Hybrid structural health monitoring using data-driven modal analysis and model-based Bayesian inference.

Jice Zeng
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HYBRID STRUCTURAL HEALTH MONITORING USING DATA-DRIVEN MODAL ANALYSIS AND MODEL-BASED BAYESIAN INFERENCE

By

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B.S., Xihua University, 2013
M.S., Chongqing Jiaotong University, 2016

A Dissertation
Submitted to the Faculty of the
J.B. Speed School of Engineering of the University of Louisville
In Partial Fulfilment of the Requirements
for the Degree of

Doctor of Philosophy
in Civil Engineering

Department of Civil and Environmental Engineering
University of Louisville
Louisville, Kentucky

December 2021
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Dr. John Kielkopf
DEDICATION

This dissertation is dedicated to my parents

Mr. Rongbiao Zeng

and

Mrs. Xiaohui Geng

who have given me invaluable educational opportunities
ACKNOWLEDGEMENTS

It is a challenging and unforgettable journey during my Ph.D. life with ups and downs, I learnt lots of perseverance and dedication to overcome different difficulties. I cannot be thankful enough for the support from my family, my professor, and my friends. Foremost, I would like to express my deep and sincere appreciation to my professor Young Hoon Kim throughout my Ph.D. study. He always gives me technical guidance and encourages me in my research. Prof. Kim is a rigorous and knowledgeable teacher whose open and critical thinking always enlightens me. Academical and emotional support from him has greatly helped my research advancement. It is a great honor to work with him.

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ABSTRACT

HYBRID STRUCTURAL HEALTH MONITORING USING DATA-DRIVEN MODAL ANALYSIS AND MODEL-BASED BAYESIAN INFERENCE

Jice Zeng

November 12, 2021

Civil infrastructures that are valuable assets for the public and owners must be adequately and periodically maintained to guarantee safety, continuous service, and avoid economic losses. Vibration-based structural health monitoring (VBSHM) has been a significant tool to assess the structural performance of civil infrastructures over the last decades. Challenges in VBSHM exist in two aspects: operational modal analysis (OMA) and Finite element model updating (FEMU). The former aims to extract natural frequency, damping ratio, and mode shapes using vibrational data under normal operation; the latter focuses on minimizing the discrepancies between measurements and model prediction. The main impediments to real-world application of VBSHM include 1) uncertainties are inevitably involved due to measurement noise and modeling error; 2) computational burden in analyzing massive data and high-fidelity model; 3) updating structural coupled parameters, e.g., mass and stiffness. Bayesian model updating approach (BMUA) is an advanced FEMU technique to update structural parameters using modal data and account for underlying uncertainties. However, traditional BMUA generally assumes mass is precisely known and only updating stiffness to circumvent the coupling effect of mass and
stiffness. Simultaneously updating mass and stiffness is necessary to fully understand the structural integrity, especially when the mass has a relatively large variation.

To tackle these challenges, this dissertation proposed a hybrid framework using data-driven and model-based approaches in two sequential phases: automated OMA and a BMUA with added mass/stiffness. Automated stochastic subspace identification (SSI) and Bayesian modal identification are firstly developed to acquire modal properties. Following by a novel BMUA, new eigen-equations based on two sets of modal data from the original and modified system with added mass or stiffness are derived to address the coupling effect of structural parameters, e.g., mass and stiffness. To avoid multi-dimensional integrals, an asymptotic optimization method and Differential Evolutionary Adaptive Metropolis (DREAM) sampling algorithm are employed for Bayesian inference. To alleviate computational burden, variance-based global sensitivity analysis to reduce model dimensionality and Kriging model to substitute time-consuming FEM are integrated into BMUA. The proposed VBSHM are verified and illustrated using numerical, laboratory and field test data, achieving following goals: 1) properly treating parameter uncertainties; 2) substantially reducing the computational cost; 3) simultaneously updating structural parameters with addressing the coupling effect; 4) performing the probabilistic damage identification at an accurate level.
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CHAPTER 1

INTRODUCTION

1.1 Research Background

Civil infrastructures such as buildings, bridges, tunnels, dams, pipelines, and other types of structures are aging and structurally deteriorating over time due to various reasons, including internal factors like deficient structural design, imperfect construction, and material defects (e.g., steel corrosion, concrete delamination), and external factors like natural disasters like hurricane, flood, and earthquake, as well as man-made disasters. The proper maintenance, management, and repair routine work are necessary to guarantee safe and reliable structural operation. Structural health monitoring (SHM) involves the periodical evaluation of an investigated structure to early capture any abnormal condition of structures. SHM is a powerful tool to provide an accurate assessment in a cost-effective manner for the present and future safety of structures and prevent extension of premature structural damage to more severe damage resulting in the whole structural collapse (Farrar and Worden, 2007, Balageas et al., 2010).

In America, a large number of civil infrastructures such as bridges, highways were constructed in the past decades to trigger economic growth. In contrast, many of them are now subjected to aging problems. American Society of Civil Engineering (ASCE, 2009) has reported that around 14% and 32% of rural roads and urban roads, and 20% of national
highways had deteriorated. The U.S. Department of Transportation also reported that 42% of all bridges, or 7.5% of nation’s bridges were defined as structurally deficient and functionally obsolete (ASCE, 2021). Figure 1.1 shows the bridge condition classification from 2009 to 2019. It is observed that the number of bridges sliding into a fair category is increasing annually, and the number of bridges labeled as good started to decrease in 2016. The bridges with fair label are a concern, as they are potentially downgraded to structurally deficient bridges. Therefore, SHM is imperative to assess structural condition and in-service safety, particularly for those structures in fair condition.

![Bridge condition classification by year](image)

Figure 1.1. Bridge condition classification by year
Note: data was reported from ASCE, 2021

The main goal of SHM is to detect structural damage at an early stage so that the prompt actions can be taken to ensure the structural integrity and normal services. It has been acknowledged that the outlines of an SHM scheme can be summarized as five levels (Chen, 2018):

- **Level 1**: damage detection, providing the indication of damage existence
- **Level 2**: damage localization, giving information of damage location
- **Level 3**: damage classification, giving information of damage pattern
- **Level 4**: damage quantification, giving information of damage severity
Level 5: damage prediction, giving information of remaining service life of structure

A wide range of non-destructive techniques (NDT) has been developed to detect damages in the past decades (Hellier, 2013). Conventional NDTs are generally local methods using 1) some special sensors; 2) visual inspection. The former, such as acoustic emission (Nair and Cai, 2010), guided waves (Cantero-Chinchilla et al., 2019), electromagnet methods (Witoś et al., 2018), and laser doppler vibrometer (Tian et al., 2019b), require that the damage location is known and sensor installation is accessible, which may not be suitable for large-scale structure. The later largely relies on inspectors’ judgement and experience, resulting in an unreliable damage detection. In addition, all these methods are labor-intensive, time-consuming and costly.

Driven by these issues, vibration-based structural health monitoring (VBSHM) is extensively investigated due to the advancement of the measurement and acquisition of vibration signals at a low cost (Fassois and Kopsaftopoulos, 2013). The fundamental concept of VBSHM is that any abnormalities that induce physical properties changes (e.g., mass and stiffness) will cause changes in modal properties (e.g., natural frequency, damping ratio, and mode shape) (Hu et al., 2015, Kong et al., 2017, Sun et al., 2017). Therefore, it is intuitive to use model features to reflect structural conditions and facilitate further damage detection. Generally, VBSHM can be categorized into two groups: non-model-based and model-based methods. Non-model-based methods have a critical drawback of only detecting damage location and cannot quantify damage severity (Huang et al., 2012). Model-based methods use measured data to update the parameterized computer-simulated models so that damage can be detected, localized, and quantified by the variation in structural parameters (Eltouny and Liang, 2021).
The general procedures of model-based VB-SHM consist of five steps (Sohn et al., 2003, Huang et al., 2012): (1) dynamic vibration measurement, e.g., displacement and acceleration; (2) modal identification through analyzing recorded vibration data; (3) characterization of an initial Finite element model (FEM) based on design and information in field test; (4) implement FE model updating (FEMU) using identified modal parameters; and (5) evaluation of structural performance using the updated finite element model.

This research only focuses on modal identification and FEMU (steps 2 and 4). Modal identification is a prerequisite for FEMU, and its accuracy directly affects the quality of model updating results. Typically, identification of modal parameters is achieved by operational modal analysis (OMA). OMA's primary advantage for civil engineering structure is to avoid interruption to the normal operation of observed structure and requiring no artificial loading (Ivanovic et al., 2000, Brownjohn et al., 2010). However, for long-term SHM, it requires a vast amount of recorded data and data analysis in a short amount of time. Therefore, it demands extensive labor work to process massive measured data and identify modal parameters by manual intervention and engineers’ experience. In fact, much user intervention on a large amount of vibration data can be an obstacle in a real application. For overcoming this, it is essential to have an automated evaluation of structural conditions in almost real-time. Therefore, the development of an automated OMA algorithm has become an attentive topic in recent years to efficiently handle continuously recorded data. In addition, modal parameters from OMA are subject to various uncertainties such as modeling error, measurement noise, and unmeasured excitation; information concerning uncertainty on modal parameters is crucial to evaluate the accuracy of modal parameters (Yuen, 2010, Au, 2017c).
At step four, FEMU is performed by utilizing identified modal parameters in OMA. It is commonly known that FEMU is a popular and promising technique in the field of VBSHM. However, the discrepancies between FE models and physical structures are always existing, and the source of discrepancies mainly results from 1) modeling error from the ideal assumption in FE modal construction; 2) statistical uncertainties in material and geometric properties; and 3) irregularities in structure (Mottershead et al., 2011, Simoen et al., 2015b, Chen, 2018). These issues may impair the quality and accuracy of numerical models. Therefore, FE model updating (FEMU) techniques are developed to calibrate and identify structural parameters by minimizing the gap between predicted data from FE models and measured vibrational data (Soize et al., 2008, Chen and Maung, 2014, Sipple and Sanayei, 2014).

One significant application of FEMU is structural damage detection and quantification; successful applications of FEMU in terms of damage detection can be found in work (Teughels and De Roeck, 2005, Fang et al., 2008, Grip et al., 2017, Alkayem et al., 2018, Das and Debnath, 2018). Beyond a wide array of available FEMUs with uncertainty identification (Mares et al., 2006, Govers and Link, 2010, Simoen et al., 2015b), the Bayesian model updating approach (BMUA) has been considered as one of the most efficient updating approaches (Beck and Katafygiotis, 1998, Sohn and Law, 2000, Vanik et al., 2000, Ching and Beck, 2004, Huang et al., 2018). BMUA can not only provide us the most optimal values of updated parameters but also give us uncertainty information on parameters. Also, BMUA can incorporate all uncertainties, including measurement and modeling errors, and all observed incomplete data. However, conventional BMUA assumes that the mass is known to only update stiffness (Yan and Katafygiotis, 2015,
Mustafa and Matsumoto, 2017, Sedehi et al., 2019). Because simultaneous identification of mass and stiffness will yield infinite combinations of mass and stiffness with the same frequency arising from the coupling effect of mass and stiffness, this assumption is questionable when a structure is experiencing damages in both mass and stiffness. Few works have attempted to address this in updating mass and stiffness with uncertainty quantification together (Cheung and Bansal, 2017, Mustafa and Matsumoto, 2017, Do and Gül, 2019).

Another challenge is that BMUA is computationally demanding due to a vast amount of FE model evaluations is required. As a result, it becomes impractical for complex and large-scale engineering structures. It has been recognized that high-fidelity modeling for complex and large-scale structures is usually necessary for a better model prediction and structural analysis, which involves hundreds of thousands of elements and nodes in commercial FEA packages. The computational time would be highly expensive if a large amount of iteration is needed. Therefore, the cost-effective model updating method is practically valuable.

1.2 Problem Statement and Motivation

Based on the above statement, it is concluded that developing an automated OMA and effective FEMU is needed to provide a reliable SHM scheme for accomplishing the condition assessment and damage detection. Various uncertainties should be considered in modal analysis and model updating to understand structural performance. To summarize, the following problems are identified from the current practice:
1. Existing modal identification methods still require much human effort and engineering judgment that impairs the accuracy of the identified modal parameter.

2. For a large amount of data during long-term SHM, existing modal identification methods still have difficulty in preforming modal identification in real-time and are computationally expensive.

3. Traditional BMUA can typically update stiffness parameters with the assumption of well-known mass. However, it is possible to have a relatively large mass variation in a real application. This area is not well understood to address the practical issues.

4. Existing FEMU methods for identifying mass and stiffness can only update structural parameters, while different sources of uncertainties are ignored or poorly estimated.

5. For large-scale structures, FEMU is computationally intensive because of complex structural model and massive model evaluations, which restricts its practical application. An efficient FEMU is highly required to be applicable for real-world cases.

### 1.3 Research Goals and Significance

The main goal in this dissertation is to develop a reliable and efficient VBSHM in the field of civil engineering. Challenges in practical applications of VBSHM include OMA with minimized human involvement, simultaneous identification of mass and stiffness, and uncertainty quantification in modal analysis and model updating. Therefore, a two-phase framework of VBSHM, namely automated modal identification and Bayesian model
updating, is proposed in this research. Figure 1.2 schematically shows the VBSHM flowchart proposed in this dissertation.

![Figure 1.2: Schematic flowchart of the proposed VBSHM](image)

Note: SSI is stochastic subspace identification; BMI is Bayesian modal identification

In summary, the research work in this dissertation has the following goals:

**Goal 1:** Develop an automated modal identification method with low human intervention. Different sources of uncertainties are considered to quantify the accuracy of modal parameters.

**Goal 2:** Address the coupling effect of mass and stiffness in BMUA to simultaneously identify mass and stiffness parameters.
Goal 3: Develop an efficient BMUA to overcome issue of demanding computational cost in complex and large-scale real-world structures.

Goal 4: Develop a stochastic model updating method that accounts for uncertainties using vibration data to detect and quantify damage.

This research attempts to deal with problems in two aspects of existing VBSHM, e.g., OMA and BMUA. Traditional OMA requires much human involvement so that the accuracy and reliability of modal analysis cannot be ensured; traditional BMUA is unable to update mass and stiffness because of the coupling effect. Also, BMUA is generally computationally expensive because of model complexity in real cases. Furthermore, uncertainties are inevitably entailed in modal analysis and model updating. It may cause incorrect damage detection results if not appropriately treating uncertainties.

The research is significant because it aims to overcome these challenges for a reliable VBSHM accounting for comprehensive uncertainties and to be computationally efficient. In short, all these aspects contribute to a real-time VBSHM and provide instructive information for structural condition assessment and damage detection. The outcomes of this research can also be integrated with some standalone programs with a user-friendly interface, which makes practical VBSHM more convenient and accessible to engineers.

1.4 Outline of the Dissertation

The outline of the dissertation is summarized as follows:

Chapter 1 introduced the research background, current challenges, and practical limitations in the field VBSHM, the research objectives, and significance.

Chapter 2 provided a comprehensive literature review related to VBSHM. Systematic introduction of output-only operational modal analysis is presented, including time domain,
frequency domain, and Bayesian-based methods. Uncertainty quantification of modal parameters by different scopes of methods is also discussed. Then an overview of VBSHM approaches is described, briefly summarizing two categories in the field of VBSHM, e.g., non-model based and model-based methods. In addition, the classification of FEMU techniques is presented, in which direct and indirect FEMU methods, deterministic and stochastic FEMU methods, are thoroughly introduced. A possible solution for uncertainties of structural parameters is also provided.

Chapters 3 and 4 aim to achieve goal 1. In Chapter 3, an automated stochastic subspace identification (SSI) method is proposed, involving a two-stage framework to automatically interpret stabilization diagram, human involvement and engineers’ judgment are significantly minimized. In the pre-processing stage, modal validation criteria and uncertainty criterion are included. In clustering stage, a novel self-adaptive clustering method and outlier detection are carried out to determine final modal parameters. The performance of the proposed automated SSI is demonstrated using two field tests, e.g., the Dowling Hall Footbridge and the Z24 bridge. Remark that in Chapter 3, Section 3.1 presented the literature review of automated SSI. Section 3.2 introduced the theoretical background of original SSI by Peeters and De Roeck (1999). The main contribution is included elaborately in Section 3.3. Section 3.4 is methodology verification.

Chapter 4 proposed an automated Bayesian modal identification (BMI) method. Two challenges in BMI are addressed, e.g., the selection of initial frequency and frequency bandwidth. Initial frequency in BMI is determined through the automated interpretation of the stabilization diagram; effective frequency bandwidths are picked by sifting frequency difference between initial frequency and identified frequency. The proposed automated
BMI is verified by a numerical example and a field test of the Z24 bridge. Remark in Chapter 4, Section 4.1 presented the literature review of automated SSI. Section 4.2 introduced the theoretical background of original BMI by Au (2012a). The main contribution is provided in detail in Section 4.3. Section 4.4 is methodology verification.

Chapters 5 and 6 aim to achieve goals 2 and 4. In Chapter 5, a novel Bayesian model updating approach with known modification (either added mass or added stiffness) is proposed to address the coupling effect of mass and stiffness for simultaneous identification of mass and stiffness. In this chapter, the objective functions are reformulated by introducing the new characteristic equations. Subsequently, the analytical formulations of the optimal parameters (natural frequency, mode shape, mass and stiffness) are derived using an asymptotic approximation method; associated uncertainty is also quantified by the inverse of Hessian matrix of objective functions. The 2D and 3D numerical examples are used to validate the performance of the proposed approach. Remark that in Chapter 5, Section 5.1 presented the literature review of Bayesian model updating. Sections 5.2 and 5.3 introduced the theoretical background of original Bayesian model updating by Yuen (2010). Sections 5.4 and 5.5 showed the main contribution of proposed Bayesian approach.

Chapter 6 presented the identification of mass and stiffness using Bayesian model updating with added mass/stiffness and Differential Evolutionary Adaptive Metropolis (DREAM) sampling algorithm. The modal parameters for the original and modified system with added mass and stiffness, e.g., natural frequency and mode shape, are firstly identified using automated modal identification. Then, the proposed Bayesian model updating framework is implemented in two cases, namely FEMU and probabilistic damage identification with different damage scenarios. The posterior distribution function is solved
by DREAM sampling method. Furthermore, probabilistic damage estimation is also provided for visualization of damage location and damage prediction. The proposed approach is validated by a numerical example of a six-story shear building and a laboratory-scale three-story shear frame. Remark that in Chapter 6, Section 6.1 presented the motivation of the proposed approach in this chapter. Section 6.2 introduced the theoretical background of Bayesian model updating using classical characteristic equation. The main contribution is provided in Section 6.3. Methodology verification is implemented in Section 6.4.

Chapter 7 developed an efficient Bayesian model updating framework for complex and large-scale structures in real world for achieving goal 4. Two time-saving strategies are proposed, e.g., variance-based global sensitivity analysis to reduce model dimensionality and Kriging model to substitute time-consuming FE model for further alleviation of computational burden. Following these strategies, Bayesian model updating is carried out for a cable-stayed pedestrian bridge. The computational cost is remarkably reduced compared to FEMU with non-time saving strategies. Remark that in Chapter 7, Section 7.1 gave possible reasons why Bayesian model updating is computationally burdensome. Section 7.2 presented the formulations of traditional Bayesian model updating. The main contribution is explicitly introduced in Section 7.3. Methodology verification is provided in Section 7.4.

Chapter 8 summarized the main conclusions and findings of this research. Suggestions and future research work are also discussed.
CHAPTER 2

LITERATURE REVIEW

2.1 Introduction

In contrast to local or visual inspection methods, global vibration-based methods have been aroused considerable interests in the past decades (Salawu, 1997, Fan and Qiao, 2011). The fundamental principal of vibration-based SHM (VBSHM) is rather intuitive. Modal properties such as natural frequency and mode shape are directly correlated with physical properties such as mass and stiffness. Changes in mass and stiffness can be reflected on quantifiable changes in modal parameters. Therefore, the use of vibrational characteristics allows for 1) convenient measurement interpretation; 2) accessibility to investigate structure; 3) effective structural condition assessment using a limited set of sensors and equipment.

Understandably, the prerequisite of VBSHM is the identification of modal parameters. In most cases, the excitation during the vibration is hardly measured with adequate energy and in a controlled way, especially in large bridges with the frequency of interest of 0~1Hz (Chen and Ni, 2018). Only vibration response induced by ambient excitation such as wind, traffic, pedestrian walking, or their combinations can be readily measured. Therefore, the output-only (response-only) measurement, also named operational modal analysis (OMA), has gained increasing attention and substantial progress in the field of civil engineering. OMA aims at accurately identifying modal parameters using output-only response, which
is measured under normal working/operational conditions (Peeters and De Roeck, 2001, Brincker and Ventura, 2015). Following modal analysis using OMA, VBSHM is conducted to evaluate current structural performance. A wealth of research has been investigated in VBSHM during the past decades. Extensive techniques and algorithms are developed for various structures, basically from structural components such as beams and plates to complicated structures such as buildings and bridges. Doebling et al. (1996) and Sohn et al. (2003) provide comprehensive review of different VBSHM methods and classification of damage detection methods using extracted response features before 1996 and between 1996 and 2001, respectively. Carden and Fanning (2004) mainly reviewed articles and papers related to VBSHM published from 1996 to 2003. More recently, an extensive review of vibration-based damage detection methods in the case of bridges between 2011 and 2017 are presented by An et al. (2019). The development and advancement of VBSHM between 2010 and 2019 were thoroughly introduced by Hou and Xia (2021), challenges and future trend in VBSHM were also discussed in their work.

This chapter starts with an overview of OMA in Section 2.2, including non-Bayesian based methods in the time domain and frequency domain and Bayesian-based methods. Then, section 2.3 focuses on VBSHM techniques, in which non-model based and model-based methods are introduced. In the following subsections, a detailed introduction of FEMU is presented, direct and indirect FEMU, deterministic and stochastic FEMU are both discussed.

2.2 Operational modal analysis (OMA)

OMA uses ambient excitation to measure vibrational response with a limited set of sensors installed on locations of interest. Because of lacking information on the input,
OMA assumes the input is zero-mean Gaussian white noise. However, this assumption is not always strict in real cases. Therefore, a qualitative evaluation of uncertainty and accuracy control of modal parameters is necessary. This section presents the literature review related to OMA techniques, which are classified as non-Bayesian based methods in the time domain and frequency domain, and Bayesian-based methods. Uncertainty assessment of modal analysis is also discussed.

2.2.1 Non-Bayesian based methods

2.2.1.1 Time domain methods

Ibrahaim time domain (ITD) method was initially developed to identify modal parameters using free decay responses; later impulse response function-based modal analysis was proposed on the basis of ITD (Ibrahaim and Mikulcik, 1977). ITD starts to transform all discrete vibration responses to mathematical matrix form, then the correlation function of the response of each degree of freedom (DOF) is obtained. As a result, the system matrix which does not rely on measurement locations can be computed using the least square method. Finally, the complex eigen solutions of the system matrix are deduced, in which natural frequency and mode shape are estimated from eigenvalue and eigenvector, respectively (Pappa and Ibrahim, 1985). ITD method is robust for highly damped systems and does not require input excitation. However, ITD has limited capability to higher frequencies; only modes in the low frequency range can be identified accurately (Siringoringo and Fujino, 2008).

NExT-ERA is essentially a combination of two techniques, e.g., Natural Excitation Technique (NExT) and Eigensystem Realization Algorithm (ERA) to accomplish modal identification within two steps. NExT is initially proposed to analyze input-output data and
modified for OMA by considering the correlation function of ambient vibration response as free-decaying sinusoids; ambient excitation can be treated as white noise (Caicedo et al., 2004). Random decrement (RD) is often used in NExT to properly treat system response proportional to correlation functions as a random decrement function (He et al., 2011). Following the step of NExT, ERA is performed to extract modal parameters. ERA was originally developed by Juang and Pappa (1985) for modal analysis and system model reduction. A linear time-invariant system is represented by a discrete state-space model in ERA; the modal properties is extracted from state-space matrices. NExT-ERA exhibits desirable performance in modal identification for complex structure. Therefore, it is considered as an efficient and robust methodology in civil engineering community. Many practical studies have been conducted to evaluate the effectiveness of NExT-ERA in the case of challenging task of identifying closely-spaced and weakly excited modes (Hosseini Kordkheili et al., 2018, Yang et al., 2019, Pan et al., 2021).

Autoregressive moving average model (ARMA) is a commonly used model in linear time-invariant systems on the basis of assuming a signal is the output of a system that is excited by Gaussian white noise, which can be used for modal analysis (Bertha and Golinkval, 2017, Huang et al., 2020). In ARMA methods, the environmental effect on the frequency can be filtered out by data normalization. A system with a transfer function is converted from the original model; the modal parameters are identified by factorization of the new system, e.g., the poles of the transfer function. ARMA has the disadvantage of high computational cost. Various factors from the operation and environmental effect must be considered, resulting in high computational cost and unfeasible in practice. Therefore, these challenges restrict ARMA’s application.
Stochastic subspace identification (SSI) is an efficient system identification method by adopting a discrete time-invariant state-space model with an input of Gaussian white noise. Two types of SSI have been developed, namely covariance-driven SSI (SSI-cov) and data-driven SSI (SSI-data). SSI-cov deals with stochastic realization problems from output-only data, which is dependent on stochastic state-space system. In SSI-cov, response data is transformed to covariance Toeplitz matrix, which is then decomposed to singular values (Peeters and De Roeck, 1999). In contrast to SSI-cov, SSI-data directly works with measured data by projecting future outputs into past outputs in the Hankel matrix without the computation of covariance matrix (Van Overschee and De Moor, 1996). Modal properties are estimated from system matrices that are obtained by Kalman state sequences. A wide range of applications has demonstrated that SSI methods are robust and efficient to perform modal analysis (Li et al., 2019, He et al., 2021, Pan et al., 2021). In both SSI methods, the stabilization diagram is usually used to distinguish physical modes from spurious modes by graphically observing the distributions of poles with different model orders. Elimination of spurious modes on the stabilization diagram involves much human interaction, resulting in demanding computational cost, especially for a vast amount of data during long-term SHM. Additionally, manually removing spurious modes is subjective and brings unreliable results.

In SSI-types methods, all identified modal parameters inevitably involve uncertainties due to various reasons, such as the finite number of data samples, measurement noise, unknown excitation, modeling error (e.g., assumption of stationary and linear structure), and imperfect digital filter. Therefore, uncertainty quantification is necessary to assess the accuracy of modal parameter estimates. Reynders et al. (2008) initially developed the
uncertainty computation based on the propagation of first-order perturbation from measured data to modal parameters. Also, some validation and application are summarized in work (Reynders et al., 2016, Pereira et al., 2020). Later, Döhler and Mevel (2013) significantly improved the computational efficiency of uncertainty calculation at multi-model orders. Döhler et al. (2013) also proposed uncertainty quantification of modal parameters from multiple measurement setups. Methods to quantify parameters’ uncertainties in the scope of SSI have been successfully applied to different types of structures (Nord et al., 2019, Reynders, 2021, Su et al., 2021); however, the uncertainty information is rarely used or does not contribute to further automated modal identification procedures.

2.2.1.2 Frequency domain methods

Peak-picking (PP) method may be the easiest way to identify modal parameters using output-only data under ambient excitation. The basic idea of the PP method is the plot of power spectral density (PSD) has salient peaks representing natural frequencies (resonances) (Naderpour and Fakharian, 2016). Therefore, natural frequencies can be simply determined by observing the peaks. The half bandwidth is used to estimate damping ratios. Associated mode shapes are calculated by the ratios of peak magnitudes at measurement moving channels to those at reference channels (Cárdenas and Medina, 2021). Although PP method has a simple implementation, it has several drawbacks. The selection of peaks largely depends on the frequency resolution in PSD and is generally visually determined, yielding subjective and unreliable results. It is also found that identified damping ratios using PP methods are inaccurate. Furthermore, in general, only well-separated modes are identified; it is difficult to precisely identify closely spaces modes.
The PP methods do not give actual mode shapes; the identified mode shapes are often called “deflection shapes” (Rainieri and Fabbrocino, 2014).

The derivatives and modifications of the PP method were developed to overcome the PP method’s limitation. Frequency domain decomposition (FDD) method firstly estimates the PSD matrix at discrete frequencies which is then decomposed by singular value decomposition (SVD). Each singular value represents a single degree of freedom (SDOF) spectral density function (SDF). Each SDF identifies natural frequency and mode shape. FDD method is in essence a SVD extension of the PP method, as it is based on diagonalization of the PSD matrix and the fact that PSD matrices are characterized by a few modes (Brincker et al., 2001). Traditionally, manual peak-picking in PP methods is required, which is vulnerable to human-induced error and impairs the identification accuracy. In recent years, automation on picking peaks has been developed to avoid human manipulation and minimize operator bias and error. Kim and Sim (2019) proposed a region-based convolutional neural network with possible object locations to observe peaks; the peak detector is trained by deep learning method. Jin et al. (2021) utilized the modified automated multiscale-based peak detection algorithm and median absolute derivation baseline correction to identify the natural frequency of stay cables. Chen et al. (2021) also presented an automated peak-picking method by introducing the peak slope; the threshold of peak slope is determined by a support-vector machine to eliminate undesirable peaks. However, these methods are only promising and robust to automatically pick well-separated peaks; it is still cumbersome to achieve an automated selection of closely-spaced peaks.
In frequency domain methods, the damping ratio is usually identified inaccurately, which may be caused by the assumption that only the data around the peak in PSD plot is utilized to calculate the SDOF-SDF system. To enhance the damping ratio estimation, frequency-spatial domain decomposition (FSDD) was developed by considering additional singular vectors at a certain frequency to better estimate the output of PSD (A Hasan et al., 2018, Hızal, 2020). FSDD substantially improves the output of PSD near the expected frequency region and filter the noise (e.g., attenuation) beyond this region by the singular vector.

### 2.2.2 Bayesian-based methods

In Bayesian-based methods, modal identification is assumed to be a probability problem that measures the plausibility of modal parameters based on given model class and measured data (Au, 2017b). With Bayesian context, identification results are represented by the posterior probability distribution function (PDF) conditional on given data and modeling assumption. Prior information is typically considered to be uninformative to ensure measured data to fully govern the posterior PDF. Hence, the posterior PDF is proportional to the likelihood function that describes the measured data distribution with respect to modal parameters. With sufficient data length, the posterior PDF is shaped as a center point, but not any standard distribution (Au et al., 2018).

Katafygiotis and Yuen are pioneers in establishing the framework of Bayesian modal identification and developed fundamental theory (Yuen et al., 2002a, Yuen and Katafygiotis, 2003). Bayesian Spectral Density Approach (BSDA) is a frequency domain Bayesian-based modal identification method based on the assumption that the spectral density function of measured data is a complex Wishart distribution (Katafygiotis and
Bayesian Time Domain Approach (BTDA) assumes the measured data follows zero-mean Gaussian distribution (Yuen and Katafygiotis, 2001). Another Bayesian-based method is Bayesian Fast Fourier Transform Approach (BFFTA) developed by Yuen and Katafygiotis (2003). BFFTA uses the statistical properties of FFT of measured data and assumes the random vector consisting of the real and imaginary part of FFT follows Gaussian distribution with zero-mean. The values of mean and covariance matrices (the inverse of Hessian matrix) of posterior PDF represent the most probable values (MPV) of modal parameters and associated uncertainty, respectively. In general, Bayesian-based methods construct objective function by taking the negative logarithm of posterior PDF, then minimizing the objective function with respect to each modal parameter, giving the MPVs. However, the main challenge of above Bayesian-based methods is that MPVs rely on solving for a multi-dimensional numerical optimization problem, uncertainty computation requires finite difference, which is highly computationally expensive and ill-posed. Therefore, Bayesian-based methods have been restricted seriously in real applications (Zhu et al., 2021b).

To address this issue, Au (2016a) proposed a fast Bayesian FFT to compute the MPVs and covariance matrix by a condensed form of the objective function and analyzing a single mode in the selected frequency band. In the framework of fast Bayesian FFT, the determination of MPVs is only associated with a four-dimensional numerical optimization problem. Five modal parameters can be well estimated by fast Bayesian FFT, i.e., natural frequency, damping ratio and mode shape, the spectral density of the modal force and that of prediction error. The covariance matrix has also been analytically formulated by the Hessian matrix of posterior PDF rather than adopting finite difference method, making it...
possible to directly quantify parameter uncertainty. Consequently, the computational effort connected with the number of measured DOFs is significantly reduced, only several seconds are needed. The fast Bayesian FFT is also extended to perform modal analysis using forced vibration data (Au and Ni, 2014) and free vibration data (Zhang et al., 2016a). Later, Li and Au (2019) applied expectation-maximization (EM) algorithm to fast Bayesian FFT so that convergency speed is noticeably improved. Zhu et al. (2019) modified the fast Bayesian FFT for buried modes identification using ambient vibration data, when the spectral contribution of a certain mode is significant around neighbor modes. Zhu et al. (2021b) and Zhu and Au (2018) proposed fast Bayesian FFT to deal with well-separated and closely spaced modes for multiple setup data and asynchronous data, respectively. Fast Bayesian FFT has been applied to a wide range of structural types for modal analysis, such as bridges (Brownjohn et al., 2018, Ni et al., 2021b), super-tall buildings (Ni et al., 2017, Zhang et al., 2019), offshore lighthouse (Brownjohn et al., 2019), monopole telecoms structures (Capilla et al., 2021), and historic twin-tower structure (Liu et al., 2021).

Generally, the initial frequency in Bayesian fast FFT has to be visually picked from the singular value (SV) spectrum. Besides, frequency bandwidth governing levels of identification uncertainty is user-defined. This has been found to be obstructed and difficult in practice because of the low signal-to-noise ratio and intensity of modes.

2.3 Vibration-based SHM methods

Generally, vibration-based damage detection methods can be classified into two groups: non-model based and model-based methods. In this section, a comprehensive overview of these two methods is provided. Corresponding advantages and disadvantages are discussed in terms of practical implementation.
2.3.1 Model-free based methods

Model-free based methods are referred to non-model based methods and generally do not require the computer-simulated model to detect damage. It straightforward uses measured vibration responses to observe deviances between current measurement and reference measurement. Identified discrepancies indicate that abnormality occurs compared to the structural normal state. Therefore, model-free based methods are also called data-driven pattern recognition methods. One of the attractive features of mode-free based methods is that it implements fast, and results are simply interpreted. Furthermore, uncertainties resulting from modeling error and the process of modal parameter identification are avoided because of model-free feature (Neves et al., 2017, Rabiepour et al., 2020). However, model-free based methods generally cannot quantify the damage extent; only damage location can be detected. Additionally, the capability of these methods largely relies on the amount of data, which is usually limited in a field test.

Research on model-free based methods has made great progress, and a wealth of methods are developed in the last decade. Shi and Qiao (2018) proposed a new surface fractal dimension (FD) method for detecting notch damage in plate-type structures. Mode shape irregularity is also identified by a modified edge perimeter dimension (EPD) based window dimension locus. The FD damage detector was formed to localize and quantify possible damage by analyzing the sudden change (the peak of FD curve) of vibration frequency and displacement mode shape within a sliding window along the structural length. The FD method enables to quickly detect damage due to its simplicity and directly working on signal rather than state-space model. The experimental results also showed FD
method was robust to measurement noise. However, the FD method limitedly detects multiple-damaged structures.

Frequency response functions (FRFs) are an extensively used model-free damage detection method. FRFs are directly derived from vibration responses of an investigated structure and enable to provide sufficient information in damage detection, such as structural behavior over a frequency range instead of a frequency point. In addition, FRFs operate without a numerical process for identifying modal properties so that uncertainties are reduced (Bandara et al., 2014). In practice, the measured response at different locations under external forces consists of FRFs. Although the type of external forces does not dominate the FRFs of a system, the information of external forces is required, which may not always be available in real measurement, especially for large-scale and in-service structure. Another obstacle is that FRFs need a large amount of data, resulting in a time-consuming configuration of data network and a computationally inefficient problem (Chen and Ni, 2018, Allemang et al., 2022).

Wavelet transform (WT) has also attracted researchers’ attention. The WT has capability of properly dealing with non-stationary data characterized by scale (frequency) and position (time) (Peng et al., 2013). Therefore, the WT method has been demonstrated as a popular damage detection method. However, the WT method has a drawback of poor frequency resolution in the region of high frequency, which limits its application in damage detection, as structural damage usually locally occurs and is probably reflected by higher modes (Wang et al., 2018).

The wavelet packet transform (WPT) can be regarded as an extension of WT. The WPT enables to extract damage features from stationary and non-stationary signals by sufficient
frequency decomposition at a local level (Pan et al., 2019). Jiang et al. (2012) proposed a complex continuous wavelet technique transform of the slope of the mode shape using Complex Gaus1 Wavelet, the cracks were localized by the modulus line and the angle line of wavelet coefficients. This method was demonstrated by detecting beam cracks in different boundary conditions, crack locations and depth. Ibáñez et al. (2015) developed a wavelet entropy-based method to detect small variations in non-stationary signals. The entropy evolution enables to detect damage in multiwire cables, including breaking a single cable and changes in the mechanical contact conditions among the wires. These methods were experimentally validated, but they require a reference structural state (healthy condition). Furthermore, sensors need to be installed at a damage location, and only damage location is identified. Asgarian et al. (2016) used the rate of signal energy of WPT as an index to detect damage for a steel jacket type offshore platform. Vibration signals measured under impact loads that periodically excite a known location are decomposed into component signals by WTP. Component energies are then computed and used as inputs in Neural Network (NN) models for different types of damage detection. One limitation of this research is excitation needs to be measured repeatedly. In addition, high-fidelity FE model for training NN model is required.

Machine-learning (ML) methods have been widely used to advance non-model based VBSHM. ML methods traditionally directly extract damage-sensitive features from measured signals, which are then incorporated into ML methods to conduct damage detection (Farrar and Worden, 2012). Chun et al. (2015) numerically investigated a steel bridge with reduced thickness of girder due to corrosion. The maximum and variance of acceleration signals are calculated as damage features that are further processed by a multi-
layer perceptron (MLP) to assess structural conditions. However, future work of validating the methodology by a real bridge is needed. Abdeljaber et al. (2017) presented an efficient and accurate damage detection method using 1D Convolutional Neutral Networks (CNNs), which allows fusing both feature identification and classification blocks into a single and compact learning body. This method also has the ability to extract optimal damage features from the raw vibrational signals and is validated by a grandstand simulator. However, the CNN parameters have to be selected in a trial-and-error manner and detect slight damage. Chun et al. (2020) proposed a damage detection method using multi-point acceleration measurement that was interpreted by a three-step Random Forest, a supervised ML method. Different damage features such as the maximum response, standard deviation, logarithmic decay rate, and natural frequency were utilized to enhance the accuracy of damage detection. The actual aluminum alloy I-beam with cracks was used to verify the method. In contrast, a large number of training data is required, which may not be practical in real cases. Paral et al. (2021) proposed a method combining the 2D CNN with Continuous Wavelet Transform (CWT) of the response signal to evaluate the health condition of steel structural connection. The method only requires global vibration signals and is validated through a steel frame with a semi-rigid joint to detect beam-column connection damage. The limitation is that only damage on structural connection is considered; new datasets are required to train CNN for different types of damage events. Abdeljaber and Avci (2021) developed a nonparametric ML-based method that training a set of unsupervised classifiers, e.g., Self-organizing Maps (SOMs), for extraction of damage features from vibration response. Finally, the damage detection was conducted by training MLP to interpret
damage features. However, an experimental verification is needed for the methodology generalization.

2.3.2 Model-based methods

Model-based methods involve computer-simulated FE models predefined and parameterized by critical physical properties, e.g., mass and stiffness. These parameters are then calibrated using measured responses to detect damage location and damage severity. The advantages of model-based methods are environmental and operational effects can be adequately accounted. Modal properties, e.g., natural frequency, damping ratio, and mode shape, are correlated with structural mass and stiffness. Therefore, structural identification by model-based methods has an unbiased interpretation of results. In addition, the updated model by measured data is quite useful for structural repair and maintenance, evaluation of structural performance, and prognosis of remaining life (Chen, 2018). Although model-based methods are promising, some issues exist in these methods, such as how to build a reliable and accurate initial model, how to choose critical parameters, the number of parameters, and what kind of measured data is needed.

A wide range of model-based method has been developed to detect damage. Frequency-based methods directly use natural frequency as damage index and can be conveniently implemented, as frequency is easier to measure from limited sensors and robust to measurement noise. Moughty and Casas (2017) summarized the advancement of frequency-based methods in damage detection and also suggested that it may not be sufficient to use frequency only to identify damage. The frequency-based method also has a critical limitation of only small frequency changes due to damage is observed and is difficult to distinguish from those from environmental variation.
Mode shape-based methods refer to the use of changes in mode shape between intact and damage structure as damage feature. Modal Assurance Criterion (MAC) (Pastor et al., 2012) and Coordinate Modal Assurance Criterion (COMAC) (Khatir et al., 2016) are two commonly used indices to detect the abnormality. MAC values and COMAC values are a similarity and a point-wise measure of two mode shapes, respectively. It has been demonstrated that changes in mode shapes are more robust and reliable to detect damage compared to frequency shifts (Fan and Qiao, 2011). Mode shape-based methods have appealing advantages of damage localization and being insensitive to environmental effect, e.g., temperature. However, these methods generally require reference structural state (healthy condition), which may not be available for in-situ structures. Furthermore, the accurate measurement of mode shapes is relatively difficult compared to the natural frequency. The mode shapes are generally measured with incomplete DOFs due to a limited set of sensors.

Many research showed that mode shape itself is not sensitive to slight damage; even mode shape is measured with an intense sensor network (Limongelli et al., 2021). Motivated by this issue, the mode shape curvature (MSC) methods were proposed as an indicator for slight damage detection. MSC was firstly defined as the second derivative of mode shapes by Pandey et al. (1991) and successfully applied for a cantilever beam and simply supported beam. Later, the efficacy of MSC methods has tested by various research work, including beam structures (Janeliukstis et al., 2017, Dahak et al., 2019), building structure (Tomaszewksa, 2010, Paral et al., 2019), and bridge (Nick and Aziminejad, 2021, Pooya and Massumi, 2021). Although MSC shows outstanding performance in damage detection, it requires the dense measurement across the structure and accurate mode shape
identification. Besides, errors in measurement of mode shapes are accumulated due to differentiation, therefore yielding large uncertainties.

Another model-based method is the flexibility method. Flexibility is defined as the inverse of stiffness. Variation in stiffness due to possible damage will induce change in flexibility. Flexibility describes the relation of modal displacement and static force and can be approximated by natural frequencies and mass-normalized mode shapes (Moughty and Casas, 2017). Bernagozzi et al. (2018) proposed a two-stage modal flexibility-based approach to detect damage using output-only vibration data without or with minimal mass information, which was demonstrated by a numerical model of a six-story shear building and laboratory-scale four-story shear frame. Le et al. (2020) presented an enhanced method to detect damage in beam-type structures by observing the changes in modal flexibility (MF) matrices, three damage locating criteria, and explicit relationship between MF-based deflection change and damage features. Wickramasinghe et al. (2020) developed and applied vertical and lateral damage indices based on modal flexibility with lower order modes to detect damage for main cables and hangers of a suspension bridge. Huang et al. (2021) introduced a two-stage damage detection method for a steel-concrete composite bridge. At the first stage, a superposition of modal flexibility curvature (SMFC) is adopted to locate the damage accurately; then damage extent is determined by constructing an objective function based on MF and the enhanced whale optimization algorithm. But the flexibility-based methods generally require knowing external excitation and modes need to be mass-normalized, which limits their application in real structures.

Model-based (parametric) ML methods are also developed to extract damage features from structural systems using input-output or output-only modal analysis. Different ML
classifiers are well trained to process the modal parameters for the evaluation of structural integrity. Betti et al. (2015) applied ANNs and genetic algorithm to detect damages of column cutting in a three-story steel frame. A feed-forward back-propagation (FFBP) network structure was used as ANN and trained to perform the classification process. It is concluded that a combination of ANN and GA is powerful for damage detection. Meruane (2016) used another classifier except for ANN, Online Sequential Extreme Learning Machine (OSELM), to classify modal parameters. The approach was experimentally validated in a rectangular beam and a mass-spring laboratory structure with various damage scenarios. Duan et al. (2019) proposed an automated damage detection method for hanger cables in a tied-arch bridge based on CNN. The raw acceleration data for the Fourier amplitude spectrum were used without pre-processing to extract modal properties. The CNN model’s construction was accomplished hierarchically, the multi-damage location and quantification was also achieved. Beheshti Aval et al. (2020) developed a signal-based damage detection method for multi-story frame subjected to an earthquake event. Hilbert vibration decomposition technique was firstly used to extract acceleration responses of the sensors with high resolution. Next, the damage patterns were classified by a two-stage artificial neural network. Sharma and Sen (2020) proposed an output-only method for assessing the joint condition in which a 1D CNN was introduced to detect deficient joints in semi-rigid frames. The CNN was also modified to automatically extract damage features from 1D, 2D, and 3D response signals. The method was numerically and experimentally validated on a steel frame structure. Fu and Jiang (2021) proposed a new intelligent data fusion system to detect various damage types for a two-span steel tubular arch bridge and seven-story steel frame, taking advantage of probabilistic neural network (PNN) and data
fusion with correlation fractal dimension (CFD). The eigen-level model and the decision-level model are included in this intelligent system. Ritto and Rochinha (2021) proposed to construct digital twins for damage detection, where a physic-based computational model was used to investigate various damage scenarios. Different ML classifiers were trained using data from built computational model. Furthermore, different model parameters were considered to generate datasets for training purposes. Although model-based ML methods have been extensively applied, the performance of these methods largely rely on the classifiers. It is also no guarantee that a certain classifier is the best choice for all damage detection. Furthermore, feature extraction process is usually computationally expensive, which hinders the practical use of ML methods.

Based on the aforementioned discussion and literature review, the current research work mainly focuses on model-based VBSHM. Among numerous approaches in model-based VBSHM, Finite element model updating (FEMU) attracts more attention because it has a simple theory and may be universally applicable for diverse structural types, damage patterns, and measured data classes. The updating results can be readily interpreted and directly used for structural damage detection, parameter identification, model response prediction, and structural failure analysis. In the following sections, a thorough overview of FEMU is provided.

