Q' = (x, s, t).$$

By (BNT) and $(M), f(Q') \neq 0$. Let $r \in J(X)$ such that $r \leq f(Q')$. Since $Q' \in \mathscr{W}$, it follows from Lemma 2.1.8 that $|K_r(Q')| \geq 2$. Note that since s and t are atoms, either $r = s$ and $K_s(Q') = \{1, 2\}$ or $r = t$ and $K_s(Q') = \{1, 3\}$. In either case, $|K_r(Q')| = 2$. By $(S), f(Q') = f(x, t, s)$ and so we may assume that $K_r(Q') = \{1, 2\}$. For any join irreducible element j, if

$$Q'' = (j, j, 0)$$

then $Q'' \in \mathscr{W}$ and so, by $(RDN), r \leq f(Q')$ implies that $j \leq f(Q'')$.

Now suppose that $P = (x_1, x_2, x_3)$ is a profile and $|K_j(P)| \geq 2$ for some $j \in J(X)$. By using (S), if necessary, we may assume that $\{1, 2\} \subseteq K_j(P)$. By using the above and (M) we get

$$j \leq f(j, j, 0) \leq f(P).$$

Hence $j \leq f(P)$ and the statements is true in the case $k = 3$.

For the rest of the argument, assume that $k \geq 4$. Let $s, s' \in X$ be distinct atoms. For the first part of the argument, we will assume that s and s' are the only atoms of X and that s and s' both are covered by a common element r. Since X is not a lattice, there exists at least one join irreducible element such that it is not less than or equal to r. We now define a bi-profile

$$U = (s, ..., s, s', ..., s')$$

for which $|K_s(U)| = \left\lfloor \frac{k}{2} + 1 \right\rfloor$ and since $k \geq 4$, $|K_s(U)| \geq 3$. Next, we define the profile

$$W = (s, ..., s, r, s', ..., s')$$

such that $K_s(U) = K_s(W)$. W is identical to U except in the $\left\lfloor \frac{k}{2} + 1 \right\rfloor$th slot, where an s has been replaced by an r. Note that since $r = s \vee s'$, r is not a join irreducible, and hence $W \in \mathscr{W}$ for all $k \geq 4$. Since $s \leq r$, we see that $U \leq W$. Since by (BNT), $f(U) \neq 0$, and since it follows from (M) that $f(U) \leq f(W)$ it is also true that $f(W) \neq 0$. For any $t \in J(X)$ such that $t \notin \{s, s'\}$, it follows that $|K_t(W)| = 0$ and by Lemma 2.1.8 then, $t \nleq f(W)$.

Let P be an arbitrary profile such that $|K_j(P)| > \frac{k}{2}$ for some $j \in J(X)$. If $s \leq f(W)$, then define the profile

$$W' = (j, ..., j, 0, ..., 0)$$

such that $K_j(W') = K_s(W)$. If $s \nleq f(W)$, then $s' \leq f(W)$ and we then define the profile

$$W' = (0, ..., 0, j, ..., j)$$

such that $K_j(W') = K_s(W)$. In either case, it follows from (RDN) that $j \leq f(W')$. Notice that $|K_j(P)| \geq |K_j(W')|$. Let P' be a permutation of P such that $K_j(W') \subseteq K_j(P')$. We see that $W' \leq P'$ so by (M), $f(W') \leq f(P')$. We conclude that $j \leq f(P')$ and by (S) then, $j \leq f(P)$.

For the final part of the argument, we will again let s and s' be two atoms of X. We will assume that there exists an element $w \in X$, distinct from the two atoms s and s', with the property that $s \leq w$ while $s' \not\leq w$. We define the profile

$$Q = (s, ..., s, s', ..., s')$$

such that $|K_s(Q)| = \left\lfloor \frac{k}{2} + 1 \right\rfloor$. Next, we define the profile

$$R = (w, s, ..., s, s', ..., s')$$

which is identical to Q except in the first slot, where an s has been replaced by a w. Since $k \geq 4$, the atom s appears at least two times in R and the atom s' appears at least once, in addition to the element w. Therefore, $R \in \mathscr{W}$. Since $Q \leq R$, it follows by (M) that $f(Q) \leq f(R)$. Since it follows from (BNT) that $f(Q) \neq 0$, we conclude that $f(R) \neq 0$. Let $u \in J(X)$ such that $u \neq s$. Since s and s' are atoms, $|K_u(R)| \leq \frac{k}{2}$ and by Lemma 2.1.8 then, $u \not\leq f(R)$. Since $f(R) \neq 0$ there must exists some $j \in J(X)$ such that $j \leq f(R)$. The only remaining possibility is that $s \leq f(R)$.

We again let P be an arbitrary profile such that $|K_j(P)| > \frac{k}{2}$ for some $j \in J(X)$. We define the profile

$$R' = (j, ..., j, 0, ..., 0)$$

such that $K_j(R') = K_s(R)$. By (RDN), $j \leq f(R')$. Notice that $|K_j(P)| \geq |K_j(R')|$. Let P' be a permutation of P such that $K_j(R') \subseteq K_j(P')$. Notice that $R' \leq P'$ so by (M), $f(R') \leq f(P')$. We conclude that $j \leq f(P')$ and by (S) then, $j \leq f(P)$. $\square$

Lemma 2.1.10. Let X be a median semilattice that is not a lattice. Let X be of height at least two and have at least two atoms. If $f : X^k \longrightarrow X$ with $k \geq 3$ satisfies (S), (M), (BNT), and (RDN), then

$$\mathrm{Maj}(P) \leq f(P).$$

Proof. This follows as a corollary from Lemma 2.1.9. $\square$

The following definition will help us to give a characterization of majority rule for an easily described class of median semilattices.

Definition 2.1.11. We say that a median semilattice X is **sufficient** if the following is true

(i) X is not a lattice, has at least two atoms, and $h(X) \geq 2$.

(ii) **either** none of the maximal elements of X are join irreducible **or** join irreducible elements that are not maximal are meet reducible.

Let $\max(X)$ denote the set of all maximal elements of X and let $M(X)$ denote the set of all meet irreducible elements of X. Then the second item in the definition of sufficient says that

$$\text{either } J(X) \cap \max(X) = \emptyset \text{ or } J(X) \backslash \max(X) \cap M(X) = \emptyset.$$

Proposition 2.1.12. Let S be a finite set with at least four elements. Then $(\mathscr{H}(S), \leq)$, the median semilattice of all hierarchies on S is sufficient.

Proof. Since $|S| \geq 4$, there exists a four element subset a, b, c, d of S. Observet that $H_{\{a,b\}} \vee H_{\{c,d\}}$ does not exist and so $(\mathscr{H}(S), \leq)$ is not a lattice. Since

$$H_{\{a,b\}} < H_{\{a,b\}} \cup \{\{c,d\}\}$$

it follows that the height of $(\mathscr{H}(S), \leq)$ is greater than or equal to two. Recall that the set of join irreducible elements of $(\mathscr{H}(S), \leq)$ is given by

$$J(\mathscr{H}(S)) = \{H_A : 1 < |A| < |S|\}.$$

Let H_A be an arbitrary join irreducible element. We will show that H_A is meet reducible. If $|A| \leq |S|-2$, then let $\{x,y\}$ be a two element subset of $S \backslash A$ and note that

$$H_A = H_A \cup \{A \cup \{x\}\} \vee H_A \cup \{A \cup \{y\}\}$$

If $|A| = S-1$, then let $\{x,y\}$ be a two element subset of A and note that

$$H_A = H_A \cup \{A \backslash \{x\}\} \vee H_A \cup \{A \backslash \{y\}\},$$

We know that every join irreducible element is meet reducible. Hence the median semilattice $(\mathscr{H}(S), \leq)$ is sufficient. $\square$

If $|S| = 3$, then the height of $(\mathscr{H}(S), \leq)$ is one. Therefore, $(\mathscr{H}(S), \leq)$ is not sufficient when $|S| = 3$.

Lemma 2.1.13. Let X be a sufficient median semilattice. Assume that $f : X^k \longrightarrow X$ with $k \geq 3$ satisfies $(S), (M), (BNT)$, and (RDN). If P is a biprofile then

$$f(P) \leq \mathrm{Maj}(P).$$

Proof. Let P be a biprofile consisting of the elements j and j', such that $|K_j(P)| > \frac{k}{2}$. Let $s \in J(X)$ such that $s \leq f(P)$. We want to show that $s \leq \mathrm{Maj}(P)$.

We claim that $s \leq j$ or $s \leq j'$. Suppose that $s \not\leq j$ and $s \not\leq j'$ and note that under this assumption $K_s(P) = \emptyset$. By Lemma 2.1.9, $j \leq f(P)$. Now $j \leq f(P)$ along with $s \leq f(P)$ implies that $j \vee s \leq f(P)$. We construct the profile P', identical to P with the exception of one slot, where a j is replaced by the element $j \vee s$. Since j, j', and $j \vee s$ are three distinct elements belonging to the profile P' it follows that $P' \in \mathscr{W}$. By Lemma 2.1.8, $P' \in \mathscr{W}$ along with $|K_s(P')| = 1 < \frac{k}{2}$ implies that $s \not\leq f(P')$. Now note that $P \leq P'$ and so, by (M), it follows that $f(P) \leq f(P')$. This contradicts our assumption that $s \leq f(P)$ and hence, we conclude that $s \leq j$ or $s \leq j'$.

Assume that $s \not\leq j$. Then $s \leq j'$. Since $j \vee s$ exists and $j \neq j \vee s$, it follows that j is not a maximal element of X. Assume that

$$J(X) \cap \max(X) = \emptyset.$$

Then there exists a maximal element t such that $j' < t$ and $t \notin J(X)$. Let Q be a profile identical to P except in one slot where j' is replaced by t and note that $P \leq Q$. Notice that $t \notin J(X)$ implies that $Q \in \mathscr{W}$. It follows from Lemma 2.1.8 that $s \not\leq f(Q)$, a contradiction since $s \leq f(P) \leq f(Q)$. This contraction implies that

$$J(X) \cap \max(X) \neq \emptyset.$$

Since j is not maximal, $j \notin M(X)$ as demanded by the sufficiency of X. Let u, u' be elements distinct from j such that $u \wedge u' = j$. Note that $s \not\leq u$ or $s \not\leq u'$, since $s \not\leq j$. Assume without loss of generality that $s \not\leq u$ and let R be a profile identical to P, except in one slot where a j is replaced by a u. Observe that j, j', and u are three distinct elements belonging to R and so $R \in \mathscr{W}$. Since $R \in \mathscr{W}$ and $|K_s(R)| < \frac{k}{2}$ it follows from Lemma 2.1.8 that $s \not\leq f(R)$. By (M), $P \leq R$ implies that $s \leq f(P) \leq f(R)$ contrary to $s \not\leq f(R)$. This last contradiction implies that $s \leq j$.

Now $|K_j(P)| > \frac{k}{2}$ implies that $j \leq \mathrm{Maj}(P)$ by the definition of Maj. Since $s \leq j$ if follows that $s \leq \mathrm{Maj}(P)$ and we are done. □

Theorem 2.1.14. *Let X be a sufficient semilattice. Furthermore, let $f : X^k \longrightarrow X$, where k is an integer greater than or equal to three. Then $f = \mathrm{Maj}$ if and only if f satisfies $(M), (S), (BNT)$, and (RDN).*

Proof. $(\Longleftarrow)$ Suppose that f satisfies (M), (S), (BNT), and (RDN). It follows from Lemma 2.1.10 that for any profile $P \in X^k$, $\mathrm{Maj}(P) \leq f(P)$. Let $P \in X^k$. Assume that $P \in X^k \backslash \mathscr{W}$. It follows from Lemma 2.1.13 that $f(P) \leq \mathrm{Maj}(P)$. Now suppose that $P \in \mathscr{W}$. By Lemma 2.1.8, if $j \in J(X)$ and $j \leq f(P)$, then $|K_j(P)| > \frac{k}{2}$. Therefore, $j \in J(X)$ and $j \leq f(P)$ imply that $j \leq \mathrm{Maj}(P)$. It now follows that $f(P) \leq \mathrm{Maj}(P)$. Hence, $f(P) \leq \mathrm{Maj}(P)$ for any profile P.

$(\Longrightarrow)$ Let $f = \mathrm{Maj}$. Let $P = (x_1, x_2, ..., x_k)$ and $Q = (y_1, y_2, ..., y_k)$.

We claim that Maj satisfies (M). Suppose that $P \leq Q$. It follows from the properties of the join that $\mathrm{Maj}(P) \leq \mathrm{Maj}(Q)$ whenever $P \leq Q$. Indeed, note that since $x_i \leq y_i$ for all $i \in K$ it follows that

$$\{j \in J(X) : |K_j(P)| > k/2\} \subseteq \{j \in J(X) : |K_j(Q)| > k/2\}. \tag{2.5}$$

Hence, it follows that the join, or least upper bound, of the right hand side of (2.5) is greater than or equal to that of the left hand side.

We claim that Maj satisfies (S). Let $P = (x_1, x_2, ..., x_k)$ and let σ be a permutation of K. Then $P_\sigma = (x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(k)})$. Note that

$$\{j \in J(X) : |K_j(P)| > k/2\} = \{j \in J(X) : |K_j(P_\sigma)| > k/2\}$$

and therefore their respective joins are also identical.

We claim that Maj satisfies (BNT). Suppose that $P \in X^k \backslash \mathscr{W}$. Every biprofile has the property that $|K_j(P)| > k/2$ for some $j \in J(X)$. Thus

$$j \leq \bigvee \{s \in J(X) : |K_s(P)| > k/2\}$$

and hence we see that $\mathrm{Maj}(P) \neq 0$.

Last, we claim that Maj satisfies (RDN). Let $P, Q \in \mathscr{W}$ and $j, j' \in J(X)$ such that $K_j(P) = K_{j'}(Q)$. Suppose that $j \leq \mathrm{Maj}(P)$. Then $k/2 < |K_j(P)| = |K_{j'}(Q)|$. We see that $j' \leq f(Q)$ and similarly, $j' \leq \mathrm{Maj}(Q)$ implies that $j \leq \mathrm{Maj}(P)$. $\square$

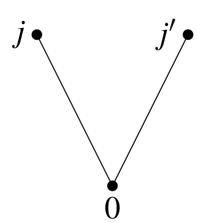


Figure 2.1: A median semilattice with two atoms and height one.

2.2 Examples and Counterexamples

We start by looking at some median semilattices that are not sufficient and give examples of non-majority consensus functions on these semilattices that satisfy axioms given in Theorem 2.1.14.

Example 2.2.1. Let X be the median semilattice shown in Figure 2.1. Notice that X is a semilattice that is not a lattice, so X satisfies the conditions of Lemmas 2.1.6 through 2.1.8. Also notice that X does not have height two, a condition for Lemmas 2.1.9, 2.1.10, and 2.1.13. Define $f : X^3 \longrightarrow X$ by

$$f(P) = \begin{cases} 0 & \text{if } P \in \mathscr{W} \\ \mathrm{Maj}(P) & \text{otherwise.} \end{cases}$$

It is clear that f satisfies $(S), (RDN)$, and (BNT). It remains to verify that f satisfies (M). Since Maj satisfies (M), f satisfies (M) for any pair of biprofiles. Since $f(P) = 0$ for any non-biprofile, f satisfies (M) for any pair of non-biprofiles. Let Q be a biprofile and R be a non-biprofile. Since $f(R) = 0$, (M) is satisfied whenever $R \leq Q$. Finally, observe that $Q \not\leq R$, since every element in a biprofile is maximal.

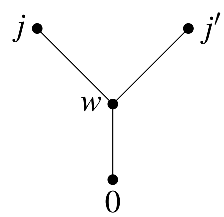


Figure 2.2: A median semilattice with a single atom.

Definition 2.2.2. Let k, q be integers such that $\frac{k}{2} < q < k$. We define $M_q : X^k \longrightarrow X$ by

$$M_q(P) = \bigvee \{ j \in J(X) : |K_j(P)| > q \}.$$

Example 2.2.3. Let X be the median semilattice shown in Figure 2.2. Notice that X satisfies the conditions of Lemmas 2.1.6 through 2.1.8. Since X does not have at least two atoms, X does not satisfy the conditions of Lemmas 2.1.9, 2.1.10, or 2.1.13. Define $f : X^3 \longrightarrow X$ by

$$f(P) = \begin{cases} M_2(P) & \text{if } P \in \mathscr{W} \\ \\ \mathrm{Maj}(P) & \text{otherwise.} \end{cases}$$

Clearly, f satisfies (S), (BNT), and (RDN). For any pair of non-biprofiles, (M) is satisfied by the properties of the join. The same is true for any pair of biprofiles. Let $P \in \mathscr{W}$ and $Q \in X^3 \backslash \mathscr{W}$. Assume that $P \leq Q$. Note that

$$\{j \in J(X) : |K_j(Q)| > 2\} \subseteq \left\{ j \in J(X) : |K_j(Q)| > \frac{3}{2} \right\}$$

and hence $M_2(P) \leq M_2(Q) \leq \mathrm{Maj}(Q)$. Now assume that $Q \leq P$. First, suppose that Q is equal to w in two slots, so that $f(Q) = w$. Since P is a non-biprofile such that $Q \leq P$ and every element in P is above w, $K_w(P) = \{1,2,3\}$ and hence, $w \leq M_2(P)$. Now suppose that $|K_j(Q)| = 2$. In this case, $P = (j, j, j)$ and $j = \mathrm{Maj}(Q) = M_2(P)$. Similarily, if $|K_{j'}(Q)| = 2$, then $P = (j', j', j')$ and $j' = \mathrm{Maj}(Q) = M_2(P)$. We conclude that f satisfies (M).

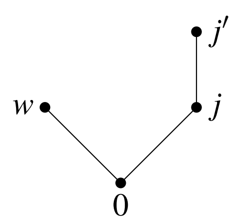


Figure 2.3: A median semilattice with a two-chain of join-irreducible elements.

Example 2.2.4. Let X be the median semilattice shown in Figure 2.3. Notice that X is a median semilattice of height at least two that has at least two atoms. Hence X satisfies the conditions of Lemmas 2.1.6 through 2.1.10, but since X is not sufficient, X does not satisfy the conditions of Lemma 2.1.13. Define $f : X^3 \longrightarrow X$ by

$$f(P) = \begin{cases} j' & \text{if } P \text{ is a permutation of } (j, j, j') \\ \mathrm{Maj}(P) & \text{otherwise.} \end{cases}$$

Clearly, (S) is satisfied. Since $f(P) = \mathrm{Maj}(P)$ for all $P \in \mathscr{W}$, we see that f satisfies (RDN). Let P be a biprofile. Whether P is a permutation of (j, j, j') or not, $f(P) \neq 0$, so f satisfies (BNT). We claim that f satisfies (M). Let Q be a permutation of (j, j, j'), so that $f(Q) = j'$. Let R be a profile such that $Q \leq R$. Either R is identical to Q, or $|K_{j'}(R)| \geq 2 > \frac{k}{2}$. We see that $f(R) = j'$ and hence that $f(Q) \leq f(R)$. Since $f(Q) = j'$ and j' is a maximal element, $f(R') \leq f(Q)$ for any profile $R' \leq Q$. It remains to consider two profiles neither of which is a permutation of (j, j, j'). Since f is identical to Maj for such profiles, f satisfies (M) for such pairs as well.

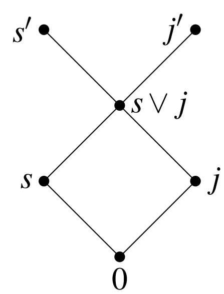


Figure 2.4: Another non-sufficient median semilattice with two atoms and of height greater than or equal to two.

Example 2.2.5. Let X be the median semilattice shown in Figure 2.4. Notice that X satisfies the conditions of Lemma 2.1.6 through 2.1.10, but X is not sufficient. Define $f : X^5 \longrightarrow X$ by

$$f(P) = \begin{cases} s \vee j & \text{if } P \text{ is a permutation of } (j, j, j, j', j') \\ \mathrm{Maj}(P) & \text{otherwise.} \end{cases}$$

Clearly, (S) and (BNT) are satisfied. Since $f(P) = \text{Maj}(P)$ for all $P \in \mathscr{W}$, (RDN) is satisfied. Let P be a permutation of (j, j, j, j', j') and assume that $P \leq Q$ for some $Q \neq P$. We claim that $f(P) \leq f(Q)$. Note that since $|K_j(Q)| = 5$, $j \leq f(Q)$. Since $P \leq Q$ and j' is a maximal element, it follows that at least two elements of Q are equal to j' and a third element of Q is greater than j and therefore also strictly greater than s. Hence, $|K_s(Q)| \geq 3$. It follows that $s \leq f(Q)$ and thus, $s \vee j \leq f(Q)$.

Again, let $P \leq Q$, where we now assume that P is not a permutation of

$$(j, j, j, j', j').$$

If Q is not a permutation of (j, j, j, j', j'), then $f(Q) = \text{Maj}(Q)$ and as shown in the proof of Theorem 14, Maj satisfies (M). Now suppose that Q is a permutation of (j, j, j, j', j'), and so $f(Q) = s \vee j$. Let $t \in J(X) \backslash \{j\}$ and note that since j is an atom and $P \leq Q$, it follows that $|K_t(P)| \leq 2$. If $P = Q$, then $f(P) = s \vee j$. If $P \neq Q$, then $f(P) = \text{Maj}(P)$ and hence, $t \not\leq f(P)$ for all $t \in J(X) \backslash \{j\}$, so $f(P) = j$ or $f(P) = 0$. In either case, $f(P) \leq f(Q)$, which is what we want to show.

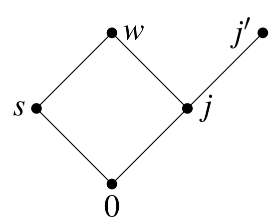


Figure 2.5: Another example of a non-sufficient median semilattice.

Example 2.2.6. Let X be the median semilattice shown in Figure 2.5. Notice that X satisfies the conditions of Lemma 2.1.6 through 2.1.10, but not Lemma 2.1.13, since X is not sufficient. Define $S = (s, ..., s, j', ..., j')$ for which $K_s(P) = \lceil \frac{k}{2} \rceil$. Let $f : X^k \longrightarrow X$

where k is an **odd** integer be defined by

$$f(P) = \begin{cases} w & \text{if } P \text{ is a permutation of } S. \\ \\ \text{Maj}(P) & \text{otherwise.} \end{cases}$$

Clearly $(S), (RDN)$, and (BNT) are all satisfied. Let P be a profile such that $P \leq S$. Suppose that $f(P) \not\leq w = f(S)$. This would mean that $f(P) = \text{Maj}(P) = j'$. However, $|K_{j'}(P)| \leq \frac{k}{2}$, so $j' \not\leq f(P)$. Now let Q be a profile such that $S \leq Q$. We claim that $w \leq f(Q)$. If $S \leq Q$ and $Q \neq S$, then Q must hold a w in at least one slot in which P holds an s. Since $K_{j'}(Q) = K_{j'}(S) = \lceil \frac{k}{2} \rceil - 1$, and $j \leq j'$, we conclude that $K_j(Q) \geq \lceil \frac{k}{2} \rceil > \frac{k}{2}$. Hence, $f(S) = \text{Maj}(S) = w$ and hence $f(S) \leq f(Q)$.

Example 2.2.7. Let X be the median semilattice shown in Figure 2.5. Let $f : X^k \longrightarrow X$ where $k \geq 4$ is an **even** integer. Assume that f satisfies (S), (M), (BNT), and (RDN). We claim that $f = Maj$.

By Lemma 2.1.10, $\text{Maj}(P) \leq f(P)$ for all $P \in X^k$. Let $P \in \mathscr{W}$ and $j \in J(X)$ such that $j \leq f(P)$. Suppose that $j \not\leq \text{Maj}(P)$, so that $|K_j(P)| \leq \frac{k}{2}$. Since $P \in \mathscr{W}$, it follows from Lemma 2.1.8 that $j \not\leq f(P)$ and therefore, $f(P) \leq \text{Maj}(P)$ for every $P \in \mathscr{W}$. It remains to show that $f(Q) = \text{Maj}(Q)$ for every biprofile Q.

We claim that for every biprofile with majority element t, i.e. the element that occurs in the majority of the slots of Q, there exists a profile $R \in \mathscr{W}$ such that $Q \leq R$ and for which t is the only element with the property that $|K_t(R)| > \frac{k}{2}$.

Suppose that j is the majority element in Q. If the remaining slots hold the element s, we construct R by replacing one j in Q with the element j', otherwise making the profile identical to Q. If the remaining slots of Q hold the element j', we construct R by letting exactly $\frac{k}{2}$ slots in R hold the element j', while $Q \leq R$. This is possible since j' covers j and more than $\frac{k}{2}$ slots of Q contain j. Note that $R \in \mathscr{W}$ and that $|K_j(R)| = k$, while $|K_{j'}(R)| = \frac{k}{2}$. Suppose that j' is the majority element in Q. Since j' is a maximal element, it follows that $f(Q) \leq \text{Maj}(Q) = j'$. Finally, suppose that s is the majority element of Q.

If the remaining slots hold j', we construct R by replacing an s with a w. If the remaining slots hold the element j, we construct R by replacing one of the j-elements with the element j'.

Now let Q be a biprofile and let $x \in J(X)$ with the property that $x \not\leq \mathrm{Maj}(Q)$. Since Q is a biprofile, there exists another element $t \neq x$ such that $|K_t(Q)| > \frac{k}{2}$. We construct R as described above. Since $R \in \mathscr{W}$ and t is the only join irreducible with the property that $|K_t(R)| > \frac{k}{2}$, it follows from Lemma 2.1.8 that $x \not\leq f(R)$. Recall that $Q \leq R$, so that by (M), $f(Q) \leq f(R)$. We see that $x \not\leq f(Q) \leq f(R)$. Hence $f(Q) \leq f(R)$, which is what we wanted to show.

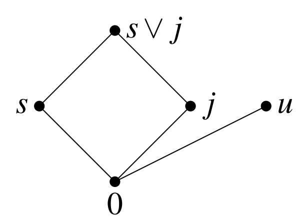


Figure 2.6: An example of a non-sufficient median semilattice for which $f = \mathrm{Maj}$ whenever f satisfies $(M), (S), (RDN)$, and (BNT).

Example 2.2.8. Let X be the semilattice in Figure 2.6. Note that X is non-sufficient since

$$J(X) \cap max(X) = \{u\} \text{ and } J(X) \backslash max(X) \cap M(X) = \{s, j\}.$$

Let $f : X^k \longrightarrow X$, where $k \geq 3$ is an integer and assume that f satisfies (S), (M), (BNT), and (RDN). By Lemma 2.1.10, $\mathrm{Maj}(P) \leq f(P)$ for all $P \in X^k$.