2.4 Finite element model updating

Finite element model updating (FEMU) constitutes another large group of vibration-based SHM methods. A finite element model (FEM) has been extensively used in addressing various challenges in engineering community: SHM (Balageas et al., 2010, Chen, 2018), risk and reliability analysis (Jensen et al., 2013, Gardoni, 2017), and structural
dynamics and response control (Xiang and Nishitani, 2015, Kim, 2019). However, the errors between analytical responses from FEMs and counterparts from real structures are always unavoidable. The sources of errors mainly stem from: (1) measurement error due to signal quality, measurement devices, and human operation, (2) modeling error such as idealization assumption, improper discretization, and (3) erroneous assumption in material properties and dimensions (Mustafa and Matsumoto, 2017, Alkayem et al., 2018). Therefore, FEMU has great demand and practical value to enhance the fidelity of FEM. FEMU is essentially the process to minimize the discrepancy between analytical prediction and test results in such a way that progressively adjusts physical parameters until FEM reproduces the measured data to a satisfactory level (Tian et al., 2021).

![Finite element model updating (FEMU)](image)

**Figure 2.1. The classification of FEMU**

Current model updating techniques can be generally categorized into two aspects: 1) direct and indirect methods, and 2) deterministic and stochastic methods. Figure 2.1 shows the classification of FEMU. Direct and indirect methods are two independent categories which are defined based on whether the FE model is parameterized and updated in an iterative manner; generally, both deterministic and stochastic methods belong to indirect
methods, in which they can be distinguished according to whether uncertainties are considered or not in an updating process. In addition, most deterministic methods can be regarded as direct methods. The comprehensive introduction of FEMU classification is presented in the following sections.

2.4.1 Direct and indirect methods

Direct methods refer to update the system mass and stiffness matrices with one step to match measured experimental data with counterparts from FEM (Mao and Dai, 2012, Yang et al., 2014, Sehgal and Kumar, 2016). Lei et al. (2012) applied a direct method to update the reduced FE model of the Canton Tower; incomplete modal data were used for model stiffness matrix updating without any model reduction or expansion techniques. Lim et al. (2016) proposed a semi-direct FEMU method to improve the reliability of FRFs. The stiffness matrix was directly updated using a matrix mixing approach; the modal damping ratios were also obtained through minimizing the FRFs error function. Kumar Bagha et al. (2020) utilized a direct updating algorithm for cantilever steel/composite beam to match measured frequency with a model-derived ones; the mass and stiffness matrix were modified by satisfying orthogonal constraints. The main advantage of direct methods is computational efficiency as they are implemented in one step. However, updating results using these methods are significantly influenced by measurement noise and model inaccuracy. In addition, direct methods require complete and accurate measurement data, although system matrices can be either condensed to the only measured DOFs or incomplete mode shapes can be expanded to the full DOFs. Furthermore, direct methods fail to preserve the physical connectivity of updated matrices, leading to loss of matrices’ symmetry and positive definition. These methods cannot reasonably reflect variations in
structural properties as they solve inverse problems in a mathematical way. Because of these drawbacks, direct methods are limitedly used in vibration-based SHM (Yang and Chen, 2012, Moravej et al., 2017).

In contrast to direct methods, indirect/iterative methods are developed to maintain the physical meaning of FEM by adjusting preselected parameters in an iterative manner until FEM reproduces the measured data with desirable accuracy. The iteration process is accomplished when the discrepancy between measured data and prediction from FEM is reduced to a tolerable level. Nonlinear functions are generally used as error functions in indirect methods based on the model and tested responses, e.g., eigenvalues and eigenvectors. The system matrices updated by the indirect methods maintain symmetric and positive definite which has a clear physical configuration and can be easily understood. But indirect methods are computationally expensive since plenty of iteration is required to ensure a good convergence. The bias for solving the problem may occur during the iteration (Sehgal and Kumar, 2016).

A sensitivity-based method is a widely used indirect method. The basic idea of sensitivity-based method is measured responses are regarded as derivatives of analytical data from FE model of an intact structure, then the optimization problem is formulated by selected error/penalty functions (Mottershead et al., 2011, Rezaiee-Pajand et al., 2020). (Petersen and Øiseth, 2017) applied a sensitivity-based method to a long-span floating pontoon bridge with considering the fluid-structure interaction; the bridge model was parameterized with 27 parameters and updated using 30 natural frequencies and mode shapes. Grip et al. (2017) proposed a new sensitivity-based method for updating a concrete plate; the total variation-based regularization method was used to more precisely localize
and quantify structural damage. However, the performance of regularization depends on the choice of regularization parameters which may not always be automatically optimized. Machado et al. (2018) presented a sensitivity-based updating framework with FRFs to update distributed and homogeneous model parameters that are spatially correlated random fields and are expanded in a spectral Karhunen-Loève (KL) decomposition; An experimental test with a 3D printing beam was used to verify this method. Cao et al. (2020) proposed a dynamic sensitivity-based model updating method to update nonlinear parameters in an oscillator, a magnetometer boom, and a cantilever beam using time-domain response data derived by a direct differentiation method. Zhu et al. (2021a) developed a substructure-based response sensitivity method to update large-scale structures; the equivalent modes were reformulated to convert higher modes to lower modes. The motion equation was also reduced and simplified to efficiently compute structural response and response sensitivity, finally speeding up the convergence of model updating. Some challenges still remained in this method. The measured responses cannot be too deviated from the analytical data, resulting in only minor damage being detected. Also, the core of this method is to calculate the derivatives of modal parameters, leading to computational inefficiency during the overall updating procedures. Furthermore, the updating results are often prone to noticeable errors due to measurement noise (Hou and Xia, 2021).

2.4.2 Deterministic and stochastic methods

Deterministic FEMU methods can be categorized as direct methods, such as matrix updating methods, or as a part of indirect methods, such as sensitivity-based methods. Generally, deterministic methods refer to calibrate structural parameters in a point-estimate
manner to minimize the difference between measured data and analytical prediction, only
a single FE model is explored, yielding unique or deterministic updating results. The
thorough overview of deterministic methods for damage detection can be seen in the work
of Aghagholizadeh and Catbas (2015). However, uncertainties are inherently inevitably
involved in FEMU due to various sources. For example, the measured data are always
exposed to uncertainty or variability originating from structural deterioration, measurement
noise, disassembly, and assembly with the need of maintenance and renovation; FE model
is afflicted with modeling error because of idealization and simplification assumption,
improper discretization, etc. (Simoen et al., 2015a). Deterministic methods cannot
appropriately account for these uncertainties, which gives unsatisfactory updating results
for complex and large-scale structures.

To capture various uncertainties, stochastic methods have developed and attracted a lot
of interest in recent years. As opposed to deterministic methods, stochastic methods do not
update structural parameters as fixed values. Instead, stochastic methods aim to update
parameters as either a range or probability distribution function (PDF); parameter
uncertainty can be straightforward quantified (Wan and Ren, 2016). Also, it attempts to
seek all plausible models during updating process with given available measured data and
provide a confidence interval of updated results, providing engineer researchers the
information of the model’s accuracy. Because it is rarely possible to confidently specify
one value for updating parameters, stochastic methods are more reliable in most cases. A
wide range of stochastic methods have developed to more accurately and reasonably
identify model parameters accounting for different source of uncertainties, including
Bayesian methods (Yuen, 2010, Ramancha et al., 2020), Monte Carlo (MC) based methods
(Lam et al., 2018, Baisthakur and Chakraborty, 2021), perturbation based methods (Huang and Chen, 2019, Chen et al., 2020), filtering methods (Astroza et al., 2019, Song et al., 2020), interval model updating methods (Chen et al., 2018, Mo et al., 2021), and covariance matrix adjustment method (Govers and Link, 2010). A comprehensive overview of stochastic methods is available in the literature (Wan and Ren, 2016, Zhao et al., 2020).

Among all stochastic methods, the Bayesian model updating approach (BMUA) has grown in popularity and prominence during the last decades due to simple theorem and intuitively appealing practical value. Different real applications using the Bayesian approach in civil engineering have demonstrated its efficiency and robustness, e.g., buildings (Simoen et al., 2013, Lam et al., 2019), bridges (Mustafa and Matsumoto, 2017, Li and Jia, 2020), lab-scaled structures (Sedehi et al., 2019, Wang et al., 2020). Beck et al. (Beck and Katafygiotis, 1998) established the fundamental theory of Bayesian model updating. Further derivatives and modifications (Yuen et al., 2006, Behmanesh et al., 2015, Mustafa and Matsumoto, 2017, Das and Debnath, 2018, Zeng and Kim, 2020) extend the Bayesian approach’s capability and efficiency. Bayesian approach characterizes to-be-updated structural parameters as random variables and formulates parametric model updating function within the Bayes’ theorem; the posterior PDF is explicitly built using prior knowledge from engineering judgment and likelihood function consisting of measured data. The key strengths of the Bayesian approach are as follows: 1) rationally and reliably handling incomplete experimental data; 2) using Bayes’ theorem, physical model parameters are characterized by the PDF; 3) only repeating straightforward model evaluations to avoid the most inverse problem’s challenges of unidentifiability, ill-posedness, and non-uniqueness (Wan and Ren, 2016).
To feasibly treat high-dimensional integrals involved in the posterior PDF for Bayesian inference, Beck and Katafygiotis (1998) employed an asymptotic approximation method that assumes the posterior PDF is unimodal and Gaussian distribution to estimate the posterior PDF of model parameters. However, the assumption does not necessarily guarantee a true physical model when a high level of modeling error and measurement noise occurs in practice, especially for multi-modal and non-Gaussian posterior (Wan and Ren, 2016, Yang and Lam, 2018a, Ni et al., 2021a). Also, given an insufficient amount of data and complex model class, model updating problems may become unidentifiable (Lam et al., 2015). Markov Chain Monte Carlo (MCMC) is a favorable alternative to infer the posterior PDF for multi-modal or unidentifiable problems, because no assumption on the model parameters is required to directly generate samples distributed as the posterior PDF. There are various MCMC techniques incorporated into Bayesian model updating, such as Metropolis Hastings (MH) algorithms (Green, 2015), Gibbs sampling (Huang and Beck, 2018), Hamiltonian Markov chains (Mao et al., 2020a), and delayed rejection adaptive Metropolis (DRAM) (Simoen et al., 2013, Wan and Ren, 2016).

Although Bayesian model updating with MCMC is powerful, it is computationally demanding because a vast amount of FE model evaluations is required. It becomes impractical for complex and large-scale engineering structures. In addition, most Bayesian approaches assume that mass is well known and invariant; only stiffness is updated with believing that mass is less critical. However, this is not always valid when noticeable variation and uncertainty in mass occurs. Structural parameters in mass and stiffness are coupled concerning the natural frequency and mode shape. Therefore, simultaneous identification of mass and stiffness can be defined as an unidentifiable problem because an
infinite combination of mass and stiffness exists and gives the same natural frequency (herein, the coupling effect of mass and stiffness) (Beck and Au, 2002). To avoid this issue, mass is usually well-estimated or exactly known for updating stiffness in traditional BMUA due to the availability of the mass information in a deterministic manner (e.g., dimensions).

2.5 Summary and Conclusions

The literature review of SHM techniques is presented in this chapter. Some difficulties in theoretical and practical aspects still remain. In this section, the technical issues based on the aforementioned introduction that need to be addressed for the transition of academia to practical applications are summarized.

In operational modal analysis:

- It has been concluded that SSI, as a popular non-Bayesian based method, is efficient and reliable in extracting modal parameters under structural operational conditions due to its simple mathematical nature and quick implementation. However, spurious modes appear using SSI because of measurement noise and assumption in the algorithm itself. It is challenging to distinguish physical modes from spurious modes with human involvement. Also, uncertainties on modal parameters should be properly treated.
- On the other hand, Bayesian-based methods exhibit outstanding performance. Physical modeling assumptions are strictly obeyed, and measured data is fully used. The modal parameter uncertainties are naturally provided based on Bayes’ theorem. However, initial frequency in this method needed to be handily picked from singular value spectrum. The selection of frequency bandwidth by trial-
and-error, which dominates uncertainty quantification, should be paid more attention.

In vibration-based SHM:

- The implementation of non-model based methods is fast, and results are easily understood. In addition, uncertainties caused by modeling error and modal parameter identification are avoided because of a model-free feature. However, only damage location can be detected by these methods. Their capability largely relies on the amount of data, which is usually limited in field tests.

- Model-based methods have pronounced advantages against non-model based methods, as structural identification is implemented with clear physical meaning. The effect of environmental and operational change is considered. The updated model is quite functional for dynamic behavior analysis and prognosis of remaining life. But model-based methods have challenges in a model establishment.

In FEMU:

- FEMU is one of the promising SHM techniques. Different sources of uncertainties are inevitably entailed in the FEMU process, including modeling error and measurement noise. If these uncertainties are not properly treated, the accuracy of structural identification and damage detection will be significantly impaired.

- Stochastic FEMU methods are more reliable compared to deterministic methods as information accounted for uncertainty is provided. It is noted that
many stochastic methods tend to underestimate uncertainties due to only considering measurement noise and ignoring modeling error.

- Bayesian model updating has a simple theorem and is intuitively appealing. It aims to update structural parameters as a distribution and hence naturally provide uncertainty information. But traditional Bayesian approach only updates stiffness with assuming mass is known to avoid the coupling effect of mass and stiffness, which is problematic, especially noticeable mass change is observed. The Bayesian approach is also computationally expensive because of many model evaluations for large-scale structures.

Driven by the issues in OMA and FEMU, the research work in this dissertation attempts to develop the automated OMA, e.g., automated SSI and automated BMI, to efficiently deal with a large amount of data and appropriately consider modal parameter uncertainties. Additionally, a novel Bayesian model updating framework was proposed to update the coupled structural parameters with high computational efficiency.
CHAPTER 3

AUTOMATED STOCHASTIC SUBSPACE IDENTIFICATION (SSI)

3.1 Introduction

Among various non-probabilistic system identification algorithms, stochastic subspace identification (SSI) has been widely applied to diverse structures to perform operational modal analysis (OMA). It offers accurate identification results and simple implementation, which are important attributes accounting for its popularity. In addition, due to SSI’s explicit mathematical nature, SSI tends to be more suited for automated modal identification. However, the major challenge in SSI is spurious modes appear in outputs. Commonly, spurious modes consist of pure mathematical (i.e., non-physical) and noise modes (Reynders et al., 2008). The most common strategy to deal with this challenge is to construct a stabilization diagram, a plot of model order vs. frequency for an extensive range of model order. In the stabilization diagram, physical modes are referred to as those poles that cross most of the model orders consistently. Therefore, physical modes should be graphically recognized and homogeneously distributed along vertical alignments in the stabilization diagram (Cabboi et al., 2017). On the contrary, spurious modes appear in the stabilization diagram in a scattered way. Spurious modes are eliminated in a manual analysis depending on empirical discovery and engineers’ judgment, which is subjective, time-consuming, and leads to incorrect modal identification.
For addressing this issue, a variety of methods are proposed in the literature to automatically interpret stabilization diagram and remove spurious modes. In general, the process can be divided into three steps:

1. Step 1: Apply the modal validation criteria to eliminate as many spurious modes as possible in the stabilization diagram
2. Step 2: Group modes with similar characteristics, i.e., frequencies, damping ratios, and mode shapes by clustering strategies
3. Step 3: Detect outliers in each cluster to improve the accuracy of modal parameters and select representative of each cluster

Several methods aiming at minimizing human involvement in the interpretation of the stabilization diagram have been developed. For example, in step one, many modal validation criteria are proposed to detect spurious modes in the stabilization diagram. These criteria include hard criteria, which yield a binary answer, and soft criteria, which yield a certain range of values. Reynders et al. (2012) thoroughly reviewed and summarized hard and soft criteria. However, conventional modal validation criteria limitedly remove a certain number of spurious modes; many spurious modes, which still remain in the stabilization diagram, affects parameter estimates' accuracy and imposes a computational burden to the following step (clustering process).

In step 2, various clustering strategies are widely employed to group modes with similar characteristics. Hierarchical clustering has been extensively applied by many researchers and is considered as the most natural approach (Reynders et al., 2012). Hierarchical clustering has a significant advantage of allowing a good selection of physical clusters. However, the main drawbacks include a user-defined tree cutoff distance and human
intervention with demanding computational cost. Furthermore, the hierarchical clustering is sensitive to outliers. Another strategy is partitioning methods, often referred to as K-means clustering (Neu et al., 2017). K-means clustering has the benefit of being fast processing. However, the number of clusters has to be predefined, and it is sensitive to cluster seeds (initial centroid). By merging the benefits of hierarchical clustering and K-means clustering to overcome some of their limitations, self-adaptive clustering is recently proposed (Cabboi et al., 2017, Fan et al., 2019). The self-adaptive clustering has outstanding features: 1) simple implementation; 2) fast computation; 3) No need for the number of clusters; 4) Clustering threshold is iteratively trained during the clustering process.

While it still starts with a user-defined clustering threshold, which requires some level of human intervention. Some methods are proposed to automatically calculate clustering threshold based on statistical properties, i.e., mean and standard derivation or median, of the distance between two closed poles in the stabilization diagram (Magalhães et al., 2009, Reynders et al., 2012, Yang et al., 2019). However, these methods do not consider uncertainty on modal parameters and inaccuracy of mode shapes. In practice, modal parameters' uncertainty is inevitable due to modeling error and measurement noise; it can be a more reasonable approach to consider uncertainty when calculating the clustering threshold. Also, measurement on mode shapes is less accurate than that on frequencies. Thus, a weighting factor can reduce the inaccuracy of mode shape difference on threshold calculation (Boroschek and Bilbao, 2019).

In step 3, some outliers are undesirably involved in identified physical clusters; this phenomenon is pronounced in a damping ratio with a scattered nature. Most outlier
detection techniques need to define limit bounds, such as box-plot outlier detection (Yang et al., 2019). A bound-free outlier detection method is needed to improve the accuracy of parameter estimations. In summary, challenges to the current automated interpretation of the stabilization diagram are listed as follows:

1. Conventional modal validation criteria are inefficient resulting in high computational cost in the clustering process.
2. The clustering threshold and distance calculation in the clustering process does not consider the uncertainty of parameters and the weighting factor.
3. Uncertainties on identified modal parameters and physical clusters are unavailable.
4. Outlier detection requires to define limit bounds.

Figure 3.1. A framework of the proposed automated approach

To address the aforementioned challenges, this chapter proposed a two-stage framework for automated OMA. Figure 3.1 shows the flowchart of the proposed framework: (1) modal analysis using covariance-driven reference-based SSI (SSI-cov/ref);
(2) two-stage automated interpretation of stabilization diagram. In the first place, SSI-cov/ref is adopted to perform modal analysis and construct a stabilization diagram. Subsequently, a two-stage automated analysis for the stabilization diagram is carried out.

At the pre-processing stage, besides applying conventional modal validation criteria, such as damping ratio check and modal complexity check, to eliminate spurious modes, a new supplementary criterion: uncertainty criterion, which is also applied for further removal of spurious modes. At the clustering stage, a novel threshold calculation, which incorporates the uncertainty of modal parameters and weighting factor, is proposed. An improved self-adaptive clustering with new distance calculation is then employed to group modes with similar characteristics and identify physical clusters. Finally, robust outlier detection is implemented to exclude outliers. The average of each cluster's elements is chosen as representative frequency, damping ratio, and mode shape. This chapter is organized as follows: in Section 3.2, the background of SSI-cov/ref is briefly introduced. In Section 3.3, a two-stage approach for proposed automated modal identification is presented. In Section 3.4, the capability of the proposed approach is validated by two field tests along with the modal tracking results. Finally, the conclusion is presented in Section 3.5.

3.2 Theoretical background of SSI

SSI has been extensively spread over the field of OMA during the past few decades accounting for its quick implementation and high accuracy. In this paper, covariance-driven reference-based SSI (SSI-cov/ref) is employed to reduce the dimensions of the output matrix and computational cost. The detailed theoretical fundamentals of SSI-cov/ref are fully described in the literature (Peeters and De Roeck, 1999). Briefly, SSI-cov/ref is
developed based on assuming a linear and stationary N degree of freedoms (DOFs) system with a dynamic motion characterized by the discrete-time state-space equation:

\[ x_{k+1} = Ax_k + \omega_k \]

\[ y_k = Cx_k + v_k \]  

where subscript \( k \) denotes time step; \( A \in \mathbb{R}^{n \times n} \) denotes system state matrix with \( n = 2N \); \( C \in \mathbb{R}^{l \times n} \) is an output matrix, \( l \) is defined as the number of measured signals; \( x_k \in \mathbb{R}^{n \times 1} \) and \( y_k \in \mathbb{R}^{l \times 1} \) are discrete-time state vector and measured response vector, respectively; \( \omega_k \in \mathbb{R}^{n \times 1} \) is a process white noise vector. \( v_k \in \mathbb{R}^{l \times 1} \) is the measurement white noise vector.

Based on the assumptions above, modal parameters can be identified by analyzing output only vibration response. SSI-cov/ref can be implemented as follows: (1) the calculation of covariance between outputs and the limited sets of reference outputs; (2) the configuration of the block Toeplitz matrix; (3) singular value decomposition (SVD) of Toeplitz matrix; (4) the computation of the observability and reference-reversed controllability matrix; (5) the identification of modal parameters from extracted matrices.

The block Hankel matrix is defined as (Peeters and De Roeck, 1999):

\[ H = \frac{1}{\sqrt{j}} \begin{bmatrix} y_0^{\text{ref}} & y_1^{\text{ref}} & \cdots & y_{i-1}^{\text{ref}} \\ y_1^{\text{ref}} & y_2^{\text{ref}} & \cdots & y_j^{\text{ref}} \\ \vdots & \vdots & \ddots & \vdots \\ y_i^{\text{ref}} & y_{i+1}^{\text{ref}} & \cdots & y_{l+j-2}^{\text{ref}} \\ y_{i+1}^{\text{ref}} & y_{i+2}^{\text{ref}} & \cdots & y_{i+j}^{\text{ref}} \\ \vdots & \vdots & \ddots & \vdots \\ y_{2i-1}^{\text{ref}} & y_{2i}^{\text{ref}} & \cdots & y_{2i+j-2}^{\text{ref}} \end{bmatrix} \]  

A block Toeplitz matrix is formed as:
\[ T_{(1|i)} = \begin{bmatrix} \hat{R}_i & \hat{R}_{i-1} & \cdots & \hat{R}_1 \\ \hat{R}_{i+1} & \hat{R}_i & \cdots & \hat{R}_2 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{R}_{2i-1} & \hat{R}_{2i-2} & \cdots & \hat{R}_i \end{bmatrix} \]  \hspace{1cm} (3.3)

where \( \hat{R}_i \) is an output correlation and computed as:

\[ \hat{R}_i = \frac{1}{Q - i} y_{(1:Q-i)} y_{(1:Q-i)}^T \]  \hspace{1cm} (3.4)

where \( Q \) is the number of time steps in a single sensor.

The block Toeplitz matrix is next decomposed by singular value decomposition (SVD):

\[ T_{(1|i)} = U \Sigma V^T \]  \hspace{1cm} (3.5)

where \( U \) and \( V \) are orthogonal matrices; \( \Sigma \) is a diagonal matrix with positive singular values.

From SVD results, the observability matrix \( O_i \) and controllability matrix \( \Gamma_i \) are written as:

\[ O_i = U_1 \Sigma^{1/2} \]

\[ \Gamma_i = \Sigma^{1/2} V_1 \]  \hspace{1cm} (3.6)

where \( U_1 \) and \( V_1 \) are singular vectors corresponding to non-zero singular values in \( \Sigma \).

The system matrix \( A \) and output matrix \( C \) are obtained by:

\[ A = \Sigma^{-1/2} U_1^T T_{(2|i)} V_1 \Sigma^{-1/2} \]

\[ C = O_i (1: \text{number of sensors}) \]  \hspace{1cm} (3.7)

where \( T_{(2|i)} \) consists of covariance elements from lag 2 to 2 \( i \), defined as:

\[ T_{(2|i+1)} = \begin{bmatrix} \hat{R}_{i+1} & \hat{R}_i & \cdots & \hat{R}_2 \\ \hat{R}_{i+2} & \hat{R}_{i+1} & \cdots & \hat{R}_3 \\ \vdots & \vdots & \ddots & \vdots \\ \hat{R}_{2i} & \hat{R}_{2i-1} & \cdots & \hat{R}_{i+1} \end{bmatrix} \]  \hspace{1cm} (3.8)
Finally, the modal parameters can be obtained from identified matrices $A$ and $C$. $A$ has an eigenvalue decomposition as:

$$A = \varphi \Lambda \varphi^T, A \varphi_i = \lambda_i \varphi_i$$

(3.9)

where $\lambda_i$ and $\varphi_i$ are the $i$-th eigenvalue and eigenvector of $A$, respectively. The modal parameters are expressed as:

$$f_i = \frac{|f_s \ln \lambda_i|}{2\pi}$$

$$\zeta_i = \frac{(\ln \lambda_i)^R}{|\ln \lambda_i|}$$

$$\varphi_i = C \times \varphi_i$$

(3.10)

where $f_i$, $\zeta_i$ and $\varphi_i$ are the $i$-th frequency (Hz), damping ratio and mode shape, respectively; $f_s$ is the sampling frequency; $(\ln \lambda_i)^R$ is the real component of $\ln \lambda_i$.

Two main SSI preparation parameters significantly affect the accuracy of identification results: (1) model order; (2) time lag, $i$. Unfortunately, the value of model order and $i$, which yield the best identification results are never known (Ubertini et al., 2013, Fan et al., 2019). In practice, it is necessary to over-specify model order to cover weakly-excited modes, but spurious modes increase with model order increasing. These spurious modes must be singled out in the subsequent procedure. On the other hand, the value of $i$ determines the size of the response covariance function. The smaller $i$ may fail to identify the fundamental mode, but the larger value of $i$ yields more spurious modes and increases computational time. The value of $i$ may be chosen at least estimated value as follows (Fan et al., 2019):

$$i \geq T_i/t$$

(3.11)
where $T_i$ denotes fundamental period, (unit: second); $t$ denotes sampling interval, (unit: second).

### 3.3 A two-stage automated modal identification

In this section, a two-stage framework for automated SSI is proposed. The flowchart of the entire automated process in detail is presented in Figure 3.2. At the pre-processing stage including conventional modal validation criteria and a new additional uncertainty criterion are included. Subsequently, the clustering stage is introduced. First, a newly proposed threshold calculation for clustering. An improved self-adaptive clustering is then employed to determine physical clusters. Finally, robust outlier detection is performed to improve the accuracy of modal parameter estimates. The pseudocode of the proposed automated SSI is provided in Appendix C.

![Figure 3.2. A flowchart of the proposed two-stage automated SSI](image)
3.3.1 The Pre-processing stage

3.3.1.1 Modal validation criteria

First of all, for civil engineering structures, a negative or high damping ratio hardly appears in practice. Therefore, a damping ratio with less than 0 and higher than 10% is discarded (Cabboi et al., 2017, Fan et al., 2019).

Additionally, two popular modal validation criteria are used to measure the complexity of mode shape vectors, namely, modal phase collinearity (MPC) and mean phase deviation (MPD). These two indicators have been utilized by various researchers to distinguish physical modes from spurious modes (Reynders et al., 2012, Neu et al., 2017). The real (Re) and imaginary (Im) part of mode shapes display a linear correlation, which can be assessed by the MPC indicator. The value of MPC for the $t$th mode shape, $\phi_t$, is expressed as (Reynders et al., 2012):

$$\text{MPC}(\phi_t) = \frac{\|\text{Re}(\bar{\phi}_t)\|^2 + \frac{1}{\alpha} \text{Re}(\bar{\phi}_t^T)\text{Im}(\bar{\phi}_t^T)(2(\alpha^2 + 1)\sin^2\gamma - 1)}{\|\text{Re}(\bar{\phi}_t)\|^2 + \|\text{Im}(\bar{\phi}_t)\|^2} \tag{3.12}$$

The $k^{th}$ component of $\bar{\phi}_t$ is given: $\bar{\phi}_{t,k} = \phi_{t,k} - \frac{\bar{\phi}_{t,k}}{L}$, $L$ is the number of components in $\phi_t$.

$$\alpha = \frac{\|\text{Im}(\bar{\phi}_t)\|^2 - \|\text{Re}(\bar{\phi}_t)\|^2}{2\text{Re}(\bar{\phi}_t^T)\text{Im}(\bar{\phi}_t^T)} \tag{3.13}$$

$$\gamma = \arctan(\|\alpha\| + \text{sign}(\alpha)\sqrt{1 + \alpha^2}) \tag{3.14}$$

MPC values are dimensionless; they lie within the range of 0 and 1. MPC value closer to 1 indicates that mode shape, $\phi_t$, is more collinear and ‘monophase,’ which is usually regarded as a physical mode.
With regard to MPD, it represents the phase degree of each identified mode shape vector. The value of MPD/90 lies between 0 and 1. A smaller quantity of MPD implies that mode shape vector is more likely to be physical. A detailed discussion can be found in Reynders et al. (2012). For the \( t \)th mode shape, \( \phi_t \), the mean phase (MP) is defined as:

\[
MP(\phi_t) = \arg \min_\theta \left( \frac{||\text{Im}(\phi_t) - \tan \theta \text{Re}(\phi_t)||^2}{1 + \tan \theta} \right)
\]

where \( \theta \) is a phase angle in degree, Equation (6) can be solved by the least square as:

\[
MP(\phi_t) = \arctan \left( \frac{V_{12}}{V_{22}} \right), USV^T = [\text{Re}(\phi_t), \text{Im}(\phi_t)]
\]

where \( USV^T \) is singular value decomposition, \( V_{12} \) and \( V_{22} \) denotes elements (1,2) and (2,2) of \( V \) matrix, respectively. Then, MPD can be determined as:

\[
\text{MPD}(\phi_t) = \frac{\sum_{k=1}^{L} \omega_k \arccos \left( \text{Re}(\phi_{t,k})V_{22} - \text{Im}(\phi_{t,k})V_{12} \right)}{\sum_{k=1}^{N} \omega_k}
\]

where \( \omega_k \) is a weighting factor that equals to the \( k^{th} \) component of the \( t \)th mode shape, \( \phi_t \).

The selection of threshold values of MPC and MPD depends on measurement conditions and dynamic vibration properties. For a structure with clear linear behavior and high signal-to-noise ratio, the threshold of MPC and MPD can be conservatively chosen as 0.7 and 0.3, which implies that modes whose MPC are less than 0.7 and MPD exceed 0.3 are regarded as spurious modes. Conversely, the threshold of MPC and MPD are chosen as 0.3 and 0.7 in the case of structures with complex behavior (Cabboi et al., 2017, Fan et al., 2019). Two representative field tests with complex measurement conditions are used to validate the methods. Thus, the values of 0.3 and 0.7 are selected as a threshold for MPC and MPD, respectively, in this study.
3.3.1.2 Uncertainty criterion

Although conventional modal validation criteria remove certain spurious modes, many spurious modes still remain in the stabilization diagram, slowing down the following process (herein clustering process). More effective validation criteria should be adopted to delete as many spurious modes as possible. This study employed supplementary uncertainty criteria at the pre-processing stage to further eliminate spurious modes.

Uncertainty on modal parameters by SSI mainly arise from five sources: (1) finite number of data sample; (2) unmeasured excitation and measurement noise modeled as white noise; (3) the assumption of linear and stationary behavior; (4) imperfect filter of data; (5) incorrect choice of model order (Reynders et al., 2008). Reynders et al. (2008) initially developed the uncertainty computation based on the propagation of first-order perturbation from measured data to modal parameters. Also, some validation and application are summarized in Reynders et al. (2016).

Later, Döhler and Mevel (2013) significantly improved the computational efficiency of uncertainty at multiple model orders, which has been applied in various structures (Döhler et al., 2013). Uncertainty quantification can provide information to measure the accuracy of identified modal parameters. It is the fact that the uncertainty of physical modes is smaller than those of spurious modes. Based on this information, coefficient of variation (COV) (standard derivation/mean) with respect to frequency may be used to distinguish physical modes from spurious modes (Döhler and Mevel, 2013).

Some research has introduced uncertainty features in the stabilization diagram, but uncertainty criterion is not used or does not contribute to further automated modal procedure. General procedures of uncertainty computation are summarized as follows:
- Input parameters: the number of block rows in Hankel matrix, \( q \); the amount of data blocks, \( n_b \); the range of model order, \((n_{min}, n_{max})\);

- Compute Hankel matrix, \( H \), system state matrix, \( A \) and output matrix, \( C \), as well as observability matrix, \( O \), based on SSI-cov/ref, then compute transform matrix, \( T \)

- Compute covariance and sensitivity of subspace matrix from SSI-cov/ref, given by \( \Sigma_{H}^{cov} \) and \( J_{O,H} \), respectively

- Compute sensitivity and covariance of system state matrix, \( A \), and output matrix, \( C \) from SSI-cov/ref, given by \( J_{A,O}, J_{C,O} \) and \( \Sigma_{A,C} \), respectively

- For each mode \( i \) at successive modal order, compute sensitivity matrix: \( J_{f_i A}, J_{\zeta_i A} \) and \( J_{\phi_i A} \). Finally, compute covariance of modal parameters, frequency, \( f_i \), damping ratio, \( \zeta_i \); mode shape, \( \phi_i \): \( \text{cov} \left( \begin{bmatrix} f_i \\ \zeta_i \end{bmatrix}, \begin{bmatrix} f_j \\ \zeta_j \end{bmatrix} \right) \) and 

\[
\text{cov} \left( \begin{bmatrix} \text{Re}(\phi_i) \\ \text{Im}(\phi_i) \end{bmatrix}, \begin{bmatrix} \text{Re}(\phi_j) \\ \text{Im}(\phi_j) \end{bmatrix} \right)
\]

Comprehensive derivation for uncertainty estimation can be found in Döhler and Mevel (2013). When the COV in the frequency is chosen as a threshold (herein, 2%), modes with the COV of frequency larger than the threshold will be discarded.

### 3.3.2 The clustering stage

The clustering stage is sequentially performed to assemble modes based on similarities in modal parameters in this section. A novel method is proposed to calculate the clustering threshold; an improved self-adaptive clustering is then used to identify physical clusters.
Finally, robust outlier detection is implemented, and each representative of modal parameters is determined.

3.3.2.1 Automated computation of clustering threshold

Typically, two kinds of thresholds are usually adopted for clustering: (1) static threshold; (2) automatically computed threshold. A static threshold relies on the engineers’ judgment. Also, during long-term health monitoring, a well-defined static threshold may be suitable for some initial datasets; however, there is no guarantee that the static threshold will be keeping appropriate for all datasets. This is more challenging in the case of handling massive datasets. In this study, a novel method is proposed to calculate the clustering threshold based on possible physical modes at the pre-processing stage. First, the mutual distance between the two modes is defined as:

\[
\text{distance} = \frac{|F_i - F_j|}{\text{max} (F_i, F_j)} + \omega (1 - \frac{(\Phi_i^T \Phi_j)^2}{(\Phi_i^T \Phi_i)(\Phi_j^T \Phi_j)})
\]

where \( F_i = f_i + 2\sigma_{f_i} \), \( F_j = f_j + 2\sigma_{f_j} \), \( f_i \) and \( f_j \) are \( i^{th} \) and \( j^{th} \) identified frequency at a pre-processing stage, respectively; \( \sigma_{f_i} \) and \( \sigma_{f_j} \) are corresponding standard derivation, respectively; \( \Phi_i = \phi_i + 2\sigma_{\phi_i} \), \( \Phi_j = \phi_j + 2\sigma_{\phi_j} \), \( \phi_i \) and \( \phi_j \) are \( i^{th} \) and \( j^{th} \) mode shapes at the pre-processing stage, respectively; \( \sigma_{\phi_i} \) and \( \sigma_{\phi_j} \) are corresponding standard derivation, respectively. \( \omega \) is a weighting factor of mode shape difference, \( \omega = \frac{(\sigma_{\phi_i} + \sigma_{\phi_j})}{2} \).

Eq. (3.18) does not consider the damping ratio difference because it is difficult to accurately measure the damping ratio in practice. In addition, there is a high probability of two different modes with a similar damping ratio. A weighting factor, \( \omega \), represents different participation for frequency difference and mode shape difference. Generally,
mode shape is measured with limited sensors, yielding missing components of mode shape; frequency is usually measured with an accurate level. Therefore, the use of $\omega$ can reduce the effect of measurement inaccuracy of mode shapes on distance calculation (Boroschek and Bilbao, 2019). An uncertainty quantification using standard derivation is used to form a weighting matrix for Finite Element Model Updating (Yang and Lam, 2018b). Similarly, this work adopted the average of the standard derivation of mode shapes to define $\omega$. Furthermore, as uncertainty on modal parameters is inevitable in practice, it is more reasonable to incorporate uncertainty in distance calculation. Here, two standard derivations are considered in Eq. (3.18).

At the next model order, the mutual distance between one mode and all other modes is computed by Eq. (3.18), then the minimum distance is determined. Assuming $n$ modes have been identified at the pre-processing stage, each mode has its minimum mutual distance with forming a minimum distance vector, $V = (d_{min}^1, d_{min}^2, d_{min}^3 \ldots d_{min}^n)$, ($n$ denotes the number of modes, $d_{min}$ denotes the minimum distance between one mode and all other modes). Finally, the sum of mean and two standard derivations of $V$ are used to compute the clustering threshold, $\bar{d}$ (Reynders et al., 2012):

$$\bar{d} = \mu + 2\sigma$$

(3.19)

Generally, modal features are usually assumed to follow Gaussian normal distribution, such as frequency, damping ratio and mode shape (MAC value) (Au, 2011b). In this study, Eq. (3.18) defines modal distance which is the sum of frequency difference and mode shape difference between two modes. Therefore, modal distance turns out to be Gaussian normal distribution. Two standard derivations in Eq. (3.19) guarantee the distance between two
modes should be captured within a 95% confidence interval with the assumption of Gaussian distribution.

3.3.2.2 Mode clustering

Mode clustering starts with a calculated threshold in former section to group individual physical modes with similar modal characteristics. This study adopts self-adaptive clustering (Cabboi et al., 2017) to accomplish automated process. But different from original work (Cabboi et al., 2017), a weighted distance with an uncertainty of modal parameters is proposed. The $i$th weighed distance at model order, $n$, is defined as:

$$d_{n,i} = \left(\frac{|\bar{F}_z - F_{n,i}|}{\bar{F}_z}\right) + c\left(1 - MAC(\tilde{\Phi}_z, \Phi_{n,i})\right)$$  \hspace{1cm} (3.20)

where $\bar{F}_z = \bar{f}_z + 2\sigma_{\bar{f}_z}$, $\tilde{\Phi}_z = \bar{\phi}_z + 2\sigma_{\bar{\phi}_z}$, $F_{n,i} = f_{n,i} + 2\sigma_{f_{n,i}}$, $\Phi_{n,i} = \phi_{n,i} + 2\sigma_{\phi_{n,i}}$. $\bar{f}_z$ and $\bar{\phi}_z$ are mean frequency and mean mode shape at the $z$th cluster, respectively; $\sigma_{\bar{f}_z}$ and $\sigma_{\bar{\phi}_z}$ are corresponding mean standard derivation at the $z$th cluster. $f_{n,i}$ and $\phi_{n,i}$ are the $i$th frequency, and mode shape at model order, $n$, respectively; $\sigma_{f_{n,i}}$ and $\sigma_{\phi_{n,i}}$ are corresponding standard derivation, respectively. $MAC$ represents the modal assurance criteria (Pastor et al., 2012). $c$ is a weighting factor to reduce the effect of inaccurate mode shape on distance calculation ($c = \frac{\sigma_{\bar{f}_z} + \sigma_{\bar{\phi}_z}}{2}$).

Figure 3.3. The flowchart of an improved self-adaptive clustering
The major benefits of clustering technique used in this study include: 1) empirically assumptions on the number of clusters are not required; 2) clustering starting threshold is calculated rather than user-defined; 3) the threshold is iteratively trained with accumulative modes; 4) simple implementation and fast computation.

Eq. (3.18) and (3.20) are similar; both consider the uncertainty of parameter estimates and the importance of mode shape difference. Figure. 3.3 shows the flowchart of an improved clustering strategy. A more detailed introduction of self-adaptive clustering is referred to Cabboi et al. (2017).

3.3.2.3 Robust outlier detection

The number of physical poles in the stabilization diagram has trends with the increase of model order, exhibiting variability of modal estimates (Neu et al., 2017). The phenomenon more frequently appears in the damping ratio because the damping ratio has a high scattered nature. Outlier detection is applied to penalize undesirable modes in the final clusters for reducing identification variance from different measurements. In this study, robust outlier detection based on the minimum covariance determinant (MCD) is employed to identify outlying values from physical clusters. A robust distance (RD) is defined as:

\[ RD(x) = d(x, \hat{\mu}_{MCD}, \hat{\Sigma}_{MCD}) \]  

(3.21)

where observation sample, \( x \), is either frequency or damping ratio in a physical cluster in our case. \( \hat{\mu}_{MCD} \) is the MCD estimates of location; \( \hat{\Sigma}_{MCD} \) is the MCD covariance estimate. Explicit derivation and introduction can be found in Hubert et al. (2017).
A robust MCD estimator based on Eq. (3.21) is very powerful to flag outliers, as RD in Equation (12). It is not sensitive to diagnostic tools' masking effect compared to statistical distance and Mahalanobis distance (Cerioli, 2010). Also, MCD has a high resistance to outliers and are more robust and efficient (Hubert et al., 2017). Furthermore, MCD has the advantage of requiring no user-defined threshold, like a box-plot method that needs to define limit bounds (Sarlo and Tarazaga, 2019). Robust outlier detection in this work can be done by the function ‘robustcov’ in MATLAB.

After outlier removal, the average frequency, damping ratio, and mode shape in each physical cluster are taken as a representative. For evaluating the quality of each identified cluster, uncertainty on the $z$th physical clusters are quantified by Euclidean norm of uncertainty on modal parameters:

$$
\sigma_z = \sqrt{\left( \bar{\sigma}_{f,z}^2 + \bar{\sigma}_{\phi,z}^2 + \bar{\sigma}_{\xi,z}^2 \right)}
$$

(3.22)

where $\sigma_z$ is the standard derivation of the $z$th clusters; $\bar{\sigma}_{f,z}$, $\bar{\sigma}_{\phi,z}$, and $\bar{\sigma}_{\xi,z}$ are the average values of standard derivations of all frequencies, damping ratios, and mode shapes in the $z$th clusters.

### 3.4 Illustrative examples

In this section, the performance of the proposed automated SSI is validated by two field tests on the bridge, namely, the Dowling Hall Footbridge located at Turfs University in the U.S. and the Z24 bridge benchmark located in Switzerland. The data are open sources, and many researchers used these data to test the algorithms in the research community.
3.4.1 Application 1: Dowling Hall Footbridge

Dowling Hall Footbridge is located at Tufts University, as shown in Figure 3.4 (a). The bridge is a two-span steel frame bridge, 144 ft (44 m) long and 12 ft (2.7 m) wide with a reinforced concrete deck. A continuous health monitoring was designed and performed on Dowling Hall Footbridge from January 2010 to May 2010. The layout of eight accelerometers is shown in Figure 3.4 (b). More details of Dowling Hall Footbridge's information can be found in Moser and Moaveni (2011). In this study, the first six modal characteristics are used as baseline results that are obtained from the literature (Moser and Moaveni, 2011) to evaluate the performance of the proposed approach.

The acceleration data used in this study are obtained from vertical measurement under ambient excitation collected in the first week at 1:00 P.M. on January 7th, 2010. The frequency range of interest is 0-30Hz. The sampling frequency is 128Hz. Preparation parameters for SSI-cov/ref in this application are: \( i = 60 \), model order \( n = 40 \sim 150 \), reference sensor = \( (1,2,3,4,5,6,7,8) \).

Figure 3.4. Description of Application 1: (a) Dowling Hall Footbridge; (b) Sensor layout (Moser and Moaveni, 2011)
3.4.1.1 Identification results

The proposed approach described in Section 3.3 is utilized to analyze measured data. Figure 3.5 (a)-(c) show modal identification results at the pre-processing stage. The singular value spectrum (appeared in the curves in Figure 3.5) is plotted below the stabilization diagram. The standard derivation (±σ) uncertainty bounds of the frequency are shown as horizontal bars.

![Figure 3.5](image)

Figure 3.5. Identification results: (a) after conventional validation criteria; (b) after uncertainty criterion (c) after improved self-adaptive clustering

Figure 3.5 (a) displays all possible physical modes remaining in the stabilization diagram after applying conventional validation criteria, e.g., damping ratio check and modal complexity check. Figure 3.5 (b) shows the stabilization diagram filtered by a
supplementary uncertainty criterion. It is observed only using conventional validation criteria, the stabilization diagram still looks busy, including lots of scattered poles, which are spurious modes. However, uncertainty criterion can eliminate as many spurious modes as possible compared to conventional validation criteria, which will speed up later automated processes. The pre-processing stage's identification results demonstrate that the uncertainty criterion is more effective than conventional validation criteria.

The clustering stage then starts with a calculated clustering threshold using Eqs. (3.18) and (3.19) based on the remaining modes in Figure 3.5 (b). The proposed method's calculated threshold in this example is 0.022, while without the weighting factor, \( \omega \), it is 0.0488. It implies the stricter threshold by Eqs. (3.18) and (3.19) that allows removing more spurious modes with keeping physical modes. Furthermore, the updated threshold by improved clustering with Eq. (3.20) is 0.0086. Still, the original work (distance calculation without weighting factor) gives the updated threshold as 0.0304, indicating that the weighted distance tends to give a smaller value of the updated threshold.

The identified modes are more consistent and stable. It may be attributed to the use of \( \omega \) can improve the accuracy of measured mode shapes by distance calculation. Figure 3.5 (c) shows the modal identification results after performing the improved self-adaptive clustering. It is observed that clustering procedures remove spurious modes, the stabilization diagram is clarified with only remaining stable modes (vertical alignments). The first six modes in the reported work (Moaveni and Behmanesh, 2012) are used as a baseline for comparison, marked as \( M_1 \) to \( M_6 \) (A total of six clusters) in Figure 3.5 (c). It is noted that \( M_5 \) and \( M_6 \) are closely spaced modes, which are a common challenge in the
OMA. The robust outlier detection is to remove outlying frequencies and damping ratios.

As seen in Figure 3.6 (b), damping ratios are tighter and more consistent.

![Figure 3.6. Damping ratio vs frequency: (a) before outlier detection; (b) after outlier detection](image)

Table 3.1 presents identified frequencies and damping ratios along with the baseline data. Identified frequencies in this work agree well with those in the literature; the maximum relative difference (2.05%) is observed in the third mode. While larger variation is found in terms of damping ratio. It is because two tests were performed at a different time. Moaveni and Behmanesh (2012) reported baseline data, measured on April 4, 2009. In this study, measured data was collected on January 7, 2010. When considered the effect of environmental variables such as temperature, it is not surprising to have these differences. The frequency is less sensitive to environmental effects than the damping ratio.

<table>
<thead>
<tr>
<th>Modes</th>
<th>Frequency (Hz)</th>
<th>Damping ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Baseline*</td>
<td>The proposed approach</td>
</tr>
<tr>
<td>1st ($M_1$)</td>
<td>4.63</td>
<td>4.63</td>
</tr>
<tr>
<td>2nd ($M_2$)</td>
<td>6.07</td>
<td>6.04</td>
</tr>
<tr>
<td>3rd ($M_3$)</td>
<td>7.07</td>
<td>7.21</td>
</tr>
<tr>
<td>4th ($M_4$)</td>
<td>8.90</td>
<td>8.95</td>
</tr>
<tr>
<td>5th ($M_5$)</td>
<td>13.13</td>
<td>13.24</td>
</tr>
<tr>
<td>6th ($M_6$)</td>
<td>13.56</td>
<td>13.46</td>
</tr>
</tbody>
</table>

Note: *: Moaveni and Behmanesh (2012)
Figure 3.7. Error bar of frequency (left, ± 2σ) and damping ratio (right, ± σ)

Table 3.2. Uncertainty of physical clusters

<table>
<thead>
<tr>
<th>No. cluster</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>S.D. (%)</td>
<td>0.184</td>
<td>0.001</td>
<td>0.430</td>
<td>0.010</td>
<td>0.119</td>
<td>2.712</td>
</tr>
</tbody>
</table>

Note: S.D. denotes standard derivation

The uncertainty on modal parameters and physical clusters are also investigated in this example. The frequency and damping ratio in each mode are plotted as an open circle overlapping the standard derivation (σ) error bar in Figure 3.7. And the uncertainty of modal frequencies is much smaller than those of damping ratios. It is often more difficult to accurately measure the damping ratio in practice. The uncertainty of identified physical clusters is also quantified by Eq. (3.22) and shown in Table 3.2. And the uncertainty of the sixth cluster is much larger than those of others, suggesting it is more challenging to identify the sixth cluster because this cluster contains weakly-excited and closely spaced modes.
Mode1: Frequency=4.632Hz, $\xi=0.808\%$

Mode2: Frequency=6.042Hz, $\xi=0.069\%$

Mode3: Frequency=7.215Hz, $\xi=0.631\%$

Mode4: Frequency=8.948Hz, $\xi=0.056\%$

Mode5: Frequency=13.244Hz, $\xi=0.117\%$

Mode6: Frequency=13.464Hz, $\xi=1.113\%$

Figure 3.8. Identified mode shapes by the proposed approach and $\pm 2\sigma$ uncertainty bounds (blue dashed lines)

Overall, the proposed approach successfully identifies six modes under ambient vibration, as shown in Figure 3.8. The first six global mode shapes with corresponding uncertainties are presented; $\pm 2\sigma$ uncertainty bounds are plotted as blue dashed lines. Identified mode shapes have good agreement with those identified in the reported work (Moaveni and Behmanesh, 2012). Modes 3 and 4 are bending-torsional mode with evident rotational motion, while only vertical deformation is found on other modes. In addition, uncertainty bounds for all modes are narrow, which concludes that the identification of mode shapes is accurate.
For continuous SHM, it is crucial to track the change of modal parameters over time. In this example, the proposed approach is applied to modal tracking with measured data collected at every 1:00 P.M. from January 5th to February 28th, in 2010 (total 55 datasets). The same procedures as the former data analysis are applied for modal tracking.

As shown in Figure 3.9, solid black lines indicate frequency estimates, and grey areas cover ±2 standard derivations. All six modes are identified and tracked for all datasets by the proposed approach. It is not surprising that frequencies varied over time, mainly
because of environmental change and ambient excitation's randomness. The frequency at mode 6 has a relatively larger variation for two months, as this mode is not excited well and unstable to environmental change. The results illustrate the proposed approach can analyze massive datasets with minimum human intervention.

3.4.1.2 Sensitivity analysis

![Figure 3.10. Frequencies at different parameters: (a) model order range sensitivity (fixed $i = 60$); (b) time lag range sensitivity (fixed $n_{\text{max}} = 100$)](image)

Two preparation parameters in SSI, e.g., maximum model order, $n_{\text{max}}$, and time lag, $i$, significantly affect identification results. The influence of $n_{\text{max}}$ and $i$ is investigated to demonstrate the proposed approach is robust to their choice. $n_{\text{max}}$ and $i$ are varied from 70 to 160 and 30 to 120 in intervals of 10, respectively. As shown in Figure 3.10, identified frequencies are almost invariant to a different choice of $n_{\text{max}}$ and $i$, suggesting the proposed approach is robust and not sensitive to these two preparation parameters. It is very difficult to identify the best $n_{\text{max}}$ and $i$ in practice (Ubertini et al., 2013, Neu et al., 2017). Thus, insensitivity to them allows to more conveniently perform automated OMA and continuous health monitoring.
On the other hand, a different choice of uncertainty threshold (COV of frequency) is utilized to evaluate its effect on identification results. The uncertainty threshold is varied from 1% to 5% in the interval of 1%. As shown in Figure 3.11, the proposed approach yields almost the same frequencies regardless of COV thresholds, indicating a COV threshold can be safely chosen in the range of $1% \leq \text{COV} \leq 5\%$.

3.4.2 Application 2: Z24 bridge

The proposed approach is also applied to the Z24 bridge benchmark to validate its performance. The Z24 bridge was built in 1963 and located in Switzerland, serving to connect Koppigen with Utzenstorf and crossing over the A1 highway (See Figure 3.12 (a)). It is a post-tensioned concrete box-girder bridge with a main span of 100 ft (30 m) and two sides span of 46 ft (14 m). Detailed Introduction of the Z24 bridge can be found in Maeck and De Roeck (2003). The Z24 bridge was demolished at the end of 1998. Before the complete demolition, a short-term progressive damage test was implemented on the bridge to investigate the effect of simulated damage on the safety of the bridge.
A total of 17 different damage scenarios were designed under full forced and ambient excitation (Reynders and Roeck, 2009). In this work, acceleration response data from the scenario of No.8 for the new reference condition under ambient excitation is used to assess the proposed approach. A total of 291 DOFs were measured (See Figure 3.12 (b)). Due to the limited number of sensors, only at most 33 DOFs were measured for each set-up. Therefore, nine measurement set-ups were recorded with most 33 sensors to have full location coverage of the whole bridge, containing five reference sensors that are common to each set-up and 28 moving sensors whose location changes with different set-ups. For the No.5 set-up, only 22 moving sensors were used. Samples of 65536 data were recorded at each set-up at a 100 Hz sampling rate.

For each dataset, preparation parameters in SSI-cov/ref are defined as: time lag is $i = 50$, model order ranges from 2 to 120, to create stabilization diagrams. Reference sensors are selected as No. 29-33 (for set-up No.5, as No. 23-27). After the stabilization diagram is created, the proposed approach is applied to automatically interpret the stabilization diagram.
3.4.2.1 Identification results

Nine stabilization diagrams corresponding to each set-up are created; results of the fifth set-up are only presented in Figure 3.13 due to space limitation in this paper. The singular value spectrum (appeared in curves in Figure 3.13) is also plotted below the stabilization diagram. ±σ (Standard derivation) uncertainty bounds of frequency are plotted as horizontal bars. Figure 3.13 (a) displays modal identification results using conventional validation criteria, many scattered poles which are definitely spurious modes, still retain in the stabilization diagram. However, the uncertainty criterion can remove most spurious modes, demonstrating that the uncertainty criterion is more effective than conventional validation criteria (See Figure 3.13 (b)).

![Figure 3.13. Pre-processing stage for No. 5 set-up: (a) after validation criteria; (b) after uncertainty criterion](image)

Based on the remaining poles in the stabilization diagram after the pre-processing stage, the clustering threshold for each measurement set-up is calculated using Eqs. (3.18) and (3.19). As shown in Figure 3.14 (a), all threshold values are significantly reduced compared to those calculated without weighting factor, as weighting factor can offset the effect of mode shape difference. Manual clustering thresholds in commercial OMA software are
usually below 0.06 (Neu et al., 2017). The threshold derived from the newly proposed method is closer to the one from manual analysis, indicating proposed method’s rationality and feasibility in practice. Mode clustering is then implemented to group physical modes. The number of scattered poles is greatly removed by the proposed approach, only remaining obvious vertical alignments in the stabilization diagram in Figure 3.14 (b).

![Figure 3.14](image1.png)

**Note:** \( \omega \) is weighting factor in Eq. (3.18)

Figure 3.14. The clustering stage for No. 5 set-up: (a) calculated clustering threshold; (b) after improved self-adaptive clustering

In addition, an improved self-adaptive clustering that considers the weighting factor of \( c \) in Eq. (3.20) tends to give a smaller updated threshold, implying identified modal parameters are more stable and consistent with each other (See Figure 3.15). The use of
the weighting factor can improve the performance of clustering. Because only the first six modes are present in baseline work (Reynders et al., 2012), the first six clusters are presented in Figure 3.14 (b), marked as $P_1$ to $P_6$. Robust outlier detection is used to identify outlying modes (See Figure 3.16). Finally, the average of modal parameters in each cluster is selected as representative.

![Figure 3.16](image)

Figure 3.16. Damping ratio vs frequency: (a) before outlier detection; (b) after outlier detection

Table 3.3 shows the sample mean and sample standard derivation of frequency and damping ratio over nine measurement set-ups obtained from the proposed approach. The standard deviation in Table 3.3 represents the setup-to-setup sample statistics among all set-ups. The calculation of sample standard derivation (S.D.) in Table 3.3 only considers the environmental change among different set-ups rather than uncertainty sources. It is seen from Table 3.3 that the damping ratio has more significant variability than frequency, implying it is more challenging to identify damping ratio in practice, as the damping ratio is sensitive to environmental change. Overall, the proposed approach's identified frequencies and damping ratios are almost identical to those from baseline work, demonstrating low demand for human intervention.
Table 3.3. Identification results

<table>
<thead>
<tr>
<th>Modes</th>
<th>Frequency (Hz)</th>
<th>Damping ratio (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Baseline*</td>
<td>S.D.</td>
</tr>
<tr>
<td>1st ($P_1$)</td>
<td>3.86</td>
<td>0.01</td>
</tr>
<tr>
<td>2nd ($P_2$)</td>
<td>4.90</td>
<td>0.01</td>
</tr>
<tr>
<td>3rd ($P_3$)</td>
<td>9.76</td>
<td>0.02</td>
</tr>
<tr>
<td>4th ($P_4$)</td>
<td>10.3</td>
<td>0.09</td>
</tr>
<tr>
<td>5th ($P_5$)</td>
<td>12.41</td>
<td>0.19</td>
</tr>
<tr>
<td>6th ($P_6$)</td>
<td>13.22</td>
<td>0.15</td>
</tr>
</tbody>
</table>

Note: *: Reynders et al. (2012).

Uncertainties of modal parameters arising from assumptions made in SSI, such as linear, stationary structural behavior, white noise, etc., are also studied. Figure 3.17 shows the variability of frequency and damping ratio from modes 1 to mode 6 across nine measurement set-ups, respectively, with open circles representing the parameter estimates and error bars covering ± $2\sigma$ standard derivations. Both frequencies and damping ratios change over time, while the damping ratios have larger uncertainties. The negative damping ratio is immaterial in Figure 3.17 (b), such as mode 5 at No. 4 set-up and mode 6 at No. 1 set-up, merely because of the Gaussian distribution approximation and the larger standard derivation. Mode 6 has relatively larger uncertainty since the mode is not excited well.

Table 3.4 presents the average of standard derivation for each cluster over nine measurement set-ups using Eq. (22). As expected, the sixth cluster has the highest uncertainty, implying it is relatively harder to identify this cluster, which is also reflected in Figure 3.14 (b) that the sixth vertical alignments from the left form at a very ambiguous peak. Generally speaking, quantities of identified frequency and damping ratio are
consistent from one to another set-up numbers, suggesting robust and fair performance on modal analysis.

![Figure 3.17](image)

**Figure 3.17.** ±2σ standard derivation error bar ratio across nine setups: (a) frequency; (b) damping ratio

**Table 3.4.** Average of standard derivation for physical clusters among nine setups

<table>
<thead>
<tr>
<th>No. cluster</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>S.D. (%)</td>
<td>0.56</td>
<td>1.37</td>
<td>3.06</td>
<td>5.25</td>
<td>10.19</td>
<td>17.61</td>
</tr>
</tbody>
</table>

The global mode shapes are directly assembled from a local one in a single dataset by multiplying by a scaling factor so that their common DOFs, at the location of reference
sensor, agree well with each other through data fitting, namely, the method of the least squares. For the sake of article spaces, detailed procedures for calculating scaling factors are referred to work (Au, 2011a).

![Mode shapes of the Z24 bridge](image)

Mode1: Frequency=3.86Hz, $\xi=0.737\%$

Mode2: Frequency=4.91, $\xi=1.376\%$

Mode3: Frequency=9.77Hz, $\xi=1.338\%$

Mode4: Frequency=10.28Hz, $\xi=1.305\%$

Mode5: Frequency=12.44Hz, $\xi=2.908\%$

Mode6: Frequency=13.25Hz, $\xi=3.540\%$

As seen in Figure 3.18, the entire six modes are successfully identified from vibration response in all the nine measurement set-ups, which are in good accordance with those in Reynders et al. (2012). Mode 1 is a typical bending mode with a symmetric shape that has
the maximum deflection at midspan. Mode 2 is the first torsional mode with a slight rotational dynamic behavior along y-axis (transverse direction). Similar to mode 2, but more significant rotation is observed on modes 3 and 4; they are another two torsional modes. Modes 5 and 6 are vertical modes with asymmetric shapes.

Furthermore, five mode shapes at the only vertical direction (corresponding z-axis in Figure 3.18) and one mode shape at the only transverse direction (corresponding y-axis in Figure 3.18) are also presented in Figures. 3.19 and 20, ±2σ uncertainty bounds are plotted as blue dashed lines. Figure 3.19 shows only mode 6 has relatively wider uncertainty intervals since it is weakly-excited, while others have narrow bounds, implying mode shapes are identified with an accurate level.

![Mode shapes](image)

Figure 3.19. Mode shapes at X-Z plane with ±2σ uncertainty bounds

![Mode shape](image)

Figure 3.20. Mode shape at X-Y plane with ±2σ uncertainty bounds (not visible)
To further investigate the performance of the proposed approach for continuous health monitoring. As seen in Table 3.5, the proposed approach is applied to eight different damage scenarios during the short-term progressive damage test.

Table 3.5. Damage scenarios during the progressive damage test in 1998

<table>
<thead>
<tr>
<th>Measurement No</th>
<th>Date</th>
<th>Scenario</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>04, August</td>
<td>First reference measurement</td>
</tr>
<tr>
<td>2</td>
<td>09, August</td>
<td>Second reference measurement</td>
</tr>
<tr>
<td>3</td>
<td>10, August</td>
<td>20mm settlement of pier</td>
</tr>
<tr>
<td>4</td>
<td>12, August</td>
<td>40mm settlement of pier</td>
</tr>
<tr>
<td>5</td>
<td>17, August</td>
<td>80mm settlement of pier</td>
</tr>
<tr>
<td>6</td>
<td>18, August</td>
<td>95mm settlement of pier</td>
</tr>
<tr>
<td>7</td>
<td>19, August</td>
<td>Tilt of foundation</td>
</tr>
<tr>
<td>8</td>
<td>20, August</td>
<td>Third reference measurement</td>
</tr>
</tbody>
</table>

Figure 3.21. Identified frequencies for different damage scenarios
A total of 72 datasets consists of nine individual measurement setups for each damage scenario. The tracked frequencies, damping ratios, and associated uncertainty are plotted in Figures 3.21 and 3.23, with sample mean (solid black lines) and two averages of standard derivation (grey shaded areas) among all measurement set-ups. It is observed that the maximum frequency happened at scenario No. 1 corresponding to undamaged condition; the minimum frequency happened at modes 1, 3, 4, and 5 in scenario No. 6 and modes 2 and 6 in scenario No. 7 for corresponding to 95 mm settlement of pier and tilt foundation.

As seen in Figure 3.22, damage scenarios in Table 3.5 have significant effect on frequency, especially when the pier is settled, and foundation is tilted. For example, frequency reduction at modes 1, 3, 4 and 5 reaches the maximum magnitude when the pier has the maximum settlement, 95 mm, ranging from 5.93% to 8.08%. On the other hand, modes 2 and 6 have the maximum frequency reduction of 9.06% and 3.66%, respectively, due to foundation’s tilt, respectively. In contrast, 95-mm pier settlement still impairs on frequency, suggesting pier and foundation may be paid more attention during SHM.

![Figure 3.22. Frequency change due to damage](image-url)
Figure 3.23 shows the variability of damping ratio is smaller than of frequency, implying damping ratio is not sensitive to global damage scenarios in Table 3.5, but the damping ratio has much larger uncertainty. The results demonstrate potential benefits to handle a large amount of data with an acceptable level of performance while reducing human involvement. Therefore, the proposed approach is suitable for continuous health monitoring and modal tracking.

3.4.2.2 Sensitivity analysis

To examine the performance of the proposed approach in case of a different combination of preparation parameters in SSI-cov/ref, e.g., the maximum mode order,
\( n_{max} \), and time lag, \( i \), the sensitivity analysis is conducted for No. 5 measurement setup in this example. \( n_{max} \) and \( i \) range from 70 to 160 and from 30 to 120, respectively.

As seen in Figure 3.24, the proposed approach has consistent behavior for identifying six modes using different SSI-cov/ref preparation parameters. Similar to Application 1, Figure 3.25 shows that any threshold between 1 and 5\% yields the same outcomes. The sensitivity analysis demonstrates that the proposed approach is insensitive to two crucial parameters in SSI-cov/ref: model order and time lag. Generally, model order is overestimated to identify weakly excited modes, yielding more spurious modes; a small value of time lag may fail to generate enough stable poles in the stabilization diagram. It is very difficult to determine the best model order and time lag in real test. The proposed approach provides more flexibility for the selection of the two parameters, significantly facilitating automated modal identification in practice.

![Figure 3.24](image)

**Figure 3.24.** Frequencies at different parameters: (a) model order range sensitivity (fixed \( i = 50 \)); (b) time lag range sensitivity (fixed \( n_{max} = 90 \))
3.4.3 Practical aspects

The proposed approach accurately identifies the modes of interest and concurrently eliminates spurious modes. The weakly excited and closely spaced modes are identified on two bridges under ambient vibration. The procedures only require a few initial parameters setting, e.g., model order range, time lag, the threshold of MPC/MPD, and uncertainty criterion. In short, the proposed approach is insensitive to these parameters, especially, two crucial parameters: model order and time lag.

Table 3.6. Recommendations on initial parameters under complex test condition

<table>
<thead>
<tr>
<th>Initial parameters</th>
<th>$n_{max}$</th>
<th>$i$</th>
<th>MPC</th>
<th>MPD</th>
<th>COV</th>
</tr>
</thead>
<tbody>
<tr>
<td>Recommendation</td>
<td>100-160</td>
<td>2-3 times of Equation (2)</td>
<td>0.3</td>
<td>0.7</td>
<td>1%-5%</td>
</tr>
</tbody>
</table>

Note: $n_{max}$ is the maximum model order; $i$ is time lag; MPC is modal phase collinearity; MPD is mean phase deviation; COV is coefficient of variation of frequency.

Some recommendations are summarized in Table 3.6. Both $n_{max}$ and $i$ are over-defined, yielding spurious modes, but the proposed approach will remove them. MPC and MPD can be regarded as standard values, with no need for adjustment. Uncertainty threshold, COV, can also be safely chosen in the range of 1-5%. In addition, two-month period data (55 datasets: Application 1) and short-term progressive damage test (72 datasets: Application 2) demonstrates the feasibility of automated OMA and modal tracking of massive data for continuous health monitoring.
3.5 Conclusions

This chapter presented a novel two-stage framework for automated OMA based on SSI-cov/ref. Firstly, a stabilization diagram is created by SSI-cov/ref. Two-stage framework, e.g., pre-processing stage and clustering stage, is then implemented to interpret the stabilization diagram with low demand of user intervention. Two field tests on the bridge are employed to validate the capability of the proposed approach. The proposed framework has a minimal user’s involvement to achieve sufficient accuracy. Therefore, the proposed work is suitable for long-term health monitoring, e.g., modal tracking. The conclusion is summarized as follows:

- The uncertainty criterion is efficient in eliminating many undesired modes at the processing stage, which speeds up the later automation process.
- A novel distance calculation with the uncertainty of modal parameters and weighting factor yields a reasonable threshold for clustering.
- An improved self-adaptive clustering is proposed based on weighted distance calculation with the uncertainty of modal parameters.
- The improved clustering strategy has following features: 1) empirically assumptions on the number of clusters are not required; 2) clustering starting threshold is calculated rather than user-defined; 3) the threshold is iteratively trained with accumulative modes; 4) simple implementation and fast computation.
- The uncertainty on modal parameters and identified physical clusters are also quantified and providing additional information about quality of identified results.
- A robust outlier detection requiring no setting of threshold improves modal parameters' accuracy.
CHAPTER 4

AUTOMATED BAYESIAN MODAL IDENTIFICATION (BMI)

4.1 Introduction

Bayesian modal identification (BMI) method has been developed progressively in recent years. BMI method has remarkable advantages, for example: (1) Psychical modelling assumptions are strictly obeyed; (2) Formulation allows to make full use of data. (3) Measured data is directly analyzed by FFT without any approximately system matrices. (4) It can identify additional two modal parameters, namely, spectral density of the modal excitation and that of prediction error, but also offers quantitative assessment of the accuracy (Au, 2011b, Au, 2012a). BMI method assumes modal identification is a probability problem which is used to measure the plausibility of identified modal parameters given model class and measured data (Cox, 1963, Beck and Katafygiotis, 1998, Jaynes, 2003). For ambient vibration, Katafygiotis and Yuen are pioneers to establish the framework of BMI and developed fundamental theory (Katafygiotis and Yuen, 2001, Yuen and Katafygiotis, 2001, Yuen et al., 2002a, Yuen et al., 2002b, Yuen and Katafygiotis, 2003). The values of mean and covariance matrix of posterior PDF represent the most probable values (MPV) of modal parameters and associated uncertainty, respectively. To more efficiently perform Bayesian OMA, Au (Au, 2011b, Au, 2012a, Au, 2012b) proposed a fast Bayesian Fast Fourier Transform (FFT) to deal with different types of modes (i.e.,
well-separated and closely spaced modes (Au, 2011b, Au, 2012b) and different types of data (i.e., single setup and multiple-setup data (Zhu et al., 2021c), synchronized and asynchronized data (Zhu et al., 2018). In fast Bayesian FFT, the MPVs and covariance matrix are computed by a condensed form of objective function and analyzing a single mode in the selected frequency band. In the framework of fast Bayesian FFT, five modal parameters can be well estimated, i.e., natural frequency, damping ratio and mode shape, power spectral density (PSD) of the modal force and that of prediction error. MPVs of modal parameters could be quickly calculated, covariance matrix has also been analytically formulated by Hessian matrix of posterior PDF rather than adopting finite difference method. Consequently, the computational effort connected with the number of measured degrees of freedom (DOFs) is significantly reduced. Later, Li and Au (2019) applied expectation-maximization (EM) algorithm to fast Bayesian FFT so that convergency speed is noticeably improved.

Although fast Bayesian FFT has been successfully applied to various civil infrastructure, such as buildings (Au et al., 2012, Zhang et al., 2016d, Zhu et al., 2018), bridges (Au and Zhang, 2011, Ni et al., 2015, Zhang et al., 2016b), TV towers (Zhang et al., 2016c), the major challenge of fast Bayesian FFT is choice of two important factors, namely, initial frequency and frequency bandwidth. The appropriate initial frequency and frequency bandwidth are the prerequisite to perform fast Bayesian FFT, poorly estimated these two factors may lead to incorrect modal identification. Traditionally, initial frequency is visually picked from singular value (SV) spectrum, relying on empirical observation and qualitative judgement. Furthermore, multiple peaks are usually displayed on SV spectrum and peaks that representing spurious modes are always inevitable. It is highly difficult to
manually select initial frequency and distinguish physical peaks from spurious peaks on SV spectrum with human effort under complex environmental condition, since at most situations, peaks on SV spectrum usually are not clearly visible and distributed intensively, even impossible to be handpicked. This situation is frequently happened to higher modes that is weakly excited and closely spaced modes.

On the other hand, different choice of frequency bandwidth may lead to different identification uncertainty and determine what data information to be used in making inference of the modal parameters (Au, 2014). When selected frequency bandwidth is narrow, identification uncertainty may be intolerably large. But when bandwidth is wide, bias of identified modal parameters could become greatly remarkable (Au, 2017a). Au (Au, 2017c) investigated the choice of frequency bandwidth and suggested that \( \kappa \) could range from 5 to 10 so that identification uncertainty will be acceptable. However, this rule of thumb may fail for modes with low signal-to-noise ratio or closely spaced modes. Band selection has been investigated in (Au, 2016b), where evidence ratios were applied to evaluate frequency bandwidth by considering maximum entropy principle to determine a representative competitive model class. It still has challenge on how to properly choose a frequency bandwidth before modal identification.

Driven by the essential demand, this chapter proposed a method to achieve the automation in Bayesian FFT modal identification as well as to automatically provide parameter uncertainty information. A stabilization diagram is firstly built and automatically interpreted, resulting in frequency clusters. The frequency representative of each cluster is then recognized as the initial frequency. Spurious modes are also cleared in this step. Next, the frequency band is picked through sifting frequency difference between
initial frequency and identified frequency, and a statistical index (e.g., mean and median) of modal parameters and associated uncertainty is chosen as representative. The proposed method is verified using a numerical example and then applied to the Z24 benchmark bridge for long-term data analysis.

This chapter is outlined as follows: a brief review of the fast Bayesian FFT is firstly presented in Section 4.2. The automated selection of initial frequency and frequency bandwidth are then presented in Section 4.3, and automated interpretation of stabilization diagram and selection of effective bandwidth factors are provided in Section 4.4. In section 4.5, a numerical study and a field test of Z24 benchmark bridge are used to illustrate the performance of the proposed method. Finally, conclusion and discussion are given in Section 4.6.

4.2 Theoretical background of fast Bayesian FFT

This section briefly reviews the Bayesian FFT formulation for modal identification. For thorough overview of original formulations, one is referred to work (Au, 2011b, Au, 2012a). In the context of Bayesian inference, unknown excitation and dynamic response are modeled as stationary stochastic process. Also, unknown modal excitation is assumed to have complex Gaussian distribution. The measured acceleration $\ddot{x}_j$ is comprised of theoretical response and prediction error:

$$\ddot{x}_j = \ddot{x}_j(\theta) + e_j$$

(4.1)

where $\ddot{x}_j(\theta) \in \mathbb{R}^n (j = 1, 2 \cdots N)$ denotes theoretical response expressed with modal parameters $\theta$, including frequency $f$, damping ratios $\zeta$, mode shapes $\Phi$, spectral density $S$ of modal excitation and that of the prediction error $S_e$, which is expected to be identified. $e_j \in \mathbb{R}^n (j = 1, 2 \cdots N)$ denotes the difference between model response and measured data
which may result from modelling error and measurement noise, $e_j$ is assumed to have complex Gaussian distribution with zero-mean. $n$ and $N$ are number of DOFs and number of sampling points, respectively. The FFT of $\ddot{x}_j$ could be defined as:

$$F_k = \sqrt{\frac{2\Delta t}{N}} \sum_{j=1}^{N} \ddot{x}_j \exp \left\{ -2\pi i \frac{(k-1)(j-1)}{N} \right\}$$

(4.2)

where $i^2 = -1$. $\Delta t$ is sampling interval. $k = 1, 2 \ldots N_q$ with $N_q = \text{int}(N/2) + 1$, $N_q$ is the Nyquist frequency, $\text{int}(\cdot)$ is the integral part.

Let $Z_k = (\text{Re} F_k; \text{Im} F_k)$ be a vector of the real and imaginary part of $F_k$. In Bayesian modal identification, only FFT data in a selected frequency band containing modes of interest are used for modal identification. Denoting such FFT data as $Z_k$. Based on Bayes’ theorem, with sufficient data, the prior PDF is non-informative, the posterior PDF is dominated by likelihood function (Au, 2012a). Therefore, within the selected frequency band, the posterior PDF of modal parameters $\theta$ given measured data $Z_k$ is proportional to likelihood function and can be expressed as:

$$P(\theta|\{Z_k\}) \propto P(\{Z_k\}|\theta) = \prod_k \pi^{-n}|C_k(\theta)|^{-1} \exp[-Z_k^T C_k(\theta)^{-1} Z_k]$$

(4.3)

where $|\cdot|$ denotes the determinant of the term $C_k$, $C_k$ is the covariance matrix of $Z_k$ and given as:

$$C_k = \frac{1}{2} \left[ \Phi \Phi^T \right] \left[ \begin{array}{cc} \text{Re} H_k & -\text{Im} H_k \\ \text{Im} H_k & \text{Re} H_k \end{array} \right] \left[ \begin{array}{cc} \Phi^T & 0 \\ 0 & \Phi^T \end{array} \right] + \frac{S_e}{2} I_{2n}$$

(4.4)

where $\Phi$ is mode shapes matrix. $S_e$ is spectral density of the prediction error. $I_{2n}$ denotes $2n \times 2n$ identity matrix. $H_k$ is the theoretical spectral density matrix of the modal acceleration response and the $(i,j)$ element of this matrix is given by:
\[ \mathbf{H}_{(i,j)} = S_{i,j}[(\beta_{ik}^2 - 1) + 2i\zeta_i\beta_{ik}]^{-1}[(\beta_{jk}^2 - 1) - 2i\zeta_j\beta_{jk}]^{-1} \]  

(4.5)

where \( \beta_{ik} = f_i/f_k \) is frequency ratio. \( f_i \) and \( f_k \) are the \( i \)th modal frequency and the FFT frequency abscissa, respectively. \( \zeta_i \) denotes the \( i \)th damping ratio; \( S_{i,j} \) is the cross spectral density between the \( i \)th and \( j \)th modal excitations.

Thus, the most probable values (MPVs) could be obtained by maximizing \( P(\{Z_k\}|\theta) \), or equivalently minimizing ‘negative log-likelihood function’ (NLLF) \( L(\theta) \):

\[
L(\theta) = -\ln p(\{Z_k\}|\theta) = nN_f \ln \pi + \sum_k \ln |C_k(\theta)| + \sum_k Z_k^T E_k(\theta)^{-1} Z_k
\]  

(4.6)

where \( N_f \) is the number of FFT points in the selected frequency band. The posterior uncertainty can be calculated from the inverse of the Hessian of the \( L(\theta) \). Note that mode shape is assumed to have a unit norm, i.e., \( \|\Phi\|^2 = \Phi^T \Phi = 1 \) to avoid unidentifiable problem. General steps for modal identification by traditional fast Bayesian FFT could be summarized as follows:

Step 1: Initial frequency hand-picked from SV spectrum

Step 2: Determination of frequency bandwidth in a trial-and-error manner

Step 3: Modal identification for \( \Phi, f, \zeta, S \) and \( S_e \) by the work (Au, 2011b, Au, 2012a)

Step 4: Uncertainty quantification by the work (Au, 2012b, Au, 2017b)

In Bayesian FFT modal identification, initial frequency and frequency band have to be set primarily for computing the MPVs. For well-separated mode, a frequency band is considered to include only one mode. Figure 4.1 shows an idealized SV spectrum for a single mode, which plots the eigenvalues of the PSD of the data with the frequency.

Supposing the data from a selected band \( f_0(1 \pm \kappa\zeta_0) \) is used for modal identification, where \( f_0 \) is the initial frequency, \( \zeta_0 \) is the initial damping ratio (always set to be 1%), \( \kappa \) is
the bandwidth factor and $T_d$ is data duration. The initial frequency is used as the initial value in computing the MPVs. Conventionally, it is manually picked as the peak, and its value is governed by the resolution of the spectrum. The frequency band is controlled by the bandwidth factor $\kappa$. The selection of the frequency band affects the FFT data used in making inference about modal parameters. It is a trade-off between the amount of available information and risk of modeling error. A larger band provides more data in the likelihood function and so that more informative for modal identification. While wide band also increases the modelling error risk since the theory assumes a constant PSD within the frequency band though this may not be true especially for a wider band.

![Figure 4.1. Idealized SV spectrum of data PSD for a well-separated mode](image)

**4.3 A two-step automation approach for fast Bayesian FFT**

In this section, a two-step automation approach for Bayesian FFT modal identification is presented, as shown in Figure 4.2. Step 1 (Section 4.3.1) aims at selecting the initial frequency based on automated interpretation of stabilization diagram which involves modal criteria and clustering. Step 2 (Section 4.3.2) targets on the selection of frequency
band, where a series of effective bandwidth factors within a predefined range are obtained. The following presents the detail of these two automation steps.

![Flowchart of the two-step automation approach](image)

Figure 4.2. The flowchart of the two-step automation approach

### 4.3.1 The selection of initial frequency

The selection of the initial frequency employs clustering-based automated interpretation of stabilization diagram, which has been introduced in detail in Chapter 3 (Section 3.3). The output of the stabilization diagram is frequency representative of each cluster, who serves as the initial frequency for the fast Bayesian FFT modal identification. Stabilization diagram, a plot of a range of model orders with frequency, is a popular tool to identify modal parameters in the class of the SSI technique. One challenge is that undesirable spurious modes may appear in the stabilization diagram, due to measurement noise and over-specified model order etc. (Zeng and Kim, 2020). In general, physical modes can be graphically distinguished since they are consistently displayed as vertical alignments in the stabilization diagram. Conversely, spurious modes form in a scattered
Figure 4.3. Flowchart of the selection of initial frequency

In pre-processing, as described in Chapter 3 (Section 3.3.1), a covariance-driven reference-based SSI (SSI-cov/ref) is performed to construct a stabilization diagram. Criteria are then applied on this diagram for initial removal of spurious modes and speed up clustering process. For instance, the damping ratios of civil structures are commonly recognized in the range of 0 to 10%, otherwise it should be discarded. Modal phase collinearity (MPC) and mean phase deviation (MPD) are two indicators to measure mode shape complexity. The MPC value closer to 1 indicates that modes tend to be physical. In contrast, a smaller value of the MPD implies that the mode shape vector is more likely to be physical. A coefficient of variation (COV = standard derivation/mean, calculated by the SSI) with respect to the identified frequency is treated as another indicator to further eliminate spurious modes. A frequency with 2% COV is chosen as the threshold, representing those modes with COV exceeded 2% should be removed.
In clustering, as described in Chapter 3 (Section 3.3.2), a self-adaptive clustering technique is carried out to automatically assemble modes with similar characteristics. The clustering process starts with the threshold calculation based on mutual distance between two possible modes. Next, the modes are iteratively grouped when their modal distances are less than the calculated threshold during the clustering process. Finally, clusters with modes exceeding one third of the total model order are kept, otherwise it should be discarded.

When the interpretation of stabilization diagram is complete, the output, which is the average frequency of each identified cluster, will be used as the initial frequency for Bayesian modal identification.

4.3.2 The selection of frequency band

In fast Bayesian FFT, only the FFT data within a selected frequency band (containing modes of interest) are used for computation of modal parameters. The selection of the frequency band affects the data information involved for computation; thus, it influences the identification accuracy. The selection of the bandwidth is a trade-off between available data for modal identification and the modeling error involved. In this section, a method for the automated selection of frequency band is presented, and its flowchart is shown in Figure 4.4.

A range of bandwidth factor \([\kappa_1, \kappa_2]\) is firstly defined with an interval of \(\Delta \kappa\). The upper bond \(\kappa_2\) is chosen from 5 to 10 to include fairly sufficient data information for making inference (Au, 2017b). The lower bond \(\kappa_1\) is chosen to be not less than 1, where \(\kappa = 1\) represents the half-power band. Each bandwidth factor gives a certain frequency band for modal identification. The frequency difference (a vector) is calculated between the initial
frequency and the identified frequency corresponding to each bandwidth factor. The frequency difference vector for each mode is defined as:

\[ \mathbf{f} = \left( \bar{f}_1, \bar{f}_2, \bar{f}_3, \cdots \frac{\kappa_2 - \kappa_1}{\Delta \kappa} \right) \]  

where \( \bar{f}_i \) is the \( i \)-th frequency difference, expressed as \( \bar{f}_i = \frac{|f_i - f_{\text{initial}}|}{f_{\text{initial}}} \) \( (i = 1, 2, \cdots (\kappa_2 - \kappa_1)/\Delta \kappa) \); \( f_{\text{initial}} \) is the initial frequency obtained from Section 4.3.1; \( f_i \) is the \( i \)-th identified frequency corresponding to the bandwidth factor \( \kappa_i \).

A series of effective bandwidth factors are next selected by sifting frequency difference. If frequency difference \( \bar{f}_i \) is less than 1% and damping ratio is positive, the corresponding bandwidth factor can be deemed as a valid one, otherwise should be discarded. Note that 1% is a common frequency tolerance for acceptance to eliminate frequency outliers (Mao et al., 2019, Tran and Ozer, 2020). Identified modal parameters corresponding to each effective bandwidth factor are stored. The average frequency and mode shape are chosen as representatives for the mode. Regarding the damping ratio, due to its dispersed nature, the median of the damping ratio is used as a representative to minimize the effect of outliers. Repeat above procedures until accomplishing identification for each mode of interest. Remark that a series of frequency bands corresponding to the bandwidth factors are automatically selected by the proposed method, which significantly reduces human-
induced uncertainty on bandwidth; while conventional way chooses the bandwidth factor in a trial-and-error manner, which is subjective and time-consuming. Improper choice of bandwidth can even yield much divergence on identification results. Integrating with automation on frequency bandwidth, fast Bayesian FFT can achieve a fully automated modal analysis.

4.4 Illustrative examples

In this section, the performance of the proposed automated fast Bayesian FFT modal identification is evaluated. A numerical example is firstly presented in Section 4.4.1 to validate the proposed method, where extremely cases such as modes with low modal signal-to-noise ratio (SNR) and closely spaced are additionally considered. A field data example of the Z24 benchmark bridge is then presented in Section 4.4.2 to demonstrate the feasibility of the proposed method under operational (complex) condition with long-term monitoring data. Section 4.4.3 provides the practical aspect of the proposed method.

4.4.1 Numerical example: mass spring-damper structure

A numerical example of three DOFs mass spring-damper structure is considered to generate synthetic data with well-separated mode and closely spaced modes, as shown in Figure 4.5. The three masses $m_i (i = 1, 2, 3)$ are assumed to be 2 kg of each. The four spring stiffnesses $k_j (j = 1, 2, 3, 4)$ are equal to 600, 50, 50, 600 N/m, respectively. The damping ratios of all modes are set to be 1%. The theoretical natural frequencies and mode shapes can then be calculated by the characteristic eigen equation, i.e., $f_1 = 1.074$ Hz, $f_2 = 2.869$ Hz, $f_3 = 2.888$ Hz. The 2nd and 3rd frequencies are closely spaced modes. Note that the frequency space index representing the relative percentage of frequency
difference is 0.68%, which can be regarded as a pair of extremely closely spaced modes with challenging identification in practice (Wu et al., 2018). The modal excitation and prediction error are modeled as i.i.d. Gaussian white noise with PSDs of $S = 1 \, (\mu g)^2 / \text{Hz}$ and $S_e = 72 \, (\mu g)^2 / \text{Hz}$, respectively. This yields a SNR ($= S/4S_e\zeta^2$) to be around 35 for all modes, which is relatively low in real test (Zhu et al., 2018) and indicates the quality of data to be poor due to high measurement noise. Data duration of 6000 seconds were generated at a sampling frequency of 100 Hz.

![Figure 4.5. Mass spring-damper structure](image)

4.4.1.1 Automated selection of initial frequency and frequency bandwidth

An SV spectrum is plotted in Figure 4.6 (a), where peaks are highlighted by red circles. Note that the 2nd and 3rd modes are closely spaced, and it is hardly to be handpicked. In the proposed method, a stabilization diagram is adopted and automatically interpreted to determine the initial frequency. To draw this diagram, the model order ranges from 20 to 100 and time lag is set to be 100. It is noticed that the model order is usually over-specified to cover weak-excited modes, but this also leads the apparent of spurious modes. The proposed method is applied to clear and interpret the stabilization diagram. As seen in Figure 4.6 (b), scattered poles (spurious modes) are shown in the full stabilization diagram, which will be removed using the proposed method. Figure 4.6 (c) shows the cleared stabilization diagram with three vertical red-circle alignments in the plot, representing three
clusters containing physical frequencies. Note that the last two alignments, corresponding to the closely spaced modes, cannot be visually distinguished. Finally, the average of the clusters is chosen as an initial frequency (black circles in Figure 4.6 (c)).

![Diagram](image)

**Figure 4.6.** (a) An SV spectrum; (b) full stabilization diagram; (c) cleared stabilization diagram; bracket: frequency band; black circle: selected initial frequency
The bandwidth factors are initially chosen in the range of (4, 6) with a step of 0.1. If the frequency difference between the initial and identified ones is less than 1% and the damping ratio is positive, the applied bandwidth factor can then be regarded as an effective one. Figure 4.7 shows effective bandwidth factors and corresponding identified frequencies for all modes. All identified frequencies are close to each other, indicating the effectiveness of that bandwidth. Compared to the conventional way of choosing the bandwidth, the proposed method is fully automated, and it gives more flexibility on the band selection. This significantly facilitates the Bayesian modal identification in practice. The selected initial frequencies and bands of the modes 1-3 are listed in Table 4.1 as below.

<table>
<thead>
<tr>
<th>Mode</th>
<th>No.1</th>
<th>No. 2</th>
<th>No.3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Initial frequency (Hz)</td>
<td>1.073</td>
<td>2.868</td>
<td>2.889</td>
</tr>
<tr>
<td>Bandwidth Lower bond (Hz)</td>
<td>1.020</td>
<td>2.725</td>
<td>2.745</td>
</tr>
<tr>
<td>Bandwidth Upper bond (Hz)</td>
<td>1.127</td>
<td>3.012</td>
<td>3.035</td>
</tr>
</tbody>
</table>

### 4.4.1.2 Identification results

The Bayesian FFT modal identification is conducted for each effective frequency bandwidth. The average of frequencies, mode shapes, and the median of damping ratios are used as representatives. Identification results are summarized in Table 4.2. The identified modal parameters are well-matched with their exact values. The coefficient of variation (c.o.v.) of the identified frequencies is less than 0.1%; while a larger c.o.v. can be observed in damping ratios, which have the same order of magnitude as modal force PSD. Mode shapes are evaluated by modal assurance criterion (MAC), which is almost equal to 1 for all modes, indicating that the identified mode shapes are close to its exact
counterpart. The mode shape c.o.v. calculated by the sum of diagonals of the posterior covariance matrix, has also the same order of magnitude. It can be demonstrated that the proposed method is capable to automatically identify modal parameters with adequate accuracy. The example also shows that the method works well in low SNR situation and for closely spaced modes.

Table 4.2. Modal identification results.

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Mode 1</th>
<th>Mode 2</th>
<th>Mode 3</th>
</tr>
</thead>
<tbody>
<tr>
<td>Frequency, $f$ (Hz)</td>
<td>Identified 1.074 (0.067)</td>
<td>2.869 (0.041)</td>
<td>2.890 (0.042)</td>
</tr>
<tr>
<td></td>
<td>Exact 1.074</td>
<td>2.869</td>
<td>2.889</td>
</tr>
<tr>
<td>Damping ratio, $\zeta$ (%)</td>
<td>Identified 1.023 (7.8)</td>
<td>0.981 (4.8)</td>
<td>1.020 (4.9)</td>
</tr>
<tr>
<td></td>
<td>Exact 1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Mode shape, MAC</td>
<td>Identified 1.000 (1.3)</td>
<td>0.998 (6.1)</td>
<td>0.998 (6.7)</td>
</tr>
<tr>
<td></td>
<td>Exact 1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Modal force PSD, $S$</td>
<td>Identified 1.034 (7.4)</td>
<td>0.978 (4.9)</td>
<td>0.965 (5.6)</td>
</tr>
<tr>
<td></td>
<td>Exact 1.000</td>
<td>1.000</td>
<td>1.000</td>
</tr>
<tr>
<td>Prediction error, $S_e$</td>
<td>Identified 70.69 (2.8)</td>
<td>73.88 (2.3)</td>
<td>73.88 (2.3)</td>
</tr>
<tr>
<td></td>
<td>Exact 72.00</td>
<td>72.00</td>
<td>72.00</td>
</tr>
</tbody>
</table>

Note: the values in parenthesis are c.o.v. (units: %); The MAC is calculated between the identified and exact mode shapes; $S$ and $S_e$ unit: $(\mu g)^2$/Hz

4.4.2 Field test: Z24 bridge

The proposed automated fast Bayesian FFT is validated by a field test of Z24 bridge which was used in Section 3.4.2. The detailed description is referred to Section 3.4.2. In this section, the initial frequency and frequency bandwidth are automatically selected in modal identification for each data set. The modal tracking by the proposed method was implemented for data recorded during a long-term monitoring. Furthermore, a short-term progressive damage detection was also carried out. The probability of frequency change due to damage was discussed.
4.4.2.1 Automated selection of initial frequency and frequency bandwidth

The plot of the SV spectrum with the first three eigenvalues of the PSD matrix for setup No. 5 is viewed in Figure 4.8 (a). The first three modes (indicated by red circles) can be clearly observed, while higher modes (highlighted in the red square area) are relatively complex to distinguish. In this example, to draw stabilization diagram, the time lag is chosen to be 50, a model order ranges from 2 to 120. Once the stabilization diagram is constructed, an automated interpretation of the stabilization diagram is applied to eliminate spurious modes on the SV spectrum. Figure 4.8 (b) shows the full stabilization diagram of setup No. 5, including numerous spurious modes. After applying the automation strategy, the stabilization diagram is re-constructed as Figure 4.8 (c), which is much apparent compared to the previous one. The first six clusters are displayed as red vertical alignments shown on Figure 4.8 (c), marked as $P_1$ to $P_6$. The average frequency of each cluster is utilized as the initial frequency for Bayesian modal identification, presented as a black circle on Figure 4.8 (c). $P_3$ and $P_4$ are regarded as closely spaced modes that are identified as a group.
Figure 4.8. (a) SV spectrum for setup No. 5; (b) full; (c) cleared (black circle: initial frequency; bracket: frequency bandwidth)

Turning attention to frequency band, shown as brackets [ ] on Figure 4.8 (c), a range of bandwidth factor [3, 7] for each mode is chosen with a step of 0.1. Effective bandwidth factors are selected by the proposed method. Figure 4.9 shows effective factors and corresponding frequency for each mode in setup No. 5. Modes 1-5 have a relatively greater number of effective factors within the range, while mode 6 has fewer effective bandwidth factors. As reflected in Figure 4.8 (a), mode 6 is ambiguous to identify. Table 4.3 summarizes the initial frequencies and frequency bandwidth factors for setup No. 5. After automated identification, modal parameters corresponding to each effective bandwidth
factor for each mode are stored; the average of frequencies and mode shapes, the median of damping ratios are used as representative.