Let $P \in \mathscr{W}$ and $j \in J(X)$ such that $j \leq f(P)$. Suppose that $j \not\leq \mathrm{Maj}(P)$, so that $|K_j(P)| \leq \frac{k}{2}$. Since $P \in \mathscr{W}$, it follows from Lemma 2.1.8 that $j \not\leq f(P)$ and therefore, $f(P) \leq \mathrm{Maj}(P)$ for every $P \in \mathscr{W}$.

Since no pair of join irreducible elements in X are comparable, there exists exactly one $t \in J(X)$ for every biprofile Q, such that $|K_t(Q)| > \frac{k}{2}$. We claim that for every biprofile with majority element t, there exists a profile $R \in \mathscr{W}$ such that $Q \leq R$ and for which t is the only element with the property that $|K_t(R)| > \frac{k}{2}$. Suppose that u is the majority element in Q. Then we construct R by replacing one of the elements Q that is covered by $s \vee j$ with the element $s \vee j$. In all other slots, R is chosen to be identical to Q. Suppose that s is the majority element in Q. Then we chose R to be identical to Q in all slots, except in one s-slot, for which we choose the element $s \vee j$. An analogous construction works when Q has t in the majority of its slots.

Let Q be a biprofile and let $x \in J(X)$ such that $x \nleq \mathrm{Maj}(Q)$. There exists a profile $R \in \mathscr{W}$ such that $Q \leq R$ and $|K_x(R)| \leq \frac{k}{2}$. By Lemma 2.1.8, $x \nleq f(R)$ and hence, by (M), $x \nleq f(Q) \leq f(R)$. We have shown that $f(Q) \leq \mathrm{Maj}(Q)$ for every biprofile Q. Hence $\mathrm{Maj} = f$, which is what we wanted to show.

CHAPTER 3
DECISIVE NEUTRALITY ON MEDIAN SEMILATTICES

The goal of this chapter to describe the class of consensus functions defined on a median semilattice that satisfy the axioms (DN), (BI), and (S). The problem originated with Monjardet's *Arrowian Characterizations of Latticial Federation Consensus Functions* (1990) [9].

3.1 Definitions and Examples

Let X be a finite median semilattice such that the cardinality of X is at least three and let $f : X^k \longrightarrow X$ be a consensus function.

Definition 3.1.1. We say that f satisfies **bi-idempotency** (BI) if for any biprofile $P, f(P) \in \{P\}$, i.e. $f(P)$ is one of the join-irreducible elements belonging to P.

This is a weaker version of the bi-idempotency condition introduced by Monjardet in 1990 [9] and a stronger condition than the biprofile nontrivial condition given in the last chapter.

Definition 3.1.2. We say that f satisfies **decisive neutrality** (DN) if for any profiles P, P' and for any $j, j' \in J(X)$,

$$K_j(P) = K_{j'}(P') \implies [j \le f(P) \iff j' \le f(P')].$$

The decisive neutrality condition given in Definition 3.1.2 is the same as Monjardet's J-decisive neutrality condition (Monjardet 1990, p. 59) [9]. Unlike Restrictive Decisive Neutrality from Chapter 2, this axiom is not restricted to non-biprofiles.

The goal of this chapter is to characterize the class of consensus functions satisfying (BI), (S), and (DN). The following three examples illustrate that these axioms do not characterize majority rule on all median semilattices.

Example 3.1.3. If X is a finite chain, then define $f : X^k \longrightarrow X$ as follows: for any profile $P = (x_1, ..., x_k)$

$$f(P) = \bigwedge_{i=1}^{k} x_i$$

Since X is a chain, $f(P)$ is equal to the smallest member of P and hence (BI) is satisfied for all profiles. Clearly, (S) is satisfied. Let $P = (x_1, x_2, ..., x_k)$ and $P' = (x'_1, x'_2, ..., x'_k)$ be profiles, and $j, j' \in J(X)$ such that $K_j(P) = K_{j'}(P')$. Suppose that $j \leq \bigwedge_{i=1}^{k} x_i = f(P)$ and note that this implies $K_j(P) = K_{j'}(P') = K$. Hence, j' is a lower bound of $\{x'_1, x'_2, ..., x'_k\}$ and we conclude that $j' \leq \bigwedge_{i=1}^{k} x'_i = f(P')$ and hence f satisfies (DN).

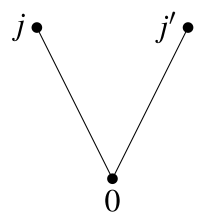


Figure 3.1: A median semilattice that is not a lattice.

Example 3.1.4. Let X be the median semilattice shown in Figure 3.1 and notice that X is a semilattice that is not a lattice. Let $P \in X^3$ and $f : X^3 \longrightarrow X$ be defined as

$$f(P) = \begin{cases} 0 & \text{if } K_t(P) = K \text{ for some } t \in J(X). \\ \mathrm{Maj}(P) & \text{otherwise.} \end{cases}$$

We claim that f satisfies (S), (BI), and (DN). Clearly (S) and (BI) are satisfied. Note that the only profiles for which $f(P) \neq \mathrm{Maj}(P)$ are the profiles (j, j, j) and (j', j', j'). Since $f(j, j, j) = f(j', j', j') = 0$, we see that (DN) is satisfied as well.

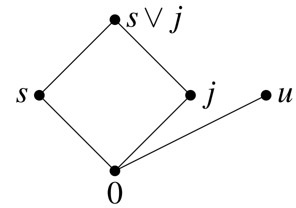


Figure 3.2: A median semilattice where every join irreducible is an atom.

Example 3.1.5. Let X be the semilattice shown in Figure 3.2. Let $P \in X^3$ and $f : X^3 \longrightarrow X$ be defined as

$$f(P) = \bigvee\{t \in J(X) : |K_t(P)| = 2\}$$

Clearly, f satisfies (S). Any biprofile has the property $|K_t(P)| = 2$ for some $t \in J(X)$, so (BI) is satisfied. Let $P, P' \in X^k$ and suppose that $K_t(P) = K_{t'}(P')$ for some $t, t' \in J(X)$. If $|K_t(P)| = |K_{t'}(P'| = 2$, it follows that $t \leq f(P)$ and $t' \leq f(P')$. Since every $w \in J(X)$ is an atom, $w \leq \bigvee\{t \in J(X) : |K_t(P)| = 2\}$ only if $w \in \{t \in J(X) : |K_t(P)| = 2\}$. Hence, f satisfies (DN).

3.2 Results

The goal of this chapter to describe the class of functions that satisfy the axioms (DN), (BI), and (S). The results are summarized in Figure 3.3. The reader should start at the top and follow either the *yes* arrow or the *no* arrow, depending on the class of semilattice of interest. If the semilattice X is not a lattice, the functions characterized by our axioms are given by Theorem 3.2.5. If X is a chain, the functions characterized by our axioms are given by Theorem 3.2.12. If X is lattice that is not a chain and not atomistic, the characterization is given by Theorem 3.2.16. Finally, if X is a lattice that is not a chain, but X is atomistic, Proposition 3.2.15 tells us that X is also Boolean. If X is Boolean with two atoms, the characterization can be found in Theorem 3.2.20. If X is Boolean with three or more atoms, the characterization is given by Theorem 3.2.19. Note

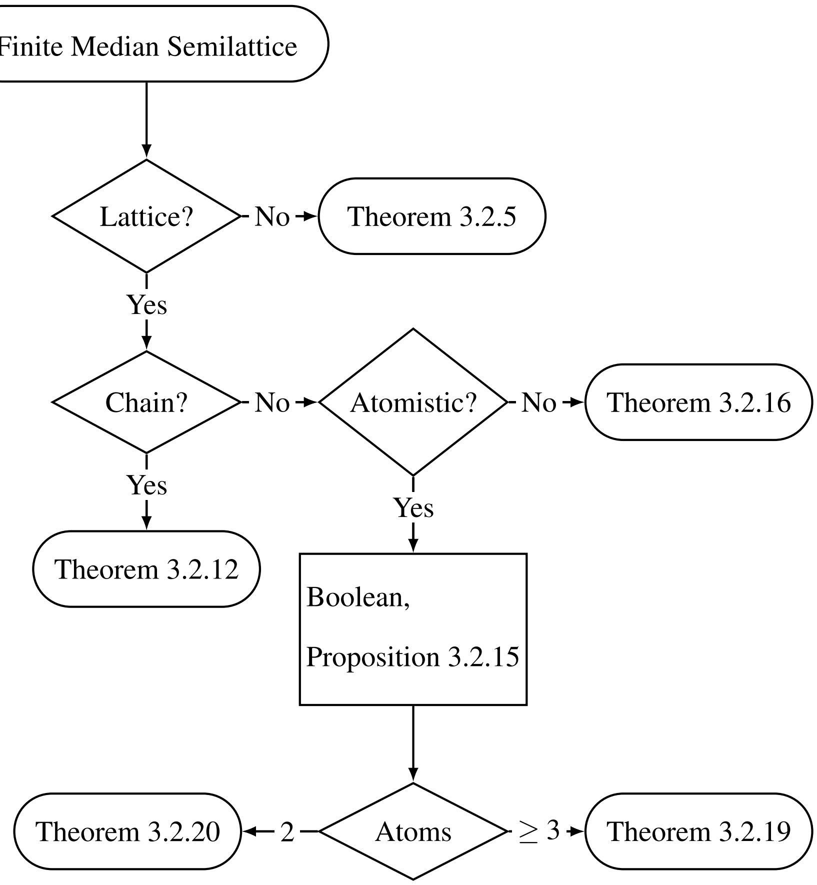


Figure 3.3: A summary of the main result of the chapter.

that this covers every finite median semilattice with cardinality at least three.

If a consensus function satisfies the decisive neutrality condition (DN), then, as shown in the next lemma, there is an associated family of sets $\mathscr{D}_f$ where $I \subseteq K$ belong to $\mathscr{D}_f$ if and only if there exists a profile P and a join irreducible j such that $K_j(P) = I$ and $j \leq f(P)$. We will call $\mathscr{D}_f$ a decisive family.

Lemma 3.2.1. Let $f : X^k \longrightarrow X$ be a consensus function where X is any finite median semilattice. If f satisfies (DN), then there exists a subset $\mathscr{D}_f$ of the power set $\mathscr{P}(K)$ such that for every profile $Q \in X^k$

$$f(Q) = \bigvee \{s \in J(X) : K_s(Q) \in \mathscr{D}_f\}.$$

Proof. Given the decisive neutral function f, we define

$$\mathscr{D}_f = \{I \subseteq K : \exists P \in X^k \text{ and } j \in J(X) \text{ s.t. } K_j(P) = I \text{ and } j \leq f(P)\}.$$

Let $Q \in X^k$ and suppose $K_j(Q) \in \mathscr{D}_f$ for some $j \in J(X)$. Since $K_j(Q) \in \mathscr{D}_f$ it follows from the definition of $\mathscr{D}_f$ that there exists $P \in X^k$ and $s \in J(X)$ such that $K_s(P) = K_j(Q)$ and $s \leq f(P)$. Since f satisfies (DN), $j \leq f(Q)$. Therefore,

$$\bigvee \{s \in J(X) : K_s(Q) \in \mathscr{D}_f\} \leq f(Q).$$

Now suppose that $j' \leq f(Q)$ for some $j' \in J(X)$. By the definition of $\mathscr{D}_f$, $K_{j'}(Q) \in \mathscr{D}_f$. Hence

$$f(Q) = \bigvee \{s \in J(X) : K_s(Q) \in \mathscr{D}_f\}.$$

$\square$

It follows from Lemma 3.2.1 that if $f : X^k \longrightarrow X$ is any consensus function satisfying (DN) and $P \in X^k$ is an arbitrary profile, then

$$j \leq f(P) \iff K_j(P) \in \mathscr{D}_f$$

for all $j \in J(X)$. In this case, we will refer to $\mathscr{D}_f$ as the decisive family determined by the decisive neutral consensus function f.

Lemma 3.2.2. Let $f : X^k \longrightarrow X$ be a consensus function where X is any finite median semilattice which is not a lattice. If f satisfies (DN) and $\mathscr{D}_f$ is the corresponding decisive family of f i.e.

$$\mathscr{D}_f = \{I \subseteq K : \exists P \in X^k \text{ and } j \in J(X) \text{ s.t. } K_j(P) = I \text{ and } j \leq f(P)\},$$

then

$$[A \in \mathscr{D}_f, B \in \mathscr{D}_f] \implies [A \cap B \neq \emptyset].$$

Proof. Assume that there exists $A, B \in \mathscr{D}_f$ such that $A \cap B = \emptyset$. Since X is not a lattice, there exists join irreducibles j and j' such that the join $j \vee j'$ does not exist. Let $P \in X^k$ be a profile such that $K_j(P) = A$ and $K_{j'}(P) = B$. By Lemma 3.2.1,

$$f(Q) = \bigvee \{s \in J(X) : K_s(P) \in \mathscr{D}_f\}.$$

Therefore, $f(P)$ is an upper bound for $\{j, j'\}$ contrary to the fact that $j \vee j'$ does not exist. $\square$

Lemma 3.2.3. Let $f : X^k \longrightarrow X$ be a consensus function where X is any finite median meet semilattice which is not a lattice. If f satisfies (DN) and (S), then $|A| > k/2$ for all $A \in \mathscr{D}_f$.