Figure 4.9. Effective bandwidth factor and corresponding frequency for Setup No.5

Table 4.3. Initial frequency and frequency bandwidth for setup No.5.

<table>
<thead>
<tr>
<th>Mode</th>
<th>No.1</th>
<th>No.2</th>
<th>No.3</th>
<th>No.4</th>
<th>No.5</th>
<th>No.6</th>
</tr>
</thead>
<tbody>
<tr>
<td>frequency (Hz)</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Lower Bandwidth (Hz)</td>
<td>3.663</td>
<td>4.651</td>
<td>9.281</td>
<td>9.888</td>
<td>11.844</td>
<td>12.934</td>
</tr>
<tr>
<td>Upper Bandwidth (Hz)</td>
<td>4.049</td>
<td>5.141</td>
<td>10.257</td>
<td>10.594</td>
<td>13.090</td>
<td>13.970</td>
</tr>
</tbody>
</table>

4.4.2.2 Identification results

Tables 4.4 and 4.5 present all the identified modal parameters, including frequencies, damping ratios, PSDs of modal force and prediction error. Note that values in Tables 4.4 and 4.5 denote setup-to-setup statistical properties, i.e., the sample mean and sample c.o.v. (sample standard derivation/sample mean) among all setups, reflecting the change of environmental condition. To be specific, the modal parameters in Tables 4.4 and 4.5 are obtained by taking the average of parameters identified in each measurement setup. The benchmark results by automated SSI (Chapter 3) are also used for comparison purpose.
The results are used to verify the performance of the proposed method. Here only frequencies, damping ratios, and mode shapes are available in the benchmark results.

Table 4.4. The sample mean of identified frequency and damping ratio of nine setups

<table>
<thead>
<tr>
<th>Mode number</th>
<th>SSI $f$ (Hz)</th>
<th>Proposed $f$ (Hz)</th>
<th>SSI sample c.o.v. (%)</th>
<th>Proposed $\zeta$ (%)</th>
<th>Proposed sample c.o.v. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.1</td>
<td>3.86</td>
<td>3.86</td>
<td>0.33</td>
<td>0.74</td>
<td>0.79</td>
</tr>
<tr>
<td>No.2</td>
<td>4.91</td>
<td>4.90</td>
<td>0.39</td>
<td>1.38</td>
<td>1.39</td>
</tr>
<tr>
<td>No.3</td>
<td>9.77</td>
<td>9.77</td>
<td>0.40</td>
<td>1.34</td>
<td>1.50</td>
</tr>
<tr>
<td>No.4</td>
<td>10.28</td>
<td>10.28</td>
<td>0.58</td>
<td>1.30</td>
<td>1.76</td>
</tr>
<tr>
<td>No.5</td>
<td>12.44</td>
<td>12.49</td>
<td>1.40</td>
<td>2.91</td>
<td>3.44</td>
</tr>
<tr>
<td>No.6</td>
<td>13.25</td>
<td>13.22</td>
<td>0.61</td>
<td>3.54</td>
<td>3.83</td>
</tr>
</tbody>
</table>

Table 4.5. The sample mean of PSD of modal force and prediction error of nine setups

<table>
<thead>
<tr>
<th>Mode number</th>
<th>modal force PSD $\sqrt{S}$</th>
<th>SSI sample c.o.v. (%)</th>
<th>prediction error PSD $\sqrt{S_e}$</th>
<th>SSI sample c.o.v. (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>No.1</td>
<td>16.73</td>
<td>84.06</td>
<td>2.22</td>
<td>54.48</td>
</tr>
<tr>
<td>No.2</td>
<td>3.46</td>
<td>61.59</td>
<td>5.84</td>
<td>95.30</td>
</tr>
<tr>
<td>No.3</td>
<td>3.02</td>
<td>84.90</td>
<td>2.32</td>
<td>38.73</td>
</tr>
<tr>
<td>No.4</td>
<td>2.09</td>
<td>79.61</td>
<td>2.32</td>
<td>38.73</td>
</tr>
<tr>
<td>No.5</td>
<td>4.88</td>
<td>36.15</td>
<td>5.47</td>
<td>45.77</td>
</tr>
<tr>
<td>No.6</td>
<td>6.53</td>
<td>81.08</td>
<td>5.85</td>
<td>39.17</td>
</tr>
</tbody>
</table>

Unit: ($\mu g$)/\(\sqrt{\text{Hz}}\).

As shown in Table 4.4, frequencies obtained from the proposed method are well-matched with the results from the SSI. Identified frequencies exhibit small variability (sample c.o.v.< 2%) for all modes, indicating a good precision. The damping ratios exhibit a larger difference between the reference and proposed method, indicating the difficulty of obtaining damping ratio in practice. The damping ratio also has a relatively larger sample c.o.v. when compared to the frequency. Table 4.5 summarizes sample mean and sample c.o.v. of the identified PSD (square root) of modal force and prediction error. It is not surprising to find pronounced variability on these two parameters, as $S$ and $S_e$ are referred
to as the intensity of environmental excitation and measurement noise, respectively. From the identification results, it can be seen that environmental change has a great influence on the identified modal parameters. Figure 4.10 shows the variation of modal SNR with respect to setups for all six modes. The modal SNR of mode 1 is overall higher than other modes. This is also reflected by the SV spectrum in Figure 4.8(a).

![Figure 4.10. The variation of modal SNR with respect to setups for all six modes](image)

Local mode shapes are identified from individual setups. Based on reference sensor locations, the global mode shapes are assembled from local ones using a least square method. The obtained global mode shapes are listed in Figure 4.11, which match with those in Section 3.4.2.1. Frequencies and damping ratios obtained through averaging among all setups are shown above each mode shape. The average of posterior c.o.v. in all setups is present in parenthesis. Mode 1 is the first bending mode with a symmetric mode shape along the vertical direction and the maximum deformation appears at the midspan. Mode 2 is a combination of a dominated translational mode in Y direction and a torsional mode. Modes 3 and 4 show vertical-torsional motion. Rotational behavior is observed with respect to Z direction due to the skewness of the bridge. Modes 5 and 6 are also bending modes
appearing as asymmetric shapes along the bridge deck with a maximum deformation at side span. Overall, all the global mode shapes are identified soundly by the proposed method, suggesting the automation method has satisfactory performance in modal analysis.

**Figure 4.11.** The global mode shapes of the first six modes
Posterior uncertainties associated with modal parameters are also investigated. Tables 4.6-4.8 show the posterior c.o.v.s of identified frequencies, damping ratios and mode shapes among the nine setups, respectively. The mean values of posterior c.o.v.s of frequencies are all near 1%; while the posterior c.o.v.s of damping ratios are much larger. The posterior c.o.v.s of mode shapes in Table 4.8 are less than 2%. Generally, the posterior c.o.v. of modal parameters in a single setup are significantly smaller than the sample c.o.v. among all setups in Tables 4.4 and 4.5. The sample c.o.v. merely reflects setup-to-setup sample statistical variability arising from the environmental condition, e.g., wind, temperature (See Tables 4.4 and 4.5). While the posterior c.o.v. represents the uncertainty of modal parameters due to measurement noise and modeling error, reflecting modal identification accuracy.

Table 4.6. Posterior c.o.v.s of frequency (%)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Setup No.</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>0.08</td>
<td>0.08</td>
</tr>
<tr>
<td>2</td>
<td>0.11</td>
<td>0.11</td>
</tr>
<tr>
<td>3</td>
<td>0.08</td>
<td>0.07</td>
</tr>
<tr>
<td>4</td>
<td>0.07</td>
<td>0.08</td>
</tr>
<tr>
<td>5</td>
<td>0.15</td>
<td>0.57</td>
</tr>
<tr>
<td>6</td>
<td>1.03</td>
<td>0.23</td>
</tr>
</tbody>
</table>

Table 4.7. Posterior c.o.v.s of damping ratio (%)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Setup No.</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>1</td>
<td>2</td>
</tr>
<tr>
<td>1</td>
<td>11.83</td>
<td>11.34</td>
</tr>
<tr>
<td>3</td>
<td>4.67</td>
<td>5.28</td>
</tr>
<tr>
<td>4</td>
<td>5.09</td>
<td>5.14</td>
</tr>
<tr>
<td>5</td>
<td>7.73</td>
<td>52.16</td>
</tr>
<tr>
<td>6</td>
<td>68.39</td>
<td>9.69</td>
</tr>
</tbody>
</table>

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Table 4.8. Posterior c.o.v.s of mode shape (%)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>Mean</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.11</td>
<td>0.13</td>
<td>0.14</td>
<td>0.07</td>
<td>0.08</td>
<td>0.11</td>
<td>0.12</td>
<td>0.20</td>
<td>0.29</td>
<td>0.14</td>
</tr>
<tr>
<td>2</td>
<td>1.44</td>
<td>1.12</td>
<td>1.09</td>
<td>2.15</td>
<td>2.07</td>
<td>1.45</td>
<td>1.19</td>
<td>1.12</td>
<td>1.57</td>
<td>1.46</td>
</tr>
<tr>
<td>3</td>
<td>1.84</td>
<td>1.69</td>
<td>0.69</td>
<td>0.87</td>
<td>0.98</td>
<td>1.23</td>
<td>1.31</td>
<td>2.18</td>
<td>2.15</td>
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<td>4</td>
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<td>3.87</td>
<td>2.10</td>
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<td>1.80</td>
<td>2.08</td>
<td>2.15</td>
<td>4.07</td>
<td>2.67</td>
</tr>
<tr>
<td>5</td>
<td>2.26</td>
<td>1.42</td>
<td>2.58</td>
<td>1.48</td>
<td>1.36</td>
<td>1.29</td>
<td>0.87</td>
<td>0.36</td>
<td>1.03</td>
<td>1.41</td>
</tr>
<tr>
<td>6</td>
<td>2.96</td>
<td>0.56</td>
<td>1.85</td>
<td>1.04</td>
<td>1.92</td>
<td>1.10</td>
<td>1.92</td>
<td>0.77</td>
<td>2.40</td>
<td>1.61</td>
</tr>
</tbody>
</table>

Figures 4.12 and 4.13 show variation of modal parameters across different setups. The values of modal parameters are represented by red dots with error bars spanning ±2 posterior standard derivations. It is found that the frequency slightly varies with setups, while the damping ratio has a larger variation. On the other hand, frequency has a much lower posterior c.o.v. compared to the damping ratio. In short, both identified frequency and damping ratio are consistent with each other among all the setups, demonstrating the robustness of the proposed method.
4.4.2.3 Modal tracking

The proposed method is applied to a one-year monitoring project from 11 November 1997 to 11 September 1998, including evaluating the environmental effect on dynamic properties and short-term progressive damage detection. A total of 49 sensors were deployed to capture environmental factors that affects structural behavior such as air/soil temperature, humidity, and wind speed. Besides, eight accelerometers were used to measure structural response every hour. One can found a thorough overview of the project in Peeters and Roeck’s work (Peeters, 2001). The same procedures as the former data analysis are applied for modal tracking.
As shown in Figures 4.14 and 4.15, the first four frequencies and damping ratios are identified using the data from 11/Nov 1997 to 24/Apr 1998, before the damage was artificially introduced. Solid black lines indicate frequency estimates, and grey areas cover ±2 standard derivations. It is noted that there are some gaps during the measurement, mainly because the monitoring system was not operating, or ambient excitation is insufficient to identify modes (especially when there is not much traffic at night). A close observation on Figure 4.14 reveals that frequencies have significant variation due to temperature change, especially at day 75-95 (see a range of red dashed lines), reaching the largest peak. This large variation is highly associated with the period of very cold temperature around −2°C− −8°C on the asphalt layer of the bridge deck, yielding increasement in stiffness and nonlinear relation between frequency and temperature, and change in boundary condition (Peeters, 2001, Worden and Cross, 2018). Figure 4.15 shows the time history of the damping ratio at the same period. It is observed that damping ratios
also vary with time, while fluctuations seem to be more stable; no remarkable peaks appear, implying in the Z24 bridge case, damping ratios are less sensitive to temperature change compared to frequencies.

![Graphs showing time history of damping ratio](image)

**Figure 4.15.** Time history of damping ratio (from November 11, 1997 to April 24, 1998)

Regarding uncertainties in natural frequencies and damping ratios, which here are represented by shaded areas in Figures 4.14 and 4.15. It clearly shows that identified damping ratios have larger uncertainties, indicating relatively low reliability and accuracy in damping ratios. The general consensus is that the damping ratio is more difficult to measure and correlate its variability with external influence parameter, e.g., temperature. These attributes restrict damage detection purpose by damping ratio analysis (Cigada et al., 2008). In contrast, frequencies are identified more accurately with smaller uncertainties, they are often strongly correlated with temperature. Frequency abnormality and stiffness or boundary condition changes due to temperature are successfully detected (red dashed lines in Figure 4.14). Finally, the proposed method enables to deal with a vast of data to
perform reliability analysis of parameter identification and detect modal parameters’ abnormality due to environmental change.

To perform the reliability analysis, the proposed method is applied to different damage scenarios (See Table 3.5). The short-term progressive damage test contains nine individual measurement setups for each damage scenario (a totally 72 datasets). The probabilistic damage detection is carried out using tracking frequencies. The MPVs of identified frequencies and associated uncertainties are used together to quantify the probability given a specific percentage of frequency shift, \(d\) (as decimals), compared with its healthy condition (measurement No. 1). Based on asymptotic Gaussian approximation, the probability of occurrence by measuring the shift of the \(l\)th modal frequency can be given by (Beck and Katafygiotis, 1998):

\[
P_{l}^{dam}(d) = P(\theta_{l}^{pd} < (1 - d)\theta_{l}^{ud}) = \int_{-\infty}^{\infty} P(\theta_{l}^{pd} < (1 - d)\theta_{l}^{ud} | \theta_{l}^{ud}) p(\theta_{l}^{ud}) d\theta_{l}^{ud}
\approx \Phi \left[ \frac{(1 - d)\theta_{l}^{*ud} - \theta_{l}^{*pd}}{\sqrt{(1 - d)^2 (\sigma_{l}^{ud})^2 + (\sigma_{l}^{pd})^2}} \right]
\]

(4.7)

where \(\Phi(\cdot)\) represents the standard Gaussian cumulative distribution function for random variables; \(\theta_{l}\) and \(\sigma_{l}\) represents the \(l^{th}\) modal frequency estimate and its standard derivation, respectively. Superscripts, \(ud\) and \(pd\), represents undamaged and possibly damaged structural state, respectively. \(d\) in Eq. (7) is defined as \((\theta_{l}^{ud} - \theta_{l}^{pd})/\theta_{l}^{ud}\).

The proposed method is then applied to automatically identify modal parameters. The tracked frequencies, damping ratios, and associated posterior uncertainties are plotted in Figure 4.16 (i series), with dots representing sample means and error bars covering two
averages of posterior standard derivations among all measurement setups. It is observed that the maximum frequency happened at scenario No. 1 (undamaged condition); the minimum frequency happened at damage scenario No. 6 for mode 1, 3, 4, and 5, damage scenario No. 7 for mode 2 and 6, indicating 95 mm settlement of pier and tilted foundation have significant effect on structural dynamic behavior of the Z24 bridge. Figure 4.16 (ii series) shows the probability of occurrence with respect to the frequency reduction due to damage; x and y axis is the percentage of frequency reduction (denoted as $d$ in Eq. (4.7)) and its occurrence probability, respectively. It is understandable based on Eq. (4.7) that the probability of occurrence is 1 indicates that the x-value of the frequency reduction is always reached with 100% probability; the x-value now is the minimum percentage of frequency change (lower bound) due to the certain damage types (herein, settlement or tilt of foundation). On the other hand, the probability of occurrence is 0 indicates that the x-value of the frequency reduction not feasible in a probabilistic estimation; the x-value now is the maximum percentage of frequency change (upper bound) due to damage. The visually separated curve from other groups of curves indicates that certain types of damage can be detectable by measuring the shift of frequencies. The bounds are meaningful to discuss the probability of damage detection given certain damage types.
Figure 4.16. Identified frequencies: frequency evolution (i series); the probability of frequency reduction (ii series)
Figure 4.16. (Continued) Identified frequencies: frequency evolution (i sereis); the probability of frequency reduction (ii sereis)

It is not surprising that 80-mm and 95-mm pier settlement, and tilted foundation have noticeable impair on frequency. For example, in the mode No. 1 (a-ii), the probability of occurrence with respect to the frequency reduction at 80-mm and 95-mm pier settlement
exhibit high probabilities (55% and 80%, respectively) when measuring a possible frequency reduction of 4% and 6% (herein \(d\) in Eq. (4.7)). In the mode No. 2 (b-ii), frequencies due to 95-mm pier settlement and tilted foundation have a possible frequency reduction of 4% and 9% with a high probability of 60% and 71%, respectively. Similar analysis for mode No. 3-No. 6 is performed.

Some interesting observations are also found in Figure 4.16 (ii series). In mode No. 1, percentage of frequency reduction within the range of (2.6%, 5%) and (5.8%, 8%) may be attributed to 80-mm and 95-mm settlement of pier, respectively (see dashed box in Figure 4.16 (a-ii)). It is expected that bounds of frequency shift due to 95 mm settlement are located at the right of that due to 80-mm settlement, since more severe pier settlement is detected. For mode No. 2, only the curve due to tilted foundation is clearly separated from others; frequency changes from 6.8% to 12.4% may result from the tilt of foundation (see dashed box in Figure 4.16 (b-ii)). While the curves due to 80-mm and 95-mm settlement are apparently different from other curves for mode No, 3, they have a certain overlapped range of frequency reduction (see dashed box in Figure 4.16 (c-ii)). Hence it may be concluded that frequency shift within the bounds of (5.6%, 10%) can be explained by either 80-mm or 95-mm pier settlement. In contrast to the first three modes, the curves are relatively not distinguishable for the rest of modes. Therefore, it is not easy to decide the bounds of frequency shift due to individual damage scenario, probably because higher modes have higher uncertainties. In short, the proposed method enables to identify the bounds of frequency shift due to damage. Therefore, possible causes of damage can be determined by observing whether the actual frequency change lies in the bounds or not.
Figure 4.17 shows the variability of damping ratio is smaller than that of frequency, implying that damping ratio is not sensitive to global damage scenarios in Table 4.9. The practical application for analyzing a large amount of data indicates that the proposed method represents a useful tool to offer robust and feasible health monitoring and modal tracking with the minimum human intervention. In addition, modal information (parameter estimates and uncertainties) using the proposed automated Bayesian method are used for the reliability analysis to detect certain type of damages.

![Graphs of damping ratio evolution at different damage scenarios](image)

**Figure 4.17.** Damping ratio evolution at different damage scenarios

### 4.4.3 Practical aspects

The proposed method provides an automation technique incorporating the fast Bayesian FFT modal identification. This is the first attempt to automatically perform modal
identification using a Bayesian approach. The method addresses two challenges on the operating of conventional Bayesian modal identification: the selection of initial frequency and bandwidth. In addition to common situation (well-separated modes with a moderate modal SNR), the weakly excited modes and closely spaced modes are also successfully identified on both numerical and field test examples. It should be mentioned that selecting initial frequency involves SSI based automated interpretation of the stabilization diagram. Frequency representatives of each cluster can then be considered as final modal estimates. In other words, the initial frequency used in Bayesian approach can also be seen as representative frequency estimates for observed structure.

Another remark is that the proposed method requires to perform the Bayesian modal identification several times to select effective bandwidth factors, which in turn increase computational cost. However, the time consumption is still acceptable in practice, since performing the fast Bayesian FFT is highly efficient (Au, 2012b). For example, in the application of Z24 bridge, it takes around 30 seconds for one mode identification, illustrating that the proposed method is still promising for automated modal analysis even for field data. Additionally, the proposed method has been applied to the one-year health monitoring project on the Z24 bridge, including evaluation of environmental effect on dynamic properties and short-term progressive damage test, results demonstrate that the proposed method has potential and feasibility for automated Bayesian modal identification and modal tracking in long-term health monitoring.

4.5 Conclusions

In this chapter, a two-step automation technique has been developed to incorporate Bayesian modal identification. A numerical example has been used for validating the
proposed method and field test on Z24 benchmark bridge with closely spaced and weakly excited modes has demonstrated the capability of the method, especially for the application in long-term data analysis. The feasible application of reliability analysis is demonstrated to detect the certain types of damage by tracking the frequencies using the proposed method. Another originality of this study is to demonstrate the reliability of automated Bayesian modal identification using long-term data to detect the certain types of damage in probabilistic manner.

Overall, the main conclusions and contributions of this study are summarized as follows:

- Compared to traditional Bayesian modal identification, initial frequency and frequency bandwidth are automatically determined, requiring minimal human interference to achieve sufficient accuracy.
- With the proposed method, a large number of measurements can be automatically treated without any loss of physical modes of interest. The evolution of modal parameters and abnormality due to environmental change can be detected.
- Modal parameters and uncertainties are automatically calculated, more conveniently serving reliability analysis and probabilistic damage detection, such as measuring the accuracy of parameter identification, providing possible early warning of certain damage type given identified bounds of frequency shift. To the best of authors’ knowledge, it is the first attempt to directly investigate the capability of Bayesian modal identification in reliability and damage detection.
- Based on the current study, the modal parameters, particularly natural frequencies are the most appropriate index to track abnormality in long-term SHM and perform the reliability analysis for identifying the certain types of damage.
Compared to automated SSI in Chapter 3, automated Bayesian modal identification makes full use of data information and strictly obeys physical modeling assumption in modal identification, e.g., measured data is directly analyzed by FFT rather than transforming to mathematical matrices in SSI. Furthermore, Bayesian method identifies additional two parameters, e.g., modal force and prediction error, the former is a measure of excitation level, their combination gives SRN, which enhances the reliability and provides more valuable information in OMA.
CHAPTER 5

BAYESIAN MODEL UPDATING WITH ASYMPTOTIC OPTIMIZATION METHOD

5.1 Introduction

Traditional Bayesian model updating approach (BMUA) has been considered as a promising and reliable model updating tool, also has many satisfactory practical experiences. Traditional BMUA adopts the classical characteristic equation:

\[(K - \lambda M)\phi = 0\]  \hspace{1cm} (5.1)

where \(M\) and \(K\) are system mass matrix and system stiffness matrix, respectively. \(\lambda\) and \(\phi\) are eigenvalues and eigenvectors, respectively. Eq. 5.1 shows that \(M\) and \(K\) are naturally coupled, updating both mass and stiffness causes an unidentifiable problem that yields infinite combinations of mass and stiffness with the same frequency (Zeng and Kim, 2020). To avoid such the coupling effect of mass and stiffness, traditional BMUA take a common assumption that the mass is known/well estimated or invariant due to possible damage to only update stiffness, believing mass is less critical. However, this assumption is questionable, especially relatively a large mass change occurs. Results in stiffness updating may be erroneous if keeping using invariable mass value.

Very few works have attempted to update both mass and stiffness. Das and Debnath (2018) proposed a BMUA combining normal and lognormal probability distribution to update both mass and stiffness. However, the coupling effect still remains. Cheung and Bansal (2017) applied the Gibbs sampling method using complex data to identify mass and
stiffness, but mass properties are well estimated with small variance. Mustafa and Matsumoto (2017) proposed the formulations for updating mass and stiffness simultaneously using the Bayesian approach, but the mass is still known well in the application to a truss bridge. Previous research still pertains to the challenges in updating both mass and stiffness simultaneously without the coupling effect of mass and stiffness.

Although the coupling effect of mass and stiffness is addressed by some researchers, uncertainties are ignored or poorly considered. Xu et al. (2018) proposed a time-domain nonlinear restoring force to identify mass and stiffness; however, an external force is required. Zhang and Li (2017) presented a loop substructure identification method for mass and stiffness, while mass at sensor location should be known. Do and Gül (2020) established a time series based model to identify mass and stiffness features. Lei et al. (2020) employed an extended Kalman filter (EKF) to determine the mass-stiffness coupled coefficient using incomplete measured data. Nevertheless, these model updating approaches cannot quantify the uncertainties of model parameters. Ding et al. (2019) proposed an evolutionary-based model updating approach to simultaneously update mass and stiffness parameters, while only partial uncertainty due to measurement noise was available. More efforts to manipulate mass and stiffness’s coupling effect and identify both mass and stiffness along with their uncertainties are greatly demanding from the practical point of view.

In this chapter, the proposed BMUA considers mass and stiffness as equivalently important and attempts to inherently address the coupling effect of mass and stiffness. The uncertainties of structural parameters arising from modeling error and measurement error are also quantified. The proposed BMUA updates mass and stiffness using output-only
vibration data. The new eigen-equations are reformulated by two measured data acquired from the original system and modified system with mass addition or stiffness addition to address the coupling effect of mass and stiffness, also giving the new prior probability density function (PDF). The objective functions are obtained by taking the negative logarithm of the posterior PDF to circumvent complex integrals. An asymptotic approximation method is then adopted to derive analytical formulations of optimal model parameters, associated uncertainty is also quantified by inverse Hessian matrix of objective function. Finally, modal parameters, e.g., frequency and mode shape, and structural parameters, e.g., mass and stiffness, are updated iteratively.

The modified system can be created by either adding mass ($\Delta m$) or adding stiffness ($\Delta k$). In the case of modified system with $\Delta m$, in a real-world setting, this can be achieved by a practical addition such as a moving truck loads or artificial dead loads on the bridge structure (Tian et al., 2019a) or adding stationary weights on buildings, which often considered in the seismic design practice (Paz and Kim, 2019). In the case of modified system with $\Delta k$, the modified system with attaching additional components was widely used for stiffness enhancement. For example, springs were attached to cantilever beam to achieve stiffness change for identification of scaling factors (Khatibi et al., 2012, López-Aenlle et al., 2012). Curved dampers (CDs) and fluids viscous dampers (FVDs) were installed in building structure to improve initial stiffness. In addition, some specially made braces allow structural system to have better seismic bearing ability, as stiffness is greatly improved.

This chapter is organized as follows: the theoretical background of BMUA and parameterization of mass and stiffness matrices are first described in Section 5.2.
Subsequently, Section 5.3 gives the way to calculate the probability of damage occurrence in Bayesian updating framework. BMUA with mass addition and stiffness addition are presented in Section 5.4 and Section 5.5, respectively, including new-eigen equations, analytical formulation of optimal parameters, and associated uncertainties. The performance of proposed BMUA for structural identification and damage detection is evaluated through two numerical examples: a six-story shear building and a three-dimensional three-story braced frame. Finally, conclusions and summaries are presented in Section 5.6.

5.2 Theoretical background of BMUA

Applying the classic Bayes' theorem, prior distribution function, and likelihood function are integrated to form the posterior PDF, given measured data. Thus, the posterior PDF is written as (Yuen, 2010):

\[ p(\Omega|D, C) = \frac{p(D|\Omega, C) \cdot p(\Omega|C)}{p(D|C)} \]  \hspace{1cm} (5.2)

where \( C \) is a model class which represents patterns of a structural model, \( \Omega \) is the vector of parameters considered in Bayesian updating process, and \( D \) is measured data. \( p(\Omega|C) \) is the prior PDF of \( \Omega \) depending on engineering judgment, \( p(D|\Omega, C) \) is called a likelihood function, reflecting the likelihood of observing measured data \( D \), when the model is characterized by parameters \( \Omega \). The denominator in Eq. (5.2), \( p(D|C) \), is a normalizing constant to ensure the posterior PDF is integrated into unity over parameter space. To simplify Eq. (5.2), the constant is denoted as \( c_0 \) in the rest of this paper. \( p(\Omega|D, C) \) represents the posterior PDF given the measurement and defined model class in advance. In this study, measured data is taken as measured eigenvalue (square of frequency) and
mode shapes for model updating. $\bm{\Omega}$ is structural physical parameters, including mass and stiffness parameters in this study. Therefore, Eq. (5.2) is reformulated:

$$
\begin{align*}
p(\lambda, \phi, \Omega | \hat{\lambda}, \hat{\phi}, C) &= c_0 p(\hat{\lambda}, \hat{\phi} | \lambda, \phi, \Omega, C) p(\lambda, \phi | \Omega, C) p(\Omega | C) \\
&= c_0 p(\hat{\lambda}, \hat{\phi} | \lambda, \phi) p(\lambda, \phi | \Omega, C) p(\Omega | C)
\end{align*}
$$

(5.3)

where $\lambda$ are updated eigenvalues; $\phi$ are updated mode shapes; $\Omega$ are certain critical parameters to be updated. $\hat{\lambda}$ are measured eigenvalues; $\hat{\phi}$ are measured mode shapes. The MPVs of updated parameters can be explored by means of maximizing posterior PDF. Procedures and Formulations in detail are presented as follows.

A linear structural model can be parameterized by model parameters based on Degree-of-freedoms (DOFs), $N_d$, and defined model class, $C$. A commonly used parameterization of stiffness matrix, $K(\theta)$, and mass matrix, $M(\beta)$, could be described as (Mustafa and Matsumoto, 2017):

$$

K(\theta) = K_0 + \sum_{l=1}^{N_\theta} \theta_l K_l \quad M(\beta) = M_0 + \sum_{m=1}^{N_\beta} \beta_m M_m
$$

(5.4)

where $\theta = [\theta_1, \theta_2, \cdots, \theta_{N_\theta}]^T$ are stiffness parameters vector; $\beta = [\beta_1, \beta_2, \cdots, \beta_{N_\beta}]^T$ are mass parameters vector. The $l$th stiffness parameter forms the $l$th elemental stiffness matrix, $K_l = \partial K / \partial \theta_l$; similarly, the $m$th mass parameter forms the $m$th elemental mass matrix, $M_m = \partial M / \partial \beta_m$. In the proposed updating framework, $\theta_l$ and $\beta_m$ will be updated to match the FEM model with the real structural model using measured data. Note that $K_0$ and $M_0$ in Eq. (5.4) are defined as constant matrices that are not dependent on model parameters. In this study, $K_0$ and $M_0$ are set as zero for the sake of convenient.
5.3 Probabilistic damage detection

The application of FE model updating directly allows a damage assessment: damage detection and quantification. To evaluate damage in stiffness and mass, the probability of damage is considered in terms of reduction of mass/stiffness parameters by a fractional level, \( d \), compared to its intact state. This probability can be computed using updated parameters and corresponding standard deviations based on asymptotic Gaussian Approximation as (Mustafa and Matsumoto, 2017, Das and Debnath, 2020):

\[
P_l^{\text{dam}}(d) = P\left( \Omega_l^{\text{ud}} < (1-d)\Omega_l^{\text{pd}} | \mathcal{C} \right) = \int_{-\infty}^{\infty} P\left( \Omega_l^{\text{ud}} < (1-d)\Omega_l^{\text{pd}} \right) p(\Omega_l^{\text{ud}} | \mathcal{C}) d\Omega_l^{\text{ud}} \\
\approx \Phi \left[ \frac{(1-d)\Omega_l^{\text{ud}} - \Omega_l^{\text{pd}}}{\sqrt{(1-d)^2(\sigma_l^{\text{ud}})^2 + (\sigma_l^{\text{pd}})^2}} \right]
\]

where \( \Phi(\cdot) \) represents the cumulative distribution function of the standard Gaussian random variable, \( \Omega_l^{\text{ud}} \) and \( \Omega_l^{\text{pd}} \) denote the most probable values of the \( l \)th mass/stiffness parameters for the intact and (possibly) damaged structures, respectively. Further, \( \sigma_l^{\text{ud}} \) and \( \sigma_l^{\text{pd}} \) are corresponding standard deviations.

5.4 Bayesian model updating with added mass \( \Delta m \)

In this section, the Bayesian model updating framework with added mass \( \Delta m \) is proposed to simultaneously update mass and stiffness. The new eigen-equations are derived to address the coupling effect of mass and stiffness. The posterior PDF is reformulated incorporating \( \Delta m \). The optimal parameters are determined by asymptotic optimization method. It should be noted that the subheading with \( \Delta m \) or ‘new’ indicates
the presented equations and formulations in this section are originally derived by authors, otherwise, references are cited accordingly.

5.4.1 Formulation of new eigen-equations with added mass $\Delta m$

Traditional Bayesian updating approach uses Eq. (5.1) as an eigen-equation to control modeling error. In the proposed Bayesian updating framework, the modified system is firstly created by adding known mass ($\Delta m$), then a new eigen-equation is introduced using two sets of measured data from unmodified system and modified system. Finally, the coupling effect of stiffness and mass is eliminated; the detailed formulation is presented as follows.

Considering an original system and modified system which is added mass, $\Delta m$, to the system, based on fundamental dynamic equations, we obtain:

$$K\phi = M\phi \lambda$$  \hspace{1cm} (5.6)

$$K\phi' = (M + \Delta m)\phi' \lambda'$$  \hspace{1cm} (5.7)

where $\lambda$ is an eigenvalue (square of frequency), and $\phi$ is an eigenvector (mode shape) before adding mass to structure; $\lambda'$, is an eigenvalue and $\phi'$ is an eigenvector after adding mass to a structure.

Premultiplying Eq. (5.7) by $\phi^T$, we have:

$$\phi^T K\phi' = \phi^T (M + \Delta m)\phi' \lambda'$$  \hspace{1cm} (5.8)

Taking the transposed matrix of Eq. (5.6) and postmultiplying the resulting matrix equation by $\phi'$,

$$\phi^T K\phi' = \lambda\phi^T M\phi'$$  \hspace{1cm} (5.9)

Subtracting Eq. (5.9) from Eq. (5.8), the following equation can be expressed as:

$$\lambda\phi^T M\phi' - \lambda'\phi^T M\phi' = \lambda'\phi^T \Delta m\phi'$$  \hspace{1cm} (5.10)
Let \( P = \phi^T M \phi' \) and \( Q = \lambda' \phi^T \Delta m \phi' \), then Eq. (5.10) may be simplified into:

\[
\lambda P - \lambda' P = Q \tag{5.11}
\]

From Eq. (5.11), \( P \) can be solved as a new term, \( P' \)

\[
P' = (\lambda - \lambda')^{-1} Q \tag{5.12}
\]

Then a new eigen-equation error for mass updating, \( ME_m \), can be expressed as:

\[
ME_m = P' - P = (\lambda - \lambda')^{-1} \lambda' \phi^T \Delta m \phi' - \phi^T M \phi' = 0 \tag{5.13}
\]

Similar procedures when updating stiffness can be performed.

Premultiplying Eq. (5.6) by \( \lambda^{-1} \), we have:

\[
\lambda^{-1} \phi^T K \phi' = \phi^T M \phi' \tag{5.14}
\]

Postmultiplying Eq. (5.8) by \( \lambda'^{-1} \), we obtain:

\[
\phi^T K \phi' \lambda'^{-1} = \phi^T (M + \Delta m) \phi' \tag{5.15}
\]

Subtracting Eq. (5.14) from Eq. (5.15), we obtain:

\[
\phi^T K \phi' \lambda'^{-1} - \lambda^{-1} \phi^T K \phi' = \phi^T \Delta m \phi' \tag{5.16}
\]

Let \( S = \phi^T \Delta m \phi' \) and \( U = \phi^T K \phi' \), then Eq. (5.16) can be simplified as:

\[
U \lambda'^{-1} - \lambda^{-1} U = S \tag{5.17}
\]

Therefore, \( U \) is expressed as a new term, \( U' \):

\[
U' = (\lambda'^{-1} - \lambda^{-1})^{-1} S \tag{5.18}
\]

Then a new eigen-equation error for stiffness updating, \( ME_k \), can be obtained:

\[
ME_k = U' - U = (\lambda'^{-1} - \lambda^{-1})^{-1} \phi^T \Delta m \phi' - \phi^T K \phi' \tag{5.19}
\]

The elimination of the coupling effect of stiffness and mass has been completed using Eqs. (5.13) and (5.19), because no stiffness information is required when using Eq. (5.13) to update mass; similarly, when using Eq. (5.19) to update stiffness. It may be mentioned that the location and quantity of added mass can be acceptable when meeting two basic
requirements in measured data: 1) there is obvious frequency shift after modification; 2) there is no significant mode shape change after modification (Brincker et al., 2004, Parloo et al., 2005, Fernández Fernández et al., 2007, López-Aenlle et al., 2010). Comprehensive instruction of constructing a modified structure, such as the magnitude of added mass, number of added mass and location of added mass, could be found in López-Aenlle et al. (2010). Further research to optimize the mass-change strategy and its uncertainty in the FEMU should be investigated in the lab and field environments.

5.4.2 Formulation of the new prior PDF with \( \Delta m \)

Assuming that \( N_m \leq N_d \) modes are measured. When updating mass, the prior PDF of all the unknown parameters is given by:

\[
p_m(\lambda, \phi, \beta | C) = p_m(\lambda, \phi | \beta, C) \cdot p_m(\beta | C) \tag{5.20}
\]

where \( \lambda = [\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N_m)}]^T \), \( \phi = [\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(N_m)}]^T \), and \( p_m(\lambda, \phi | \beta, C) \) is constructed by choosing Gaussian PDF as a probability model for the eigen-equation error.

\[
p_m(\lambda, \phi | \beta, C) = c_0 \exp \left[ -\frac{\| (\lambda - \lambda')^{-1} \lambda' \phi^T \Delta m \phi' - \phi^T M(\beta) \phi' \|^2}{2\sigma_{eq}^2} \right] \tag{5.21}
\]

where \( c_0 \) is normalizing constant, \( \sigma_{eq}^2 \) is preselected an eigen-equation error variance, \( \| . \| \) denotes the Euclidean norm of a vector. Also Eq. (5.21) can be simplified as:

\[
p_m(\lambda, \phi | \beta, C) = c_0 \exp \left[ -\frac{1}{2} J_{g,m}(\lambda, \phi; \beta) \right] \tag{5.22}
\]

where

\[
J_{g,m}(\lambda, \phi; \beta) = T_m^T \Sigma_{eq}^{-1} T_m \tag{5.23}
\]

where \( T_m = (\lambda^{(m)} - \lambda^{(m)}')^{-1} \lambda^{(m)} \phi^{(m)}^T \Delta m \phi^{(m)}' - \phi^{(m)}^T M(\beta) \phi^{(m)}' \), and \( \Sigma_{eq} = \sigma_{eq}^2 I \), is a prior covariance matrix, and \( I \) is the identity matrix. The term of \( \Sigma_{eq} \) provides
treatment for modeling error, since eigen-equation is never exact in practice. The \( p_m(\boldsymbol{\beta} | C) \) can be taken as a Gaussian distribution with \( \boldsymbol{\beta}^n \) representing the nominal values of mass parameters and with covariance matrix, \( \Sigma_\beta \). Defining \( \Sigma_\beta = \sigma_\beta^2 \mathbf{I} \), \( \sigma_\beta \) are chosen to be large variances. Therefore, \( p(\boldsymbol{\beta} | C) \) has the expression as:

\[
p_m(\boldsymbol{\beta} | C) = \exp \left[ - \frac{\| \boldsymbol{\beta} - \boldsymbol{\beta}^n \|^2}{2 \sigma_\beta^2} \right]
\]  

(5.24)

Finally, plugging Eq. (5.22) and (5.24) into Eq. (5.20), the prior PDF for mass updating is obtained:

\[
p_m(\lambda, \phi, \beta | C) = c_0 \exp \left[ - \frac{1}{2} \mathcal{J}_{gm}(\lambda, \phi; \beta) \right] \cdot \exp \left[ - \frac{\| \boldsymbol{\beta} - \boldsymbol{\beta}^n \|^2}{2 \sigma_\beta^2} \right]
\]  

(5.25)

5.4.3 Formulation of likelihood function

To construct a likelihood function, we firstly introduce a measurement error, \( \epsilon \):

\[
\begin{bmatrix}
\hat{\lambda} \\
\hat{\Psi}
\end{bmatrix} = \begin{bmatrix}
\lambda \\
L_0 \phi
\end{bmatrix} + \epsilon
\]  

(5.26)

where \( \epsilon \) is chosen as a Gaussian distribution with zero mean and covariance matrix, \( \Sigma_\epsilon \). \( \hat{\Psi} \) gives measured mode shapes and \( \hat{\lambda} \) gives corresponding measured eigenvalues from tested data. \( L_0 \) is a selection matrix of ‘1s’ or ‘0s’ used for mapping predicted mode shapes with their observed counterparts. Accordingly, the likelihood function of mass updating can be expressed as in Eq. (5.27):

\[
p_m(\hat{\lambda}, \hat{\Psi} | \lambda, \phi, \beta, C) = p_m(\hat{\lambda}, \hat{\Psi} | \lambda, \phi) = \exp \left[ - \frac{\| \hat{\lambda} - \lambda \|^2}{2 \Sigma_\epsilon} \right]
\]  

(5.27)
It is easy to find from Eq. (5.27) that the likelihood function finally becomes the form of a Gaussian distribution with mean \([\lambda^T, (L_0 \phi)^T]^T\) and covariance matrix, \(\Sigma_e\).

### 5.4.4 Formulation of the new posterior PDF with \(\Delta m\)

The posterior PDF consists of a prior PDF and a likelihood function, as shown in Eq. (5.3). Plugging Eq. (5.25) and (5.27) to Eq. (5.3), the posterior PDF of mass updating may be rewritten as:

\[
p_m(\lambda, \phi, \beta | \hat{\lambda}, \hat{\psi}, \mathcal{C}) = c_0 \exp \left[-\frac{1}{2} \left\| \frac{\hat{\lambda} - \lambda}{\Sigma_e} \right\|_2^2 - \frac{1}{2} J_{gm}(\lambda, \phi; \beta) - \frac{\| \beta - \beta^\eta \|_2^2}{2 \sigma_{\beta}^2} \right] \tag{5.28}
\]

The most probable values of the unknown parameters can be found by maximizing this PDF. The objective function is defined by taking a negative logarithm of a posterior PDF without including the constant that does not depend on the uncertain parameters. Then the objective function is minimized instead of maximizing posterior PDF. The objective function, including known added mass, is given by Eq. (5.29):

\[
J_m(\lambda, \phi, \beta) = \frac{1}{2} (\beta - \beta^\eta)^T \Sigma_{\beta}^{-1} (\beta - \beta^\eta)
\]

\[
+ \frac{1}{2 \sigma_{\delta}^2} \sum_{m=1}^{N_m} \left\| (\lambda^{(m)} - \lambda^{(m)})^{-1} \lambda^{(m)} \phi^{(m)} \Delta m \phi^{(m)} - \phi^{(m)^T} M(\beta) \phi^{(m)} \right\|_2^2
\]

\[
+ \frac{1}{2} \left[ \frac{\hat{\lambda}}{\hat{\psi}} - L_0 \phi \right]^T \Sigma_e^{-1} \left[ \frac{\hat{\lambda}}{\hat{\psi}} - L_0 \phi \right] \tag{5.29}
\]

Similarly, when using the same procedures, the prior PDF for the stiffness updating is obtained:

\[
p_k(\lambda, \phi, \theta | \mathcal{C}) = c_0 \exp \left[-\frac{1}{2} J_{g,k}(\lambda, \phi; \theta) \cdot \exp \left[-\frac{(\theta - \theta^\eta)^2}{2 \sigma_{\theta}^2} \right] \right] \tag{5.30}
\]

where \(\lambda\) and \(\phi\) are updated eigenvalues and eigenvector, respectively. \(\theta\) is updated stiffness parameters, \(\mathcal{C}\) is defined earlier, and \(J_{g,k}(\lambda, \phi; \theta)\) is defined as:
\[
J_{g,k}(\lambda, \phi; \theta) = c_0 T_k^T \Sigma_{eq}^{-1} T_k \tag{5.31}
\]

where \( T_k = \left( \lambda^{(m)-1} - \lambda^{(m)-1} \right)^{-1} \phi^{(m)} \Delta m \phi^{(m)} - \phi^{(m)} K(\theta) \phi^{(m)}. \)

The likelihood function is shown as:

\[
p_k(\lambda, \bar{\psi} | \lambda, \phi, \theta, C) = p(\lambda, \bar{\psi} | \lambda, \phi) = \exp \left[ -\frac{\left\| \bar{\lambda} \bar{\psi} - L_0 \phi \right\|^2}{2 \Sigma_e} \right] \tag{5.32}
\]

The objective function of stiffness updating with added mass is represented as in Eq. (5.33):

\[
J_k(\lambda, \phi, \theta) = \frac{1}{2} (\theta - \theta^n)^T \Sigma_{\theta}^{-1} (\theta - \theta^n) + \frac{1}{2} \sum_{m=1}^{N_m} \left\| (\lambda^{(m)-1} - \lambda^{(m)-1})^{-1} \phi^{(m)} \Delta m \phi^{(m)} - \phi^{(m)} K(\theta) \phi^{(m)} \right\|^2
\]

\[
+ \frac{1}{2} \left[ \frac{\bar{\lambda}}{\bar{\psi}} - \frac{\lambda}{\phi} \right]^T \Sigma_e^{-1} \left[ \frac{\bar{\lambda}}{\bar{\psi}} - \frac{\lambda}{\phi} \right] \tag{5.33}
\]

5.4.5 Optimization framework with \( \Delta m \)

The modal parameters and structural parameters are updated by minimizing objective functions in Eq. (5.29) and (5.33). It may be mentioned that these objective functions are quadratic with respect to \( \lambda, \phi, \beta, \) or \( \theta \) if the other two parameters are fixed. Then, the partial derivatives of the objective function with respect to updated parameters \( (\lambda, \phi, \beta, \theta) \) are considered to be zero. The sign of * in the following sections represents updated values. By minimizing the objective function, \( J_m(\lambda, \phi, \beta) \) in Eq. (5.29) with respect to \( \phi \), the optimal vector of \( \phi^*_m \) can be obtained:

\[
\phi^*_m = \left[ s_{eq}^{-2} G_{\phi,m}^T G_{\phi,m} + L_0^T (\Sigma_e^{-1})_{22} L_0 \right]^{-1} L_0^T \left[ (\Sigma_e^{-1})_{21} (\bar{\lambda} - \bar{\lambda}^*) + (\Sigma_e^{-1})_{22} \bar{\psi} \right] \tag{5.34}
\]

where \((\Sigma_e^{-1})_{21}\) and \((\Sigma_e^{-1})_{22}\) are referred to left bottom and right bottom sub-matrices of \( \Sigma_e^{-1} \), respectively. Symmetric matrix, \( G_{\phi,m} \), is given by:
\[ \mathbf{G}_{\phi_m} = \text{diag}(\mathbf{\phi}^{(N_m)^T} (\mathbf{\lambda}^{(N_m)^*} - \mathbf{\lambda}^{(N_m)^*})^{-1} \mathbf{\lambda}^{(N_m)^*} \Delta \mathbf{m} - \mathbf{M}^*) \]_{N_m \times N_d N_m} \tag{5.35} \]

where \( \mathbf{M}^* = \mathbf{M} (\mathbf{\beta}) \) is a mass-parameterized system matrix. The symbol ‘\( \text{diag} \)’ represents a diagonal matrix.

Similarly, by minimizing the objective function \( J_m (\mathbf{\lambda}, \mathbf{\phi}, \mathbf{\beta}) \) in Eq. (5.29) with respect to \( \mathbf{\lambda} \), the optimal vector of \( \mathbf{\lambda}^* \) can be obtained:

\[
\mathbf{\lambda}^* = \left( \frac{\sigma_{eq}^2 \mathbf{G}_{\lambda_m} + (\Sigma_{\epsilon^{-1}})^{11}}{\sigma_{eq}^2 \mathbf{G}_{\lambda_m} + (\Sigma_{\epsilon^{-1}})^{11}} \right)^{-1} \left( \frac{\sigma_{eq}^2 \mathbf{G}_{\lambda_m} + (\Sigma_{\epsilon^{-1}})^{11}(\Psi - \mathbf{L}_0 \mathbf{\phi}^*)}{\sigma_{eq}^2 \mathbf{G}_{\lambda_m} + (\Sigma_{\epsilon^{-1}})^{11}(\Psi - \mathbf{L}_0 \mathbf{\phi}^*)} \right)^{-1}
\tag{5.36} \]

where \((\Sigma_{\epsilon^{-1}})^{11}\) and \((\Sigma_{\epsilon^{-1}})^{12}\) are the left top and right top of sub-matrices of \( \Sigma_{\epsilon^{-1}} \). Symmetric matrix, \( \mathbf{G}_{\lambda_m} \) is obtained as:

\[
\mathbf{G}_{\lambda_m} = \sigma_{eq}^2 \text{diag}(\mathbf{\phi}^{(N_m)^T} \mathbf{M}^* \mathbf{\phi}^{(N_m)^*}) \]_{N_m \times N_m} \tag{5.37} \]

By minimizing the objective function \( J_m (\mathbf{\lambda}, \mathbf{\phi}, \mathbf{\beta}) \) in Eq. (5.29) with respect to \( \mathbf{\beta} \), the optimal vector \( \mathbf{\beta}^* \) can be obtained:

\[
\mathbf{\beta}^* = \left( \frac{\sigma_{eq}^2 \mathbf{G}_{\beta} \mathbf{G}_{\beta}^T \mathbf{b}_m + \Sigma_{\beta^{-1}}}{\sigma_{eq}^2 \mathbf{G}_{\beta} \mathbf{G}_{\beta}^T \mathbf{b}_m + \Sigma_{\beta^{-1}}} \right)^{-1} (\sigma_{eq}^2 \mathbf{G}_{\beta} \mathbf{b}_m + \Sigma_{\beta^{-1}} \mathbf{\beta}^*)
\tag{5.38} \]

where the matrix \( \mathbf{G}_{\beta} \) and vector \( \mathbf{b}_m \) are represented as in Eq. (5.39) and Eq. (5.40),

\[
\mathbf{G}_{\beta} = \left[ \mathbf{\phi}^{(N_m)^T} \mathbf{M}_1 \mathbf{\phi}^{(N_m)^*} \ldots \mathbf{\phi}^{(N_m)^T} \mathbf{M}_{N_\beta} \mathbf{\phi}^{(N_m)^*} \right]_{N_m \times N_\beta}
\tag{5.39} \]

Where

\[
\mathbf{b}_m = \left[ \left( \mathbf{\lambda}^{(N_m)^*} - \mathbf{\lambda}^{(N_m)^*} \right)^{-1} \mathbf{\phi}^{(N_m)^*} \mathbf{\phi}^{(N_m)^*} \mathbf{\phi}^{(N_m)^*} \mathbf{\phi}^{(N_m)^*} \mathbf{\phi}^{(N_m)^*} \right]_{N_d N_m \times 1}
\tag{5.40} \]

When it comes to stiffness updating, by minimizing the objective function \( J_k (\mathbf{\lambda}, \mathbf{\phi}, \mathbf{\theta}) \) in Eq. (5.33) with respect to \( \mathbf{\lambda}, \mathbf{\phi}, \mathbf{\theta} \), the optimal vector of \( \mathbf{\phi}_k^* \) is given by:

\[
\mathbf{\phi}_k^* = \left[ \sigma_{eq}^2 \mathbf{G}_{\phi,k} \mathbf{G}_{\phi,k}^T \mathbf{L}_0^T (\mathbf{\Sigma}_{\epsilon^{-1}})_{22} \mathbf{L}_0 \right]^{-1} \mathbf{L}_0^T (\mathbf{\Sigma}_{\epsilon^{-1}})_{21} (\mathbf{\lambda}^* - \mathbf{\lambda}^*) + (\mathbf{\Sigma}_{\epsilon^{-1}})_{22} \mathbf{\psi}
\tag{5.41} \]

where
where $K^* = K(\theta)$ is the stiffness-parameterized system matrix.

The optimal vector of $\lambda_k^*$ is given by:

$$
\lambda_k^* = \left[ \sigma_{eq}^2 G_{\lambda,k} + (\Sigma_\epsilon^{-1})_{11} \right]^{-1} \left[ \sigma_{eq}^2 \left( \lambda^{(N_m)} + \Delta m - K^* \phi^{(N_m)} \right) + (\Sigma_\epsilon^{-1})_{11} \lambda + (\Sigma_\epsilon^{-1})_{12} (\bar{\beta} - L_0 \phi^*) \right]
$$

(5.43)

where

$$
G_{\lambda,k} = \sigma_{eq}^2 \text{diag} \left[ \left( \phi^{(N_m)} K^* - \lambda^{(N_m)} \phi^{(N_m)} \Delta m \phi^{(N_m)} \right) \right]_{N_m \times N_m}
$$

(5.44)

The optimal vector of $\theta^*$ is given by:

$$
\theta^* = \left( \sigma_{eq}^2 G_{\theta} G_{\theta} + \Sigma_{\theta}^{-1} \right)^{-1} \left( \sigma_{eq}^2 G_{\theta}^T b + \Sigma_{\theta}^{-1} \theta^\eta \right)
$$

(5.45)

where

$$
G_{\theta} = \left[ \left( \phi^{(N_m)} K_1 \phi^{(N_m)} + \phi^{(N_m)} K_2 \phi^{(N_m)} + \cdots + \phi^{(N_m)} K_{N_\theta} \phi^{(N_m)} \right) \right]_{N_m \times N_\theta}
$$

(5.46)

$$
b_k = \left[ \left( \lambda^{(N_m)} \phi^{(N_m)} \Delta m \phi^{(N_m)} - \phi^{(N_m)} K_0 \phi^{(N_m)} \right) \right]_{N_m \times 1}
$$

(5.47)

From the literature review, the way to obtain the optimal values of parameters is using $\phi^*, \lambda^*, \beta^*$ and $\theta^*$ in an iterative manner (Yuen, 2010, Mustafa and Matsumoto, 2017, Das and Debnath, 2020). It is commonly known that mode shapes are usually measured with incomplete DOFs due to the limited accessibility of sensors and frequencies may be measured with relatively high accuracy. Therefore, the optimization is implemented here in the sequence of $(\phi^*, \lambda^*, \beta^*)$ or $(\phi^*, \lambda^*, \theta^*)$. Figure 5.1 shows the iterative procedure for updating mass and stiffness parameters. First set initial values of $\beta^\eta$, $\theta^\eta$, and measured $\hat{\lambda}$, respectively, the iterative procedure consists following steps:
• Update the system mode shapes, $\phi_{m}^{(m)*}$ using Eq. (5.34) (mass updating), $\phi_{k}^{(m)*}$ using Eq. (5.41) (stiffness updating), $m = 1, 2, 3 ..., N_{m}$.

• Update the system eigenvalues, $\lambda_{m}^{(m)*}$ using Eq. (5.36) (mass updating), $\lambda_{k}^{(m)*}$ using Eq. (5.43) (stiffness updating), $m = 1, 2, 3 ..., N_{m}$.

• Update the model parameter, mass parameter, $\beta^*$ and $\theta^*$, using Eq. (5.38) and (5.45), respectively.

• Iterate the steps 1, 2 and 3 until the model parameters, $\beta^*$ and $\theta^*$, satisfy some convergence criterion. Herein, when the updated parameters start to remain closed to 0.0001 difference, the iteration stops.

Figure 5.1. Flowchart of iterative procedure in the proposed BMUA with $\Delta m$

Note: the initial parameters are assumed to be 1–2 times exact value (mass and stiffness).
5.4.6 Uncertainty quantification with $\Delta m$

The posterior PDF can be well approximated by a Gaussian distribution with a mean at optimal parameters and a covariance matrix, $\Gamma$, that equals the inverse of the Hessian matrix of the objective function. The expression of the covariance matrix of the objective function of $J_m(\lambda, \phi, \beta)$ in Eq. (5.29) for mass updating is expressed as:

$$\Gamma(\lambda, \phi, \beta) = \begin{bmatrix} \sigma_{eq}^{-2} G_\lambda + (\Sigma_\epsilon^{-1})_{11} & \sigma_{eq}^{-2} L_1 + (\Sigma_\epsilon^{-1})_{12} L_0 & \sigma_{eq}^{-2} L_2 \\ \sigma_{eq}^{-2} G_\phi^T G_\phi + L_0^T (\Sigma_\epsilon^{-1})_{22} L_0 & -2\sigma_{eq}^{-2} L_3 \\ \text{sym} \end{bmatrix}^{-1}$$  \hspace{1cm} (5.48)

where $L_1$ is given by:

$$L_1 = \text{diag} \left( \begin{bmatrix} \lambda^{(N_m)^*} \phi^{(N_m)^*}^T M^* - \lambda^{(N_m)^*} \phi^{(N_m)^*}^T (M^* + \Delta m) \end{bmatrix}_{N_{m} \times N_{d} N_{m}} \right)$$  \hspace{1cm} (5.49)

The $lth$ column of $L_2$ is given by:

$$L_2 = \left[ \begin{bmatrix} \lambda^{(N_m)^*} - \lambda^{(N_m)^*} \phi^{(N_m)^*}^T M_l \phi_{m^*} \end{bmatrix}_{N_{m} \times 1} \right]_{N_{d} N_{m} \times 1}$$  \hspace{1cm} (5.50)

The $lth$ column of $L_3$ is given by:

$$L_3 = \left[ \phi^{(N_m)^*} \left( \phi^{(N_m)^*} \left( \begin{bmatrix} \lambda^{(N_m)^*} - \lambda^{(N_m)^*} \end{bmatrix}_{N_{m} \times 1} \right)^{-1} \lambda^{(N_m)^*} \Delta m - M^* \right)^T M_l \phi_{m^*} \right]_{N_{d} N_{m} \times 1}$$  \hspace{1cm} (5.51)

The covariance matrix of the objective function of $J_k(\lambda, \phi, \theta)$ in Eq. (5.33) is expressed as:

$$\Gamma(\lambda, \phi, \theta) = \begin{bmatrix} \sigma_{eq}^{-2} G_\lambda + (\Sigma_\epsilon^{-1})_{11} & \sigma_{eq}^{-2} L_1 + (\Sigma_\epsilon^{-1})_{12} L_0 & \sigma_{eq}^{-2} L_2 \\ \sigma_{eq}^{-2} G_\phi^T G_\phi + L_0^T (\Sigma_\epsilon^{-1})_{22} L_0 & -2\sigma_{eq}^{-2} L_3 \\ \text{sym} \end{bmatrix}^{-1}$$  \hspace{1cm} (5.53)

where $L_4$ is given by:

$$L_4 = \text{diag} \left( \begin{bmatrix} \lambda^{(N_m)^*} \phi^{(N_m)^*}^T K^* - \lambda^{(N_m)^*} \phi^{(N_m)^*}^T (K^* + \lambda^{(N_m)^*} \Delta m) \end{bmatrix}_{N_{m} \times N_{d} N_{m}} \right)$$  \hspace{1cm} (5.54)

The $lth$ column of $L_5$ is given by:
\[ L_5 = \left[ (\lambda^{(N_m)} - \lambda^{(N_m)})^T \phi^{(N_m)} K_i \phi_i^* \right]_{N_m \times 1} \]  

5.4.7 Illustrative examples

In this section, the performance of the proposed BMUA is validated by two simulated-data examples: a) Six-story shear building; b) Three-dimensional three-story braced frame. In the presented approach, both mass and stiffness parameters are considered as model parameters to be updated. Defining parameters to be updated as a ratio between updated mass/stiffness parameters and exact mass/stiffness parameters: \( \theta = K_u/K_e, \beta = M_u/M_e \), where \( K_u \) and \( M_u \) are updated stiffness and mass, respectively. \( K_e \) and \( M_e \) are FEM stiffness and mass, respectively. For undamaged cases, \( \theta \) and \( \beta \) should be unity. Moreover, a comparative study is carried out for different damage scenarios to compare the proposed Bayesian updating approach against the conventional Bayesian updating approach. It may be mentioned that all the information of DOFs is assumed to be obtainable in these two examples.

5.4.7.1 Example 1: six-story shear building

The system chosen for this example is a six DOFs structure, as shown in Figure. 5.2.
and has following properties: The mass per floor is taken to be $M_i = 2 \text{ kg} \ (i = 1, 2, \cdots, 6)$, while the inter-story stiffness is chosen to be $K_i = 100 \text{ KN/m} \ (i = 1, 2, \cdots, 6)$. The total height of this building is 10 m. Therefore, there are a total of 6 mass and stiffness parameters to be updated. The modified system is created by adding 0.035 kg to each floor in this example, two sets of simulated measured data of unmodified and modified system are obtained. Also, measurement noise is considered by adding zero-mean Gaussian noise with 1% coefficient of variation (COV) to extracted frequencies and mode shape to simulate more realistic measurements. Based on the literature review (Yuen, 2010, Au, 2011b), identified COV of frequencies by Fast FFT Bayesian modal identification in field tests are much smaller than 1%, many of them even are less than 0.1%. Therefore, 1% COV of frequencies is reasonable.

![Figure 5.2. Six-story shear building](image)

**FE model updating using incomplete modes**

To evaluate the capability of handling uncertainty induced by incomplete data of the proposed approach, the shear building is updated by the proposed Bayesian approach with incomplete modes under an intact condition. The initial values of mass and stiffness parameters for each floor are taken as 2, which is significantly overestimated by 100% comparing with exact values 1. The performance of updated frequencies using the proposed
approach is shown in Table 5.1. It has been observed that updated frequencies are very close to actual values using the proposed approach. Even if only the first four modes are used to update the model, the error is less than 1%.

Table 5.1. Actual and updated frequencies using incomplete modes (Hz)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Actual</th>
<th>4 modes</th>
<th>5 modes</th>
<th>6 modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.2938</td>
<td>0.2934</td>
<td>0.2939</td>
<td>0.2938</td>
</tr>
<tr>
<td>2</td>
<td>0.8613</td>
<td>0.8614</td>
<td>0.8616</td>
<td>0.8612</td>
</tr>
<tr>
<td>3</td>
<td>1.3702</td>
<td>1.3689</td>
<td>1.3706</td>
<td>1.3699</td>
</tr>
<tr>
<td>4</td>
<td>1.7857</td>
<td>1.7839</td>
<td>1.7867</td>
<td>1.7844</td>
</tr>
<tr>
<td>5</td>
<td>2.0795</td>
<td>2.0796</td>
<td>2.0802</td>
<td>2.0798</td>
</tr>
<tr>
<td>6</td>
<td>2.2315</td>
<td>2.2320</td>
<td>2.2327</td>
<td>2.2314</td>
</tr>
</tbody>
</table>

Figure 5.3 shows the comparison between updated mode shapes and actual mode shapes. The updated mode shapes obtained from incomplete modes have good agreement with the actual mode shape. On the other hand, identified mass parameters, stiffness parameters, and corresponding standard derivation (S.D.) are presented in Table 5.2. The updated mass and stiffness parameters calculated from the proposed Bayesian approach are consistent with actual values of mass and stiffness. Moreover, by using covariance matrices in Eqs. (5.48) and (53), the standard derivation which evaluates uncertainty is estimated.
The uncertainty reduces as the number of measured modes increases.

Table 5.2. Actual and updated mass and stiffness parameters with incomplete modes

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>6modes</th>
<th>5modes</th>
<th>4modes</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>1.0000</td>
<td>1.0071</td>
<td>1.011</td>
<td>1.0198</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.9939</td>
<td>0.991</td>
<td>0.991</td>
<td>0.9862</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.0033</td>
<td>0.9904</td>
<td>0.9866</td>
<td>0.9876</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>0.9976</td>
<td>0.9904</td>
<td>0.9866</td>
<td>0.9876</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.9996</td>
<td>1.0215</td>
<td>1.0178</td>
<td>1.0178</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>1.0014</td>
<td>0.981</td>
<td>0.9876</td>
<td>0.9876</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.9912</td>
<td>1.0068</td>
<td>1.0071</td>
<td>1.0071</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1.0061</td>
<td>0.9946</td>
<td>0.9919</td>
<td>0.9919</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>1.0026</td>
<td>0.9986</td>
<td>1.0005</td>
<td>1.0005</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>1.0024</td>
<td>0.9988</td>
<td>1.0009</td>
<td>1.0009</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>1.0063</td>
<td>0.9946</td>
<td>0.9925</td>
<td>0.9925</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>0.9912</td>
<td>1.0066</td>
<td>1.0066</td>
<td>1.0066</td>
</tr>
</tbody>
</table>

**Probabilistic damage detection**

Damage is created artificially by changing critical structural parameters such as flexural stiffness of $EI$ ($E$ is young’s modulus, $I$ is the moment of inertial), unit mass. Different damage scenarios are considered to assess the performance of the proposed approach. Details of various damage scenarios are presented in Table 5.3. The negative sign represents mass/stiffness loss with respect to undamaged values in this table. Also, these damage scenarios are selected based on the increasing severity of the damage. Regarding the selection of initial values for damaged cases, since no prior information of damaged structure is available, initial values of both mass and stiffness parameters are taken as 1, which assumes that the observed structure has no damage.

Table 5.3. Damage scenarios

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Mass change</th>
<th>Stiffness change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-10% (1\textsuperscript{st} floor), -20% (3\textsuperscript{rd} floor)</td>
<td>-10% (2\textsuperscript{nd} floor), -20% (4\textsuperscript{th} floor)</td>
</tr>
<tr>
<td>2</td>
<td>-30% (2\textsuperscript{nd} floor), -40% (5\textsuperscript{th} floor)</td>
<td>-30% (3\textsuperscript{rd} floor), -40% (6\textsuperscript{th} floor)</td>
</tr>
</tbody>
</table>
In this example, six modes (including frequencies and mode shapes) are used as measured data for damage detection. When updating stiffness parameters, mass parameters are known and considered as undamaged in the conventional BMUA. Similarly, when updating mass parameters, stiffness parameters are known and undamaged in conventional BMUA. However, no prior information of mass and stiffness is required when using the proposed Bayesian approach.

Table 5.4. Actual and updated frequencies (Hz)

| Mode | Damage No.1 | | | | Damage No.2 | | | |
|------|-------------|------------------|------------------|------|------------------|------------------|------|
|      | Actual      | Updated          | Updated          |      | Actual           | updated          |      |
| 1    | 0.2886      | 0.2882           | 0.3167           |      | 0.3003           | 0.3013           | 0.3293 |
| 2    | 0.8570      | 0.8564           | 0.9117           |      | 0.8828           | 0.8838           | 0.9536 |
| 3    | 1.3920      | 1.3925           | 1.4628           |      | 1.2923           | 1.2943           | 1.3531 |
| 4    | 1.7340      | 1.7335           | 1.7764           |      | 1.6914           | 1.6934           | 1.8313 |
| 5    | 2.1038      | 2.1033           | 2.1753           |      | 2.0849           | 2.0899           | 2.1923 |
| 6    | 2.2130      | 2.2124           | 2.7229           |      | 2.2668           | 2.2628           | 2.6369 |

Figure 5.4. Comparison of actual and updated frequency: (a) Damage No.1; (b) Damage No.2
Table 5.4 and Figure 5.4 show identified frequencies using the proposed Bayesian approach match well with their actual values in two damage cases. However, the conventional Bayesian gives us obvious bias in terms of updated frequencies, particularly 23.0% error and 16.3% error in the 6th frequency for damage case No.1 and No.2, respectively. Figure 5.5 illustrates that identified mode shapes using the proposed approach are almost identical to actual mode shapes, while mode shapes obtained from conventional Bayesian have deviated much from actual mode shapes. On the other hand, the identified mass, stiffness parameters, and corresponding standard derivation obtained from the proposed and conventional approach for all damage scenarios are presented in Tables 5.5 and 5.6.
Table 5.5. Actual and updated structural parameters for damage No.1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Conventional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.9000</td>
<td>0.9006</td>
<td>0.0169</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.0091</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.8000</td>
<td>0.8005</td>
<td>0.0142</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>1.0000</td>
<td>0.9993</td>
<td>0.0295</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>1.0000</td>
<td>1.0012</td>
<td>0.0476</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>1.0000</td>
<td>0.9993</td>
<td>0.0290</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0286</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.9000</td>
<td>0.9901</td>
<td>0.0109</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>1.0000</td>
<td>0.9996</td>
<td>0.0286</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>0.8000</td>
<td>0.8000</td>
<td>0.0159</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>1.0000</td>
<td>0.9997</td>
<td>0.0106</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>1.0000</td>
<td>1.0005</td>
<td>0.0331</td>
</tr>
</tbody>
</table>

Table 5.6. Actual and updated structural parameters for damage No.2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Conventional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0096</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.7000</td>
<td>0.7000</td>
<td>0.0095</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.0000</td>
<td>1.0001</td>
<td>0.0108</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.0038</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.6000</td>
<td>0.6001</td>
<td>0.0029</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.0023</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.0005</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>0.7000</td>
<td>0.7000</td>
<td>0.0045</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>1.0000</td>
<td>1.0004</td>
<td>0.0089</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>1.0000</td>
<td>0.9997</td>
<td>0.0067</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>0.6000</td>
<td>0.5999</td>
<td>0.0044</td>
</tr>
</tbody>
</table>

Figure 5.6 shows the severity of damage of structural parameters nearly matches the assumed values for all damaged scenarios using the proposed Bayesian approach. Also, as the extent of damage increases (from damage No.1 to damage No.2), the error difference between the actual and estimate values increases (from the error of 21.8% to 66.6%). In
other words, the conventional Bayesian approach seems to fail to detect damage in this example, which results from the assumption that mass/stiffness is invariable due to damage when updating stiffness/mass. Change of mass/stiffness due to damage may not be ignored. Otherwise, it will have a significant bias on results. In the proposed Bayesian updating framework, no assumption is made, the issue of the coupling effect of mass and stiffness is tackled by using data from unmodified and modified systems, then both mass and stiffness can be successfully updated. This example demonstrates that the performance of the proposed Bayesian approach is quite superior to the conventional Bayesian approach in terms of updating both mass and stiffness.

Figure 5.6. Comparison of updated parameters. Damage No.1: a) mass; (b) stiffness; Damage No.2: (c) mass; (d) stiffness
Once the extent of the damage for mass and stiffness and their standard derivation are identified, Eq. (5.5) is used to estimate the probabilities of damage of all structural parameters. As shown in Figure 5.7, for example, the first mass parameter and the second stiffness parameter in damage No.1 have possible damage both 10% with a probability of 56.3% and 59.8%, respectively.

![Diagram](image)

Figure 5.7. Probability of damage: damage No.1: (a) mass (b) stiffness; damage No.2: (c) mass (d) stiffness

Also, the fifth mass parameter and the third stiffness parameter in damage No.2 have possible damage 40% and 30% with a probability of 62.3% and 62.9%, respectively. Fig. 8 suggests that the proposed Bayesian updating approach is very sensitive because it not only localizes damage but also quantifies damage. Therefore, the proposed Bayesian approach has much potential for damage detection.
5.4.7.2 Example 2: three-dimensional three-story braced shear frame

For the second example, a three-story braced frame is investigated to validate the proposed Bayesian approach. The diagram and plan view are presented in Figure 5.8. The floor mass is taken to be \( M=10^4 \) kg for each floor, giving three mass parameters to be updated. Four stiffness parameters are considered in \( x \) and \( y \) direction for each floor to give a total of twelve stiffness parameters. \( \theta_{4(i-1)+1} = K_{i,+x}, \theta_{4(i-1)+2} = K_{i,+y}, \theta_{4(i-1)+3} = K_{i,-x}, \theta_{4(i-1)+4} = K_{i,-y}, \) \( i = 1,2,3, \) where \( i \) denotes story number and \( +x, +y, -x, \) and \( -y \) denote the direction of the outer face. The actual values of lateral stiffness are \( K_{i,+x} = K_{i,-x} = 50000 \) KN/M, \( K_{i,+y} = K_{i,-y} = 40000 \) KN/M. The modified system is created by adding \( 3.5\% \times M = 350 \) kg on each floor. Missing modes are always existing in the real structure. Therefore, the first six modes of the unmodified and modified system are considered as measured data. Additionally, similar to previous example, zero-mean Gaussian noise with 1% COV of modal parameters is added to measured frequencies and mode shapes. The same noise level is applied to both the conventional and proposed methods.

![Diagram and plan view of investigated model](image)

Figure 5.8. Diagram and plan view of investigated model
Probabilistic damage detection

Two damage scenarios are considered with different severity: higher designated number indicates more severity. Table 5.7 shows the different levels of damage. Similar with previous example, initial values of both mass and stiffness parameters for damage detection are taken as 1 for each damage case.

Table 5.7. Damage scenarios

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Mass change</th>
<th>Stiffness change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-15% (1st floor)</td>
<td>-15% (2nd floor, +x face)</td>
</tr>
<tr>
<td></td>
<td>-20% (1st floor)</td>
<td>-20% (1st floor, −x face)</td>
</tr>
<tr>
<td>2</td>
<td>-30% (2nd floor)</td>
<td>-30% (2nd floor, −y face)</td>
</tr>
<tr>
<td></td>
<td>-30% (3rd floor)</td>
<td>-40% (3rd floor, +x face)</td>
</tr>
</tbody>
</table>

A comparative study is implemented to compare the proposed Bayesian approach against the conventional Bayesian approach. The mass is assumed to be known (even for the damaged mass) when updating stiffness using a conventional Bayesian approach; similarly, stiffness is assumed to be known (even for damaged stiffness) when updating mass using the conventional Bayesian approach. However, no mass or stiffness information is needed when using the proposed Bayesian approach.

Table 5.8. Actual and updated frequencies (Hz) for damage scenarios

| Mode | Damage No.1 | | Damage No.2 | | |
|------|-------------|-------|-------------|-------|
|      | Proposed    | Conventional | Proposed    | Conventional | |
|      | approach     | approach   | approach     | approach   | |
| Mode | Actual | Updated | Updated | Actual | updated | Updated |
| 2    | 6.9097 | 7.1314 | 7.4463 | 8.0189 | 8.0482 | 7.6733 |
| 4    | 17.5213 | 17.6475 | 18.3255 | 20.1385 | 20.2384 | 21.5418 |
| 6    | 25.6451 | 25.829 | 27.7781 | 28.8006 | 29.1327 | 30.1786 |
The performance of updated frequencies using the proposed approach and the conventional approach is shown in Table 5.8 and Figure 5.9 for all damage scenarios. It is clearly observed that updated frequencies obtained from the proposed approach show considerable matching with actual values for all cases, but updated frequencies by conventional approach are far from actual values. Also, diagonal modal criteria (MAC) values between actual and updated mode shapes using two approaches are evaluated, as shown in Table 5.9 for all damage scenarios.

Table 5.9. MAC values for damage scenarios

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Damage No.1</th>
<th></th>
<th>Damage No.2</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Proposed approach</td>
<td>Conventional approach</td>
<td></td>
<td>Proposed approach</td>
</tr>
<tr>
<td>1</td>
<td>0.9998</td>
<td>0.9987</td>
<td></td>
<td>0.9997</td>
</tr>
<tr>
<td>2</td>
<td>0.9999</td>
<td>0.9989</td>
<td></td>
<td>0.9998</td>
</tr>
<tr>
<td>3</td>
<td>0.9998</td>
<td>0.9988</td>
<td></td>
<td>0.9999</td>
</tr>
<tr>
<td>4</td>
<td>0.9999</td>
<td>0.9992</td>
<td></td>
<td>0.9996</td>
</tr>
<tr>
<td>5</td>
<td>0.9997</td>
<td>0.9989</td>
<td></td>
<td>0.9998</td>
</tr>
<tr>
<td>6</td>
<td>0.9997</td>
<td>0.9972</td>
<td></td>
<td>0.9997</td>
</tr>
</tbody>
</table>

Obviously, the proposed Bayesian approach gives more accurate MAC values than the conventional Bayesian approach. Particularly, the conventional Bayesian approach gives
0.9560 of MAC value of the fifth mode in damage No. 2. Updated mass, stiffness parameters, and their corresponding standard derivation using two approaches are presented in Table 5.10 and Table 5.11. It was found in Table 5.10 (Damage No. 1) and that the parameters in mass and stiffness, $\beta_1$ and $\theta_{1,-y}$ are 1.30 and 1.27, in Table 5.11 (Damage No. 2), the parameters in stiffness, $\theta_{3,+y}$ and $\theta_{3,-y}$ are 1.45 and 1.44 by the conventional Bayesian approach compared to the target value of 1.

Table 5.10. Actual and updated structural parameters for damage No.1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Conventional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.8500</td>
<td>0.8456</td>
<td>0.0015</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.0000</td>
<td>0.9874</td>
<td>0.0019</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.0000</td>
<td>1.0131</td>
<td>0.0014</td>
</tr>
<tr>
<td>$\theta_{1,+x}$</td>
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<td>1.0183</td>
<td>0.0098</td>
</tr>
<tr>
<td>$\theta_{1,+y}$</td>
<td>1.0000</td>
<td>1.0032</td>
<td>0.0100</td>
</tr>
<tr>
<td>$\theta_{1,-x}$</td>
<td>1.0000</td>
<td>1.0026</td>
<td>0.0099</td>
</tr>
<tr>
<td>$\theta_{1,-y}$</td>
<td>1.0000</td>
<td>0.9967</td>
<td>0.0100</td>
</tr>
<tr>
<td>$\theta_{2,+x}$</td>
<td>0.8500</td>
<td>0.8443</td>
<td>0.0187</td>
</tr>
<tr>
<td>$\theta_{2,+y}$</td>
<td>1.0000</td>
<td>0.9992</td>
<td>0.0120</td>
</tr>
<tr>
<td>$\theta_{2,-x}$</td>
<td>1.0000</td>
<td>1.0122</td>
<td>0.0199</td>
</tr>
<tr>
<td>$\theta_{2,-y}$</td>
<td>1.0000</td>
<td>0.9935</td>
<td>0.0139</td>
</tr>
<tr>
<td>$\theta_{3,+x}$</td>
<td>1.0000</td>
<td>0.9963</td>
<td>0.0150</td>
</tr>
<tr>
<td>$\theta_{3,+y}$</td>
<td>0.8000</td>
<td>0.7842</td>
<td>0.0182</td>
</tr>
<tr>
<td>$\theta_{3,-x}$</td>
<td>1.0000</td>
<td>0.9922</td>
<td>0.0101</td>
</tr>
<tr>
<td>$\theta_{3,-y}$</td>
<td>1.0000</td>
<td>0.9796</td>
<td>0.0168</td>
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</table>
Table 5.11. Actual and updated structural parameters for damage No.2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Conventional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.8000</td>
<td>0.7898</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.7000</td>
<td>0.6895</td>
<td>0.0090</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.7000</td>
<td>0.6929</td>
<td>0.0020</td>
</tr>
<tr>
<td>$\theta_{1,+x}$</td>
<td>1.0000</td>
<td>0.9879</td>
<td>0.0135</td>
</tr>
<tr>
<td>$\theta_{1,+y}$</td>
<td>1.0000</td>
<td>0.9723</td>
<td>0.0241</td>
</tr>
<tr>
<td>$\theta_{1,-x}$</td>
<td>0.8000</td>
<td>0.7955</td>
<td>0.0154</td>
</tr>
<tr>
<td>$\theta_{1,-y}$</td>
<td>1.0000</td>
<td>1.0286</td>
<td>0.0123</td>
</tr>
<tr>
<td>$\theta_{2,+x}$</td>
<td>1.0000</td>
<td>1.0194</td>
<td>0.0167</td>
</tr>
<tr>
<td>$\theta_{2,+y}$</td>
<td>1.0000</td>
<td>0.9768</td>
<td>0.0145</td>
</tr>
<tr>
<td>$\theta_{2,-x}$</td>
<td>1.0000</td>
<td>1.0145</td>
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</tr>
<tr>
<td>$\theta_{2,-y}$</td>
<td>0.7000</td>
<td>0.7062</td>
<td>0.0181</td>
</tr>
<tr>
<td>$\theta_{3,+x}$</td>
<td>0.6000</td>
<td>0.6155</td>
<td>0.0215</td>
</tr>
<tr>
<td>$\theta_{3,+y}$</td>
<td>1.0000</td>
<td>0.9976</td>
<td>0.0158</td>
</tr>
<tr>
<td>$\theta_{3,-x}$</td>
<td>1.0000</td>
<td>0.9845</td>
<td>0.0332</td>
</tr>
<tr>
<td>$\theta_{3,-y}$</td>
<td>1.0000</td>
<td>0.9890</td>
<td>0.0194</td>
</tr>
</tbody>
</table>

Figure 5.10 shows that updated mass/stiffness parameters agree well with actual values by the proposed approach for all damage scenarios. The updated error is less than 3%. However, updated mass/stiffness is obtained from conventional Bayesian have remarkable error (mass: up to 30%, stiffness: up to 44%). In other words, the conventional Bayesian approach gives false damage alarm, which may attribute to the assumption that mass/stiffness change due to damage is ignored. The assumption in conventional BMUA is questionable to update stiffness when the mass has significantly changed. This example demonstrates that the proposed Bayesian approach has a quite better level of performance when compared with the conventional Bayesian approach.
Figure 5.10. Comparison of updated parameters. Damage No.1 (a) mass; (b) stiffness
Damage No.2 (c) mass; (d) stiffness

Based on identified mass/stiffness parameters and their standard derivation, the probability of damage for all parameters are evaluated using Eq. (5.5). Figure 5.11 shows that the first mass parameter and stiffness parameter of +y face on the third floor in damage No.1 have possible damage 15% and 20% with a probability of 72.32% and 76.19%, respectively. Furthermore, the second mass parameter and stiffness parameter +x face of the third floor in damage No.2 have possible damage 30% and 40% with a probability of 84.81% and 78.61%, respectively.
Figure 5.11. Probability of damage. Damage No.1: (a) mass (b) stiffness; damage No.2: (c) mass (d) stiffness

5.5 Bayesian model updating with added stiffness $\Delta k$

In this section, the Bayesian model updating framework with added stiffness $\Delta k$ is proposed. Adding known stiffness is used as a surrogate way of adding mass to construct the modified system. Similar to the BMUA with mass addition, another new eigen-equations embedding $\Delta k$ is derived to solve the coupled parameters. The posterior PDF is the reformulated based on new prior PDF and likelihood function. The optimal parameters are finally obtained by asymptotic optimization method. It should be noted that the subheading with $\Delta k$ or ‘new’ indicates the presented equations and formulations in this section are originally derived by authors, otherwise, references are cited accordingly.
5.5.1 Formulation of new eigen-equations with added stiffness $\Delta k$

Two systems, namely, original, and modified systems with stiffness addition, $\Delta k$, are considered together for fundamental structural dynamics, we have:

$$K\phi = M\phi\lambda$$  \hspace{1cm} (5.57)

$$(K + \Delta k)\phi' = M\phi'\lambda'$$  \hspace{1cm} (5.58)

where $\lambda$ and $\phi$ are eigenvalue and mode shape before modification, respectively. $\lambda'$ and $\phi'$ are eigenvalue and mode shape after modification.

$\phi'^T$ is premultiplied in Eq. (5.57) by

$$\phi'^T K\phi = \phi'^T M\phi\lambda$$  \hspace{1cm} (5.59)

The transposed matrix of Eq. (5.58) is calculated, then $\phi$ is postmultiplied in the resulting equation,

$$\phi'^T (K + \Delta k)\phi = \phi'^T M\phi\lambda'$$  \hspace{1cm} (5.60)

Subtracting Eq. (5.59) from Eq. (5.60), it gives:

$$\phi'^T M\phi\lambda' - \phi'^T M\phi\lambda = \phi'^T \Delta k\phi$$  \hspace{1cm} (5.61)

Defining $H = \phi'^T M\phi$ and $Z = \phi'^T \Delta k\phi$, Eq. (5.61) is rewritten as:

$$H\lambda' - H\lambda = Z$$  \hspace{1cm} (5.62)

Therefore, $H$ is solved as another expression, $H'$:

$$H' = (\lambda' - \lambda)^{-1}\phi'^T \Delta k\phi$$  \hspace{1cm} (5.63)

Finally, eigen-equation error, $ME_m$, is reformulated when updating mass:

$$ME_m = H - H' = \phi'^T M\phi - (\lambda' - \lambda)^{-1}\phi'^T \Delta k\phi$$  \hspace{1cm} (5.64)

Similar derivation procedures for stiffness updating are employed:

$\phi'^T$ is premultiplied in Eq. (5.57)
$$\phi^T K \phi = \lambda \phi^T M \phi \quad (5.65)$$

The transposed matrix of Eq. (5.58) is firstly calculated, then postmultiplying resulting equation by $\phi$,

$$\phi^T (K + \Delta k) \phi = \lambda' \phi^T M \phi \quad (5.66)$$

Premultiplying Eqs. (5.65) and (5.66) by $\lambda^{-1}$ and $\lambda'^{-1}$, respectively:

$$\lambda^{-1} \phi^T K \phi = \phi^T M \phi \quad (5.67)$$
$$\lambda'^{-1} \phi^T (K + \Delta k) \phi = \phi^T M \phi \quad (5.68)$$

Subtracting Eq. (5.67) from Eq. (5.68), it gives:

$$\lambda^{-1} \phi^T K \phi - \lambda'^{-1} \phi^T K \phi = \lambda'^{-1} \phi^T \Delta k \phi \quad (5.69)$$

Defining $Y = \lambda'^{-1} \phi^T \Delta k \phi$ and $W = \phi^T K \phi$, hence Eq. (5.69) is simplified as:

$$\lambda^{-1} W - \lambda'^{-1} W = Y \quad (5.70)$$

Thus, $W$ has a new expression, $W'$:

$$W' = (\lambda^{-1} - \lambda'^{-1})^{-1} \lambda'^{-1} \phi^T \Delta k \phi \quad (5.71)$$

Then, eigen-equation error, $ME_k$, is reformulated when updating stiffness:

$$ME_k = W - W' = \phi^T K \phi - (\lambda^{-1} - \lambda'^{-1})^{-1} \lambda'^{-1} \phi^T \Delta k \phi \quad (5.72)$$

Finally, the coupling effect could be addressed using the new eigen-equations, e.g., Eqs. (5.64) and (5.72), because mass updating by Eq. (5.64) does not require any stiffness information. Similarly, when updating stiffness by Eq. (5.72). It should be noticed that two fundamental rules for creating a modified system using added known stiffness: (1) noticeable frequency change is observed between the original system and modified system; (2) mode shapes after modification change slightly (Coppotelli, 2009, Khatibi et al., 2012, López-Aenlle et al., 2012). Some recommendations for creating a modified structure with added stiffness could be found in the work (Khatibi et al., 2012). Further research for
stiffness-change optimization strategy may be studied under lab or field test conditions, which is beyond the scope of this work.