Proof. Let $A \in \mathscr{D}_f$ and suppose that $|A| \leq k/2$. Let $j \in J(X)$ and $P \in X^k$ such that $K_j(P) = A$. Let P_σ be a permutation of P such that $K_j(P_\sigma) \subseteq K \backslash A$. Note that it follows from Lemma 3.2.2 that

$$[A \in \mathscr{D}_f, B \in \mathscr{D}_f] \implies [A \cap B \neq \emptyset].$$

Since f satisfies (S), $K_j(P_\sigma) \in \mathscr{D}_f$, a contradiction since $K_j(P_\sigma) \cap K_j(P) = \emptyset$. $\square$

The following example illustrates the importance of symmetry.

Example 3.2.4. Let X be a finite meet semilattice and suppose $f : X^k \longrightarrow X$ is the dictatorship function $f(P) = x_1$ for all $P = (x_1, x_2, \ldots, x_k)$. Then f satisfies (DN) and (BI).

Proof. Let $P \in X^k$. Since $f(P) = x_1 \in \{P\}$, (BI) is satisfied. Suppose that $j \leq f(P) = x_1$ so that $\{1\} \subseteq K_j(P)$. Let $j' \in J(X), P' \in X^k$, and suppose that $K_{j'}(P') = K_j(P)$. It follows that $\{1\} \subseteq K_{j'}(P')$, so $j' \leq f(P')$, and therefore (DN) is satisfied. $\square$

We are now ready for the the first theorem of the chapter. When X is not a lattice, our axioms characterize a class consisting of two functions, one of which is majority rule.

Theorem 3.2.5. *Let X be a median semilattice which is not a lattice. If $f : X^k \longrightarrow X$ satisfies $(DN), (BI)$, and (S), then either*

$$f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)| < k\}$$

or

$$f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)|\}.$$

Proof. We claim that

$$\{I : |I| > k/2\}\backslash\{K\} \subseteq \mathscr{D}_f \subseteq \{I : |I| > k/2\}.$$

Since f satisfies (DN) and (S), $\mathscr{D}_f \subseteq \{I : |I| > k/2\}$ follows directly from Lemma 3.2.3. Let $A \in \{I : |I| > k/2\}\backslash\{K\}$. Since X is not a lattice, there exist join irreducibles j and j' such that $j \vee j'$ does not exist. Let P be the biprofile with $\{P\} = \{j, j'\}$ and $K_j(P) = A$. Then $K_{j'}(P) = K\backslash A$. Since $|K\backslash A| \leq k/2$ it follows from Lemma 3.2.3 that $K\backslash A \notin \mathscr{D}_f$. By the remark right after the proof of Lemma 3.2.1, $j' \not\leq f(P)$. Thus, by (BI), $f(P) = j$ and so $A \in \mathscr{D}_f$. If

$$\mathscr{D}_f = \{I : |I| > k/2\}\backslash\{K\}$$

then

$$f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)| < k\}$$

follows from Lemma 3.2.1. If

$$\mathscr{D}_f = \{I : |I| > k/2\}$$

then

$$f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)|\}$$

follows from Lemma 3.2.1. $\square$

The next definition, which can be found in [9] is an unanimity condition with respect to join irreducibles. As we shall see, this condition together with (DN), (S), and (BI) precisely characterizes majority rule on non-lattice median semilattices.

Definition 3.2.6. We say that f satisfies **J-unanimity** (JU) if for any profile P and any $s \in J(X)$, $K_s(P) = K$ implies $s \leq f(P)$.

The following result is related to Corollary 7.4 and Proposition 7.3 in Monjardet (1990) [9].

Corollary 3.2.7. *Let X be a median semilattice that is not a lattice. Then $f : X^k \longrightarrow X$ satisfies (DN), (BI), (S), and (JU) (J-unanimity) if and only if f is majority rule.*

Proof. By the definition of J-unanimity, $K \in \mathscr{D}_f$ and so by Theorem 3.2.5, f is majority rule. $\square$

Definition 3.2.8. A decisive family $\mathscr{D}_f$ is called a **federation** if

$$[A \in \mathscr{D}_f, A \subseteq B] \implies B \in \mathscr{D}_f.$$

Examples of consensus functions with decisive families that are federations include the dictatorship function and majority rule.

Lemma 3.2.9. Let X be a finite median semilattice such that there exists $x, y \in J(X)$ with $x < y$. If $f : X^k \longrightarrow X$ satisfies (DN), then the decisive family $\mathscr{D}_f$ is a federation.

Proof. Let $I \subseteq J \subseteq K$, $x, y \in J(X)$ with $x < y$, and suppose that $I \in \mathscr{D}_f$. Let $P = (x_1, x_2, ..., x_k)$ be a profile such that $x_i = y$ for all $i \in I$, $x_i = x$ for all $i \in J \backslash I$, and zero everywhere else. Since f satisfies (DN), $I \in \mathscr{D}_f$ and $K_y(P) = I$ implies that $y \leq f(P)$. Since $x < y$, $x < f(P)$ and so $K_x(P) = J \in \mathscr{D}_f$. $\square$

The following definition will be used in Theorem 3.2.12 which describes all functions that satisfy $(DN), (S)$, and (BI) when X is a chain.

Definition 3.2.10. When the decisive family of a function f is given by

$$\mathscr{D}_f = \{I \subseteq K : |I| \geq \ell\}$$

where ℓ is an integer, we denote this family by $\mathscr{D}_\ell$.

Proposition 3.2.11. Let X be chain such that $|X| = 3$. If $f : X^k \longrightarrow X$ with $k \geq 3$ satisfies (DN), (S), and (BI), then there exists an integer ℓ such that $0 \leq \ell \leq k$ and

$$\mathscr{D}_f = \mathscr{D}_\ell.$$

Proof. Since f satisfies (S), $I \in \mathscr{D}_f$ implies that

$$\{J \subseteq K : |J| = |I|\} \subseteq \mathscr{D}_f.$$

Let ℓ be the smallest integer such that $|I| = \ell$ for some $I \in \mathscr{D}_f$. By Lemma 3.2.9, $\mathscr{D}_f$ is a federation and thus

$$\mathscr{D}_f = \{I \subseteq K : |I| \geq \ell\}.$$

□

In the case of $\ell = 0$, $f : X^k \longrightarrow X$ with $k \geq 3$ is defined by

$$f(P) = 1$$

for all $P \in X^k$. Then f satisfies (DN), (S), and (BI) and

$$\mathscr{D}_f = \{I \subseteq K : |I| \geq 0\}.$$

Theorem 3.2.12. *Let X be a finite chain such that $|X| \geq 4$. If $f : X^k \longrightarrow X$ with $k \geq 3$ satisfies (DN), (S), and (BI), then there exists an integer ℓ such that $1 \leq \ell \leq k$ and*

$$\mathscr{D}_f = \mathscr{D}_\ell.$$

If $\ell = 1$, then

$$f(x_1, \ldots, x_k) = \bigvee_{i=1}^{k} x_i.$$

If $\ell = k$, then

$$f(x_1, \ldots, x_k) = \bigwedge_{i=1}^{k} x_i.$$

Proof. Since f satisfies (S), $I \in \mathscr{D}_f$ implies that

$$\{J \subseteq K : |J| = |I|\} \subseteq \mathscr{D}_f.$$

Let ℓ be the smallest integer such that $|I| = \ell$ for some $I \in \mathscr{D}_f$. By Lemma 3.2.9, $\mathscr{D}_f$ is a federation and thus

$$\mathscr{D}_f = \{I \subseteq K : |I| \geq \ell\}.$$

Assume that $\ell = 0$ and let Q be an arbitrary profile. For any $j \in J(X)$, $|K_j(Q)| \geq 0$ and so $j \leq f(Q)$. Therefore, $f(Q) = 1$ for any profile Q. Since X has at least four elements, there exists a biprofile that does not contain the element 1. Since f satisfies (BI), $f(Q) = 1$ for all Q is a contradiction. It now follows that $1 \leq \ell \leq k$.

Suppose that $\ell = 1$. Let $P = (x_1, ..., x_k) \in X^k$ and let $x = \bigvee_{i=1}^{k} x_i$. Since X is a chain, $x \in \{P\}$, $|K_x(P)| \geq 1$ and hence $x \leq f(P)$. For any $x' \in J(X)$ such that $x' > x$, $|K_{x'}(P)| = 0$ and and therefore, $x' \not\leq f(P)$ so that $f(P) = x$.

Suppose that $\ell = k$. Let $P = (x_1, ..., x_k) \in X^k$ and let $y = \bigwedge_{i=1}^{k} x_i$. Note that $y' \geq y$ for all $y' \in \{P\}$, so $|K_y(P)| = k$ and hence, $y \leq f(P)$. For any element $y' > y$, $|K_{y'}(P)| < k$ and thus $y' \not\leq f(P)$, i.e. $f(P) = y$. $\square$

Lemma 3.2.13. Let X be a median semilattice such that there exists $x, y \in J(X)$ with $x \parallel y$. If $f : X^k \longrightarrow X$ with $k \geq 3$ satisfies (DN) and (BI), then for any $I \subseteq K$ such that $1 \leq |I| < k$ and $|I| \neq k/2$,

$$|\mathscr{D}_f \cap \{I, K \backslash I\}| = 1.$$

Proof. Let $I \subseteq K$ such that $1 \leq |I| < k$ and $|I| \neq k/2$ and let $x, y \in J(X)$ with $x \parallel y$. There exists a biprofile P with $\{P\} = \{x, y\}$, $K_x(P) = I$ and $K_y(P) = K \backslash I$. Since f satisfies (BI) and $x \parallel y$, it follows that $f(P) = x$ and $f(P) \neq y$, or $f(P) \neq x$ and $f(P) = y$. Since f satisfies (DN), either $I \in \mathscr{D}_f$ and $K \backslash I \notin \mathscr{D}_f$, or $I \notin \mathscr{D}_f$ and $K \backslash I \in \mathscr{D}_f$. $\square$

Definition 3.2.14. The elements x, x' in a distributive lattice X are *complements* if $x \wedge x' = 0$ and $x \vee x' = 1$. A *boolean lattice* is a distributive lattice with at least two elements in which every element has a complement.

Proposition 3.2.15. Every finite, atomistic, distributive lattice with more than two elements is Boolean.

Proof. Let X be such a lattice. Let $x \in X$ such that $x \neq 0$ and $x \neq 1$. Let $x' = \bigvee\{a \in J(X) : a \nleq x\}$ and note that $x = \bigvee\{a \in J(X) : a \leq x\}$. We claim that x' is the complement of x. Since X is a lattice, the element

$$x \vee x' = \bigvee\{a \in J(X) : a \leq x\} \vee \bigvee\{a \in J(X) : a \nleq x\} = \bigvee J(X)$$

exists and is equal to 1. Let

$$y = x \wedge x' = \bigvee\{a \in J(X) : a \leq x\} \wedge \bigvee\{a \in J(X) : a \nleq x\}$$

and note that $y = \bigvee\{a \in J(X) : a \leq y\}$. Since $y \leq x \wedge x'$, $y \leq x$ and hence y is the join of a subset of $\{a \in J(X) : a \leq x\}$. Similarly, since $y \leq x'$, y is the join of a subset of $\{a \in J(X) : a \nleq x\}$. Since the intersection of these two sets is empty, $y = \vee\emptyset = 0$. We conclude that x' is the complement of x.

Since X is a distributive lattice where every element distinct from 0 and 1 has a complement, it follows from the definition of Boolean lattice that X is Boolean. $\square$

We now introduce the *ceiling function*. For any $x \in \mathbb{R}$,

$$\lceil x \rceil = \inf\{n \in \mathbb{Z} : n \geq x\}.$$

In particular, if $k \geq 2$ is an integer, then

$$\left\lceil \frac{k}{2} \right\rceil = \begin{cases} \frac{k}{2} & \text{if } k \text{ is even} \\ \\ \frac{k+1}{2} & \text{if } k \text{ is odd.} \end{cases}$$

The following theorem characterizes the class of functions that satisfy (D), (BI), and (S) in the case of X being a lattice, but not a chain, nor atomistic. Just like in Theorem 3.2.5, the characterized class consists of two functions, one of which is majority rule.

Theorem 3.2.16. *Suppose that X is a finite distributive lattice with $|X| \geq 3$ such that X is not a chain nor an atomistic lattice. If $f : X^k \longrightarrow X$ satisfies (DN), (BI), and (S), then*

$$\mathscr{D}_f = \mathscr{D}_\ell$$

where

$$\ell = \left\lceil \frac{k}{2} \right\rceil \text{ or } \ell = \left\lceil \frac{k+1}{2} \right\rceil.$$

When k is odd or $\ell = \lceil \frac{k+1}{2} \rceil$, f is majority rule.

Proof. Let $I \in \mathscr{D}_f$ and assume that $|I| < k/2$. Since X is not not atomistic, X has at least one pair of comparable join irreducible elements. Therefore, since f satisfies (DN), Lemma 3.2.9 implies that $\mathscr{D}_f$ is a federation. Consequently, if $J \subseteq K$ satisfies $|J| = |K \backslash I|$ and $I \subseteq J$, then $J \in \mathscr{D}_f$. Since f satisfies (S), $J \in \mathscr{D}_f$ implies that $K \backslash I \in \mathscr{D}_f$. So both I and $K \backslash I$ belong to $\mathscr{D}_f$. This contradiction shows that if $I \in \mathscr{D}_f$, then $|I| \geq k/2$.

Since f satisfies (S) and, as mentioned above, $\mathscr{D}_f$ is a federation, it follows that

$$\mathscr{D}_f = \mathscr{D}_\ell$$

for some integer $\ell \geq k/2$. Next, let J be any subset K such that $|J| = \lceil \frac{k+1}{2} \rceil$. Since $|J| \neq k/2$ and $|K \backslash J| < k/2$ it follows from Lemma 3.2.9 that $J \in \mathscr{D}_f$ and $K \backslash J \notin \mathscr{D}_f$. Hence,

$$\ell \leq \left\lceil \frac{k+1}{2} \right\rceil$$

and we are done. □

Definition 3.2.17. A subset $\mathscr{I}$ of K is *bicomplete* if

$$|\mathscr{I} \cap \{i, k-i\}| = 1$$

for all $i \in [1, k/2) \cap \mathbb{Z}$. For example, if $K = \{1,2,3,4,5\}$, then $\{1,3\}$ and $\{1,3,5\}$ are bicomplete. For any subset $\mathscr{I}$ of K, let

$$\mathscr{D}_\mathscr{I} = \{I \subseteq K : |I| \in \mathscr{I}\}.$$

Lemma 3.2.18. Let $f : X^k \longrightarrow X$ be a consensus function where X is any finite median semilattice containing two noncomparable join irreducible elements. If f satisfies (S), (DN), and (BI), then $\{|I| : I \in \mathscr{D}_f \text{ and } I \neq \emptyset\}$ is a bicomplete subset of K.

Proof. Let $i \in [1, k/2)$ and choose $I \subseteq K$ such that $|I| = i$. By Lemma 3.2.13,

$$|\mathscr{D}_f \cap \{I, K \backslash I\}| = 1.$$

If $I \in \mathscr{D}_f$ and $K\backslash I \notin \mathscr{D}_f$, then $i \in \{|I| : I \in \mathscr{D}_f \text{ and } I \neq 0\}$ and, since f satisfies (S) and $|K\backslash I| = k-i, k-i \notin \{|I| : I \in \mathscr{D}_f \text{ and } I \neq 0\}$. On the other hand, $I \notin \mathscr{D}_f$ and $K\backslash I \in \mathscr{D}_f$ implies that $k-i \in \{|I| : I \in \mathscr{D}_f \text{ and } I \neq 0\}$ and $i \notin \{|I| : I \in \mathscr{D}_f \text{ and } I \neq 0\}$. Hence $\{|I| : I \in \mathscr{D}_f \text{ and } I \neq 0\}$ is a bicomplete subset of K. $\square$

Theorems 3.2.19 and 3.2.20 handle the case when X is a boolean lattice.

Theorem 3.2.19. *Let X be a Boolean, distributive lattice with at least three atoms. If $f : X^k \longrightarrow X$ satisfies (DN), (BI), and (S), then there exists a bicomplete subset $\mathscr{I}$ such that*

$$\mathscr{D}_f = \mathscr{D}_{\mathscr{I}}.$$

Conversely, if $\mathscr{I}$ is a bicomplete subset of K and

$$f(P) = \bigvee\{j \in J(X) : |K_j(P)| \in \mathscr{I}\}$$

for all $P \in X^k$ then f satisfies (DN), (BI), and (S).

Proof. Assume that f satisfies (DN), (BI), and (S). Since f satisfies (DN) it follows from Lemma 3.2.1 that there exists a subset $\mathscr{D}_f$ of $P(K)$ such that

$$f(P) = \bigvee\{s \in J(X) : K_s(P) \in \mathscr{D}_f\}$$

for any $P \in X^k$. Moreover, as noted right after the proof of Lemma 3.2.1,

$$s \leq f(P) \text{ if and only if } K_s(P) \in \mathscr{D}_f$$

for any $P \in X^k$.

Assume that $\emptyset \in \mathscr{D}_f$. Let P be a biprofile. Since X contains at least three atoms, there exists $j \in J(X)$ such that $j \notin \{P\}$. Note that $K_j(P) \neq \emptyset$ and so $\emptyset \in \mathscr{D}_f$ implies that $j \leq f(P)$. Since X is atomistic it follows that $f(P) \notin \{P\}$ which contradicts the fact that f satisfies (BI). Thus $\emptyset \notin \mathscr{D}_f$.

Since f satisfies $(DN), (BI)$, and (S) we can apply Lemma 3.2.18 to conclude that $\{|I| : I \in \mathscr{D}_f \text{ and } I \neq \emptyset\}$ is a bicomplete subset of K. Since $\emptyset \notin \mathscr{D}_f$ it follows that

$$\{|I| : I \in \mathscr{D}_f \text{ and } I \neq \emptyset\} = \{|I| : I \in \mathscr{D}_f\}$$

Let $\mathscr{I}$ denote the bicomplete subset $\{|I| : I \in \mathscr{D}_f\}$, let $A \in \mathscr{D}_{\mathscr{I}} = \{I \subseteq K : |I| \in \mathscr{I}\}$, and note that there exists $I \in \mathscr{D}_f$ such that $|I| = |A|$. Since f satisfies (S) and (DN), it follows that $A \in \mathscr{D}_f$. Let $B \in \mathscr{D}_f$ and note that since $|B| \in \mathscr{I}, B \in \mathscr{D}_{\mathscr{I}}$. Since $\mathscr{D}_f \subseteq \mathscr{D}_{\mathscr{I}}$ and $\mathscr{D}_{\mathscr{I}} \subseteq \mathscr{D}_f$,

$$\mathscr{D}_f = \mathscr{D}_{\mathscr{I}}.$$

Let

$$f(P) = \bigvee \{j \in J(X) : |K_j(P)| \in \mathscr{I}\}$$

where $\mathscr{I}$ is a bicomplete subset of K. Clearly, (S) is satisfied. We claim that (DN) is satisfied. Let $P, Q \in X^k$ and $j, s \in J(X)$ such that $K_j(P) = K_s(Q)$, with $j \leq f(P)$. Since $j \in J(X)$, $j \leq \bigvee\{j \in J(X) : |K_j(P)| \in \mathscr{I}\}$ implies that $j \in \{j \in J(X) : |K_j(P)| \in \mathscr{I}\}$ and thus $|K_j(P)| \in \mathscr{I}$. Since $K_s(Q) = K_j(P), |K_s(Q)| \in \mathscr{I}$, and hence $s \leq f(P)$.

Finally, we claim that f satisfies (BI). Let P be a biprofile with $\{P\} = \{j, s\}$ and $|K_j(P)| < k/2$ and observe that $|K_s(P)| = k - |K_j(P)|$. Since $\mathscr{I}$ is bicomplete, $|K_{j'}(P)| \in \mathscr{I}$ implies that $k - |K_j(P)| \notin \mathscr{I}$. Hence, we either have $j \leq J(P)$ and $s \nleq f(P)$, or $j \nleq J(P)$ and $s \leq f(P)$. Since j and s are atoms and $\{P\} = \{j, s\}$, we conclude that $f(P) \in \{P\}$, which is what we wanted. $\square$

The next theorem covers the case where X is a lattice with four elements, two of which are atoms. Note that a chain with three or more elements is non-boolean.

Theorem 3.2.20. *Let X be a Boolean lattice with two atoms. If $f : X^k \longrightarrow X$ satisfies $(S), (DN)$, and (BI), then there exists a bicomplete subset $\mathscr{I}$ of K such that*

$$\mathscr{D}_f = \mathscr{D}_{\mathscr{I}} \text{ or } \mathscr{D}_f = \mathscr{D}_{\mathscr{I} \cup \{0\}}$$

Conversely, if $\mathscr{I}$ is a bicomplete subset of K and

$$f(P) = \bigvee \{ j \in J(X) : |K_j(P)| \in \mathscr{I} \}$$

for all $P \in X^k$ or if

$$f(P) = \bigvee \{ j \in J(X) : |K_j(P)| \in \mathscr{I} \cup \{0\} \}$$

for all $P \in X^k$, then f satisfies (DN), (BI), and (S).

Proof. Assume that f satisfies (DN), (BI), and (S). Since f satisfies (DN), it follows from Lemma 3.2.1 that there exists a subset $\mathscr{D}_f$ of $\mathscr{P}(K)$ such that

$$f(P) = \bigvee \{ s \in J(X) : K_s(P) \in \mathscr{D}_f \}$$

for any $P \in X^k$. Since f satisfies $(DN), (BI)$, and (S) we can apply Lemma 3.2.18 to conclude that $\{|I| : I \in \mathscr{D}_f \text{ and } I \neq \emptyset\}$, which we denote by $\mathscr{I}$, is a bicomplete subset of K. Let $A \in \mathscr{D}_{\mathscr{I}} = \{I \subseteq K : |I| \in \mathscr{I}\}$, and note that there exists $I \in \mathscr{D}_f$ such that $|I| = |A|$. Since f satisfies (S) and (DN), it follows that $A \in \mathscr{D}_f$. Let $B \in \mathscr{D}_f$ and note that since $|B| \in \mathscr{I} \cup \{0\}, B \in \mathscr{D}_{\mathscr{I} \cup \{0\}}$. Since $\mathscr{D}_f \subseteq \mathscr{D}_{\mathscr{I} \cup \{0\}}$ and $\mathscr{D}_{\mathscr{I}} \subseteq \mathscr{D}_f$,

$$\mathscr{D}_f = \mathscr{D}_{\mathscr{I}} \text{ or } \mathscr{D}_f = \mathscr{D}_{\mathscr{I} \cup \{0\}}.$$

Let $\mathscr{I}$ be a bicomplete subset of K and define $f : X^k \longrightarrow X$ as

$$f(P) = \bigvee \{ j \in J(X) : |K_j(P)| \in \mathscr{I} \cup \{0\} \}.$$

Let j and j' be the two atoms belonging to X. Clearly f satisfies (S). We claim that f satisfies (BI). Let P be a biprofile with $\{P\} = \{j, j'\}$ and $0 < |K_j(P)| < k/2$ and observe that

$|K_{j'}(P)| = k - |K_j(P)|$. Since $\mathscr{I}$ is bicomplete, $|K_{j'}(P)| \in \mathscr{I}$ implies that $k - |K_j(P)| \notin \mathscr{I}$. Hence, we either have $j \leq J(P)$ and $j' \not\leq f(P)$, or $j \not\leq J(P)$ and $j' \leq f(P)$. Since j and j' are atoms and $\{P\} = \{j, j'\}$, we conclude that $f(P) \in \{P\}$. We claim that f satisfies (DN). Let $P, Q \in X^k$ and $j, s \in J(X)$ such that $K_j(P) = K_s(Q)$, with $j \leq f(P)$. Since $j \in J(X)$, $j \leq \bigvee\{j \in J(X) : |K_j(P)| \in \mathscr{I} \cup \{0\}\}$ implies that $j \in \{j \in J(X) : |K_j(P)| \in \mathscr{I} \cup \{0\}\}$ and thus $|K_j(P)| \in \mathscr{I} \cup \{0\}$. Since $K_s(Q) = K_j(P), |K_s(Q)| \in \mathscr{I} \cup \{0\}$, and hence $s \leq f(P)$.

For the case of $f(P) = \bigvee\{j \in J(X) : |K_j(P)| \in \mathscr{I}\}$ for all $P \in X^k$, we refer to the proof of Theorem 3.2.19. $\square$

How many different functions satisfy the condtions given in Theorem 3.2.20 for a given profile length k? This question is answered in Proposition 3.2.23.

Lemma 3.2.21. Let $k \geq 3$ be an interger. Define

$$A = \left\{1, 2, \ldots, \frac{k-1}{2}, k\right\}$$

when k is odd,

$$A = \left\{1, 2, \ldots, \frac{k}{2}, k\right\}$$

when k is even. The function $f : \mathscr{P}(A) \longrightarrow \mathscr{P}(K)$ defined by

$$f(I) = \left\{i \in \mathbb{Z} : i \in I \text{ or } \frac{k}{2} \leq i \leq k-1 \text{ and } k - i \notin I\right\}$$

is one-to-one.

Proof. Let $I \subseteq A$ and $J \subseteq A$ such that $f(I) = f(J)$. Let $a \in I$. Since $a \in I, a \in f(I) = f(J)$. This mean that either $a \in J$ or $k/2 \leq a \leq k-1$. Since $a \in I$, $a \notin [k/2, k-1]$ and hence $a \in J$. By symmetry, $a \in J$ implies $a \in I$ and therefore $I = J$. $\square$

Let $K = \{1, 2, \ldots, k\}$ with $k \geq 3$ and let $\mathscr{B}(A)$ be the set of bicomplete subsets of K. In addition, let

$$A = \left\{1, 2, \ldots, \left\lceil \frac{k-1}{2} \right\rceil, k\right\}$$

and notice that A is proper subset of K.

Lemma 3.2.22. The function $f : \mathscr{P}(A) \longrightarrow \mathscr{B}(K)$ defined by

$$f(J) = \{j \in K : j \in J \text{ or } j = k - i \text{ for some } i \in A \backslash J\}$$

for all $J \in \mathscr{B}(K)$ is a well-defined bijection. In fact, the inverse map g is defined by

$$g(I) = I \cap A$$

for all $I \in \mathscr{B}(K)$.

Proof. To show that f is well-defined, let $J \in \mathscr{P}(A)$ and show that $f(J)$ is a bicomplete subset of K. Let $i \in [1, k/2) \cap \mathbb{Z}$. We claim that $|\{i, k-i\} \cap f(J)| = 1$.