5.5.2 Formulation of the new prior PDF with Δk

Assume $N_m$ modes are measured. The prior PDF in the proposed BMUA is chosen as when updating parameters, $\beta$:

$$ p_m(\lambda, \phi, \beta | C) = p_m(\lambda, \phi | \beta, C) \cdot p_m(\beta | C) \tag{5.73} $$

where $\lambda = [\lambda^{(1)}, \lambda^{(2)}, \ldots, \lambda^{(N_m)}]^T$ and $\phi = [\phi^{(1)}, \phi^{(2)}, \ldots, \phi^{(N_m)}]^T$ are eigenvalues, eigenvector to be updated, respectively. $p_m(\lambda, \phi | \beta, C)$ is formulated using Gaussian PDF with new eigen-equation error in Eq. (5.64):

$$ p_m(\lambda, \phi | \beta, C) = c_0 \exp \left[ -\frac{\| \phi'^T M \phi - (\lambda' - \lambda)^{-1} \phi'^T \Delta k \phi \|^2}{2\sigma_{eq}^2} \right] \tag{5.74} $$

where $c_0$ denotes a constant; sign of $\| . \|$ denotes mathematical Euclidean norm. $\sigma_{eq}^2$ denotes a defined variance of eigen-equation error; Eq. (5.74) is simplified as:

$$ p_m(\lambda, \phi | \beta, C) = c_0 \exp \left[ -\frac{1}{2} J_{g,m}(\lambda, \phi; \beta) \right] \tag{5.75} $$

where

$$ J_{g,m}(\lambda, \phi; \beta) = T_m^T \Sigma_{eq}^{-1} T_m \tag{5.76} $$

where $T_m = \phi'^{(m)^T} M \phi^{(m)} - (\lambda'^{(m)} - \lambda^{(m)})^{-1} \phi'^{(m)^T} \Delta k \phi^{(m)}$. $\Sigma_{eq} = \sigma_{eq}^2 I$, denotes the covariance matrix in prior PDF; $I$ denotes an identity matrix. $\Sigma_{eq}$ arises from modeling error between theoretical and target FE models. Another term of $p_m(\beta | C)$ in Eq. (5.73) is approximated by a Gaussian distribution, which has a mean value of $\beta^0$ (nominal value).
and covariance matrix, \( \Sigma_\beta = \sigma_\beta^2 I \), \( \sigma_\beta \) is selected as a large variance to let \( p_m(\beta|C) \) be a non-informative prior (Yuen, 2010). Hence, \( p_m(\beta|C) \) may be expressed as:

\[
p_m(\beta|C) = \exp \left[ -\frac{\| \beta - \beta^\eta \|^2}{2\sigma_\beta^2} \right] \quad (5.77)
\]

Combining Eqs. (5.75) and (5.77), then the prior PDF in Eq. (5.73) is rewritten as:

\[
p_m(\lambda, \phi, \beta|C) = c_0 \exp \left[ -\frac{1}{2} J_{gm}(\lambda, \phi; \beta) \right] \cdot \exp \left[ -\frac{\| \beta - \beta^\eta \|^2}{2\sigma_\beta^2} \right] \quad (5.78)
\]

### 5.5.3 Formulation of likelihood function

How good the FE model’s response agrees well with measurement can be reflected by the likelihood function. Assuming a measurement error, \( \epsilon \):

\[
\begin{bmatrix}
\hat{\lambda} \\
\hat{\Phi}
\end{bmatrix} = \begin{bmatrix}
\lambda \\
L_0 \phi
\end{bmatrix} + \epsilon
\quad (5.79)
\]

where Gaussian distribution is assigned to \( \epsilon \). \( \hat{\lambda} \) denotes measured eigenvalues; \( \hat{\Phi} \) denotes measured mode shapes. \( L_0 \) consists of ‘1s’ or ‘0s’ to match measured partial mode shapes with theoretical counterparts. Therefore, the likelihood function is expressed as:

\[
p_m(\hat{\lambda}, \hat{\Phi}|\lambda, \phi, \beta, C) = p_m(\hat{\lambda}, \hat{\Phi}|\lambda, \phi) = \exp \left[ -\frac{\| \hat{\lambda} - \lambda \|^2}{2\Sigma_e} \right] \quad (5.80)
\]

\( \Sigma_e \) in Eq. (5.80) is a measured covariance matrix that can be obtained by Bayesian modal analysis (Au, 2017a), reflecting the effect of measurement noise on identified frequencies and mode shapes.
5.5.4 Formulation of the new posterior PDF with $\Delta k$

Based on the formulation of prior PDF in Eq. (5.78) and likelihood function Eq. (5.80), the posterior PDF for mass updating can be written according to Eq. (5.2):

$$p_m(\lambda, \phi, \beta|\hat{\lambda}, \hat{\phi}, \hat{C}) = c_0 \exp \left[ -\frac{1}{2} \| \hat{\lambda} - L_0 \phi \| \frac{2}{2} \| \beta - \beta^\eta \| \right]$$  (5.81)

Generally, it requires multidimensional integrals to obtain the MPVs of structural parameters in Eq. (5.81). However, due to structural complexity in real-world, it is impractical to directly perform multidimensional integrals (Beck and Katafygiotis, 1998). The asymptotic approximation method is an efficient alternative to avoid this issue (Beck and Katafygiotis, 1998). Specifically, the posterior PDF’s negative logarithm is used as an objective function. Therefore, MPVs can be found by minimizing the objective functions, analytical formulations of model parameters and uncertainty can be conveniently derived. The objective function with added stiffness when updating mass parameters is expressed as:

$$J_m(\lambda, \phi, \beta) = \frac{1}{2} (\beta - \beta^\eta)^T \Sigma_\beta^{-1} (\beta - \beta^\eta)$$

$$+ \frac{1}{2} \sum_{m=1}^{N_m} \left( \phi^{(m)} \right)^T M \phi^{(m)} - \left( \lambda^{(m)} - \lambda^{(m)} \right)^{-1} \phi^{(m)} \Delta k \phi^{(m)} \right]^2$$  (5.82)

In terms of updating stiffness, $\theta$, the same derivation procedures are employed; the prior PDF has an expression as Eq (5.83):

$$p_k(\lambda, \phi, \theta|C) = c_0 \exp \left[ -\frac{1}{2} J_{g,k}(\lambda, \phi; \theta) \right] \cdot \exp \left[ -\frac{(\theta - \theta^\eta)^2}{2\sigma^2} \right]$$  (5.83)
where \( \lambda \) are updated eigenvalues; \( \phi \) are updated eigenvector. \( J_{g,k}(\lambda, \phi; \theta) \) is given by:

\[
J_{g,k}(\lambda, \phi; \theta) = c_0 T_k^T \Sigma_{eq}^{-1} T_k
\]  
(5.84)

where \( T_k = \phi^{(m)^T} K \phi^{(m)} - (\lambda^{(m)^{-1}} - \lambda^{(m)^{-1}})^{-1} \lambda^{(m)^{-1}} \phi^{(m)^T} \Delta k \phi^{(m)}. \)

The likelihood function is given by:

\[
p_k(\hat{\lambda}, \hat{\psi}|\lambda, \phi, \theta, C) = p(\hat{\lambda}, \hat{\psi}|\lambda, \phi) = \exp \left[ -\frac{\left\| \hat{\lambda} - \lambda - L_0 \phi \right\|}{2\Sigma_{\varepsilon}} \right] \]  
(5.85)

The objective function with added stiffness is expressed as:

\[
J_k(\lambda, \phi, \theta) = \frac{1}{2} (\theta - \theta^n)^T \Sigma_0^{-1} (\theta - \theta^n) + \frac{1}{2} \sigma_{eq}^2 \sum_{m=1}^{N_m} \left\| \phi^{(m)^T} K \phi^{(m)} - (\lambda^{(m)^{-1}} - \lambda^{(m)^{-1}})^{-1} \lambda^{(m)^{-1}} \phi^{(m)^T} \Delta k \phi^{(m)} \right\|^2 + \frac{1}{2} \left[ \frac{\hat{\lambda}}{\Sigma_{\varepsilon}} \right] \Sigma_{\varepsilon}^{-1} \left[ \frac{\hat{\lambda}}{\Sigma_{\varepsilon}} - \frac{\lambda}{\Sigma_{\varepsilon}} \right] \]  
(5.86)

5.5.5 Optimization framework with \( \Delta k \)

The MPVs of updated parameters are obtained using the asymptotic approximation method for objective functions in Eqs. (5.82) and (5.86). The symbol * below denotes updated value. The optimal \( \phi_{m}^* \) are obtained via optimizing \( J_m(\lambda, \phi, \beta) \) with respect to \( \phi \):

\[
\phi_{m}^* = \left[ \sigma_{eq}^{-2} G_{\phi,m}^T G_{\phi,m} + L_0^T (\Sigma_{\varepsilon}^{-1})_{22} L_0 \right]^{-1} L_0^T [(\Sigma_{\varepsilon}^{-1})_{21} (\hat{\lambda} - \lambda^*) + (\Sigma_{\varepsilon}^{-1})_{22} \hat{\psi}] \]  
(5.87)

where \( (\Sigma_{\varepsilon}^{-1})_{21} \) is left bottom sub-matrix of \( \Sigma_{\varepsilon}^{-1} \); \( (\Sigma_{\varepsilon}^{-1})_{22} \) is right bottom sub-matrix of \( \Sigma_{\varepsilon}^{-1} \);

\( G_{\phi,m} \) is defined as:

\[
G_{\phi,m} = diag\left( \phi^{(N_m)^* T} \left( M^* - \left( \lambda^{(N_m)^*} - \lambda^{(N_m)^*} \right)^{-1} \Delta k \right) \right)_{N_m \times N_d N_m} \]  
(5.88)

where the sign of ‘\( diag \)’ indicates diagonal matrix, \( M^* = M(\beta) \).
The same procedures with respect to \( \lambda \), the optimal \( \lambda^* \) is obtained as:

\[
\lambda_{m}^* = \left[ \sigma_{eq}^{-2} G_{\lambda,m} + (\Sigma_{e}^{-1})_{11} \right]^{-1} \left[ \sigma_{eq}^{-2} \left( \phi'_{(N_m)^*}^T M^* \phi_{(N_m)^*} - \phi'_{(N_m)^*}^T \Delta \kappa \phi_{(N_m)^*} \right) 
+ (\Sigma_{e}^{-1})_{11} \hat{\lambda} + (\Sigma_{e}^{-1})_{12} (\hat{\psi} - L_0 \phi^*) \right] 
\]

(5.89)

Where \( (\Sigma_{e}^{-1})_{12} \) is right top sub-matrix of \( \Sigma_{e}^{-1} \); \( (\Sigma_{e}^{-1})_{11} \) is left top sub-matrix of \( \Sigma_{e}^{-1} \);

Define \( G_{\lambda,m} \) as:

\[
G_{\lambda,m} = \sigma_{eq}^{-2} diag \left( \phi'_{(N_m)^*}^T M^* \phi_{(N_m)^*} \right)_{N_m \times N_m} 
\]

(5.90)

The same procedures with respect to \( \beta \), the optimal \( \beta^* \) are obtained as:

\[
\beta^* = \left( \sigma_{eq}^{-2} G_{\beta} + \Sigma_{\beta}^{-1} \right)^{-1} \left( \sigma_{eq}^{-2} G_{\beta} b_m + \Sigma_{\beta}^{-1} \beta \gamma \right) 
\]

(5.91)

where \( G_{\beta} \) and \( b_m \) are defined as:

\[
G_{\beta} = \left[ \phi'_{(N_m)^*}^T M_1 \phi_{(N_m)^*} \ldots \phi'_{(N_m)^*}^T M_N \phi_{(N_m)^*} \right]_{N \times N} 
\]

(5.92)

\[
b_m = \left[ \left( \lambda^{(N_m)^*} - \lambda^{(N_m)^*} \right)^{-1} \phi'_{(N_m)^*}^T \Delta \kappa \phi_{(N_m)^*} - \phi'_{(N_m)^*}^T M_0 \phi_{(N_m)^*} \right]_{N \times 1} 
\]

(5.93)

When updating stiffness, the optimal \( \phi_{k}^* \) is obtained via optimizing \( J_k (\lambda, \phi, \theta) \) with respect to \( \phi \) as:

\[
\phi_{k}^* = \left[ \sigma_{eq}^{-2} G_{\phi,k} T G_{\phi,k} + L_0^T (\Sigma_{e}^{-1})_{22} L_0 \right]^{-1} L_0^T (\Sigma_{e}^{-1})_{21} (\hat{\lambda} - \lambda^*) + (\Sigma_{e}^{-1})_{22} \hat{\psi} 
\]

(5.94)

where

\[
G_{\phi,k} = diag \left( \phi'_{(N_m)^*}^T \left( K^* - \left( \lambda^{(N_m)^*} - \lambda^{(N_m)^*} \right)^{-1} \lambda^{(N_m)^*} \Delta \kappa \right) \right)_{N \times N} 
\]

(5.95)

where \( K^* = K(\theta) \).

The optimal \( \lambda_{k}^* \) is expressed as:

\[
\lambda_{k}^* = \left[ \sigma_{eq}^{-2} G_{\lambda,k} + (\Sigma_{e}^{-1})_{11} \right]^{-1} \left[ \sigma_{eq}^{-2} \left( \lambda^{(N_m)^*} \phi_{(N_m)^*}^T \phi_{(N_m)^*} \right) 
+ (\Sigma_{e}^{-1})_{11} \hat{\lambda} + (\Sigma_{e}^{-1})_{12} (\hat{\psi} - L_0 \phi^*) \right] 
\]

(5.96)
where
\[
G_{\lambda k} = \sigma_{eq}^{-2} \text{diag} \left( \phi^{(N_m)^* T} (K^* + \Delta \bar{k}) \phi^{(N_m)^*} \right)_{N_m \times N_m}
\] (5.97)

The optimal \( \theta^* \) is obtained as:
\[
\theta^* = \left( \sigma_{eq}^{-2} G_{\theta} G_{\theta} + \Sigma_{\theta}^{-1} \right)^{-1} (\sigma_{eq}^{-2} G_{\theta} b + \Sigma_{\theta}^{-1} \theta^*)
\] (5.98)

where
\[
G_{\theta} = 
\begin{bmatrix}
\phi^{(N_m)^* T} K_1 \phi^{(N_m)^*} & \phi^{(N_m)^* T} K_2 \phi^{(N_m)^*} & \cdots & \phi^{(N_m)^* T} K_{N_g} \phi^{(N_m)^*}
\end{bmatrix}_{N_m \times N_{\theta}}
\] (5.99)
\[
b_k = 
\begin{bmatrix}
\lambda^{(N_m)^* -1} - \lambda^{(N_m)^* -1}
\end{bmatrix}_{N_m \times 1}
\]
(5.100)

The optimization work is performed in order of \((\phi^*, \lambda^*, \beta^*)\) or \((\phi^*, \lambda^*, \theta^*)\). Figure 5.12 presents the iterative procedures for parameters identification. Initial \( \beta^*, \theta^* \) and \( \lambda^* \) are defined as nominal values, respectively: \( \beta^\eta, \theta^\eta, \) and measured \( \lambda. \) Note that the magnitude of initial mass and stiffness are typically chosen as one to two times of exact values. In this study, iterative work starts with updating mode shape without initial values. The iterative procedures are shown as follows:

- Find updated mode shapes, \( \phi_{m}^{(m)*} \) by Eq. (5.87) (updating mass); \( \phi_{k}^{(m)*} \) by Eq. (5.94) (updating stiffness), \( m = 1, 2, 3 \ldots, N_m. \)
- Find updated eigenvalues, \( \lambda_{m}^{(m)*} \) by Eq. (5.89) (updating mass); \( \lambda_{k}^{(m)*} \) by Eq. (5.96) (updating stiffness), \( m = 1, 2, 3 \ldots, N_m. \)
- Find updated mass and stiffness parameters, \( \beta^* \) and \( \theta^*. \) by Eqs. (5.91) and (5.98), respectively.
• Repeat the above steps until structural parameters, $\beta^*$ and $\theta^*$, to meet defined convergence criterion: the iteration work stops when the difference of updated parameters remains about 0.0001.

![Schematic diagram of the proposed BMUA with $\Delta k$](image)

**Figure 5.12. Schematic diagram of the proposed BMUA with $\Delta k$**

### 5.5.6 Uncertainty quantification with $\Delta k$

When sufficient measured data is available, a Gaussian distribution can reasonably approximate the posterior PDF. The mean and covariance matrix of the Gaussian distribution can be represented by the MPVs of updated parameters and the Hessian matrix’s inverse of the objective function, respectively. The covariance matrix could quantify the uncertainty of model parameters. The covariance matrix of $f_m(\lambda, \phi, \beta)$ when updating mass is described as:

$$\Gamma(\lambda, \phi, \beta) = \begin{pmatrix} \lambda^2 & \phi^2 & \beta^2 \\ \phi \lambda & \phi \lambda & \phi \beta \\ \beta \lambda & \beta \lambda & \beta \beta \end{pmatrix}$$

(5.103)
\[
\begin{bmatrix}
\sigma_{eq}^2 G_{\lambda, m} + (\Sigma_\varepsilon^{-1})_{11} & \sigma_{eq}^2 L_1 + (\Sigma_\varepsilon^{-1})_{12} L_0 & \sigma_{eq}^2 L_2 \\
\sigma_{eq}^2 G_{\phi}^T G_{\phi} + L_0^T (\Sigma_\varepsilon^{-1})_{22} L_0 & \sigma_{eq}^2 L_3 \\
\text{sym} & \text{sym} & (\sigma_{eq}^2 G_{\beta}^T G_{\beta} + \Sigma_\beta^{-1})
\end{bmatrix}^{-1}
\]

where \( L_1 \) is given by:

\[
L_1 = \text{diag}(\lambda^{(N_m)^*} \phi^{(N_m)^*}_m M^* + \phi^{(N_m)^*}_m \Delta k - \lambda^{(N_m)^*}_m \phi^{(N_m)^*}_m M^*)_{N_m \times N_d N_m} \tag{5.104}
\]

\( L_2 \) and \( L_3 \) are defined as:

\[
L_2 = \left[ (\lambda^{(N_m)^*}_m - \lambda^{(N_m)^*}_m) \phi^{(N_m)^*}_m M^*_t \phi^*_m \right]_{N_m \times 1} \tag{5.105}
\]

\[
L_3 = \left[ \phi^{(N_m)^*}_m (M^* - (\lambda^{(N_m)^*}_m - \lambda^{(N_m)^*}_m)^{-1} \Delta k) M^*_t \phi^*_m \right]^T_{N_d N_m \times 1} \tag{5.106}
\]

The covariance matrix of \( J_k(\lambda, \phi, \theta) \) when updating stiffness is described as:

\[
\Gamma(\lambda, \phi, \theta) = \left[
\begin{bmatrix}
\sigma_{eq}^2 G_{\lambda} + (\Sigma_\varepsilon^{-1})_{11} & \sigma_{eq}^2 L_4 + (\Sigma_\varepsilon^{-1})_{12} L_0 & \sigma_{eq}^2 L_5 \\
\sigma_{eq}^2 G_{\phi}^T G_{\phi} + L_0^T (\Sigma_\varepsilon^{-1})_{22} L_0 & \sigma_{eq}^2 L_6 \\
\text{sym} & \text{sym} & (\sigma_{eq}^2 G_{\beta}^T G_{\beta} + \Sigma_\beta^{-1})
\end{bmatrix}^{-1}
\right]
\tag{5.107}
\]

where \( L_4 \) is given by:

\[
L_4 = \text{diag}(\lambda^{(N_m)^*}_m \phi^{(N_m)^*}_m (K^* + \Delta k) - \lambda^{(N_m)^*}_m \phi^{(N_m)^*}_m K^*)_{N_m \times N_d N_m} \tag{5.108}
\]

\( L_5 \) and \( L_6 \) are defined as:

\[
L_5 = \left[ (\lambda^{(N_m)^*}_m - \lambda^{(N_m)^*}_m) \phi^{(N_m)^*}_m K_t \phi^*_m \right]_{N_m \times 1} \tag{5.109}
\]

\[
L_6 = \left[ \phi^{(N_m)^*}_m (K^* - (\lambda^{(N_m)^{-1}} - \lambda^{(N_m)^{-1}}) \lambda^{(N_m)^*}_m \Delta k) K_t \phi^*_m \right]^T_{N_d N_m \times 1} \tag{5.110}
\]
When the covariance matrix, $\Gamma$, is calculated, standard derivations of updated parameters are equivalent to the root of the diagonal values of $\Gamma$.

### 5.5.7 Illustrative examples

To evaluate the proposed BMIUA with added stiffness, FE models are constructed in a MATLAB environment for two numerical examples with different degrees of complexity. In the present work, it is convenient to use a ratio between real and theoretical parameters as an updating index: $\theta = K_r/K_t, \beta = M_r/M_t$, where $K_r$ and $K_t$ are target and theoretical stiffness; $M_r$ and $M_t$ are target and theoretical mass. In the case of a healthy condition, $\theta$ and $\beta$ are unity. Additionally, a comparative investigation is implemented for various damage cases to demonstrate the proposed BMIUA outperforms the traditional one. Note that before system updating, measured mode shapes have to be normalized or scaled by mass-normalized FE model method (Rezaiee-Pajand et al., 2020) or scaling methods (Khatibi et al., 2012, López-Aenlle et al., 2012) to ensure tested and analytical mode shapes are comparative. Also, due to limited sensors installed in practice, measured mode shapes are usually incomplete and only available with a few DOFs related to the sensor location. Therefore, it is desirable to expand reduced measured mode shapes onto complete mode shapes with full DOFs in these two examples by mode shape expansion techniques (Chen, 2010, Chen et al., 2012).
5.5.7.1 Example 1: six-story shear building

The shear building sketch is shown in Figure 5.13, modeled as a six-DOFs structure with a total height of 30 m. The mass per floor and inter-story stiffness are designed as $M_j = 20 \text{ Kg (} j = 1, 2, \cdots 6 \text{)}$ and $K_j = 12000 \text{ KN/m (} j = 1, 2, \cdots 6 \text{)}$, respectively. This example gives six mass/stiffness parameters to be updated. Suppose the shear building under laboratory condition requires retrofits to increase inter-story stiffness and reduce lateral displacement due to long-term use, original structure (Figure 5.13 (a)) also needs to be updated to detect structural abnormality. The curved damper has superior performance to enhance stiffness and reduce inter-story drift. Also, the curved damper is practically convenient, it can be easily inspected and repaired on structural maintenance (Fathizadeh et al., 2021). Thus, two curved dampers at each floor are installed, create a modified system, as shown in Figure 5.13 (b). In this example, assuming that each floor is provided the equivalent stiffness addition of $\Delta k_j = 420 \text{ KN/m (} j = 1, 2, \cdots 6 \text{)}$ by curved dampers, weight of each curved damper is ignored. Hence, the FE model will be updated by two groups of simulated measured data acquired from original and modified systems. Gaussian white noise with zero-mean and 1% coefficient of variation (COV) is considered on
measured data.

**FE model updating using incomplete modes**

Only incomplete modes are available in practice due to the difficulty of the identification of higher modes and limited number of sensors. For mimicking the field conditions using incomplete modes, the proposed BMUA is used to update the six-story shear building model with a different number of modes in case of a healthy condition. The initial value for each mass and stiffness parameter is identically taken twice as the exact value of the unity. Table 5.12 shows updated frequencies by the proposed approach with a different number of modes. It is seen that updated frequencies match well with actual counterparts. The first four modes are graphically compared in Figure 5.14 between actual mode shapes and updated counterparts. The updated mode shapes obtained from a different mode number coincide with the actual ones, indicating the proposed approach's robustness with incomplete measured data. Table 5.13 lists the results of updating mass and stiffness and their standard derivations (S.D.). The proposed BMUA can accurately identify mass and stiffness parameters (only error of less than 2% is found). Additionally, the S.D. representing the uncertainties tends to be reduced as the number of modes used to update the model increases.

### Table 5.12. Results of updated frequencies (Hz)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Actual</th>
<th>Four modes</th>
<th>Five modes</th>
<th>Six modes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>Updated</td>
<td>Updated</td>
</tr>
<tr>
<td>1</td>
<td>1.0177</td>
<td>1.0018</td>
<td>1.0015</td>
<td>1.0184</td>
</tr>
<tr>
<td>2</td>
<td>2.9838</td>
<td>2.9680</td>
<td>2.9645</td>
<td>2.9821</td>
</tr>
<tr>
<td>3</td>
<td>4.7465</td>
<td>4.7199</td>
<td>4.7147</td>
<td>4.7467</td>
</tr>
<tr>
<td>4</td>
<td>6.1858</td>
<td>6.1731</td>
<td>6.1628</td>
<td>6.1849</td>
</tr>
<tr>
<td>5</td>
<td>7.2035</td>
<td>7.1720</td>
<td>7.1791</td>
<td>7.2045</td>
</tr>
<tr>
<td>6</td>
<td>7.7303</td>
<td>7.7198</td>
<td>7.7258</td>
<td>7.7303</td>
</tr>
</tbody>
</table>
Figure 5.14. Updated mode shapes using incomplete modes

Table 5.13. Results of updated structural parameters

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Four modes</th>
<th></th>
<th>Five modes</th>
<th></th>
<th>Six modes</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Updated</td>
<td>S.D.</td>
<td>Updated</td>
<td>S.D.</td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.9855</td>
<td>0.0101</td>
<td>0.9895</td>
<td>0.0067</td>
<td>0.9897</td>
<td>0.0037</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.0128</td>
<td>0.0090</td>
<td>1.0111</td>
<td>0.0059</td>
<td>1.0071</td>
<td>0.0034</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.9875</td>
<td>0.0100</td>
<td>0.9905</td>
<td>0.0066</td>
<td>0.9922</td>
<td>0.0027</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>1.0182</td>
<td>0.0088</td>
<td>1.0108</td>
<td>0.0058</td>
<td>1.0071</td>
<td>0.0039</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.9925</td>
<td>0.0102</td>
<td>0.9915</td>
<td>0.0033</td>
<td>0.9962</td>
<td>0.0027</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>1.0198</td>
<td>0.0089</td>
<td>1.0138</td>
<td>0.0058</td>
<td>1.0044</td>
<td>0.0054</td>
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<td>$\theta_1$</td>
<td>1.0000</td>
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<td>0.0030</td>
<td>0.9967</td>
<td>0.0001</td>
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<tr>
<td>$\theta_2$</td>
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<td>0.0147</td>
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<td>$\theta_3$</td>
<td>1.0143</td>
<td>0.0204</td>
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<td>0.0176</td>
<td>1.0047</td>
<td>0.0034</td>
</tr>
<tr>
<td>$\theta_4$</td>
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<td>1.0019</td>
<td>0.0176</td>
<td>1.0038</td>
<td>0.0030</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>0.9894</td>
<td>0.0177</td>
<td>0.9989</td>
<td>0.0146</td>
<td>1.0032</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>0.9998</td>
<td>0.0191</td>
<td>0.9992</td>
<td>0.0031</td>
<td>0.9984</td>
<td>0.0012</td>
</tr>
</tbody>
</table>

Probabilistic damage detection

Change in flexural stiffness, $EI$ (Elastic modulus multiples by the second moment of inertia), and unit mass are used to simulate damage cases. Table 5.14 shows the different damage cases considered in this example. The negative sign denotes the reduction of mass/stiffness. Damage detection is usually conducted without any prior information on structural parameters. Thus, unity is defined as the initial value for each mass and stiffness.
parameter, representing an assumption of an initially healthy structural state. When detecting damage, measured data here includes six eigenvalues (square of frequency) and mode shapes. It should be mentioned that mass property is precisely known and deemed as invariable when updating stiffness in traditional BMUA. A similar assumption is made when updating mass parameters in traditional BUMA. Nevertheless, mass or stiffness's assumption is not required when applying the proposed BMUA.

Table 5.14. Damage cases

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Mass change</th>
<th>Stiffness change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-20% (5th floor)</td>
<td>-10% (1st floor), -20% (3rd floor)</td>
</tr>
<tr>
<td>2</td>
<td>-30% (2nd floor), -20% (5th floor)</td>
<td>-20% (2nd floor), -30% (4th floor), -40% (6th floor)</td>
</tr>
</tbody>
</table>

Table 5.15. Frequencies (Hz) comparison by the proposed and traditional approach

<table>
<thead>
<tr>
<th>Mode</th>
<th>Damage No.1 Proposed approach</th>
<th>Damage No.1 Traditional approach</th>
<th>Damage No.2 Proposed approach</th>
<th>Damage No.2 Traditional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Actual</td>
<td>Updated</td>
<td>Updated</td>
<td>Actual</td>
</tr>
<tr>
<td>1</td>
<td>1.0036</td>
<td>1.0182</td>
<td>0.9288</td>
<td>0.9944</td>
</tr>
<tr>
<td>2</td>
<td>2.9411</td>
<td>2.9379</td>
<td>2.6810</td>
<td>2.8743</td>
</tr>
<tr>
<td>4</td>
<td>6.2419</td>
<td>6.2414</td>
<td>5.7089</td>
<td>5.5866</td>
</tr>
<tr>
<td>6</td>
<td>7.9430</td>
<td>7.9511</td>
<td>7.0902</td>
<td>7.4084</td>
</tr>
</tbody>
</table>

Figure 5.15. Updated frequencies: (a) Damage case No.1; (b) Damage case No.2
The updated frequencies by the proposed BMUA have a highly acceptable agreement, less than 0.02, with actual values for two damage cases, as shown in Table 5.15 and Figure 5.15. In contrast, significant discrepancies between identified frequencies by the traditional BMUA and actual ones are observed: for damage case of No. 1, 10.74% bias at the 6th frequency; for damage case of No. 2, 14.26% bias at the 4th frequency.

An excellent coincidence between actual mode shapes and the proposed approach's identified mode shapes is exhibited in Figure 5.16; nevertheless, mode shapes acquired by traditional Bayesian greatly differ from actual mode shapes. Tables 5.16 and 5.17 present structural parameters, e.g., mass and stiffness, and corresponding uncertainties obtained from the proposed and the traditional BMUA in two damage cases. It is seen from Tables 5.16 and 5.17 and Figure 5.17 that identified the reduction in mass and stiffness parameters is almost identical to the actual values in terms of all damaged cases by the proposed BMUA, indicating the proposed BMUA has outstanding performance on damage detection. However, actual and identified values using the traditional BMUA have a large difference, e.g., the maximum error is 20% and 66% for both damage cases, respectively, (see bold values in Tables 5.16 and 5.17). It can be concluded that the traditional BMUA cannot
detect damage induced by both reductions in mass and stiffness. It attributes to the required assumption that at least one of mass and stiffness is assumed to be known and unchanged due to damage to avoid the coupling effect.

Figure 5.17. Damage case No.1: (a) mass (b) stiffness; Damage case No.2: (c) mass (d) stiffness
Table 5.16. Results of updated structural parameters for damage case No.1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>1.0000</td>
<td>1.0002</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.0019</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.0000</td>
<td>1.0001</td>
<td>0.0017</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.0019</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.8000</td>
<td>0.7984</td>
<td>0.0018</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>1.0000</td>
<td>0.9999</td>
<td>0.0009</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.9000</td>
<td>0.8962</td>
<td>0.0019</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1.0000</td>
<td>1.0014</td>
<td>0.0013</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>0.8000</td>
<td>0.8014</td>
<td>0.0002</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>1.0000</td>
<td>1.0013</td>
<td>0.0009</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>1.0000</td>
<td>1.0014</td>
<td>0.0022</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>1.0000</td>
<td>1.0014</td>
<td>0.0022</td>
</tr>
</tbody>
</table>

Table 5.17. Results of updated structural parameters for damage case No.2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>1.0000</td>
<td>0.9965</td>
<td>0.0061</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.7000</td>
<td>0.6991</td>
<td>0.0105</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.0000</td>
<td>0.9969</td>
<td>0.0085</td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>1.0000</td>
<td>1.0010</td>
<td>0.0055</td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>0.8000</td>
<td>0.7955</td>
<td>0.0024</td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>1.0000</td>
<td>0.9998</td>
<td>0.0006</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>1.0000</td>
<td>1.0028</td>
<td>0.0025</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>0.8000</td>
<td>0.7958</td>
<td>0.0012</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>1.0000</td>
<td>1.0009</td>
<td>0.0012</td>
</tr>
<tr>
<td>$\theta_4$</td>
<td>0.7000</td>
<td>0.7012</td>
<td>0.0011</td>
</tr>
<tr>
<td>$\theta_5$</td>
<td>1.0000</td>
<td>1.0022</td>
<td>0.0008</td>
</tr>
<tr>
<td>$\theta_6$</td>
<td>0.6000</td>
<td>0.5979</td>
<td>0.0007</td>
</tr>
</tbody>
</table>

This example illustrates that variation in mass or stiffness reflecting the damage extent cannot be ignored; if not, significant bias on updating results can mislead engineers'
judgment. In the proposed BMUA, mass and stiffness's coupling effect is addressed by employing two groups of data acquired from two systems: original and modified systems with stiffness addition. As a result, a successful updating in mass and stiffness is achieved. In summary, the example of a six-story shear building demonstrates the proposed BMUA is superior to traditional BMUA in identifying mass and stiffness.

The probability of damage is calculated based on the MPVs of mass and stiffness and corresponding the value of S.D. using Eq. (5.5). As seen in Figure 5.18, for damage case of No. 1, probabilistic curves of the mass parameter at the fifth floor ($\beta_5$) and stiffness parameter ($\theta_3$) at the third-floor exhibit high probabilities (77.88% and 80.66%, respectively) of having a possible reduction of both 20%. Regarding damage case of No. 2, mass on the second floor ($\beta_2$) and stiffness on the fourth floor ($\theta_4$) have a possible reduction of 30% with a high probability of 70.72% and 73.1%, respectively. The proposed BMUA exhibits excellent performance for damage detection; both localization and quantification of damage are successfully identified, showing real potential applications.
Figure 5.18. Probabilistic curves: damage case No.1: (a) mass (b) stiffness; damage case No.2: (c) mass (d) stiffness

5.5.7.2 Example 2: three-dimensional three-story braced shear frame

A three-dimensional three-story shear building is utilized to evaluate the proposed BMUA under more complex condition, where includes damage detection and comparative investigation between the traditional and the proposed BMUA
The diagram of the structural model is shown in Figure 5.19. The mass at each floor is designed as $M = 10^4$ Kg, giving three to-be-updated mass parameters. Four stiffness at each floor are assumed, giving twelve to-be-updated stiffness parameters as $\theta_{4(j-1)+1} = K_{j,+x}$, $\theta_{4(j-1)+2} = K_{j,+y}$, $\theta_{4(j-1)+3} = K_{j,-x}$, $\theta_{4(j-1)+4} = K_{j,-y}$, $j = 1,2,3$, $j$ denotes number of story; $+x$, $-x$, $+y$ and $-y$ are directions of structural outer face. Nominal magnitudes of inter-story stiffness are assumed to be $K_{j,+y} = K_{j,-y} = 40000$ KN/m and $K_{j,+x} = K_{j,-x} = 50000$ KN/m.

The damaged structure after earthquake is repaired using a typical retrofitting technique. Suppose the frame model is subject to unknown seismic activities, yielding serious structural damage, such as cracks and bearing deterioration, further impairing stiffness and ductility. Therefore, it is essential to repair the structure and strengthen its resistance capability to avoid any collapse. Herein, Buckling-restrained brace (BRB) is a useful seismic retrofit equipment and can provide additional stiffness and energy dissipation capacity (Saingam et al., 2020). In this work, twelve BRBs marked as red color are welded on the structure at four directions, shown in Figure 5.19 (b). Assuming each BRB installed

---

**Figure 5.19.** Diagram of steel frame model: (a) original system and plan view; (b) modified system with BRBs (marked as red color)
at $x$ or $y$ direction has the same sectional and material properties, to provide the same stiffness addition, $\Delta k_{j,+x} = \Delta k_{j,-x} = 1750 \text{ KN/m}$, $\Delta k_{j,+y} = \Delta k_{j,-y} = 1400 \text{ KN/m}$ on each floor, the weight of each BRB is ignored here. In this example, measured data contains the first six eigenvalues and mode shapes in the original and modified system with stiffness addition. Gaussian white noise with zero-mean and 1% COV of modal parameters is considered on measured data for both the proposed BMUA and traditional BMUA.

**Probabilistic damage detection**

Alterations in mass and stiffness parameters are used to mimic damage cases (see Table 5.18). The unity is defined as the initial value for each mass and stiffness parameter when detecting damage.

Table 5.18. Damage cases

<table>
<thead>
<tr>
<th>Case No.</th>
<th>Mass change</th>
<th>Stiffness change</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>-20% (1st floor)</td>
<td>-15% (2nd floor, $+x$ face)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-25% (3rd floor, $+y$ face)</td>
</tr>
<tr>
<td></td>
<td></td>
<td>-20% (1st floor, $+x$ face)</td>
</tr>
<tr>
<td>2</td>
<td>-20% (2nd floor)</td>
<td>-30% (2nd floor, $-x$ face)</td>
</tr>
<tr>
<td></td>
<td>-30% (3rd floor)</td>
<td>-40% (3rd floor, $+y$ face)</td>
</tr>
</tbody>
</table>

A comparative investigation is also carried out to compare the proposed BMUA against the traditional counterpart. The traditional BMUA is performed by assuming that mass is known and undamaged when only updating stiffness. Similarly, stiffness is known and undamaged when only updating mass by the traditional BMUA. However, neither mass nor stiffness information is needed when applying the proposed BMUA.
Table 5.19. Frequencies (Hz) comparison by the proposed and traditional approach

<table>
<thead>
<tr>
<th>Mode</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Actual</td>
<td>Updated</td>
<td>Updated</td>
<td>Actual</td>
</tr>
<tr>
<td>1</td>
<td>6.3474</td>
<td>6.3474</td>
<td>5.8556</td>
<td>7.1355</td>
</tr>
<tr>
<td>4</td>
<td>17.8686</td>
<td>17.8686</td>
<td>18.8641</td>
<td>17.8885</td>
</tr>
<tr>
<td>5</td>
<td>20.9130</td>
<td>20.9130</td>
<td>22.6068</td>
<td>20.4540</td>
</tr>
</tbody>
</table>

Figure 5.20. Updated frequencies: (a) Damage case No.1; (b) Damage case No.2

Table 5.20. MAC values by two approaches

<table>
<thead>
<tr>
<th>Mode number</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.9999</td>
<td>0.9974</td>
<td>1.0000</td>
<td>0.9989</td>
</tr>
<tr>
<td>2</td>
<td>1.0000</td>
<td>0.9994</td>
<td>1.0000</td>
<td>0.9968</td>
</tr>
<tr>
<td>3</td>
<td>0.9999</td>
<td>0.9988</td>
<td>1.0000</td>
<td>0.9984</td>
</tr>
<tr>
<td>4</td>
<td>1.0000</td>
<td>0.9987</td>
<td>1.0000</td>
<td>0.9891</td>
</tr>
<tr>
<td>5</td>
<td>0.9992</td>
<td>0.9994</td>
<td>1.0000</td>
<td>0.9962</td>
</tr>
<tr>
<td>6</td>
<td>0.9981</td>
<td>0.9945</td>
<td>0.9999</td>
<td>0.9955</td>
</tr>
</tbody>
</table>

Table 5.19 and Figure 5.20 present updated frequencies by the proposed and traditional
BMUA in terms of two damage cases. Frequencies are updated with a considerably accurate level using the proposed approach for all damage cases. However, frequencies are updated by the traditional approach with significant error. Table 5.20 lists diagonal values of modal assurance criterion (MAC) for mode shapes by the proposed and traditional BMUA in terms of different damage cases. It is observed that the proposed approach's mode shapes are consistent with actual ones, while the traditional Bayesian approach provides biased MAC values. For example, the traditional approach yields a MAC value of 0.9891 at the fourth mode for the damage case of No. 2; a MAC value of 1.0000 is obtained by the proposed approach.

Table 5.21. Results of updated structural parameters for damage case No.1

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>S.D.</td>
</tr>
<tr>
<td>β₁</td>
<td>0.8000</td>
<td>0.8003</td>
<td>0.0014</td>
</tr>
<tr>
<td>β₂</td>
<td>1.0000</td>
<td>1.0008</td>
<td>0.0029</td>
</tr>
<tr>
<td>β₃</td>
<td>1.0000</td>
<td>0.9997</td>
<td>0.0015</td>
</tr>
<tr>
<td>θ₁⁺⁺ₓ</td>
<td>1.0000</td>
<td>0.9861</td>
<td>0.0020</td>
</tr>
<tr>
<td>θ₁⁺⁺ᵧ</td>
<td>1.0000</td>
<td>0.9943</td>
<td>0.0035</td>
</tr>
<tr>
<td>θ₁⁻⁻ₓ</td>
<td>1.0000</td>
<td>0.9833</td>
<td>0.0019</td>
</tr>
<tr>
<td>θ₁⁻⁻ᵧ</td>
<td>1.0000</td>
<td>1.0052</td>
<td>0.0036</td>
</tr>
<tr>
<td>θ₂⁺⁺ₓ</td>
<td>0.8500</td>
<td>0.8485</td>
<td>0.0022</td>
</tr>
<tr>
<td>θ₂⁺⁺ᵧ</td>
<td>1.0000</td>
<td>1.0046</td>
<td>0.0035</td>
</tr>
<tr>
<td>θ₂⁻⁻ₓ</td>
<td>1.0000</td>
<td>1.0014</td>
<td>0.0027</td>
</tr>
<tr>
<td>θ₂⁻⁻ᵧ</td>
<td>1.0000</td>
<td>0.9952</td>
<td>0.0039</td>
</tr>
<tr>
<td>θ₃⁺⁺ₓ</td>
<td>1.0000</td>
<td>1.0054</td>
<td>0.0041</td>
</tr>
<tr>
<td>θ₃⁺⁺ᵧ</td>
<td>0.7500</td>
<td>0.7481</td>
<td>0.0011</td>
</tr>
<tr>
<td>θ₃⁻⁻ₓ</td>
<td>1.0000</td>
<td>1.0102</td>
<td>0.0039</td>
</tr>
<tr>
<td>θ₃⁻⁻ᵧ</td>
<td>1.0000</td>
<td>0.9994</td>
<td>0.0014</td>
</tr>
</tbody>
</table>

The MPVs and associated standard derivation (S.D.) by two Bayesian approaches are shown in Tables 5.21 and 5.22. The proposed approach achieves satisfactory updating in
mass and stiffness parameters. In contrast, the traditional Bayesian approach poorly updates mass and stiffness parameters. For example, it was found that for damage case of No. 1 in Table 5.21, the mass parameter, $\beta_1$, and stiffness parameter, $\theta_{1,-y}$, are updated as 0.0109 and 1.3921 (target: 0.8000 and 1.0000), respectively; for damage case of No. 2 in Table 5.22, mass, and stiffness parameter, $\beta_3$ and $\theta_{3,-y}$, are updated as 0.0078 and 1.3603 (target: 0.7000 and 1.0000). As seen in Figure 5.21, the mass/stiffness parameters identified by the proposed approach highly agree well with target values for considered damage cases. It only observes an error of less than 3%. However, mass/stiffness parameters are updated with much discrepancy by the traditional Bayesian; even some updated results provide unacceptable values, such as too small values in the updated mass parameters. Like the first example, the traditional Bayesian approach fails to update mass and stiffness parameters, indicating poor or false damage detection. It can be attributed to an assumption of ignoring variation in mass or stiffness induced by damage, suggesting mass and stiffness’s coupling effect existing in traditional Bayesian governs the accuracy of updated results when simultaneously updating mass and stiffness.

Table 5.22. Results of updated structural parameters for damage case No.2

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Actual</th>
<th>Proposed approach</th>
<th>Traditional approach</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>1.0000</td>
<td>0.9997</td>
<td>0.0025</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>0.8000</td>
<td>0.8001</td>
<td>0.0009</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>0.7000</td>
<td>0.6974</td>
<td>0.0051</td>
</tr>
<tr>
<td>$\theta_{1,+x}$</td>
<td><strong>0.8000</strong></td>
<td><strong>0.7961</strong></td>
<td>0.0036</td>
</tr>
<tr>
<td>$\theta_{1,+y}$</td>
<td>1.0000</td>
<td>1.0017</td>
<td>0.0035</td>
</tr>
<tr>
<td>$\theta_{1,-x}$</td>
<td>1.0000</td>
<td>0.9986</td>
<td>0.0041</td>
</tr>
<tr>
<td>$\theta_{1,-y}$</td>
<td>1.0000</td>
<td>0.9978</td>
<td>0.0039</td>
</tr>
<tr>
<td>$\theta_{2,+x}$</td>
<td>1.0000</td>
<td>0.9938</td>
<td>0.0048</td>
</tr>
<tr>
<td>$\theta_{2,+y}$</td>
<td>1.0000</td>
<td>1.0265</td>
<td>0.0041</td>
</tr>
</tbody>
</table>
The MPVs of mass and stiffness under the damaged condition and corresponding uncertainties are utilized to compute the probability given a certain damage level. Figure 5.22 illustrates that for damage case of No.1, mass parameter, $\beta_1$, and stiffness parameter,
\( \theta_3, +y \) have high probability (61.22\% and 85.62\%, respectively) of having possible damage 20\% and 25\%. For damage case of No. 2, mass parameter, \( \beta_2 \), and stiffness parameter, \( \theta_3, +y \) have possible reduction of 20\% and 40\% with high probability of 83.62\% and 76.38\%, respectively.

\[ \text{(a) } \quad \text{(b) } \quad \text{(c) } \quad \text{(d)} \]

Figure 5.22. Probabilistic curves: damage case No.1: (a) mass (b) stiffness; damage case No.2: (c) mass (d) stiffness

### 5.6 Conclusions

This chapter presented a new Bayesian updating framework using only output-only vibration data of two structural systems: the original system and the modified system by adding known mass (\( \Delta m \)) (Section 5.4) and adding known stiffness (Section 5.5). Two numerical examples illustrate that the proposed BMUA with added either mass or stiffness
has an advantage against the traditional BMUA in identifying mass and stiffness. In the traditional BMUA, at least one of mass and stiffness is accurately known, but this assumption is quite questionable; Traditional Bayesian also ignores variation in mass or stiffness induced by possible damage. In contrast, the new BMUA with added mass or stiffness does not involve any assumption on mass and stiffness properties before model updating. In short, the proposed BMUA enables us to deal with the coupling effect of mass and stiffness so that successfully identifying mass and stiffness.

The main conclusions are presented as follows:

- The proposed updating approach can update both mass and stiffness with quite an acceptable level of performance. This indicates that the proposed approach can solve the issue of the coupling effect in mass and stiffness updating.
- A comparative study was performed between the proposed approach and the conventional approach. The proposed updating framework provides highly accurate and reliable updating results as compared with the conventional approach.
- By measuring only output-only vibration data in the original and modified systems, the damage detection capability was examined in various damage scenarios. It was found that the proposed Bayesian approach can give us a reasonable probability estimation of damage for diverse structural parameters.
6.1 Introduction

In Chapter 5, a new Bayesian model updating framework was proposed to update mass and stiffness with addressing the coupling effect of mass and stiffness for 2D and 3D numerical shear structure (Chapter 5). It also demonstrated that classical Bayesian updating work cannot update both mass and mass stiffness, simultaneously, when the coupling effect exists. The coupling effect was successfully addressed using two sets of vibration data acquired from two systems: original and modified with added known mass/stiffness. The asymptotic approximation method was employed to circumvent high-dimensional integrals involved in the posterior PDF for Bayesian inference. The analytical formulations of optimal model parameters are derived by the linear optimization method; associated uncertainties are quantified by an inverse Hessian matrix of the objective function. However, the asymptotic approximation method assumes that parameters have unimodal and Gaussian distribution that does not necessarily guarantee an actual physical model when a high level of modeling error and measurement noise occur in practice, especially for multi-modal and non-Gaussian posterior (Wan and Ren, 2016, Yang and Lam, 2018a). Also, an insufficient amount of data and complex model class may lead to an unidentifiable problem.
One promising way to solve multi-modal and unidentifiable problems is using Markov Chain Monte Carlo (MCMC) to generate samples to approximate the posterior PDF. The high-dimensional integrals in the Bayesian approach can be reasonably calculated. Another attractive feature in MCMC does not require the assumption on the physical model and accurately represents the posterior PDF. Various MCMC techniques have been developed for posterior distribution sampling (Simoen et al., 2013, Green, 2015, Wan and Ren, 2016, Huang and Beck, 2018, Mao et al., 2020b). These methods adopt a single Markov chain to draw samples, it has demonstrated a limited capability to treat high-dimensional, multi-modal, and flat manifold PDFs. Therefore, they have a relatively low convergence rate and cannot guarantee adequate exploration in parameter space for a target PDF (Vrugt, 2016).

This chapter proposed using the Differential Evolution Adaptive Metropolis (DREAM) (Vrugt, 2016) algorithm to proceed with the distribution estimate. DREAM is essentially a multi-chain (multiple Markov chains) MCMC that runs different paths in parallel to target the posterior PDF. It combines different powerful strategies, including a genetic algorithm for population evolution (Price et al., 2006), self-adaptive randomized subspace sampling, and outlier chain detection (Vrugt et al., 2009a), to quickly achieve convergence and seek the best solution by running multiple Markov chains. A wide range of applications has shown that DREAM exhibits excellent performance for complex problems with high-dimensionality, nonlinearity, numerous peaks, and large uncertainties in different research fields, including hydrology (Vrugt et al., 2009b, Shafii et al., 2014), chemistry (DeCaluwe et al., 2014, Gentsch et al., 2014), geophysics (Lochbühler et al., 2015, Zhai et al., 2021) and renewable energy technique (Zhang et al., 2021), etc. However, to the authors’ best knowledge, DREAM has not been investigated in SHM for civil infrastructures. The
current study attempts to explore the efficacy of DREAM in Bayesian model updating approach (BMUA).

The new characteristic equations are constructed by two sets of vibration data measured from the original and modified system with added known mass/stiffness. The posterior PDF is reformulated by measured modal data and predicted counterparts from the new characteristic equations. The DREAM algorithm is then employed to generate samples for approximation of the posterior PDF. The proposed BMUA simultaneously identifies the mass and stiffness; their uncertainties are also straightforward provided by the estimated PDF. A numerical study on a ten-story shear building and an experimental study on a three-story aluminum frame small-scale model are used at intact and damaged structural states to verify the accuracy and feasibility of the proposed method.

The outline of this chapter is listed as follows. The background of classical BMUA is first described in Section 6.2. Section 6.3 presents the methodology of the proposed BMUA, in which the new characteristic equations, strategies of mass addition, and DREAM algorithm are introduced explicitly. Section 6.4 shows one illustrative example to validate the methodology using a numerical example, followed by the validation of laboratory-scale testing. Probabilistic damage detection is also performed. The comparison of BMUA with mass addition and stiffness addition is discussed in Section 6.5. Finally, conclusions and summaries are provided in Section 6.6.

6.2 The classical vibration-based Bayesian model updating

The strength of the BMUA lies in that it uses both the prior information (existing structural knowledge) and measured data (new structural knowledge) to estimate the
posterior PDF. In other words, the Bayesian approach updates the prior PDF by measured data, yielding the posterior PDF.