If $i \in J$, then $i \in f(J)$ since $J \subseteq f(J)$. Notice that $i \in [1, k/2)$ implies that $k/2 < k-1 < k$ and so $k - i \notin A$. Since $J \subseteq A$ it follows that $k - i \notin J$. Next, assume that $k - i = k - i'$ for some $i' \in A \backslash J$. Then $i = i'$ leading to the contradiction $i' \in J$. Hence $k - i \notin f(J)$.

If $i \notin J$, then $i \in A \backslash J$ and so $k - i \in f(J)$. Assume $i = k - i'$ for some $i' \in A \backslash J$. Then $i + i' = k$. Since $i \in [1, k/2) \cap \mathbb{Z}$ it follows that $i' > k/2$. Since $i' \in A, i' > k/2$ implies that $i' = k$. Now $i' = k$ along with $i + i'$ forces $i = 0$ contrary to $i \geq 1$. This contradiction shows that $i \neq k - i'$ for all $i' \in A \backslash J$. This fact, along with $i \notin J$, implies that $i \notin f(J)$. Thus, $|\{i, k-i\} \cap f(J)| = 1$ and so f is well-defined.

Let $I \in \mathscr{B}(K)$ and note that $g(I) = I \cap A \subseteq A$, so the map g is well defined. Our next goal is to show that g is the inverse of f.

Let $I \in \mathscr{B}(K)$ and note that

$$f(g(I)) = f(I \cap A) \supseteq I \cap A.$$

Now let $j \in I$ and $j \notin A$. Then

$$\frac{k}{2} < j < k \text{ and } 1 < k - j < \frac{k}{2}.$$

So $k-j \in A$. Moreover, since I is a bicomplete subset of K,

$$|\{j, k-j\} \cap I| = 1.$$

Thus, $j \in I$ implies that $k-j \notin I$. Since $k-j \in A \backslash (I \cap A)$ it follows that

$$j = k-(k-j) \in f(I \cap A).$$

Thus, $I \subseteq f(g(I))$.

If $x \in f(g(I))$, then, by the definition of f, either $x \in g(I) = I \cap A$ and so $x \in I$ or $x = k-i$ for some $i \in A \backslash (I \cap A)$. For the latter case, $i \in I$ along with I being a bicomplete subset of K implies that $k-i \in I$ and so $x \in I$. We can now conclude that

$$f(g(I)) = I.$$

Let $I_1, I_2 \in \mathscr{B}(K)$ such that $g(I_1) = g(I_2)$. We see that $f(g(I_1)) = f(g(I_2))$ which implies that $I_1 = I_2$, and hence g is one-to-one. Let $J \in \mathscr{P}(A)$ and note that the element $f(J) \in \mathscr{B}(A)$ maps to J. Therefore, g onto and since g is one-to-one and onto, g is a bijection. (I would like to discuss this part) $\square$

Proposition 3.2.23. The number of functions characterized in Theorem 3.2.19 is 2^h where $h = \lceil \frac{k+1}{2} \rceil$.

Proof. For a given k, the number of consensus function characterized in Theorem 3.2.19 is equal to the number of bicomplete subsets of K. Let f be defined as in Lemma 3.2.22 and note that since $f : \mathscr{P}(A) \longrightarrow \mathscr{B}(K)$ is a bijection,

$$|\mathscr{B}(K)| = |\mathscr{P}(A)| = 2^{|A|}.$$

Recall that

$$A = \left\{1, 2, \ldots, \left\lceil \frac{k-1}{2} \right\rceil, k\right\}$$

and hence $|A| = \lceil \frac{k+1}{2} \rceil$. $\square$

Lemma 3.2.24. Let $f : X^k \longrightarrow X$ be a consensus function where X is any finite median semilattice. If f satisfies (M) and there exists a bicomplete subset $\mathscr{I}$ such that $D_f = \mathscr{D}_{\mathscr{I}}$, then $[1, k/2) \cap \mathscr{I} = \emptyset$.

Proof. Assume $c \in [1, k/2) \cap \mathscr{I}$. Since $\mathscr{I}$ is bicomplete, $k - c \notin \mathscr{I}$. Let $P, Q \in X^k$ such that $|K_j(P)| = c$, $|K_j(Q)| = k - c$, and $P \leq Q$. Note that $j \leq f(P)$ while $j \not\leq f(Q)$ and hence $f(P) \not\leq f(Q)$, a contradiction since f satisfies (M). □

Definition 3.2.25. A consensus function $f : X^k \longrightarrow X$ is a **quota rule** if there exists an integer ℓ in the interval $[1, k]$ such that for any profile P,

$$f(P) = \bigvee \left\{ j \in J : |K_j(P)| \geq \ell \right\}.$$

In other words, $\mathscr{D}_f = \mathscr{D}_\ell$. We refer to ℓ as the **threshold** of the quota rule f.

Definition 3.2.26. If k is even and X is a lattice, then $f : X^k \longrightarrow X$ is **weak majority rule** if for any profile P,

$$f(P) = \bigvee \{ j \in J(X) : |K_j(P)| \geq k/2 \}.$$

Corollary 7.4 in [9] contains an error. The claim is that that conditions (2) and (4) from that corollary are equivalent. However, this equivalence is true only for non-lattice median semilattices. The next result corrects this error. It is a complete characterization of consensus functions on a median semilattice that satisfy the four axioms of decisive neutrality, bi-idempotence, symmetry, and monotonicity.

Theorem 3.2.27. *Assume that X is a finite median semilattice such that $|X| \geq 3$ and let $f : X^k \longrightarrow X$ be a consensus function on X with $k \geq 3$.*

1. *If X is not a lattice, then f satisfies (DN), (BI), (S), and (M) if and only if f is majority rule.*

2. *If X is a lattice and not a chain, then f satisfies (DN), (BI), (S), and (M) if and only if f is majority rule, or weak majority rule and k is even.*

3. *If X is a chain, then f satisfies (DN), (BI), (S), and (M) if and only if either f is the constant function with output 1 and $|X| = 3$ or if f is a quota rule with threshold ℓ for some integer ℓ beloning to the interval $[1,k]$.*

Proof. 1. If f is majority rule, then f satisfies (DN), (BI), (S), and (M). The proofs for each of the axoims are analogous to those in found in the proof of Theorem 2.1.14.

If X is not a lattice, then by Theorem 3.2.5, (DN), (BI), and (S) imply that

$$f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)| < k\}$$

or

$$f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)|\}.$$

Suppose that $f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)| < k\}$. Let $j \in J(X)$, $P = (j, j, ..., j, 0)$ and $P' = (j, j, ..., j)$. Note that $j \nleq f(P')$ whereas $j \leq f(P)$. Since $P < P'$, monotonicity implies that $f(P) \leq f(P')$, which is a contradiction. Hence $f(P) = \bigvee\{t \in J(X) : k/2 < |K_t(P)|\}$, i.e. f is majority rule.

2. As mentioned above, if f is majority rule, then f satisfies (DN), (BI), (S), and (M). This is also true if f is weak majority rule. Recall that no element in a biprofile can be in exactly half the slots.

If X is a Boolean lattice with two atoms, Theorem 3.2.20 implies that

$$\mathscr{D}_f = \mathscr{D}_{\mathscr{I}} \text{ or } \mathscr{D}_f = \mathscr{D}_{\mathscr{I}\cup\{0\}}.$$

where $\mathscr{I}$ is a bicomplete subset of K. Recall that A subset $\mathscr{I}$ of K is bicomplete if

$$|\mathscr{I} \cap \{i, k-i\}| = 1$$

for all $i \in [1, k/2) \cap \mathbb{Z}$.

Suppose that the empty set is a decisive set. Let $P_0 = (0,0,...,0)$ and note that $f(P_0) = \bigvee J(X) = 1$ since $K_j(P_0) = \emptyset$ for all $j \in J(X)$. Then, by monotonicity, $f(P) = 1$ for all $P \in X^k$ since P_0 is below every other profile. Since $\mathscr{I}$ is bicomplete, there exists $m \in K$ such that $m \notin \mathscr{I}$. Let $Q \in X^k$ and $j \in J(X)$ such that $|K_j(Q)| = m$. Then $j \nleq f(Q)$, which is a contradiction since $f(Q) = 1$.

Since the empty set is not a decisive set, both Theorem 3.2.19 and Theorem imply 3.2.20 that

$$f(P) = \bigvee \{j \in J(X) : |K_j(P)| \in \mathscr{I}\}$$

where $\mathscr{I}$ is a bicomplete subset of K, when X is boolean with at least two atoms.

By Lemma 3.2.24, $[1,k/2) \cap \mathscr{I} = \emptyset$. If k is odd, then, $\frac{k+1}{2}$ is the smallest member of $\mathscr{I}$ and since $|\mathscr{I} \cap \{i, k-i\}| = 1$, it follows that $\mathscr{I} = \{\frac{k+1}{2}, \frac{k+1}{2} + 1, ..., k\}$, which makes f majority rule. If k is even, $\frac{k}{2}$ may or may not belong to $\mathscr{I}$. If $\frac{k}{2} \in \mathscr{I}$, $\mathscr{I} = \{\frac{k}{2}, \frac{k}{2} + 1, ..., k\}$, i.e. f is weak majority rule. If $\frac{k}{2} \notin \mathscr{I}$, f is majority rule.

If X is a lattice, but not a chain and not Boolean, it follows from directly from Theorem 3.2.16 that f is either majority rule, or f is weak majority rule and k is even.

3. Clearly, the quota rule satisfies (DN), (BI), (S), and (M).

   The other direction follows from Proposition 3.2.11 and Theorem 3.2.12.

□

CHAPTER 4
SPLIT DECISIVE NEUTRALITY ON MEDIAN GRAHPS

There is a nice connection between median graphs and median semilattices. Namely, the covering graph of a median semilattice is a median graph and any median graph is the covering graph of some median semilattice (see Bandelt *Discrete Ordered Sets whose covering graphs are median* (1984) [2]). The goal of this chapter is to use this connection to possibly extend the main results of Chapter 3. We are able to make some progress on this problem.

4.1 Median Graphs

Definitions 4.1.1, 4.1.2, 4.1.3, and part of Definition 4.1.4 come from [5].

Definition 4.1.1. A **graph** G is a finite nonempty set V of objects called **vertices** (the singular is **vertex**) together with a nonempty set E of two-element subsets of V called **edges**. If uv is an edge of G, then u and v are **adjacent vertices.**

Definition 4.1.2. For two (not necessarily distinct) vertices u and v in a graph G, a $u-v$ **walk** W in G is a sequence of vertices in G, beginning with u and ending at v such that the consecutive vertices in W are adjacent in G. A walk in a graph G with no vertex repeated is called a **path**.

Definition 4.1.3. Two vertices u and v are **connected** in a graph G if G contains a $u-v$ path. The graph G itself is connected if every two vertices of G are connected.

Definition 4.1.4. A **cycle** C_n in a graph G is a walk in which no edge is repeated, and in which the vertices can be labeled $v_1, v_2, ..., v_n$, and whose edges are v_1, v_n and v_i, v_{i+1} for $i = 1, 2, ..., n-1$. A **tree** is a connected graph that has no cycles.

Definition 4.1.5. In a connected graph $G = (V, E)$, the **distance** between two vertices x and y, denoted $d(x, y)$, is the length of the shortest path from x to y. A shortest path between x and y is called an x, y**-geodesic.**

Definition 4.1.6. Let $G = (V, E)$ be a simple, finite, and connected graph. For any $u, v \in V$,

$$\mathrm{I}_G(u, v) = \{x \in V : d(u, v) = d(u, x) + d(x, v)\}.$$

$\mathrm{I}_G(u, v)$, or just $\mathrm{I}(u, v)$, is called the **interval** between u and v.

Definition 4.1.7. A *median graph* $G = (V, E)$, is a connected graph such that for everv $u, v, w \in V$, $\mathrm{I}(u, v) \cap \mathrm{I}(u, w) \cap \mathrm{I}(v, w)$ contains a unique element. We will use the notation

$$\mathrm{I}(u, v, w) = \mathrm{I}(u, v) \cap \mathrm{I}(u, w) \cap \mathrm{I}(v, w).$$

Every finite tree is a median graph. Let G be the binary tree in Figure 4.1. Notice that for $\gamma, \delta, \zeta \in G$,

$$\mathrm{I}(\delta, \varepsilon) \cap \mathrm{I}(\delta, \zeta) \cap \mathrm{I}(\varepsilon, \zeta) = \{\beta\}$$

i.e. the intersection of the three pairs of intervals is a single element. This is true for any triplet of vertices and hence G is a median graph.

The graph in Figure 4.2, C_5, is not a median graph. Notice that

$$\begin{aligned}
\mathrm{I}(\alpha, \beta) &= \{\alpha, \beta\} \\
\mathrm{I}(\alpha, \omega) &= \{\alpha, \delta, \omega\} \\
\mathrm{I}(\beta, \omega) &= \{\beta, \gamma, \omega\}
\end{aligned}$$

The intersection

$$\mathrm{I}(\alpha, \beta) \cap \mathrm{I}(\alpha, \omega) \cap \mathrm{I}(\beta, \omega) = \emptyset$$

and hence C_5 is not a median graph.

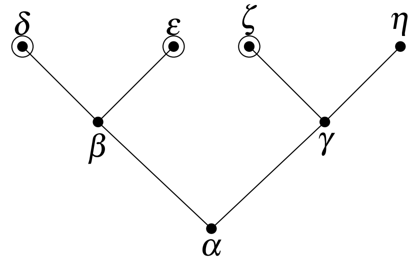


Figure 4.1: A binary tree. The median of $\{\delta, \varepsilon, \zeta\}$ is β.