In Chapter 5, the posterior PDF of uncertainty parameter $\mathbf{\Omega}$, including mass and stiffness parameters is given by Eq. (5.2). In many cases, the selection of prior PDF depends on engineers’ judgment and physical meaning. The uniform distribution is widely used as the uninformative prior PDF to ensure the measured data entirely dominates Bayesian inference and minimizes the effect of prior information. The term of $P(D|C)$ in Eq. (5.2) is a normalizing constant so that the posterior PDF can be integrated to unity over the parameter space, which is given by,

$$P(D|C) = \int P(\mathbf{\Omega}|C)P(D|\mathbf{\Omega}, C) \, d\mathbf{\Omega}$$

(6.1)

The likelihood function, $P(D|\mathbf{\Omega}, C)$, describes how likely the measurements are reproduced from a model parameterized by a set of $\mathbf{\Omega}$. Considering an uninformative prior PDF, the posterior PDF is proportional to the likelihood function:

$$P(\mathbf{\Omega}|D, C) = c_0 P(D|\mathbf{\Omega}, C)$$

(6.2)

where $c_0$ represents a constant value to reflect both $P(D|C)$ and $P(\mathbf{\Omega}|C)$.

Generally, for vibration-based system identification, the common measured data in the likelihood function consists of measured natural frequencies and mode shapes. Then, two error functions (EF) of a given one mode, $m$, are adopted to formulate the likelihood function, namely frequency EF and mode shape EF (Yuen et al., 2006). Frequency EF, $\varepsilon_{f,m}$, is defined as:

$$\varepsilon_{f,m} = \tilde{f}_m - f_m(\mathbf{\Omega})$$

(6.3)

where $\tilde{f}_m$ is the $m$th measured frequency, $f_m(\mathbf{\Omega})$ is the $m$th calculated frequency in a model given a set of $\mathbf{\Omega}$. 
Mode shape EF, $\varepsilon_{ms,m}$, is defined as:

$$
\varepsilon_{ms,m} = \bar{\phi}_m - L_0 \phi_m(\Omega)
$$

(6.4)

where $\bar{\phi}_m$ and $\phi_m(\Omega)$ are measured mode shape and calculated one of the $m$th mode, respectively. $L_0$ consists of '1s' or '0s' to match measured partial mode shapes with theoretical counterparts. Note all mode shapes are normalized to unity norm to map them in the same context.

With the assumption that $\varepsilon_{f,m}$ and $\varepsilon_{ms,m}$ follow zero-mean Gaussian distribution, then the posterior PDF in Eq. (5.2) is rewritten as follows:

$$
P(\Omega|D, C) = c_0 \exp \left( -\frac{1}{2\kappa^2} J(\Omega) \right)
$$

(6.5)

The objective function, Eq. (6.6), can evaluate the accuracy of predicted natural frequency and mode shape obtained from new characteristic equations against the measured data.

$$
J(\Omega) = \sum_{m=1}^{n} \left[ (\bar{f}_m - f_m(\Omega))^2 + \left( (\bar{\phi}_m - L_0 \phi_m(\Omega))^T \left( \bar{\phi}_m - L_0 \phi_m(\Omega) \right) \right) \right]
$$

(6.6)

where $\kappa$ is an uncertainty parameter of prediction error. In the current study, the variances of the measured frequency and mode shape are used as $\kappa^2$. $\kappa$ consists of $\sigma_{f,m}$ and $\sigma_{ms,m}$; $\sigma_{f,m}$ and $\sigma_{ms,m}$ are the standard derivation of the $m$th measured frequency and mode shape, respectively. These two weighting factors can be identified by either Bayesian modal analysis (Au, 2011b) or stochastic subspace identification (SSI) with uncertainty analysis (Zeng and Kim, 2021), rather than manually tuning.

For avoiding intractable high-dimensional integrals, MCMC is employed to approximate the posterior PDF in Eq. (6.5) without any assumption on a model by iteratively drawing samples from the target distribution. Classical BMUA calculates
theoretical frequency, \( f_m(\Omega) \), and mode shape, \( \phi_m(\Omega) \) in Eq. (6.6), given a set of \( \Omega \) using the classical characteristic equation \( (K - \lambda M)\phi = 0 \). Understandably, simultaneous updating stiffness and mass yield an unidentifiable problem due to the coupling effect of mass and stiffness. The infinite sets of mass and stiffness derive the same frequency so that correct model updating cannot be achievable. The new characteristic equations with added mass will substitute classical ones and address the coupling effect in the next section.

6.3 The formulations of a new vibration-based Bayesian model updating

New characteristic equations with added known mass are first presented in Section 6.3.1 to address the coupling effect of mass and stiffness. The mass-adding strategies are discussed in Section 6.3.2, including the number, location, and magnitude of added mass. The DREAM algorithm, a multi-chain MCMC to approximate the posterior PDF, is presented in section 6.3.3.

6.3.1 New characteristic equations with added mass

The original and modified systems with added mass, \( \Delta m \), are merged into one equation based on the fundamentals of structural dynamics. The core idea of addressing the coupling effect of mass and stiffness is to eliminate either mass or stiffness when updating each of them. For example, first, characteristic equations for the original and modified systems are expressed as:

\[
K\phi = M\phi\lambda
\]  
\[
K\phi' = (M + \Delta m)\phi'\lambda'
\]

where \( \lambda \) and \( \phi \) are eigenvalues (square of natural frequencies) and mode shapes before modification; \( \lambda' \) and \( \phi' \) are eigenvalues and mode shapes after modification.
Second, the new eigen-equation error with added mass when updating mass is derived as:

\[(\lambda - \lambda')^{-1}\lambda' \phi'^T \Delta m \phi - \phi'^T M \phi = 0\] (6.9)

For the sake of simplicity, more details can be found in Section 5.4.1. Eq. (6.9) can be rewritten as:

\[(\lambda' \phi'^T \Delta m - \lambda \phi'^T M + \lambda' \phi'^T M)\phi = 0\] (6.10)

Define \(A = \lambda' \phi'^T \Delta m + \lambda' \phi'^T M\), \(B = \phi'^T M\), then Eq. (6.10) is expressed as:

\[(A - \lambda B)\phi = 0\] (6.11)

Similarly, when updating stiffness, the new eigen-equation error is shown as (details are in Section 5.5.1):

\[(\lambda''^{-1} - \lambda^{-1})^{-1} \phi'^T \Delta m \phi - \phi'^T K \phi\] (6.12)

Eq. (6.12) can be rewritten as:

\[(\lambda''^{-1} \phi'^T K - \lambda^{-1} \phi'^T K - \phi'^T \Delta m)\phi = 0\] (6.13)

Define \(E = \lambda''^{-1} \phi'^T K - \phi'^T \Delta m\), \(F = \phi'^T K\), then Eq. (6.13) is expressed as:

\[(F - \lambda E)\phi = 0\] (6.14)

Eqs. (6.11) and (6.14) are defined as the new characteristic equations to replace the classical one \((K - \lambda M)\phi = 0\). It is noted that the two new characteristic equations have the same formats as the generalized eigenvalue problem, \(\lambda\) and \(\phi\) can be easily solved in mathematics or solver in the computer program, such as ‘eig’ function in MATLAB.

Two new characteristic equations eliminate the coupling effect of mass and stiffness. For example, mass updating by using Eq. (6.11) does not require any stiffness information. Likewise, updating stiffness does not require any mass information by using Eq. (6.14).

For output-only modal analysis, the mode shapes are not mass-normalized, and only unscaled mode shapes are identified because of unknown excitation forces. Before the
model updating, the measured mode shapes have to be normalized by either the mass-change scaling method (López-Aenlle et al., 2010) or stiffness-change scaling method (Khatibi et al., 2012) to ensure measured and predicted mode shapes are comparative. In this chapter, the mass-change scaling method is adopted to calculate scaled mode shapes. In addition, because only a few DOFs are available related to the sensor location, limited sensors in practice usually lead to incomplete measured mode shapes. Therefore, mode shape expansion techniques (Chen et al., 2012) can expand measured mode shapes to complete mode shapes of full DOFs.

6.3.2 Strategy of mass addition

The optimized mass-change strategy has been comprehensively discussed in (López-Aenlle et al., 2010), including mass magnitude, number of added mass, and locations of added mass. Generally, two criteria for creating a modified system with added known mass are required: Step 1) noticeable frequency change is observed between the original system and modified system; Step 2) mode shapes after modification change slightly.

The frequency change and mass addition are correlated by natural frequencies in the original and modified systems. Considering a structure with multiple DOFs, the relation between added mass and frequency shift can be expressed as (López-Aenlle et al., 2010):

\[
\frac{\Delta f}{f} = 1 - \sqrt{1 + \frac{\Delta M}{M^*}}
\]

where \( \Delta f = f' - f \) is the frequency change after adding mass; \( f \) and \( f' \) are the natural frequencies in the original and modified systems, respectively; \( \Delta M = \psi^T \Delta m \psi; M^* = \psi^T M \psi \), where \( \psi \) is unscaled mode shape in the original system; \( \Delta m \) is a diagonal matrix with main diagonal are added mass; \( M \) is a mass matrix in the original system. Eq. (6.15)
allows us to determine expected added mass when the terms of $\Delta f$, $f$, and $M^*$ are known; $f$ and $\psi$ are identified by modal analysis, the analytical mass is used as $M^*$. Based on (López-Aenlle et al., 2010), we select a frequency ratio, $f/f'$ to determine $\Delta M$ in Eq. (6.15). Note that the selection of ratio depends on the expected accuracy in modal analysis and mode shape normalization. Finally, the magnitude of added mass can be estimated using Eq. (6.15).

The number of added masses depends on the number of modes to identify in modal analysis. Ideally, the added or attached mass should be as many as possible. López-Aenlle et al. (2010) recommended that the number of added masses should be at least the number of peaks and valleys of each mode shape. To optimize the location of added mass, the most significant frequency shift can occur when the mass is attached to the peaks and valleys of mode shape, while the frequency shift is minimal when mass is attached to the nodal positions.

6.3.3 DREAM algorithm

The posterior PDF needs high-dimensional integrals that is impractical for complex structures. In the present work, the DREAM algorithm proposed by Vrugt et al. (2009a) is used to approximate the posterior PDF by generating samples based on a differential evolutionary algorithm. Compared to other single-chain MCMC methods, the DREAM has the appealing feature of running multiple chains simultaneously to explore global solutions. The DREAM algorithm uses randomized subspace sampling to automatically tune the mean and variance of the proposal distribution. Therefore, it is highly robust to the selection of the prior distribution.
The removal of outlier chain and crossover schemes are also used to expedite convergence to a target distribution. Practical applications exhibited high efficiency and accuracy in the sampling for the problems having high-dimensionality, nonlinearity, numerous peaks, and local optima. Theoretical background and detailed MATLAB procedures in DREAM can be found in (Vrugt et al., 2009a, Vrugt, 2016). The flowchart of the DREAM algorithm is also shown in Figure 6.1. The main implementation steps of the DREAM algorithm are summarized as follows:

**Step 1:** Initialize the problem dimension $N$, the number of Markov chains $P$, unknown parameter vector, $\mathbf{\Omega}_i^j$ ($i = 1, 2, 3 \ldots, N; j = 1, 2, 3 \ldots, P$), and the maximum iteration, $I_{max}$. $\gamma$ individual samples for each chain are randomly
generated from the selected prior distribution as initial values, such as $\Omega_1^i, \Omega_2^i, \ldots, \Omega_P^i$.

**Step 2:** A mutation operation is performed to generate candidate samples at each parameter sample of each iteration for the $k$th Markov chain. Crossover operation is then used to iteratively update current candidate samples from the mutation process based on crossover probability $CR$ within the range of $[0,1]$.

**Step 3:** Calculate the posterior probability and acceptance rate of updated candidate samples at the $s$th iteration:

$$
\alpha(\Omega_{i,s}^j, w_{i,s+1}^j) = \begin{cases} 
\min \left( \frac{p(w_{i,s+1}^j | D)}{p(\Omega_{i,s}^j | D)}, 1 \right) ; & p(\Omega_{i,s}^j | D) > 0 \\
1 ; & p(\Omega_{i,s}^j | D) < 0
\end{cases}
$$

(6.16)

where $\Omega_{i,s}^j$ and $w_{i,s+1}^j$ are the samples at the $s$th iteration and $(s + 1)$th iteration, respectively; $\alpha(\Omega_{i,s}^j, w_{i,s+1}^j)$ is the acceptance rate; $p(w_{i,s+1}^j | D)$ and $p(\Omega_{i,s}^j | D)$ are the posterior probability of $w_{i,s+1}^j$ and $\Omega_{i,s}^j$, respectively. $D$ is the measured data.

**Step 4:** Determine whether accepting or rejecting the samples of $w_{i,s+1}^j$. If $\alpha(\Omega_{i,s}^j, w_{i,s+1}^j) > u$, $u$ is randomly generated from a uniform distribution $U(0,1)$.

Then, accept a new sample of $w_{i,s+1}^j$, otherwise reject and keep the iteration.

**Step 5:** Remove the outlier chain using the Inter-Quartile-Range (IQR) statistical method (Vrugt et al., 2009a). Specifically, $H$ is firstly defined as the mean of the logarithm of the posterior distribution of the last half samples in each chain. $H = \frac{Q_3 - Q_1}{2}$ is calculated, where $Q_1$ and $Q_3$ are the lower and upper quartile of the $P$ chains. Chains with $H < Q_1 - 2 \cdot \text{IQR}$ are detected as aberrant ones. Note removal
of outlier chain is necessary, as outlier chains will impair the distribution estimate and slow down the evolution so that reaching a good convergence is impossible. In addition, outlier chains frequently present in high-dimensional problems and tend to be stuck in local optima, resulting in a biased distribution (Vrugt et al., 2009a).

**Step 6:** The iteration process stops when Markov chains converge to the target posterior distribution. Otherwise, repeat steps 2-5. DREAM algorithm uses Gelman-Rubin statistics, scale reduction factor $R_{stat}$ (Gelman and Rubin, 1992), as a convergence criterion to determine whether the calculation terminates or not. DREAM algorithm stipulates that if $R_{stat} < 1.2$ for all unknown parameters, a stable posterior PDF is achieved. Note the value of 1.2 has been demonstrated as a robust indication to officially declare stationary and reliable convergence (Vrugt et al., 2009a). $R_{stat}$ has an expression as follows:

$$R_{stat} = \sqrt{\frac{\gamma - 1}{\gamma} + \frac{P + 1}{P \cdot Z}}$$  \hspace{1cm} (6.17)

where $\gamma$ is the number of iteration samples of each chain; $P$ is the number of Markov chains used for sampling; $Z$ is the mean of the variance of total $P$ Markov chains; the ratio of $B/\gamma$ is the variance of the mean of $P$ parallel Markov chains.

In summary, the proposed BMUA addresses the coupling effect of mass and stiffness by using two sets of measurements from the original and modified system with added mass. Two new characteristic equations (herein, Eqs. (6.11) and (6.14)) substitute the classical one. Figure 6.2 shows the flowchart of the proposed method. First, the natural frequencies and mode shapes of the original and modified system are identified using the output-only modal analysis method. Note that mode shapes need to be normalized by the mass-change
scaling method before updating mass and stiffness. Second, the objective functions in Eq. (6.6) with measurements in the original system are used to measure the accuracy of analytical frequencies and mode shapes satisfying with new characteristic equations, e.g., Eqs. (6.11) and (6.14). Third, the DREAM is used to approximate the posterior PDF and estimate the quantity of interests (PDF, mean, and variance. The procedures of updating mass and stiffness are independent and individually implemented. Therefore, the coupling effect has been removed in the entire updating process.

Figure 6.2. The flowchart of the proposed Bayesian model updating

6.4 Illustrative examples

The efficacy of the proposed BMUA is evaluated by a numerical example in Section 6.4.1, followed by an experimental test with a laboratory-scale three-story shear frame in Section 6.4.2.
6.4.1 Numerical example: a ten-story shear building

The ten-story shear building sketch is shown in Figure 6.3, modeled as a ten-DOFs structure. Assume the connection between column and floor is rigid; mass and stiffness at each floor are uniformly distributed. Also, suppose one sensor is installed on each floor to measure all modal displacements in each mode shape. Lumped mass is used and taken as $M_i = 25 \text{ kg}, i = 1,2,\ldots,10$. While the inter-story stiffness at each floor is taken as $K_i = 1.5 \times 10^6 \text{ N/m}, i = 1,2,\ldots,10$. We define stiffness coefficient (SC) as $\theta_i = K_i^a/K_i$, and mass coefficient (MC) $\beta_i = M_i^a/M_i$, where $K_i^a$ and $K$ are the $i$th actual and theoretical stiffness, respectively; $M_i^a$ and $M$ are the $i$th actual and theoretical mass, respectively, resulting in a total of 20 coefficients to be updated.

The FE model of this shear building is constructed based on fundamental structural dynamics using MATLAB. The natural frequencies and mode shapes for the original structure can be obtained by the eigenvalue problem so that the first six natural frequencies are $5.827, 17.350, 28.486, 38.985, 48.613,$ and $57.156 \text{ Hz}$. To create a modified structure, we first select a frequency ratio, $f/f'$, of $1.02$. Using Eq. (6.15), the magnitude of added
mass can be estimated as 1 kg. For the sake of simplicity, each floor has the equivalent mass addition by concrete blocks with the weight of 1 kg, as shown in Figure 6.3 (b). Gaussian white noise with zero-mean and 1% coefficient of variation (COV) is considered and added to the exact frequency and mode shape for all the modes of interest. Mass and stiffness are updated by two sets of simulated measured data acquired from original and modified systems.

**FE model updating**

In the first case, no modeling error is assumed between the actual structure and the FE model. Also, the structure is healthy by setting all $\theta$ and $\beta$ as unity. The first six modes are assumed as available measured data. The DREAM algorithm is used to generate samples for estimating of the posterior PDF. Every sample will yield the analytical frequencies and mode shapes using new characteristic equations. Initial settings in DREAM are defined as: ten Markov chains are run parallelly with 6,000 samples per chain; initial values for 10 mass coefficients and 10 stiffness coefficients are set as a range of $[0.5, 1.5]$.

Figure 6.4 shows the results of updated coefficients. Figure 6.4 a-i and a-ii are trace plots of one Markov chain that show how each mass and stiffness coefficient are updated with samples, respectively. All the parameters achieved a stable state. Figure 6.4 b-i and b-ii display the variation of the convergence diagnosis for mass and stiffness updating, respectively. The scale reduction factor, $R_{stat}$, assesses whether the Markov chain converges or not. The $R_{stat}$ of each parameter quickly decays below 1.2, satisfying DREAM's convergence criterion and attains the stationary posterior distribution. The last 30,000 samples herein of ten Markov chains are used to calculate the quantity of interests of all parameters, such as mean and standard derivations.
Figure 6.4. Trace plots and $R_{stat}$ in healthy scenario: (a-i): Trace plots of ten MCs; (a-ii) Trace plots of ten MCs SCs; (b-i): $R_{stat}$ of MCs (b-ii): $R_{stat}$ of SCs

The results of updated coefficients are listed in Table 6.1, including mean and standard derivation (S.D.). The identified mean values exhibit an excellent agreement with actual counterparts. The errors and standard derivations for all coefficients are small; the maximum error of 2.08% is observed. The histograms of the marginal distribution of ten SCs and MCs are shown in Figure 6.5; red curves represent a fitted distribution based on mean and standard derivation. Each histogram has a clear peak and is well-approximated by Gaussian distribution. Overall, each parameter is reasonably identified as the correct values and has a fairly good convergence.
Figure 6.5. Histograms of updated coefficients: (a) MCs; (b) SCs

Table 6.1. Results of updated coefficients

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual</th>
<th></th>
<th>Updated</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Mean</td>
<td>S.D. (%)</td>
<td>Error (%)</td>
</tr>
<tr>
<td>$\beta_1$</td>
<td>0.9831</td>
<td>1.31</td>
<td>1.69</td>
<td></td>
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</tr>
<tr>
<td>$\beta_2$</td>
<td>0.9922</td>
<td>1.27</td>
<td>0.78</td>
<td></td>
<td></td>
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<tr>
<td>$\beta_3$</td>
<td>0.9997</td>
<td>1.81</td>
<td>0.03</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>1.0013</td>
<td>1.71</td>
<td>0.13</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_5$</td>
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<td>2.01</td>
<td>1.32</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_6$</td>
<td>1.0000</td>
<td>1.49</td>
<td>1.02</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_7$</td>
<td>1.0023</td>
<td>1.68</td>
<td>0.23</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_8$</td>
<td>0.9795</td>
<td>2.02</td>
<td>2.05</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_9$</td>
<td>1.0051</td>
<td>1.56</td>
<td>0.51</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\beta_{10}$</td>
<td>1.0016</td>
<td>1.29</td>
<td>0.16</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.9842</td>
<td>1.01</td>
<td>1.58</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>
The updated frequencies and MAC values are summarized in Table 6.2. It is observed that updated frequencies are almost the same as actual ones; the relative error is less than 1%. The values of the Modal assurance criterion (MAC) (Pastor et al., 2012) that reflect the similarity of updated and actual mode shapes are also close to unity. It is worth mentioning that the higher modal parameters from the 7th to 10th order are not used in the updating process, but they are still successfully identified.

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Actual Frequency (Hz)</th>
<th>Frequency (Hz)</th>
<th>MAC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td>Updated</td>
<td>Error (%)</td>
</tr>
<tr>
<td>1</td>
<td>5.827</td>
<td>5.803</td>
<td>0.41</td>
</tr>
<tr>
<td>2</td>
<td>17.350</td>
<td>17.275</td>
<td>0.43</td>
</tr>
<tr>
<td>3</td>
<td>28.486</td>
<td>28.490</td>
<td>0.02</td>
</tr>
<tr>
<td>4</td>
<td>38.985</td>
<td>38.920</td>
<td>0.17</td>
</tr>
<tr>
<td>5</td>
<td>48.613</td>
<td>48.580</td>
<td>0.07</td>
</tr>
<tr>
<td>6</td>
<td>57.156</td>
<td>56.854</td>
<td>0.53</td>
</tr>
<tr>
<td>7</td>
<td>64.422</td>
<td>64.171</td>
<td>0.39</td>
</tr>
<tr>
<td>8</td>
<td>70.248</td>
<td>70.225</td>
<td>0.03</td>
</tr>
<tr>
<td>9</td>
<td>74.506</td>
<td>74.521</td>
<td>0.02</td>
</tr>
<tr>
<td>10</td>
<td>77.099</td>
<td>76.936</td>
<td>0.21</td>
</tr>
</tbody>
</table>

Table 6.1. Results of updated coefficients (continued)

| θ | |  |  |  |
|---|---|---|---|
| 2 | 0.9911 | 1.66 | 0.89 |
| 3 | 0.9956 | 1.63 | 0.44 |
| 4 | 0.9899 | 1.48 | 1.01 |
| 5 | 0.9972 | 1.87 | 0.28 |
| 6 | 0.9963 | 2.19 | 0.37 |
| 7 | 0.9830 | 1.49 | 1.70 |
| 8 | 0.9914 | 2.79 | 0.86 |
| 9 | 1.0132 | 1.90 | 1.32 |
| 10 | 0.9792 | 1.69 | 2.08 |
Probabilistic damage detection

In the second case, the probabilistic damage detection is performed to detect simulated damage location and extent by the proposed VBMU. Damage extent is defined as the change in mass/stiffness coefficients at a specific floor. The damage scenario considered in this example is shown in Table 6.3. The negative sign denotes the reduction of mass/stiffness. The model at healthy condition is assumed known. The unity of MCs and SCs represents a healthy state.

Table 6.3. Damage scenarios

<table>
<thead>
<tr>
<th>Parameters</th>
<th>Mass</th>
<th>Stiffness</th>
</tr>
</thead>
<tbody>
<tr>
<td>Reduction in percent (location)</td>
<td>-10% (2\textsuperscript{nd} floor)</td>
<td>-10% (3\textsuperscript{rd} floor)</td>
</tr>
<tr>
<td></td>
<td>-20% (6\textsuperscript{th} floor)</td>
<td>-20% (6\textsuperscript{th} floor)</td>
</tr>
<tr>
<td></td>
<td>-30% (9\textsuperscript{th} floor)</td>
<td>-30% (9\textsuperscript{th} floor)</td>
</tr>
</tbody>
</table>

Figure 6.6. Trace plots and $R_{stat}$ in damages scenario: (a-i): Trace plots of ten MCs; (a-ii) Trace plots of ten MCs SCs; (b-i): $R_{stat}$ of MCs (b-ii): $R_{stat}$ of SCs
The same modified system is created by adding mass as described in previous case, and the same measurement points and vibration data are selected to identify the damage, e.g., the first six frequencies and mode shapes. In addition, the initial settings in DREAM are the same as the healthy example. Figure 6.6 shows the results of damage detection. Figure 6.6 a-i and a-ii are trace plots of one of ten Markov chains for mass and stiffness coefficients, respectively, visualizing that all the coefficients stably converge. In Figure 6.6 b-i and b-ii, the convergence criterion, $R_{stat}$ is less than 1.2, indicating the stationary Markov chains are achieved. The last 30,000 samples are used to calculate the mean and standard derivation of all coefficients.

Table 6.4 lists identified coefficients and their standard (S.D.) derivations in damage scenarios. It is observed that all updated mass and stiffness coefficients are almost identical to actual values. The maximum errors for all coefficients are less than 2% except $\theta_{10}$ with the error of 2.48%, revealing an outstanding performance in damage localization and quantification on both mass and stiffness. Figure 6.7 shows the histograms of mass and stiffness coefficients; the red curves are fitted Gaussian distribution based on samples. The Gaussian distribution can desirably approximate the posterior PDF. It is also found that the fitted curves in stiffness coefficients are relatively wider spreading than those in mass coefficients, demonstrating that identified stiffness has larger uncertainty than mass.

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Actual</th>
<th>Updated</th>
<th>Mean</th>
<th>S.D. (%)</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>1.0000</td>
<td>1.0140</td>
<td>1.31</td>
<td>1.40</td>
<td></td>
</tr>
<tr>
<td>$\beta_2$</td>
<td><strong>0.9000</strong></td>
<td><strong>0.8885</strong></td>
<td>1.11</td>
<td>1.28</td>
<td></td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.0000</td>
<td>1.0133</td>
<td>1.26</td>
<td>1.33</td>
<td></td>
</tr>
<tr>
<td>$\beta_4$</td>
<td>1.0000</td>
<td>1.0026</td>
<td>1.30</td>
<td>0.26</td>
<td></td>
</tr>
<tr>
<td>$\beta_5$</td>
<td>1.0000</td>
<td>1.0130</td>
<td>1.15</td>
<td>1.30</td>
<td></td>
</tr>
</tbody>
</table>
Table 6.4. Results of updated coefficients for damage scenario (continued)

<table>
<thead>
<tr>
<th></th>
<th>$\beta_6$</th>
<th>$\beta_7$</th>
<th>$\beta_8$</th>
<th>$\beta_9$</th>
<th>$\beta_{10}$</th>
<th>$\theta_1$</th>
<th>$\theta_2$</th>
<th>$\theta_3$</th>
<th>$\theta_4$</th>
<th>$\theta_5$</th>
<th>$\theta_6$</th>
<th>$\theta_7$</th>
<th>$\theta_8$</th>
<th>$\theta_9$</th>
<th>$\theta_{10}$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0.8000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.7000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.9000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.8000</td>
<td>1.0000</td>
<td>1.0000</td>
<td>0.7000</td>
<td>1.0000</td>
</tr>
<tr>
<td></td>
<td>0.7932</td>
<td>0.9876</td>
<td>1.0126</td>
<td>0.6887</td>
<td>0.9847</td>
<td>1.0171</td>
<td>1.0196</td>
<td>0.8891</td>
<td>1.013</td>
<td>1.0147</td>
<td>0.7865</td>
<td>1.0175</td>
<td>1.0177</td>
<td>0.6943</td>
<td>1.0245</td>
</tr>
<tr>
<td></td>
<td>0.98</td>
<td>1.28</td>
<td>1.34</td>
<td>0.87</td>
<td>1.33</td>
<td>1.45</td>
<td>1.48</td>
<td>1.30</td>
<td>1.52</td>
<td>1.35</td>
<td>1.16</td>
<td>1.45</td>
<td>1.40</td>
<td>0.99</td>
<td>1.42</td>
</tr>
<tr>
<td></td>
<td>0.85</td>
<td>1.24</td>
<td>1.26</td>
<td>1.61</td>
<td>1.53</td>
<td>1.71</td>
<td>1.96</td>
<td>1.21</td>
<td>1.30</td>
<td>1.47</td>
<td>1.69</td>
<td>1.75</td>
<td>1.77</td>
<td>0.81</td>
<td>2.45</td>
</tr>
</tbody>
</table>

Figure 6.7. Histograms of coefficients for damage scenario: (a) MCs; (b) SCs
Table 6.5. Results of updated frequencies and MAC values for damage scenario

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Actual Frequency (Hz)</th>
<th>Updated Frequency (Hz)</th>
<th>Error (%)</th>
<th>MAC</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>5.937</td>
<td>5.976</td>
<td>0.65</td>
<td>1.0000</td>
</tr>
<tr>
<td>2</td>
<td>17.149</td>
<td>17.223</td>
<td>0.44</td>
<td>1.0000</td>
</tr>
<tr>
<td>3</td>
<td>27.968</td>
<td>28.083</td>
<td>0.41</td>
<td>0.9999</td>
</tr>
<tr>
<td>4</td>
<td>37.528</td>
<td>37.646</td>
<td>0.31</td>
<td>0.9998</td>
</tr>
<tr>
<td>5</td>
<td>48.866</td>
<td>48.992</td>
<td>0.26</td>
<td>0.9998</td>
</tr>
<tr>
<td>6</td>
<td>56.971</td>
<td>57.204</td>
<td>0.41</td>
<td>0.9993</td>
</tr>
<tr>
<td>7</td>
<td>66.449</td>
<td>66.822</td>
<td>0.56</td>
<td>0.9992</td>
</tr>
<tr>
<td>8</td>
<td>71.327</td>
<td>71.519</td>
<td>0.27</td>
<td>0.9987</td>
</tr>
<tr>
<td>9</td>
<td>74.604</td>
<td>74.831</td>
<td>0.30</td>
<td>0.9984</td>
</tr>
<tr>
<td>10</td>
<td>77.088</td>
<td>77.434</td>
<td>0.45</td>
<td>0.9985</td>
</tr>
</tbody>
</table>

The updated frequencies and MAC values in the damage scenario are derived using updated mass and stiffness coefficients, as shown in Table 6.5. All errors are less than 1% indicating the efficacy of damage detection. Although incomplete modal information, e.g., only the first six modes were used, all the frequencies and MAC values are identified.

![Figure 6.8. Probabilistic damage curves: (a) MCs; (b) SCs](image)

The probabilistic damage curves are also plotted using identified coefficients and uncertainty information by Eq. (5.5), as displayed in Figure 6.8. It is found that curves at the damaged location are distinguishable from those at a healthy location by observing the
curve’s distance from healthy cases. Furthermore, some quantities can be interpreted from curves. For example, mass on the sixth floor ($\beta_6$) and stiffness on the ninth floor ($\theta_9$) have a possible reduction of 20% and 30% with a high probability of 83.2% and 81.6%, respectively. Thus, the proposed VBMU exhibits excellent performance on damage detection on mass and stiffness; both damage location and severity are successfully identified.

6.4.2 Experimental test: a three-story shear frame

The experimental test was performed to verify the accuracy and efficacy of the proposed BMUA for both mass and stiffness identification. A shear building, made of aluminum, has a height and width of 914 and 305 mm, respectively. All the plates and columns have the same geometric properties. The length and width of a plate are both 305 mm with a 25 mm thickness. The column has the length, width, and thickness of 254, 25, and 6 mm, respectively. The initial Young’s modulus and mass density of the aluminum are estimated as 69 GPa and 2,700 kg/m$^3$, respectively. The shear building is modeled as a three-DOF structure using the MATLAB program shown in Appendix D based on the dimensions and material properties.

Free vibration test was performed by inducing the excitation using a rubric hammer. The hammer impacted the structure on the top floor. Horizontal responses were measured by the three IMI 603C01 accelerometers fixed with magnets in the middle of the left side of each floor plate; the associated LabVIEW data acquisition software was used to process the measured signal. In the measurement, ten-second data were recorded with a sampling frequency of 2,000 Hz. The acceleration at each floor is also preprocessed by a low-pass
filter with a cut-off frequency of 50.2 Hz, and down sampled to 200 Hz to identify the frequencies of interest and remove noise from high frequencies.

![Figure 6.9. Test setup of the shear building: (a) original system; (b) modified system with concrete block](image)

The automated stochastic subspace identification (SSI) and Bayesian modal identification developed in Chapters 3 and 4 are used to identify modal parameters, e.g., natural frequencies and mode shapes, and associated uncertainties. Uncertainties on modal parameters measure modal parameters’ accuracy and can be used as weighting factors, such as $\kappa$ in Eq. (6.5). The identified frequencies by automated SSI and Bayesian modal identification are shown in Table 6.7, in which are consistent with each other. Figure 6.9 (a) shows the experimental setup in the laboratory for the original system at the Civil and Environmental Engineering at the University of Louisville. To create the modified system, the ratio of frequency in the original system to that in the modified system is assumed to be 1.04; the magnitude of mass addition is then estimated as 0.545 kg using Eq. (6.15). Therefore, the concrete block with a weight of 0.545 kg is added to each floor, as shown in Figure 6.9 (b). The same measurement and modal identification are carried out for the
modified system. Before model updating and damage detection, mode shapes in two systems are normalized by the mass-change scaling method. Similar to the numerical example, it is convenient to use mass and stiffness coefficients as updating indices. Each floor has a representative value of mass and stiffness coefficients, giving a total of six coefficients to be updated, e.g., $\beta_1, \beta_2$, and $\beta_3$ (mass coefficients) and $\theta_1, \theta_2$, and $\theta_3$ (stiffness coefficients) with labeling a subscript number from the bottom floor (1) to the top floor (3).

**FE model updating**

In the first case, the natural frequencies and mode shapes in the original and modified system under the healthy state are used to update the model. The initial settings in DREAM are as follows: ten Markov chains are simultaneously run to generate a total of 20,000 samples (2,000 samples per chain); all mass and stiffness coefficients have initial values ranging from 0.5 to 1.5. The number of samples designed for the experimental test is less than that in the numerical example, because we have fewer coefficients to update in this test.

Figures 6.10 (a) and 6.11 (a) are the trace plots of the variation of mass and stiffness coefficients, respectively, as samples increases in one Markov chain. The stable convergence of each coefficient is visually observed. The rest of the figures in Figures 6.10 and 6.11 give the updating results over ten Markov chains and convergence diagnosis. The sample mean of mass and stiffness coefficients in each Markov chain and $\pm 2 \cdot \text{S.D.}$ are shown in Figures 6.10 (b) and 6.11 (b), respectively. The mean value of each coefficient is almost identical to one another among ten chains, indicating that the updating results are reliable and accurate. The convergence diagnosis, $R_{stat}$, shown in Figure 6.10 (c) and 6.11
are a useful graphical tool to evaluate convergence state. The resulting plots of $R_{stat}$ that quickly decrease below 1.2, indicating that the sampling process is performed to achieve the stationary Markov chain. Herein, the last 10,000 samples are used to calculate the quantities of interest, mean and standard derivation.

Figure 6.10. Results of updated mass: (a) trace plot; (b) square: the sample mean of each chain, error bar: $\pm 2 \cdot$ S.D.; (c) convergence diagnosis, $R_{stat}$
Figure 6.11. Results of updated stiffness: (a) trace plot; (b) square: the sample mean of each chain, error bar: ±2 · S.D.; (c) convergence diagnosis, $R_{stat}$

Figure 6.12. Density distribution of updated coefficients
The density distribution of each coefficient estimated by the Gaussian kernel estimator (GKE) is also presented in Figure 6.12. It can be seen that the density distributions of $\beta_1$, $\beta_3$, $\theta_1$, and $\theta_3$ are non-Gaussian and multi-modal, indicating in practice, stiffness and mass do not necessarily follow the Gaussian distributions. The estimated distributions illustrate that DREAM is appropriate to approximate the distribution with non-normal shape and multi-peaks. Furthermore, except $\beta_2$ and $\theta_2$ which are distributed over a narrow region, the coefficients have a wide-ranging distribution, suggesting they have larger uncertainties (see S.D. in Table 6.6).

Table 6.6. Results of updated coefficients under healthy condition

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>Initial</th>
<th>Updated</th>
<th>Change (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta_1$</td>
<td>1.000</td>
<td>1.234</td>
<td>23.41</td>
</tr>
<tr>
<td>$\beta_2$</td>
<td>1.186</td>
<td>1.234</td>
<td>18.64</td>
</tr>
<tr>
<td>$\beta_3$</td>
<td>1.000</td>
<td>1.124</td>
<td>12.37</td>
</tr>
<tr>
<td>$\theta_1$</td>
<td>0.800</td>
<td>0.800</td>
<td>-20.05</td>
</tr>
<tr>
<td>$\theta_2$</td>
<td>1.245</td>
<td>1.068</td>
<td>24.52</td>
</tr>
<tr>
<td>$\theta_3$</td>
<td>1.068</td>
<td>0.880</td>
<td>6.80</td>
</tr>
</tbody>
</table>

Table 6.7. Results of updated frequencies and MAC values under healthy condition

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>Frequency (Hz)</th>
<th>MAC</th>
<th>SSI Actual</th>
<th>Bayesian</th>
<th>Initial</th>
<th>Error (%)</th>
<th>Updated</th>
<th>Error (%)</th>
<th>Initial</th>
<th>Updated</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.66</td>
<td>0.9882</td>
<td>7.75</td>
<td>8.667</td>
<td>13.20</td>
<td>7.78</td>
<td>1.63</td>
<td>0.9979</td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>22.47</td>
<td>0.9938</td>
<td>22.50</td>
<td>24.213</td>
<td>7.76</td>
<td>22.23</td>
<td>1.08</td>
<td>0.9972</td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>33.77</td>
<td>0.9954</td>
<td>33.86</td>
<td>34.856</td>
<td>3.22</td>
<td>34.00</td>
<td>0.68</td>
<td>0.9965</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Table 6.6 shows updated mass and stiffness coefficients and their S.D. The updated frequencies and MAC values are tabulated in Table 6.7. All the MCs increase but $\theta_1$ decreases. The model updating aims to match measured responses with analytical counterparts. In this case, measured frequencies are overall smaller than those in the FE
model (see Table 6.7). From fundamental structural dynamics, frequency is proportional to stiffness but inversely proportional to mass. Therefore, stiffness and mass have to decrease and increase, respectively, to match measured frequencies with those in the FE model. The frequency errors of all modes are significantly reduced, and MAC values are updated to be close to 1.0. These values demonstrate satisfactory updating model results.

**Probabilistic damage detection**

Two damage scenarios are intentionally introduced with increasing severity in the shear building by reducing the thickness of column and increasing the weight of the floor, as shown in Table 6.8, the positive/negative sign denotes the increasing/reduction. The thickness of one column at the second and third floor is reduced by 50%, resulting in a 21.8% stiffness reduction in the corresponding floor; A concrete block with the weight of 1.54 kg is added to the second and third floor to mimic mass change due to damage, which produces 21.5% mass increase in the corresponding floor.

<table>
<thead>
<tr>
<th>Table 6.8. Damage scenarios</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>Notation</strong></td>
</tr>
<tr>
<td>-----------------</td>
</tr>
<tr>
<td>D1</td>
</tr>
<tr>
<td>D2</td>
</tr>
<tr>
<td></td>
</tr>
</tbody>
</table>

The concrete block with a weight of 0.545 kg (the same as healthy condition) is added to each floor to construct the modified structure for both damage scenarios. The same measurement procedures were performed for two damage scenarios. Figure 6.13 shows the experimental setup of two damage scenarios. Modal analysis is also implemented by automated SSI and Bayesian modal identification to extract natural frequencies and mode shapes of the original and modified system in two damage scenarios. The proposed method is then performed to identify mass and stiffness coefficients based on the updated FE model
(healthy condition). In the updating process, the DREAM algorithm generates samples to target the posterior PDF by the same initial settings as before.

Figure 6.13. Test setup: (a) D1; (b) D1 with added mass; (c) D2; (d) D2 with added mass.

Figures 6.14-17 show the updated results of mass and stiffness in two damage scenarios. Figures (a)s in Figure 6.14-17 are trace plots that show the iteration of each coefficient with samples increasing; stable convergence is achieved in trace plots. The updating results of mass and stiffness coefficients over ten Markov chains are presented in Figures (b)s in Figure 6.14-17. It is seen that all coefficients in both damage scenarios are identified as consistent with each other among ten Markov chains, indicating reliable and accurate identification. In addition, the convergence diagnosis, $R_{stat}$ quickly decays below 1.2 for all coefficients, demonstrating that the stationary convergence is reached. In damage
detection, the last 10,000 samples are used to calculate the mean values and standard derivations. Note the mean values under the healthy condition are used as baselines, so the undamaged floor has the mass and stiffness coefficients with unity value.

Figure 6.14. Results of updated mass in D1: (a) trace plot; (b) square: the sample mean of each chain, error bar: $\pm 2 \cdot$ S. D.; (c) convergence diagnosis, $R_{stat}$
Figure 6.15. Results of updated stiffness in D1: (a) trace plot; (b) square: the sample mean of each chain, error bar: $\pm 2 \cdot \text{S.D.}$; (c) convergence diagnosis, $R_{\text{stat}}$

Figure 6.16. Results of updated mass in D2: (a) trace plot; (b) square: the sample mean of each chain, error bar: $\pm 2 \cdot \text{S.D.}$; (c) convergence diagnosis, $R_{\text{stat}}$
Figure 6.17. Results of updated stiffness in D2: (a) trace plot; (b) square: the sample mean of each chain, error bar: $\pm 2 \cdot \text{S.D.}$; (c) convergence diagnosis, $R_{\text{stat}}$

Figure 6.18 shows the density distribution estimated by Gaussian kernel estimator. It is observed that some coefficients exhibit multi-modal features and are non-Gaussian shaped. Especially, $\theta_1$ and $\theta_3$ in damage scenario No. 1 (D1) and $\beta_1$, $\beta_2$ and $\theta_3$ in damage scenario No. 2 (D2). It indicates that structural parameters do not always follow Gaussian distribution, the asymptotic optimization method may not be suitable to estimate the posterior PDF with a non-Gaussian shape. However, the proposed method is able to approximate non-Gaussian distribution with an accurate level. It is also found that the larger uncertainties are revealed in some coefficients, such as $\beta_2$ and $\theta_1$ in both damage scenarios (S.D. ranges from 7.9% to 16.2%). Their distributions are flatter and spread across a relatively wider region. While the distributions of other coefficients are concentrated in a narrow region and have pronounced peaks, meaning these coefficients are more certain (S.D. ranges from 1% to 10.1%).
The identified damage severities of two damage scenarios are shown in Figure 6.19.

The identified damage severities of $\beta_3$ and $\theta_3$ in damage scenario No. 1 (D1) are 23.35% and 24.72%, respectively, which is close to actual values of 21.5% for $\beta$ and 21.8% for $\theta$; the identified damage severities of $\beta_2$, $\beta_3$, and $\theta_2$, $\theta_3$ in damage scenario No. 2 (D2) are 24.55%, 23.14%, 19.18% and 20.18% respectively, which also agree well with actual values of 21.5% for $\beta$ and 21.8% for $\theta$. The false damage detection is only observed with
less than 4%. These results demonstrate the efficacy of the proposed method in both mass and stiffness updating and achieves damage localization and quantification.

Figure 6.19. Identified damage severity: (a) D1; (b): D2

Table 6.9. Results of updated frequencies and MAC values in two damage scenarios

<table>
<thead>
<tr>
<th>Damage scenario</th>
<th>Mode No.</th>
<th>Measured</th>
<th>Frequency (Hz)</th>
<th>MAC</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td></td>
<td></td>
<td>Updated</td>
<td>Error (%)</td>
</tr>
<tr>
<td>D1</td>
<td>1</td>
<td>7.140</td>
<td>7.234</td>
<td>1.31</td>
</tr>
<tr>
<td></td>
<td>2</td>
<td>19.800</td>
<td>19.420</td>
<td>1.92</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>32.311</td>
<td>32.680</td>
<td>1.14</td>
</tr>
<tr>
<td></td>
<td>1</td>
<td>6.905</td>
<td>6.922</td>
<td>0.25</td>
</tr>
<tr>
<td>D2</td>
<td>2</td>
<td>19.641</td>
<td>20.244</td>
<td>2.98</td>
</tr>
<tr>
<td></td>
<td>3</td>
<td>29.043</td>
<td>29.512</td>
<td>1.59</td>
</tr>
</tbody>
</table>
The updated frequencies and MAC values are calculated using identified mass and stiffness coefficients, as shown in Table 6.9. It is observed that all modal parameters in both damage scenarios are in accordance with measured counterparts, indicating the FE model is successfully updated by the proposed method. Based on identified mean values of mass and stiffness coefficients and their uncertainties under healthy and damaged state, the probabilistic damage curves can be plotted, as shown in Figure 6.20. It is worth mentioning that the negative, $d$ represents mass/stiffness increase, and vice versa. For damage scenario No.1 (D1), $\beta_3$ and $\theta_3$ have a probability (63.3% and 67.5%, respectively) of having possible damage $23.35\%$ and $24.72\%$. For damage scenario No. 2 (D2), $\beta_2$ and $\theta_2$ have a possible change of $24.55\%$ and $19.18\%$ with a probability of $62.1\%$ and $67.7\%$. 

Figure 6.20. Probabilistic damage curves: (a): D1; (b): D2
respectively. In practice, damage can be detected by probabilistic curves, because the curves related to damage location are generally easily distinguished from the ones related to healthy location. For example, the curves of $\beta_3$ and $\theta_3$ in D1 are clearly separated from others, indicating the location corresponding to $\beta_3$ and $\theta_3$ (herein is the third floor) may have certain damage. Similar observation is found in D2. The proposed method can detect damage in mass and stiffness along with location and severity in a probabilistic manner. The engineers can be informed that some repairing work may be necessary at certain location.