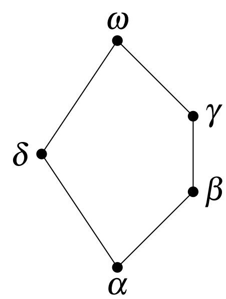


Figure 4.2: C_5.

Definition 4.1.8. Let $G = (V, E)$ be a median graph and choose any vertex z belonging to V. We define $\leq_z$ as follows:

$$x \leq_z y \text{ if } x \in \mathrm{I}(z, y)$$

for all $x, y \in V$.

Lemma 4.1.9. The pair $(V, \leq_z)$ is a partially ordered set with minimal element z.

Proof. For any vertex $x \in V$ we know that $x \in \mathrm{I}(z, x)$. Thus $x \leq_z x$ for all $x \in V$ and hence $\leq_z$ is reflexive. We claim that $\leq_z$ is antisymmetric. Let $u, v \in V$ such that $u \leq_z v$ and $v \leq_z u$. Now observe that since $u \leq_z v$ and $v \leq_z u$, $u \in \mathrm{I}(v, z)$ and $v \in \mathrm{I}(u, z)$, and thus $\{u, v\}$ is a subset of both $\mathrm{I}(v, z)$ and $\mathrm{I}(u, z)$ and hence it also follows that $\{u, v\} \subseteq \mathrm{I}(u, v) \cap \mathrm{I}(u, z) \cap \mathrm{I}(v, z)$. Since V is a median graph,

$$|\mathrm{I}(u, v) \cap \mathrm{I}(u, z) \cap \mathrm{I}(v, z)| = 1$$

which implies that u and v is the same element. We claim that $\leq_z$ is transitive. Let $u,v,w \in V$ such that $u \leq_z v$ and $v \leq_z w$ and hence

$$d(z,v) = d(z,u) + d(u,v)$$
$$d(z,w) = d(z,v) + d(v,w)$$

We combine these and apply the triangle inequality:

$$d(z,w) = d(z,u) + d(u,v) + d(v,w) \geq d(z,u) + d(u,w) \geq d(z,w).$$

Since $d(z,w) = d(z,u) + d(u,w)$, u belongs to the interval $\mathrm{I}(z,w)$ and thus $u \leq_z w$, which is what we wanted to show. □

It turns out that the partially ordered set $(V, \leq_z)$ is a meet semilattice where, for any x and $y \in V, x \wedge_z y$ is the unique element belonging to $\mathrm{I}(x,y) \cap \mathrm{I}(x,z) \cap \mathrm{I}(y,z)$. Moreover, based on the work of Sholander [13], it can be verified that $(V, \leq_z)$ is a median semilattice. To emphasize the previous fact we now state it as a theorem.

Theorem 4.1.10. *For any median graph $G = (V,E)$ and for any vertex z belonging to V, the partially ordered set $(V, \leq_z)$ is a median semilattice.*

We omit the proof of Theorem 4.1.10 since it is a well known fact.

Definition 4.1.11. For any median graph $G = (V,E)$ and for any vertex $z \in V$, let $J_z(V)$ be the set of join irreducible elements belonging to the median semilattice $(V, \leq_z)$.

Definition 4.1.12. Let $k \geq 3$ be an integer and $K = \{1,2,...,k\}$. For any $s \in J_z(V)$ and any profile $P = (x_1,x_2,...,x_k)$, let $K_s(P) = \{i \in K : s \leq_z x_i\}$.

Definition 4.1.13. For the semilattice $(V, \leq_z)$ and any profile P of length k, we define $\mathrm{Maj}_z : V^k \longrightarrow V$ by

$$\mathrm{Maj}_z(P) = \bigvee \{j \in J_z(V) : |K_s(P)| > k/2\}.$$

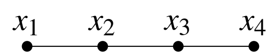


Figure 4.3: A 4-path.

Let $G = (V,E)$ be the path in Figure 4.3. Consider the profile $P = (x_1, x_2, x_3, x_4)$. First, consider the semilattice $(V, \leq_{x_2})$ and note that $J_{x_2} = \{x_1, x_3, x_4\}$ so $\mathrm{Maj}_{x_2}(P) = x_2$. Second, consider the semilattice $(V, \leq_{x_3})$ and note that $J_{x_3} = \{x_1, x_2, x_4\}$, so $\mathrm{Maj}_{x_3}(P) = x_3$. Hence $\mathrm{Maj}_{x_3}(P) \neq \mathrm{Maj}_{x_2}(P)$.

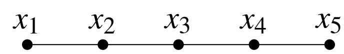


Figure 4.4: A 5-path.

Let $G = (V,E)$ be the path in Figure 4.4. Consider the profile $P = (x_1, x_2, x_3, x_4, x_5)$. Note that that $\mathrm{Maj}_{x_i}(P) = x_3$ regardless of our choice of zero element x_i.

We will now introduce the concept of a *split*, which will be central to this chapter.

Definition 4.1.14. If $G = (V,E)$ is a median graph and $xy \in E$, then *the split induced by the edge xy* is the bipartition $\{V_{xy}, V_{yx}\}$ of V.

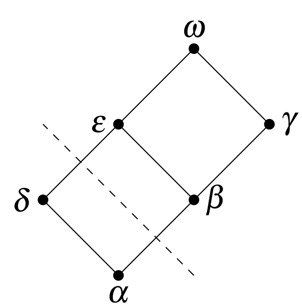


Figure 4.5: A split induced by the edge $\alpha\beta$.

Figure 4.5 shows an example of a split. The split induced by the edge $\alpha\beta$ partitions the semilattice into $\{\alpha, \delta\}$ and $\{\beta, \gamma, \varepsilon, \omega\}$. Note that the choice of the edge $\delta\varepsilon$

would induce the same split. We will now relate the concept of splits to consensus functions.

Let $G=(V,E)$ be a median graph and $f:V^k \longrightarrow V$ a consensus function on G.

Definition 4.1.15. We say that f satisfies *Split Decisive Neutrality* (SDN) if for any splits $\{A,V\backslash A\}$ and $\{B,V\backslash B\}$ and for any profiles $P=(x_1,...,x_k),P'=(x'_1,...,x'_k):$

$$\{i:x_i\in A\}=\{i:x'_i\in B\} \implies [f(P)\in A \iff f(P')\in B].$$

Example 4.1.16. The function $f:V^k \longrightarrow V$ defined by $f(P)=x_1$ for all profiles $P=(x_1,...,x_k)$ satisfies (SDN).

Proof. Let $\{A,V\backslash A\}$ and $\{B,V\backslash B\}$ be splits. Let $P=(x_1,...,x_k)$ and $P'=(x'_1,...,x'_k)$ such that

$$\{i:x_i\in A\}=\{i:x'_i\in B\}.$$

Suppose $f(P)\in A$. Then $x_1\in A$ and $x'_1\in B$. Thus $f(P')\in B$ and hence f satisfies (SDN). □

Example 4.1.17. If $f:V^k \longrightarrow V$ is a constant function, then f does not satisfy (SDN).

Proof. Assume that $f(P)=v$ for all profiles P and let $u\in V$ be adjacent to v. So $uv\in E$. Let $A=V_{vu}$ and $B=V_{uv}$ and note that $\{A,B\}$ is the split induced by the edge uv. Consider the constant profiles $P=(x_1,...,x_k)$ and $P'=(x'_1,...,x'_k)$ such that $x_i=v$ and $x'_i=u$ for $i=1,...,k$. Observe that

$$\{i:x_i\in A\}=K=\{i:x'_i\in B\}.$$

Since $f(P)=v\in A$ and $f(P')=v\notin B$ it follows that f does not satisfy (SDN). □

Recall the definition of decisive neutrality. We say that $f:X^k \longrightarrow X$, where X is a median semilattice, satisfies **decisive neutrality** (DN) if for any profiles P,P' and for any $j,j'\in J(X)$,

$$K_j(P) = K_{j'}(P') \implies [j \leq f(P) \iff j' \leq f(P')].$$

Definition 4.1.18. For $W \subseteq V$ and $x \in V \backslash W$, the vertex $z \in W$ is a *gate* for x if $z \in \mathrm{I}(x,w)$ for all $w \in W$.

The next theorem is Theorem C in McMorris, Mulder, Powers (2006) [10].

Theorem 4.1.19. *Let $G = (V,E)$ be a median graph and let z be any vertex of G. For any split G_1, G_2 of G with z in G_1, the gate s of z in G_2 is the unique join-irreducible in G_2 in the median semilattice $(V, \leq_z)$.*

4.2 Results and Examples

The following result is the main result of the chapter. It establishes a connection between (SDN) and the (DN) condition from chapter 2.

Theorem 4.2.1. *If V is a median graph, $f : V^k \longrightarrow V$ satisfies (SDN), and $z \in V$, then*

$$f : (V, \leq_z)^k \longrightarrow (V, \leq_z).$$

satisfies (DN).

Proof. Let $P = (x_1, \ldots, x_k)$ and $P' = (x'_1, \ldots, x'_k)$ be profiles and j a join irreducible such that $j \leq_z f(P)$. Suppose that

$$K_j(P) = K_{j'}(P').$$

Let s denote the element covered by j and consider the split induced by the edge sj. Similarly, let s' denote the element covered by j' and consider the split induced by the edge $s'j'$.

By letting G_1 be V_{sj} and G_2 be V_{js} it follows from Theorem 4.1.19 that j is a gate for V_{js} and hence $t \in V_{js}$ implies $j \in \mathrm{I}(z,t)$. Moreover, $j \in \mathrm{I}(z,t)$ implies $j \leq_z t$ by definition, so $t \in V_{js}$ implies $j \leq_z t$.

If $t \in V$ satisfies $j \leq_z t$ then $s \leq_z j \leq_z t$ and hence $j \in \mathrm{I}(s,t)$ and hence $d(t,j) < d(t,s)$, which in turn means that $t \in V_{js}$. Hence

$$t \in V_{js} \iff j \leq_z t$$

for all $t \in V$. Similarly,

$$t \in V_{j's'} \iff j' \leq_z t. \tag{4.1}$$

This implies that

$$\{i : x_i \in V_{js}\} = K_j(P)$$

and similarily

$$\{i : x'_i \in V_{j's'}\} = K_{j'}(P').$$

Since

$$K_j(P) = K_{j'}(P')$$

we see that

$$\{i : x_i \in V_{js}\} = \{i : x'_i \in V_{j's'}\}.$$

Since $f(P) \in V_{js}$ it follows from (SDN) that $f(P') \in V_{j's'}$ and since $j' \in V_{j's'}$, equation (4.1) implies that $j' \leq f(P')$. □

Definition 4.2.2. We say that $f : V^k \longrightarrow V$ satisfies *Symmetry* (S) if for any profile $P = (x_1, ..., x_k)$ and any permutation $\sigma : K \longrightarrow K$,

$$f(P_\sigma) = f(x_{\sigma(1)}, x_{\sigma(2)}, ..., x_{\sigma(k)}) = f(P).$$

Theorem 4.2.3. *Assume that $G = (V,E)$ is a median graph and that $f : V^k \longrightarrow V$ is a consensus function on G. If f satisfies (SDN) and (S), then, for any splits $\{A, V \backslash A\}$ and $\{B, V \backslash B\}$ and for any profiles P, P' :*

$$|\{i : x_i \in A\}| = |\{i : x'_i \in B\}| \text{ and } f(P) \in A \implies f(P') \in B.$$

Proof. Let $\{A, V\backslash A\}$ and $\{B, V\backslash B\}$ be splits. Let $P = (x_1, ..., x_k)$ and $P' = (x'_1, ..., x'_k)$ such that

$$|\{i : x_i \in A\}| = |\{i : x'_i \in B\}|.$$

Suppose that $f(P) \in A$. Let $I = \{i : x_i \in A\}$ and $J = \{i : x'_i \in B\}$. Let σ be a permutation $\sigma : K \longrightarrow K$ that maps I onto J. Then $x_i \in A$ if and only if $x_{\sigma(i)} \in B$.

Note that since f satisfies (SDN), $f(P) \in A$ implies $f(P'_\sigma) \in B$. Since f satisfies (S), $f(P') = f(P'_\sigma)$ and hence $f(P') \in B$, which is what we wanted. $\square$

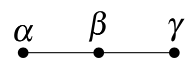


Figure 4.6: The path in Example 4.2.4 and Definition 4.2.7

We will now provide some examples to illustrate (SDN).

Example 4.2.4. Let $G = (V, E)$ be a path with vertices α, β, γ, where β is the middle vertex. Let $f : V^3 \longrightarrow V$ be defined by

$$f(P) = \begin{cases} \alpha & P = (\alpha, \alpha, \alpha) \\ \gamma & P = (\gamma, \gamma, \gamma) \\ \beta & \text{else} \end{cases}$$

for any profile P. The function f does not satisfy (SDN). To see this, let $P = (\alpha, \alpha, \gamma) = (x_1, x_2, x_3)$ and $P' = (\gamma, \beta, \alpha) = (x'_1, x'_2, x'_3)$ and observe that

$$\{i : x_i \in \{\alpha\}\} = \{i : x'_i \in \{\beta, \gamma\}\} = \{1, 2\}.$$

However, $f(P) = \beta \notin \{\alpha\}$ while $f(P') = \beta \in \{\beta, \gamma\}$.

Example 4.2.5. Let $G = (V, E)$ be a path with vertices α, β, as in as in Figure 4.7. Let $f : V^k \longrightarrow V$, where $k > 1$, be defined by

$$f(P) = \begin{cases} \alpha & \text{if } |\{i : x_i = \alpha\}| = k \\ \beta & \text{otherwise} \end{cases}$$

Figure 4.7: The path in Examples 4.2.5 and 4.2.6

We claim that f does not satisfy (SDN). Let $P = (\alpha, \alpha, \beta)$ and $P' = (\beta, \beta, \alpha)$, so that $f(P) = f(P') = \beta$. Note that if $A = \{\alpha\}$ and $B = \{\beta\}$, then

$$\{i : x_i \in A\} = \{i : x_i' \in B\}$$

and $f(P) \in B$, while $f(P) \notin A$.

Example 4.2.6. Let $G = (V, E)$ be a path with vertices α, β, as in Figure 4.7. Let $f : V^k \longrightarrow V$, where $k > 1$, be defined by

$$f(P) = \begin{cases} \alpha & x_i = \alpha \,\forall i \in K \\ \beta & x_i = \beta \,\forall i \in K \\ \beta & x_1 = \alpha \text{ and } \exists i \in \{2, \ldots, k\} \text{ such that } x_i = \beta \\ \alpha & x_1 = \beta \text{ and } \exists i \in \{2, \ldots, k\} \text{ such that } x_i = \alpha. \end{cases}$$

We claim that f satisfies (SDN). Let $\{A, V \backslash A\}$ and $\{B, V \backslash B\}$ be splits of G and $P = (x_1, x_2, \ldots, x_k)$ and $P' = (x_1', x_2', \ldots, x_k')$ profiles such that

$$\{i : x_i \in A\} = \{i : x_i' \in B\}.$$

Assume that $f(P) \in A$. If $A = B$, then $P = P'$ and so $f(P') = f(P) \in A$. If $A \neq B$ then

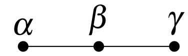
$$\{i : x_i = \alpha\} = \{i : x_i' = \beta\}.$$

If $\{i : x_i = \alpha\} = \{i : x_i' = \beta\} = K$, then $f(P) = \alpha$ and hence $A = \{\alpha\}$. Since $A \neq B$, $B = \{\beta\}$ and hence $f(P') \in B$. If $\{i : x_i = \beta\} = \{i : x_i' = \alpha\} = K$, then, by the same argument, $f(P) \in A$ and $f(P') \in B$.

If $\{i : x_i = \alpha\} = \{i : x_i' = \beta\} = D$, where $D \neq K$ and $D \neq \emptyset$, then either $x_1 = \alpha$ and $x_1' = \beta$ or $x_1 = \beta$ and $x_1' = \alpha$. In either case, $f(P) \in A$ implies that $f(P) \in B$.

For the reminder of the chapter, $G = (V, E)$ will be a path with three vertices.

Definition 4.2.7. Let $G = (V, E)$ be a path with vertices α, β, γ, where β is the middle vertex.



Let $k \geq 3$ be an odd integer. Define $M : V^k \longrightarrow V$ as follows: for profiles $P = (x_1, x_2, ..., x_k)$,

$$M(P) = \begin{cases} \alpha & \text{if } |\{i : x_i = \alpha\}| > k/2 \\ \gamma & \text{if } |\{i : x_i = \gamma\}| > k/2 \\ \beta & \text{otherwise} \end{cases}$$

M is our version of majority rule.

Claim 4.2.8. M satisfies (SDN).

Proof. We claim that for any profile $P = (x_1, x_2, ..., x_k)$ and for any split $\{A, V \backslash A\}$ of G,

$$M(P) \in A \iff |\{i : x_i \in A\}| > k/2.$$

Assume that $M(P) \in A$. If $M(P) = \alpha$, then $\alpha \in A$ and so $A = \{\alpha\}$ or $A = \{\alpha, \beta\}$. By the definition of M, $M(P) = \alpha$ implies that $|\{i : x_i = \alpha\}| > k/2$ and so $|\{i : x_i \in A\}| > k/2$. A similar argument shows that if $M(P) = \gamma$, then $|\{i : x_i \in A\}| > k/2$. The final case is when $M(P) = \beta$ and so $A = \{\alpha, \beta\}$ or $A = \{\beta, \gamma\}$. By the definition of M, $M(P) = \beta$ implies that α can occur at most $(k-1)/2$ times and γ can occur at most $(k-1)/2$ times, in P. Hence $|\{i : x_i \in V \backslash A\}| < k/2$ and $|\{i : x_i \in A\}| > k/2$.

Assume that $|\{i : x_i \in A\}| > k/2$. If $A = \{\alpha\}$, then $M(P) = \alpha$ by the definition of M and hence $M(P) \in A$. If $A = \{\alpha, \beta\}$, then $|\{x_i \in P : x_i = \gamma\}| < k/2$ so that $M(P) \neq \gamma$ and hence $M(P) \in \{\alpha, \beta\}$. With this established, we are ready to show that M satisfies (SDN). $\square$

Proof. Let $\{A, V\backslash A\}$ and $\{B, V\backslash B\}$ be splits of G and $P = (x_1, x_2, ..., x_k)$ and $P' = (x'_1, x'_2, ..., x'_k)$ profiles such that

$$\{i : x_i \in A\} = \{i : x'_i \in B\}.$$

Assume that $M(P) \in A$. As shown above, this implies that $|\{i : x_i \in A\}| \geq k/2$ and since $\{i : x_i \in A\} = \{i : x'_i \in B\}$, it follows that $|\{i : x'_i \in B\}| \geq k/2$ and so, $M(P) \in B$. $\square$

Theorem 4.2.9. *Assume that $f : V^k \longrightarrow V$, with $k \geq 3$ odd satisfies (SDN) and (S). If $G = (V, E)$ is the path shown in Figure 4.6 then $f = M$.*

Proof. For any vertex $v \in V$ and for any profile $P = (x_1, x_2, ..., x_k) \in V^k$ let

$$K_v(P) = \{i : x_i = v\}$$

and

$$f^{-1}(v) = \{Q \in V^k : f(Q) = v\}.$$

The $K_v(P)$ notation should not be confused with similar notation used in the lattice context.

By Theorem 4.2.1, $f : (V, \leq_\beta)^k \longrightarrow (V, \leq_\beta)$ satisfies (DN). By assumption, $f : (V, \leq_\beta)^k \longrightarrow (V, \leq_\beta)$ satisfies symmetry (S). Since the median semilattice $(V, \leq_\beta)$ is not a lattice, it follows from Lemma 3.2.3 that

$$P \in f^{-1}(\alpha) \implies |K_\alpha(P)| > \frac{k}{2}$$

and

$$Q \in f^{-1}(\gamma) \implies |K_\gamma(Q)| > \frac{k}{2}.$$

We will now prove these implications without Lemma 3.2.3. It is enough to show that the second implication holds. Assume that there exists a profile Q such that $Q \in f^{-1}(\gamma)$ and $|K_\gamma(Q)| \leq \frac{k}{2}$. Let $R = (x_1, ..., x_k)$ be a profile such that

$$|K_\alpha(R)| = |K_\gamma(R)| = |K_\gamma(Q)|.$$

Let $\{A, V\backslash A\}$ be the split such that $A = \{\alpha\}$, and let $\{B, V\backslash B\}$ be the split of V with $B = \{\gamma\}$. Using these splits and the profiles Q and R, observe that

$$|K_\gamma(Q)| = |K_\alpha(R)| \text{ and } f(Q) = \gamma \implies f(R) = \alpha$$

by Theorem 4.2.3. Using the split $\{A, V\backslash A\}$ and $\{B, V\backslash B\}$ along with just the profile R, observe that

$$|K_\alpha(R)| = |K_\gamma(R)| \text{ and } f(R) = \alpha \implies f(R) = \gamma$$

by a second application of Theorem 4.2.3. Since it is impossible to have $f(R) = \alpha$ and $f(R) = \gamma$ we know that for any profile $Q, Q \in f^{-1}(\gamma)$ implies that $|K_\gamma(Q)| > \frac{k}{2}$.

We will now show the reverse implication. It is enough to show that if $|K_\alpha(Q)| > \frac{k}{2}$, then $f(Q) = \alpha$. Assume that $f(Q) \neq \alpha$ and let $R' = (x'_1, \ldots, x'_k)$ be the profile such that

$$|K_\alpha(R')| = |K_\alpha(Q)| \text{ and } |K_\beta(R')| = \emptyset.$$

Using the split $\{A, V\backslash A\}$ with $A = \{\alpha\}$ and the profiles Q and R', observe that

$$|K_\alpha(R')| = |K_\alpha(Q)| \text{ along with } f(Q) \notin A \implies f(R') \notin A$$

by Theorem 4.2.3. So $f(R') \in V\backslash A = \{\beta, \gamma\}$. Using the splits $\{V\backslash A, A\}$ and $\{B, V\backslash B\}$ with $B = \{\gamma\}$ and the profile $R' = (x'_1, \ldots, x'_k)$ observe that

$$\{i : x'_i \in V\backslash A\} = \{i : x'_i \in B\}.$$

Since f satisfies (SDN) and $f(R') \in V\backslash A$ it follows that $f(R') \in B$. But $f(R') \in B$ with $B = \{\gamma\}$ implies that $R' \in f^{-1}(\gamma)$. Therefore, by the previous argument, $|K_\gamma(Q)| > \frac{k}{2}$, contrary to $|K_\alpha(R')| = |K_\alpha(Q)| > \frac{k}{2}$, then $f(Q) = \alpha$.

Recall the definition of M,

$$M(P) = \begin{cases} \alpha & \text{if } |\{i : x_i = \alpha\}| > k/2 \\ \gamma & \text{if } |\{i : x_i = \gamma\}| > k/2 \\ \beta & \text{otherwise.} \end{cases}$$

Let $P = (x_1, ..., x_k)$ and assume that $|\{i : x_i = \alpha\}| > \frac{k}{2}$. As shown above, $f(P) = \alpha$. Similarly, $|\{i : x_i = \gamma\}| > \frac{k}{2}$ implies $f(P) = \gamma$. Assume instead that $|\{i : x_i = \alpha\}| \leq \frac{k}{2}$ and $|\{i : x_i = \gamma\}| \leq \frac{k}{2}$. As shown above, this implies that $P \notin f^{-1}(\alpha)$ and $P \notin f^{-1}(\gamma)$, and hence $f(P) \in f^{-1}(\beta)$, i.e. $f(P) = \beta$. Hence $f = M$. $\square$

While we have made some progress in our effort to characterize majority rule on median graphs, a lot of work remains to be done. The following conjecture suggests a possible first extension of the chapter.

Conjecture 4.2.1. *Let $G = (V, E)$ be a finite path. Assume that $f : V^k \longrightarrow V$, with $k \geq 3$ odd satisfies (SDN) and (S). Then $f = M$*

CHAPTER 5
CONCLUSIONS

In the introductory chapter, we describe the work done by McMorris and Powers on Majority Rule on Hierarchies [7]. In their paper, they show that the four conditions *Restrictive Decisive Neutral (RDN), Symmetry (S), Monotonicity (M),* and *Biprofile Non-trivial (BNT)* characterize majority rule on hierarchies. The main result of Chapter 2 is an extension of the McMorris and Powers' theorem to a large class of median semilattices which we call sufficient.

In Chapter 3, we examine Monjardet's work on consensus functions defined on meet semilattices given in [9]. We characterize all consensus functions that satisfy the conditions *Decisive Neutrality (DN), Bi-idempotency (BI), and Symmetry (S)*, for median semilattice of cardinality at least three. For the concluding theorem of the chapter, we add the fourth condition of *Monotonicity (M)*. This theorem corrects Monjardet's mistake in [9].

In Chapter 4, we give a natural extension of the the decisive neutrality condition from median semilattices to median graphs, called *Split Decisive Neutrality*. At the end of the chapter, we prove that majority rule is the only consensus function defined on a path with three vertices that satisfies split decisive neutrality and symmetry. This is a modest result and a lot of work remains to be done.

5.1 Future Work

The following definition comes from [11].

Definition 5.1.1. Let $G = (V,E)$ be a connected graph and $P = (x_1, x_2, ..., x_k) \in V^k$. For a vertex v of a V we write $D(v,P) = \sum_{i=1}^{k} d(v,x_i)$. A vertex x minimizing this distance sum is called a **median** of P. The **median function Med** on a G is the consensus function given by

$$\mathrm{Med}(P) = \{v | v \text{ is a median of } P\}.$$

Chapter four ends with Conjecture 4.2.1 and as mentioned, this is a possible extension of the thesis: *Let $G = (V,E)$ be a finite path. Assume that $f : V^k \longrightarrow V$, with $k \geq 3$ odd satisfies (SDN) and (S). Then $f = \mathrm{Med}$.*

Next, a natural extension would be the following conjecture:

Conjecture 5.1.1. *Let $G = (V,E)$ be a finite tree. Assume that $f : V^k \longrightarrow V$, with $k \geq 3$ odd satisfies (SDN) and (S). Then $f = \mathrm{Med}$.*

It has been suggested that intermediate conjectures also could include the covering graphs of the partially ordered set of all hierarchies with three and four element sets, respectively. Finally,

Conjecture 5.1.2. *Let $G = (V,E)$ be a finite median graph. Assume that $f : V^k \longrightarrow V$, with $k \geq 3$ odd satisfies (SDN) and (S). Then $f = \mathrm{Med}$.*

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