6.5 Comparison of Bayesian model updating with added mass and added stiffness

In this subsection, Bayesian model updating with added known stiffness is proposed to update both mass and stiffness parameters. In section 6.5.1, the characteristic equations with stiffness addition are derived. Subsequently, the proposed BMUA with added stiffness are applied for laboratory-scale three-story shear building in Section 6.5.2. In addition, the comparative study is investigated between BMUA with added mass and added stiffness.

6.5.1 Characteristic equations of BMUA with added stiffness

Similar to Section 6.3.1, the new characteristic equations with stiffness addition rather than mass addition are derived to address the coupling effect.

When updating mass, from Eq. (5.64), we obtain:

$$(C - \lambda D)\phi = 0$$  \hspace{1cm} (6.18)

where $C = \lambda' \phi'^T M - \phi'^T \Delta k, D = \phi'^T M$. Derivation in detail can be found in Eqs. (5.59) ~ (5.64) in Section 5.5.1.

When updating stiffness, from Eq. (5.72), we have:
\[(G - \lambda L)\phi = 0\]  \hspace{1cm} (6.19)

where \(G = \lambda'\phi'^T K, \quad L = \phi'^T K + \phi'^T \Delta k\). Derivation in detail can be found in Eqs. (5.65) ~ (5.72) in Section 5.5.1.

Eqs. (6.18) and (6.19) are new characteristic equations incorporating added known stiffness \(\Delta k\), which can be solved by ‘eig’ function in MATLAB. The coupling effect is inherently addressed, as updating either mass or stiffness, another parameter information is not required.

Remark adding stiffness is practically feasible, such as installation of additional structural components (e.g., braces, dampers or springs) (Khatibi et al., 2012, Saingam et al., 2020, Kazemi et al., 2021), shear tab connectors at bolted joints (FEMA, 2006). The magnitude of added stiffness could be conveniently determined, when the sectional and geometric properties of added components at design stage are available, like Young’s modulus and moment of inertial or using the equivalent stiffness. In addition, Before the model updating, the measured mode shapes have to be normalized stiffness-change scaling method (Khatibi et al., 2012) to map measured and predicted mode shapes.

In BMUA framework with added stiffness, the model outputs are predicted by the two new characteristic equations instead of classical one \(((K - \lambda M)\phi = 0)\). It is analogous to BMUA with added mass, the posterior PDF in Eq. (6.5) is reformulated by the measurements and model predictions. Finally, the DREAM algorithm is applied to approximate the posterior PDF, giving the quantity of interests, e.g., mean and variance.

6.5.2 Experimental test

The same laboratory-scale shear structure as described in Section 6.4 was used to validate the performance of the proposed BMUA with added stiffness. The same
measurement was conducted to acquire accelerations under free vibration after the structure was excited by a rubric hammer, as shown in Figure 6.21. The modified system was created by installing an additional column at each floor (see red circle in Figure 6.21 (a)). Each added column has the same material properties and half thickness of the column in original system, which yielding the 3.12% stiffness addition at each floor. The recorded data for two systems was processed to extract the natural frequencies and mode shapes by the automated SSI.

![Image](image1.png)

(a) Original system  
(b) Modified system with added columns

Figure 6.21. Test setup of the shear building: (a) original system; (b) modified system with added columns

The identified natural frequencies and mode shapes for original and modified system are used to update the structure under healthy condition. The initial settings in DREAM are ten Markov chains are considered to generate a total of 20,000 samples (2,000 samples per chain); all mass and stiffness coefficients have initial values ranging from 0.5 to 1.5.

Figure 6.22 shows the trace plots of updating mass and stiffness. It is visually observed that Each coefficient finally reaches a stable convergence. It is also found that the stiffness coefficients exhibit more fluctuation than mass coefficients. Density distributions are
estimated by Gaussian kernel estimator and shown in Figure 6.23. Each mass coefficient has a sharp peak and are well estimated by Gaussian distribution. However, the distribution of all SCs spread in a wide region, $\theta_1$ and $\theta_3$ also have multi-modal shape, indicating the SCs are more uncertain and more difficult to identify compared with MCs.

Figure 6.22. Updated results: (a) trace plot of mass; (b) trace plot of stiffness
Table 6.10. Results of updated coefficients under healthy condition

<table>
<thead>
<tr>
<th>Coefficients</th>
<th>BMUA with $\Delta m$ (Table 6.6)</th>
<th>BMUA with $\Delta k$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mean</td>
<td>S.D. (%)</td>
</tr>
<tr>
<td>$\beta_1$</td>
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<td>10.36</td>
</tr>
<tr>
<td>$\beta_2$</td>
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</tr>
<tr>
<td>$\theta_3$</td>
<td>1.068</td>
<td>17.81</td>
</tr>
</tbody>
</table>

Table 6.10 listed the updated mass and stiffness coefficients by the BMUA with added stiffness and added mass. Apparently, discrepancy is observed in updated coefficients. It may be explained by the fact that in the proposed two BMUA, $\Delta m$ and $\Delta k$ need to be known prior to model updating, to some extent, their information directly affects the implementation of addressing the coupling effect and controls the quality of updating mass and stiffness. The magnitude of $\Delta m$ can be conveniently and accurately estimated by some devices, e.g., scale. But it is not the case for $\Delta k$ estimation. In this work, $\Delta k$ is provided by added columns. Theoretically, stiffness of each column can be calculated by $12EI/L^3$ under the assumption of fixed boundary condition, where $E$, $I$, and $L$ are young’s modulus, moment of inertial, and length of added column, respectively. However, two issues may not be ignored. In real structure, it is hard to guarantee the connection is completely fixed; Additionally, $E$, $I$, and $L$ inevitably have uncertainties due to manufacture, inaccurate material information, etc. Therefore, $\Delta k$ cannot be well estimated and have larger uncertainties compared to $\Delta m$. When using the erroneous $\Delta k$ estimation, it is not surprising to have biased updating results. BMUA with added mass gives more reliable identification of mass and stiffness.
6.6 Conclusions

This chapter proposed a novel vibration-based Bayesian model updating approach to simultaneously identify structural mass and stiffness. In this work, the coupling effect of mass and stiffness is successfully addressed using two sets of vibration data from original and modified system with added known mass/stiffness. The posterior PDF is approximated by DREAM sampling method instead of asymptotic optimization method. Following conclusions from numerical examples and experimental tests are summarized as follows:

- The results in numerical example and experimental test illustrate that the proposed approach can simultaneously identify structural mass and stiffness with an accurate level and their uncertainties by addressing the coupling effect of mass and stiffness.
- In experimental test, some mass coefficients exhibit larger uncertainties, indicating the effect of mass on structural integrity cannot be ignored, and the assumption of mass is known and invariant in classical Bayesian approach may be questionable when noticeable change in mass is observed, such as 21.5% mass increase in damage scenarios in this test for mimicking the mass change due to unknown damages.
- The results in experimental test reveal the structural parameters, e.g., mass and stiffness, do not always follow Gaussian distribution. Thus, the asymptotic approximation method may not be suitable for this situation. The DREAM algorithm runs multiple Markov chains in parallel and sufficiently seek all possible solutions, resulting in high capability to treat the posterior PDF with high-dimensionality, multi-modality, and numerous peaks.
• The probabilistic damage detection is also implemented by the proposed Bayesian approach. The results in experimental test demonstrate that the proposed approach enables to reliably and accurately identify damage location and severity. In addition, the probabilistic damage curves allow engineers to quickly localize damage, indicating the proposed approach is practically valuable.

• The comparison of updated results by BMUA with added mass and added stiffness are discussed. Some discrepancies are observed. Because it is difficult to accurately estimate $\Delta k$ in practice due to larger uncertainties in $\Delta k$, resulting in biased updating results. But BMUA with added mass gives more reliable identification of mass and stiffness because of well-estimated $\Delta m$. 
CHAPTER 7

BAYESIAN MODEL UPDATING FOR COMPLEX STRUCTURES USING SURROGATE MODEL

7.1 Introduction

In Chapters 5 and 6, Bayesian model updating has been demonstrated its efficacy and robustness. Although Bayesian model updating with MCMC is promising, it is computationally demanding due to a vast amount of FE model evaluations are required. As a result, it becomes impractical for complex and large-scale engineering structures. Therefore, surrogate models are potential alternatives to relieve the computational burden. The comprehensive introduction of surrogate models is referred to Xia et al. (2021). However, research on Bayesian model updating with surrogate models is still not sufficiently explored to overcome challenges. For example, Wan and Ren (2016) applied the Bayesian approach to a real-world cable-stayed bridge with the Gaussian process model. However, only measured frequency is used as input, which may result in inaccurate parameter identification. Jensen et al. (2017) used the Bayesian approach and Kriging model to update a numerical example of a two-story reinforced concrete structure. Mao et al. (2020a) updated a long-span suspension bridge using the Bayesian approach with Hybrid Monte Carlo (HMC) and Kriging model. Pepi et al. (2019) employed the Metropolis Hastings (MH) sampler in the Bayesian approach with the Kriging model to update only two structural parameters in a cable-stayed footbridge. In contrast, either HMC
or MH is a single-chain method, as aforementioned, which has a slow or even incorrect convergence and is easily stuck in a local optimum. More efforts to extend and apply the Bayesian model updating with a surrogate model in civil engineering are needed for practical application in complex structures.

An additional time-saving strategy is sensitivity analysis investigating how input parameters affect model outputs and excluding insensitive parameters (Saltelli et al., 2004). As a consequence, model dimensionality can be substantially reduced. Global sensitivity analysis (GSA) has been widely used in large structures as it enables quantifying the percentage of the uncertainty of model outputs arising from the uncertainty of input parameters. The GSA has various methods including variance-based GSA (Saltelli et al., 2004), moment-dependent method (Borgonovo et al., 2012), Fourier amplitude sensitivity test (FAST) method (Tarantola and Mara, 2017), Morris method (King and Perera, 2013), etc. The present work uses variance-based GSA to drop non-influential parameters. The variance-based GSA can evaluate the effect of the entire parameter space on model outputs and assess the effect of parameters’ interaction. For the review of variance-based GSA, readers can refer to Chen et al. (2005). However, traditional variance-based GSA using Monte Carlo simulation (MCS) and FEM has a critical drawback of low efficiency. Typically, it requires no less than 10,000 model evaluations for accurate results (Burnaev et al., 2017), resulting in high computational cost and seriously limiting its practical application with high fidelity models.

Driven by these issues, this chapter proposed an efficient vibration-based Bayesian model updating approach (BMUA). Dynamic modal data, namely frequencies and mode shapes, are used together to update the model. DREAM is adopted to estimate the posterior
PDF in Bayesian model updating; A fast-running Kriging model is used as a surrogate model of the traditional one to enhance computational efficiency. In addition, a variance-based GSA combining with the Kriging model is applied to reduce computational cost in parameter selection.

This chapter is organized as follows. Section 7.2 provides the brief review of the Bayesian model updating formulations. Section 7.3 introduces two time-saving strategies of alleviating computational burden, including the variance-based GSA and Kriging model. Section 7.4 presents a field test of a cable-stayed pedestrian bridge for the real application. Finally, conclusions and contributions are discussed in Section 7.5.

7.2 Formulations of Bayesian model updating

In Chapters 5 and 6, detailed formulations of BMUA have been introduced. Different from previous chapters, this chapter employs two fractional error functions (FEF) of a given one mode, $m$, to formulate the likelihood function, namely frequency FEF and mode shape FEF (Lam et al., 2015). Frequency FEF is defined as:

$$
\varepsilon_{f,m} = \frac{\hat{f}_m - f_m(\Omega)}{\hat{f}_m}
$$

(7.1)

where $\hat{f}_m$ is the $m$th measured frequency, $f_m(\Omega)$ is the $m$ th calculated frequency in a model given $\Omega$.

Mode shape FEF is defined as:

$$
\varepsilon_{ms,m} = \sqrt{\left(1 - \left|\tilde{\phi}_m \phi_m(\Omega)\right|^2\right)}
$$

(7.2)

where $\tilde{\phi}_m$ and $\phi_m(\Omega)$ are the $m$th measured and calculated mode shape, respectively. Note all mode shapes are normalized to unity norm to map them in the same context.
Assuming that frequency and mode shape FEF in Eqs. (7.1) and (7.2) follow zero-mean Gaussian distribution, then the posterior PDF is rewritten as follows:

$$P(\Omega|D,C) = c_0 \exp \left( -\frac{1}{2\kappa^2} J(\Omega) \right)$$  \hspace{1cm} (7.3)

$$J(\Omega) = \sum_{m=1}^{n} \left[ \left( \frac{\tilde{f}_m - f_m(\Omega)}{\bar{f}_m} \right)^2 + \left( 1 - |\bar{\Phi}_m^T \Phi_m(\Omega)|^2 \right) \right]$$ \hspace{1cm} (7.4)

where $\kappa$ is an uncertainty parameter of prediction error. In current study, the variances of measured frequency and mode shape are used as $\kappa^2$. $\kappa$ consists of $\sigma_{f,m}$ and $\sigma_{ms,m}$; $\sigma_{f,m}$ and $\sigma_{ms,m}$ are standard derivation of the $m$th measured frequency and mode shape, respectively. In this study, the DREAM algorithm combining parallel multi-chain and evolutionary concepts are adopted to sample the posterior PDF in Eq. (7.3).

### 7.3 Time-saving strategies

Thousands of model analyses are required in DREAM to ensure a stable convergence for Bayesian model updating, leading to high computational costs. Two strategies are adopted to overcome the difficulty in computational demands. Firstly, the Kriging model is introduced to substitute the time-consuming FE model in Section 7.3.1. The variance-based GSA has then presented to select the most influential model parameters in Section 7.3.2.

#### 7.3.1 Kriging model

In the past decades, the Kriging model (also called the Gaussian process model) has been extensively used in engineering communities (Wang et al., 2017, Bhosekar and Ierapetritou, 2018, Alizadeh et al., 2020), because it accurately provides not only a model prediction at design points but also a prediction uncertainty to measure the model reliability.
Another advantage of the Kriging model is various correlation functions are embedded to represent complex structures; thus, it is suitable for nonlinear problems. Essentially, the Kriging model is an interpolation emulator that combines a polynomial regression and Gaussian random process as follows (Simpson et al., 2001):

\[ Y(x) = F^T(x)\beta + Z(x) \]  \hfill (7.5)

where \( x = [x_1 \cdots x_l] \) is a structural parameter vector, \( l \) is the number of parameters; \( Y = [y_1 \cdots y_p] \) is a model response vector, such as frequency and mode shape, \( p \) is the number of responses of interest. \( F(x) = [f_1(x) \cdots f_q(x)]^T \) consists of \( q \) polynomial regression functions, showing the global trends of the predicted model. In this study, the quadratic polynomial is used as a regression function; \( \beta = [\beta_1 \cdots \beta_q]^T \) is a regression coefficient vector; the term of \( Z(x) \) is a stationary Gaussian process error with zero mean and variance, \( \sigma^2 \), reflecting the local deviation of the Kriging predictor. The non-zero covariance matrix of \( Z(x) \) is given by:

\[ \text{cov}[z(x_i), z(x_j)] = \sigma^2 R \]  \hfill (7.6)

where \( R \) is a correlation matrix with symmetric elements, \( R_{ij}(x_i, x_j) \); \( x_i \) and \( x_j \) are two random training points.

When constructing a Kriging model, correlation function, \( R_{ij}(x_i, x_j) \) needs to be user-defined. Some literature discussed the effect of different correlation functions on the prediction accuracy of a Kriging model (Mao et al., 2020a). \( R_{ij}(x_i, x_j) \) is typically expressed as:

\[ R_{ij}(x_i, x_j) = \prod_{k=1}^l R_k \left( \alpha_k, x_i^k - x_j^k \right) \]  \hfill (7.7)
where $l$ is the number of variables; $\alpha_k$ is a correlation coefficient that quantifies the contribution of each input $x_i$; $x_i^k$ and $x_j^k$ are the $k$th coordinates of points, $x_i$ and $x_j$, respectively. It has been demonstrated that the Gaussian correlation function has a fairly good smooth and differentiable surface (Mao et al., 2020a). In this study, the Gaussian correlation function is used to construct a Kriging model. Therefore, Eq. (7.7) is rewritten as:

$$R_{ij}(x_i, x_j) = \prod_{k=1}^{l} \exp\left(-\alpha_k \exp| x_i^k - x_j^k |^2 \right)$$  \hspace{1cm} (7.8)$$

For a set of training samples, a correlation coefficient, $\alpha$, and variance, $\sigma^2$, can be estimated by maximum likelihood estimation (MLE) (Martin and Simpson, 2005):

$$\max \eta(\alpha) = -\frac{1}{2} (m_s \ln(\sigma^2) + \ln| R |)$$  \hspace{1cm} (7.9)$$

where $m_s$ is the number of training samples; Both $\sigma^2$ and $R$ are the function of $\alpha$; $\text{sign} |\bullet|$ denotes the determinant of a matrix. Explicit procedures for solving Eq. (7.9) are referred to Simpson et al. (2001).

When the correlation function is decided and its coefficient is estimated, the interpolation of the Kriging model can start for untried sample points with unbiased approximation. The predicted response, $\hat{y}(x)$, is given by:

$$\hat{y}(x) = f^T(x)\tilde{\beta} + H^T(x)R^{-1}(Y - F^T\tilde{\beta})$$  \hspace{1cm} (7.10)$$

where $\tilde{\beta} = (F^T R^{-1} F)^{-1} F^T R^{-1} Y$ is obtained by applying the least square method to $F \tilde{\beta} \equiv Y$; $H(x) = [R(x, x_1), R(x, x_2), \cdots, R(x, x_{m_s})]^T$ is a vector representing the correlation between the number of $m_s$ training samples and prediction points. Meanwhile, the
prediction error of $\hat{y}(x)$ can be estimated as $\delta_y(x) = \sigma^2(1 + u^T(F^T R^{-1} F)^{-1} u - H^T R^{-1} H)$, where $u = F^T R^{-1} H - f$.

Overall, the Kriging model is constructed when the correlation coefficient is estimated. The model prediction at all tried samples equal to exact values because of the interpolation characteristic in the Kriging model. Remark that training data to construct the Kriging model should be collected, including training inputs and corresponding model outputs. The training inputs are generated by a popular space-filling design method, Latin hypercube sampling (LHS), originally proposed by McKay (1992) in statistics. LHS enables us to randomly produce points with low discrepancy and uniformly falling in hypercube through sampling from multi-dimensional distribution. Assume $X$ design variables, the probability distribution of each variable is stratified to $P$ equal and non-overlapped subintervals within the defined bounds. The samples are randomly partitioned into each subinterval in LHS. One appealing feature in LHS is that the sample points are well-spread and un-grouped in the parameter space. The general formulation of sample point $X$ using LHS is given by (McKay, 1992):

$$X = \frac{\pi + U}{P} \quad (7.11)$$

where $P$ is the number of sample point; $\pi$ is a stratification of sequence $(0,1,\ldots,P-1)$; $U$ is a random value from a uniform distribution $(0,1)$ The model outputs corresponding to each sample point are then derived by a commercial FEA package, such as ANSYS. The Kriging model used in this study is constructed by the Design and Analysis of Computer Experiments (DACE) MATALB toolbox (Lophaven et al., 2002).

Before using the Kriging model in model updating, it is necessary to evaluate the accuracy of the Kriging model. The performance of model prediction by the Kriging model
at untried data points is assessed using two criteria, namely mean square error (MSE) and a coefficient of determination, $R^2$ (Jensen et al., 2017). MSE illustrates the discrepancy between a Kriging model and a FE model. Thus, the value of MSE is closer to 0; the Kriging model is more reliable. In contrast, $R^2$, which ranges typically from 0 to 1, is close to 1, suggesting that Kriging's prediction matches the actual counterpart.

7.3.2 Variance-based global sensitivity analysis (GSA)

Variance-based GSA is employed in this study to identify essential structural parameters and eliminate non-influential ones before FEMU. Assume that a structural model has the input-output relation of $Y(x)$. $Y$ is output response; $x = (x_1, x_2, \ldots, x_d)$ is a $d$-dimensional input vector. Suppose all the input parameters are mutually independent based on the decomposition of the total variance in model outputs, the variance $V$ of $Y(x)$ can be written as (Saltelli et al., 2004):

$$V(Y) = \sum_{i=1}^{d} V_i + \sum_{i<j}^{d} V_{ij} + \cdots + V_{12\ldots d}$$

(7.12)

where $V(\bullet)$ and $E(\bullet)$ are the variance and expectation operators, respectively. For example, the first-order partial variance, $V_i = V(E(Y|x_i))$ and second-order partial variances, $V_{ij} = V\left(E(Y|x_i,x_j)\right) - V_i - V_j$, and higher-order ones can be summed. Eq. (7.12) illustrates how each input parameter and interaction effect of inputs contribute to the total variance of model output. $V_i$ and $V_{ij}$ in the first-order and the second-order partial variance are main variance contribution of $x_i$ and interaction variance contribution between $x_i$ and $x_j$, respectively.
The right side of Eq. (7.12) is divided by \( V \), resulting in the variance-based global sensitivity indices:

\[
1 = \sum_{i=1}^{d} S_i + \sum_{i<j}^{d} S_{ij} + \cdots + S_{12\cdots d}
\]  \hspace{1cm} (7.13)

where \( S_i = V_i/V \), defined as the main or first-order sensitivity index, quantifying the main contribution by a single input, \( x_i \), and \( S_{12\cdots d} = V_{12\cdots d}/V \), defined as the interaction sensitivity index, quantifying the percentage of total output variance contributed by the combination of all the inputs. For accounting for the overall contribution of \( x_i \), including individual contribution of \( x_i \) and its joint effect with other inputs, the total sensitivity index, \( S_{T_i} \), is defined as:

\[
S_{T_i} = 1 - \frac{V_{-i}}{V}
\]  \hspace{1cm} (7.14)

where \( V_{-i} = V\left(E(Y|x_{-i})\right) \) is the total contribution to \( V(Y) \) attributed to all input variables excluding \( x_i \). The use of \( S_i \) and \( S_{T_i} \) can effectively measure the importance of the \( i \)th input to a model response. The higher value of \( S_i \) and \( S_{T_i} \) is, the more important corresponding input is. It is worth mentioning that \( S_{T_i} \) contains both main effect and interaction effect with other inputs with respect to \( x_i \). If \( S_{T_i} \) equals to \( S_i \), illustrating there is no interaction effect between \( x_i \) and other inputs, and vice versa.

The variance terms of in Eq. (7.14) can be calculated from MCS (Saltelli et al., 2004). However, it requires a large amount of FE model evaluations, generally in the order of \( 10^4 \), to guarantee a satisfactory convergence. The Kriging model is used as a surrogate of the FE model to efficiently perform variance-based GSA. Figure 7.1 shows how to select significant parameters by the proposed variance-based GSA using the Kriging model.
Firstly, sample points (input) for all the possible parameter candidates are prepared by the LHS method; the corresponding model outputs at sample points are then generated by FE model analysis. Next, the Kriging model is constructed using collected training data (inputs and outputs), as described in Section 7.3.1. Finally, the task of variance-based GSA is implemented based on MCS using the Kriging model. Consequently, nonsignificant parameters are selected and discarded for following model updating.

![Figure 7.1. Flowchart of the proposed variance-based GSA](image)

In summary, the proposed Bayesian model updating consists of two main stages. The stage one involves two cost-effective strategies: variance-based GSA in Section 7.3.2 and Kriging modeling in Section 7.3.1. The stage two is the process of Bayesian model updating using the DREAM sampling algorithm. At first, all possible parameter candidates, $X_{total}$, are initially selected; non-influential parameters are eliminated by variance-based GSA, resulting in selected significant parameters, $X_{selected}$. Then, the Kriging model is built with respect to $X_{selected}$. Next, the posterior PDF in Eq. (7.3) is formulated using measured data and prediction from the Kriging model. DREAM algorithm is next adopted...
to generate samples to approximate the posterior PDF. Finally, the stationary Markov chains give us the quantity of interest, such as PDF, mean, and coefficient of variation. The flowchart of the proposed updating framework is shown in Figure 7.2.

![Flowchart of the proposed model updating framework](image)

**Figure 7.2.** Flowchart of the proposed model updating framework

It should be noted that the Kriging model is constructed twice in the proposed framework. First, one Kriging model is constructed with all the possible parameter candidates to enhance computational efficiency in GSA. Second, another is constructed with selected significant parameters, which is used in Bayesian model updating.

**7.4 Application example: a cable-stayed pedestrian bridge**

**7.4.1 Bridge description**

The cable-stayed pedestrian bridge (Figure 7.3) studied in this work, located in Wuhan in China, has three spans and a single pylon with a steel box girder. Figure 7.4 shows the configuration of the bridge. The bridge has a total length of 86.3 m and a width of 7 m, with a center span of 45 m. The center span of the bridge is composed of U-shaped and
straight segments, and the right span is skewed. The vertical steel pylon with a 16.1-m height is situated inside the center span; one T-shaped pier is below the pylon. The four parallel stay cables at each side of the pylon are anchored to connect the pylon with the bridge deck; each cable has a diameter of 115 mm.

Figure 7.3. General overview of the cable-stayed pedestrian bridge: (a) top view; (b) front view (photo by a collaborator, Prof. Qing)

Figure 7.4. Configuration of the pedestrian bridge (unit: m; N denotes cable): (a) elevation; (b) plan
Figure 7.5. The Finite-element model of the cable-stayed pedestrian bridge

The FEM of the bridge is established using the FEA package ANSYS. Shell elements (SHELL181) are used to model the main beam and pylon; Link elements (LINK180) are adopted to simulate stay cables. For non-structural components, e.g., the railing system, its stiffness contribution can be ignored due to slender and small properties, but mass contribution should be included. Therefore, the railing system is modeled by shell elements and added to the bridge deck. In summary, this bridge model consists of 39,388 nodes and 39,390 elements. The resulting FEM is shown in Figure 7.5. The Ansys Parametric Design Language (APDL) program is shown in Appendix E.

7.4.2 Operational modal analysis

Operational modal analysis (OMA) is carried out to extract dynamic modal parameters of the cable-stayed pedestrian bridge, i.e., natural frequency, damping ratio, and mode shape. OMA has been received considerable attention, because it avoids any interruption of normal operation and does not require artificial excitation. Instead, natural excitation, such as human walking, wind, traffic, etc., is used during the vibration test. The five wireless accelerometers are available and installed on two sides of the bridge deck in a vertical direction. Due to a limited number of sensors, seventeen measurement setups were
deployed to cover all locations of interest, containing three reference sensors and two roving sensors at each setup. As a result, a total of 53 locations were recorded at a sampling frequency of 200 Hz; the time duration for each setup is 15 minutes. The equipment during the field test is shown in Figure 7.6 (a) and (b). Figure 7.6 (c) displays the measurement sensor layout.

![Figure 7.6. Field test: (a) data acquisition; (b) wireless accelerometer; (c) sensor layout (units: m; △ and ○denote roving and reference sensor, respectively)](image)

The automated SSI is employed to identify modal parameters. The spurious modes are automatically eliminated, which is computationally effective and more reliable, especially when there are many data and multiple measurement setups. Details on automated SSI are referred to Chapter 3. Before applying the automated SSI, the collected data were pre-processed by a lowpass filter and cut-off frequency of 14.2 Hz, then down sampled to 50 Hz to only consider the frequency of interest and remove noise from high frequencies. The
input parameters for SSI are chosen as follows: the model order ranges from 20 to 90; the
time lag is set as 50. For the sake of space, only a stabilization diagram for setup No.5 is
presented. Figure 7.7 shows the full and cleared stabilization diagram. All spurious modes
(scattered circles) are automatically removed; physical modes appeared as vertical
alignments are remained in Figure 7.7(b). Finally, eight modes are successfully identified,
including five bending modes and three torsional modes.

![Figure 7.7. The stabilization diagram for setup No.5: (a) full; (b) cleared](image)

<table>
<thead>
<tr>
<th>Mode</th>
<th>Frequency (Hz)</th>
<th>FEM</th>
<th>SSI</th>
<th>$\sigma_f$ (%)</th>
<th>Error (%)</th>
<th>MAC</th>
<th>$\sigma_{ms}$ (%)</th>
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<td>B4</td>
<td>7.734</td>
<td>5.902</td>
<td>1.57</td>
<td>31.04</td>
<td>0.8576</td>
<td>4.57</td>
<td></td>
</tr>
<tr>
<td>T2</td>
<td>8.098</td>
<td>6.518</td>
<td>1.99</td>
<td>24.24</td>
<td>0.9479</td>
<td>2.02</td>
<td></td>
</tr>
<tr>
<td>B5</td>
<td>10.004</td>
<td>9.262</td>
<td>3.26</td>
<td>8.01</td>
<td>0.8130</td>
<td>2.47</td>
<td></td>
</tr>
<tr>
<td>T3</td>
<td>13.714</td>
<td>12.869</td>
<td>5.05</td>
<td>6.57</td>
<td>0.8183</td>
<td>6.16</td>
<td></td>
</tr>
</tbody>
</table>

Note: B denotes bending mode; T denotes torsional mode; $\sigma_f$ and $\sigma_{ms}$ are standard
derivations of measured frequency and mode shape, respectively.
Figure 7.8. Comparison of measured and FEM derived mode shapes
Table 7.1 compares measured frequencies from SSI with those from the FEM. MAC values between analytical and measured mode shapes are also presented. Figure 7.8
compared mode shapes from measurement and the FEM. A significant difference in frequencies is observed; frequency error of B2, T1, B3, B4, and T2 ranges from 24.24% to 39.8%. Measured frequency overall smaller than those from FEM, indicating the real structural is softer than the FEM. In addition, although similar mode patterns between measured and analytical mode shapes are observed, some MAC values are only around 0.8, such as B3, B5, and T3. Therefore, it is essential to update the model of the cable-stayed pedestrian bridge.

7.4.3 Bayesian model updating

Prior to model updating, variance-based GSA is applied to measure the importance of parameters to be updated based on the individual contribution to the total variance of model response. For the pedestrian bridge, material properties (elastic modulus $E$, mass density $\rho$) of different structural components, and the initial tension strain of stay cables, $S_{tr}$, are considered as updating parameters. Table 7.2 summarizes a total of 14 parameter candidates in GSA, including 12 parameters about material properties and 2 initial tension strains of cables N3 and N6 at two sides of the pylon.

<table>
<thead>
<tr>
<th>Structural component</th>
<th>Parameter</th>
<th>Symbol</th>
<th>Nominal value</th>
<th>Unit</th>
<th>Decision</th>
</tr>
</thead>
<tbody>
<tr>
<td>Pylon</td>
<td>Elastic modulus</td>
<td>$E_1$</td>
<td>202</td>
<td>GPa</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Mass density</td>
<td>$P_1$</td>
<td>7900</td>
<td>kg/m$^3$</td>
<td>✓</td>
</tr>
<tr>
<td>T-shaped pier</td>
<td>Elastic modulus</td>
<td>$E_2$</td>
<td>202</td>
<td>GPa</td>
<td>×</td>
</tr>
<tr>
<td></td>
<td>Mass density</td>
<td>$P_2$</td>
<td>7900</td>
<td>kg/m$^3$</td>
<td>×</td>
</tr>
<tr>
<td>Straight segment</td>
<td>Elastic modulus</td>
<td>$E_3$</td>
<td>202</td>
<td>GPa</td>
<td>✓</td>
</tr>
<tr>
<td></td>
<td>Mass density</td>
<td>$P_3$</td>
<td>7900</td>
<td>kg/m$^3$</td>
<td>✓</td>
</tr>
<tr>
<td>Skewed span</td>
<td>Elastic modulus</td>
<td>$E_4$</td>
<td>202</td>
<td>GPa</td>
<td>×</td>
</tr>
</tbody>
</table>
Table 7.2. Parameter candidates in a global sensitivity analysis (continued)

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value 1</th>
<th>Value 2</th>
<th>Unit</th>
<th>Kept?</th>
</tr>
</thead>
<tbody>
<tr>
<td>Mass density U-shaped segment</td>
<td>$P_4$</td>
<td>7900</td>
<td>kg/m$^3$</td>
<td>×</td>
</tr>
<tr>
<td>Elastic modulus U-shaped segment</td>
<td>$E_5$</td>
<td>202</td>
<td>GPa</td>
<td>✓</td>
</tr>
<tr>
<td>Mass density Stay cables</td>
<td>$P_5$</td>
<td>7900</td>
<td>kg/m$^3$</td>
<td>✓</td>
</tr>
<tr>
<td>Elastic modulus Stay cables</td>
<td>$E_6$</td>
<td>195</td>
<td>GPa</td>
<td>✓</td>
</tr>
<tr>
<td>Mass density Stay cables</td>
<td>$P_6$</td>
<td>7900</td>
<td>kg/m$^3$</td>
<td>×</td>
</tr>
<tr>
<td>Initial strain of cable N3</td>
<td>$S_{t1}$</td>
<td>8.30×10^{-4}</td>
<td>—</td>
<td>×</td>
</tr>
<tr>
<td>Initial strain of cable N6</td>
<td>$S_{t2}$</td>
<td>9.60×10^{-4}</td>
<td>—</td>
<td>×</td>
</tr>
</tbody>
</table>

Note: ✓ denotes the parameter kept; × denotes the parameter removed.

Figure 7.9. Sensitivity index of each modal frequency
To improve computational efficiency in GSA, a Kriging model is constructed with respect to the 14 parameter candidates in Table 7.2. LHS generates three hundred sample points, and corresponding modal frequencies are derived from a FEM, yielding 300 training data sets to construct a Kriging model. Once the Kriging modeling is complete, additional $2 \times 10^4$ samples and responses from the Kriging model are used based on MCS for GSA. The significant parameters are chosen by the first order, $S_i$ and total $S_{T_i}$ sensitivity indices for the subsequent Bayesian model updating. The sensitivity results of each frequency by the proposed variance-based GSA are shown in Figure 7.9. Additional findings from Figure 7.9 are summarized as follows:
• Interaction effects among 14 parameters are not salient, since $S_i$ and $S_{Ti}$ are almost identical. The subtraction of $(S_{Ti} - S_i)$ reflects the total interaction effects of the $i$th parameter with others.

• For the structural components of the pylon, straight, and U-shaped segment, material properties, e.g., elastic modulus and mass density, have a significant effect on the frequency responses. This can be explained by that these components play an important role in the bridge's operational vibration.

• For the stay cables, the initial strain and mass density have little effect on all the frequency responses, but elastic modulus has significant effect on only fourth natural frequency.

• Parameters $E_1$ and $P_1$ about the pylon have negligible effect on most frequencies, but considerably contribute to the eighth natural frequency.

Based on findings from Figure 7.9, parameters with lower sensitivity indices are removed (the threshold of sensitivity index is defined as 0.2 here). However, elastic modulus and mass density of the pylon, the straight span, and the skewed span are retained because of their pronounced contribution to frequency responses. Also, the elastic modulus of stay cable apparently affects the fourth frequency, hence it is considered as an updating parameter. Therefore, 7 parameters, namely, $E_1, E_3, E_5, E_6, P_1, P_3$, and $P_5$ as shown in last column in Table 7.2, are selected in Bayesian model updating.
Following the selected parameters by the GSA, Bayesian model updating is carried out. Similar to the numerical example, we defined stiffness coefficient (SC), $\theta_i = \tilde{E}_i/E_i$, $i = 1, 3, 5, 6$, $\tilde{E}_i$ and $E_i$ are actual and nominal elastic modulus, respectively; mass coefficient (MC), $\beta_j = \tilde{P}_j/P_j$, $j = 1, 3, 5$, $\tilde{P}_j$ and $P_j$ are actual and nominal mass density, respectively. The range of SCs and MCs are set as $0.7 \leq \theta_i \leq 1.3$ and $0.7 \leq \beta_j \leq 1.3$ to ensure the physical meaning. The Kriging model is then firstly constructed with respect to seven coefficients to substitute complex FE model in ANSYS. A total of 350 samples (300 for training and 50 for accuracy validation) are generated by LHS, corresponding frequency and mode shape responses are derived from the FEM. Finally, 16 Kriging models (8 for frequencies and 8 for MAC values) are constructed. Figure 7.10 shows the response surfaces of the second frequency and MAC values with respect to $\theta_1$ and $\theta_3$. As expected, the surface of MAC value is more complex than that of frequency, since it is relatively more difficult to measure mode shape compared with frequency.

A total of 50 sets of training data are used to verify the accuracy of the built Kriging models. The MSE and $R^2$ values of all frequencies ($f_1 - f_9$) and MAC values ($\text{MAC}_1 - \text{MAC}_8$) in Figure 7.11 are closed to zero and unity, respectively, indicating the Kriging models exhibit high accuracy.
The measured frequencies and mode shapes as well as their uncertainties are used in the objective function in Eq. (7.3). The model responses are predicted from the Kriging model instead of FEM. Then DREAM algorithm is applied to generate samples to approximate a posterior PDF. The input parameters in DREAM are defined as: initial values of all the coefficients range from 0.7 to 1.3; ten Markov chains are simultaneously ran with 6000 samples per chain. The results are shown in Figure 7.12. Figure 7.12 (a) is a trace plot of one chain for all SCs and MCs, giving the visual sense that all the coefficients stably converge. In Figure 7.12 (b), the convergence criterion $R_{stat}$ of all coefficients is less than 1.2 and close to zero around 10,000 and 20,000, respectively, indicating the stationary Markov chains are achieved. Compared to the numerical example, more iteration samples are needed to reach a stable posterior PDF due to the real-world application is more complicated than the numerical one. The observation that five out of ten Markov chains achieve convergence and have similar iteration performance in Figure 7.12 (c) demonstrates the updated results are reliable.
Figure 7.12. Results of updated SCs and MCs: (a) trace plot of coefficients; (b) variation of convergence diagnosis $R_{stat}$; (c) trace plots of five out of ten Markov chains

The coefficients are multiplied by analytical elastic modulus and mass density, yielding the actual values of material parameters. The histograms of actual elastic modulus and mass density and corresponding fitted distributions (red curves) are displayed in Figure 7.13. Interestingly, some parameters can be well fitted by normal distribution, but the distribution of parameters $E_3$, $E_6$ and $P_1$ exhibit non-Gaussian feature with a long tail. It is also observed that distributions of parameters $E_6$ and $P_1$ are concentrated in a very narrow region, but others are relatively wide spreading, indicating $E_6$ and $P_1$ have smaller uncertainties (C.O.V of 0.25% and 0.22%, respectively) compared with other parameters (see Table 7.3).
The updated parameters are tabulated in Table 7.3. The negative and positive sign in the last column denotes decrease and increase, respectively. It is observed that all elastic modulus decreased, and mass density increased (except for $P_3$). The model updating aims to match measured responses with analytical counterparts, measured frequencies in the pedestrian bridge are overall smaller than those in FEM (see Table 7.1). It is understandable that frequency is proportional to elastic modulus but inversely proportional to mass density based on fundamental structural dynamics. Therefore, elastic modulus and mass density has to decrease and increase, respectively, in order to have an agreement between measured frequencies and those in FEM in this case. The absolute change in most parameters is over 20%, similar updating results can be found in Brownjohn and Xia (2000) and Jaishi and
It is also worth mentioning in this study that changes in material parameters does not represent their actual variations in bridge due to any types of damage. These changes only reflect the modeling error between the FE model and the real structure, possibly attributing to the idealization and assumption in FE model, such as inaccurate boundary condition and geometry, and limited discretization. Hence, the updated parameters can be seen as the “equivalent” elastic modulus and “equivalent” mass density.

The mean values in Table 7.3 are used to calculate updated frequencies and MAC values. The frequency errors and MAC values between updated and initial model for eight modes are shown in Figure 7.14, from which the frequency errors remarkably decreased after updating. For instance, the errors decreased substantially from 24.9% to 4.7% for B2, from 28.1% to 3.8% for T1, from 39.8% to 5.8 for B3, from 31% to 3% for B4, and from 24.2% to 2.7% for T2, respectively. Regarding MAC values, they are closer to unity after updating, suggesting mode shapes derived from FEM match better with measured ones. Especially, MAC value increases from 0.8168 to 0.9087 for B3, from 0.8130 to 0.9143 for B5, and from 0.8183 to 0.8948 for T3. In short, the proposed Bayesian model updating framework enhances the accuracy of FEM and gives an excellent agreement with measurement.

![Figure 7.14. Comparison of modal parameters between initial and updated model: (a) frequency error; (b) MAC values](image)

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Turning attention to the required computational effort for this complex and large-scale cable-stayed pedestrian bridge, we used a desktop with Intel(R) Core(TM) i5-4460 CPU@3.2GHz and RAM memory of 8GB to proceed the proposed updating framework under the Windows 10 operational environment. The computational cost is compared in two aspects: 1) variance-based GSA with Kriging model; 2) Bayesian model updating with Kriging model and DREAM. The use of Kriging model in these two aspects aims to improve efficiency for GSA and model updating, respectively. It should be noted that the time cost of proposed framework is unbearable by direct FEM analysis using the personal computer, so we estimated the whole time via multiplying iteration number by the spent time in a single FEM run. The time of one FEM evaluation in ANSYS for this pedestrian bridge is about 1.5 minutes. In GSA, the total time for parameter selection using Kriging model is about 6 hrs (including time for training data) for 20,000 iterations; without a Kriging model, it requires around 21 days; in Bayesian model updating, the total consumed time of 60,000 iterations with a Kriging model is about 9 hrs (including time for training data), while the required time directly using FEM is about 125 days. Table 7.4 lists the whole-time cost in terms of GSA and updating work. It shows that the computational cost directly using FEM-based Bayesian model updating is unaffordable and impractical. It has been recognized that a high-fidelity modeling for complex and large-scale structures is usually necessary for a better model prediction and structural analysis, which involves hundreds of thousands of elements and nodes in commercial FEA packages. The computational time would be highly expensive if a large amount of iteration is needed. In this context, a fast-running Kriging model is a promising alternative of time-consuming FEM for dealing with computational issue.
Table 7.4. Time cost of in the cable-stayed pedestrian bridge (Unit: hours)

<table>
<thead>
<tr>
<th>Model type</th>
<th>GSA</th>
<th>Bayesian updating</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>FEM</td>
<td>504</td>
<td>3000</td>
<td>3504</td>
</tr>
<tr>
<td>Kriging model</td>
<td>6</td>
<td>9</td>
<td>15</td>
</tr>
</tbody>
</table>

7.5 Conclusions

In this chapter, a new Bayesian model updating framework is proposed and consists of two stages. Stage one aims to prepare for Bayesian model updating and provides two time-saving strategies, involving variance-based GSA for dropping insignificant parameters to reduce model dimensionality, and Kriging modeling to substitute FE model and further improve computational efficiency. Stage two is the implementation of Bayesian model updating with a multi-chain DREAM algorithm. A real-world application of a cable-stayed pedestrian bridge demonstrated that the proposed updating framework gives satisfactory results with much-reduced time cost. The main conclusions and contributions are summarized as follows:

- Variance-based GSA is used for parameter selection in FEMU, uncertainties and interaction effects among parameters are both considered. Traditional GSA based on MCS using FE model is computationally intensive. The use of Kriging model rather than FE model in GSA greatly reduced computational cost and makes GSA feasible in practice.

- The time-consuming high-fidelity FE model cannot achieve efficient model updating with the context of many model evaluations. The Kriging model is an effective alternative to relief computational burden while maintaining accuracy.

- In DREAM algorithm, multiple Markov chains are run in parallel to sufficiently seek the best solution in parameter space, leading to a fast convergence rate and
accurate parameter identification. In addition, a solid convergence diagnosis is provided in DREAM to determine whether Markov chains are stationary or not.

- A real-world complex and large-scale cable-stayed pedestrian bridge demonstrated the proposed updating framework has desirable performance in parameter identification and uncertainty quantification, indicating the proposed method is suitable for the real applications.
CHAPTER 8
CONCLUSIONS AND FUTURE WORK

The research work in this dissertation aims to develop an efficient and robust vibration-based structural health monitoring (SHM) framework for civil engineering structures. The presented work mainly contributes to two areas: (1) operational modal analysis (OMA) using output-only system identification methods based on vibrational measurements; (2) Bayesian model updating and probabilistic damage detection using modal data. This chapter reviews the summary and discussions of this dissertation. Potential future work associated with current research is also provided.

8.1 Conclusions

Challenges in practical vibration-based SHM are 1) time-consuming modal parameter identification with much human interaction during continuous monitoring; 2) uncertainties on modal parameters; 3) simultaneous identification of mass and stiffness, coupling effect; 4) a considerable amount of uncertainties in Bayesian model updating; 4) computational demand of Bayesian model updating for complex and large-scale structures. In this dissertation, a two-phase vibration-based SHM framework are proposed to address these challenges. Phase one focuses on developing an automated operational modal identification method using stochastic subspace identification (SSI) and Bayesian modal identification (BMI). modal parameters’ uncertainties are also accounted. This phase mainly provides modal data that will be used in the model updating process in phase two.
In phase two, a new Bayesian model updating approach (BMUA) is proposed to identify simultaneously mass and stiffness by addressing the coupling effect based on extracted modal properties in phase one. Uncertainties of structural parameters are reasonably provided. Besides, some strategies are proposed to expedite BMUA for high-fidelity structures. The major contributions and findings are summarized as follows:

**Phase one: modal parameter identification (prepare for model updating in phase two)**

In SSI, it is labor-intensive to distinguish physical modes from spurious modes with human intervention in a stabilization diagram. Additionally, the elimination of spurious modes by visual observation tends to yield incorrect and unreliable identification results. During continuous monitoring with a vast of measured data, this way is also less impractical. Chapter 3 presented an automated SSI to interpret the stabilization diagram with minimum human effort. Modal validation criteria and an additional uncertainty criterion are employed to initially remove as many spurious modes as possible. A novel threshold calculation for clustering is proposed with incorporating the uncertainty of modal parameters and the weighting factor. An improved self-adaptive clustering with new distance calculation is used to group physical modes, followed by the final step of robust outlier detection to select outlying modes. Two benchmark field tests of Dowling Hall Footbridge and a post-tensioned concrete bridge (Z24 bridge) are used to verify the proposed approach. A modal tracking was used for continuously measured data for demonstrating the applicability of the approach. The proposed framework has minimal user’s involvement in achieving sufficient accuracy. Therefore, the proposed work can be suitable for long-term health monitoring, e.g., modal tracking.
In BMI, the manual operation includes the selection of initial frequency, which is often visually picked from a singular value spectrum, and frequency bandwidth which is chosen with the consideration of a trade-off between the data used for making inference and modeling error involved. The above procedures have limited the application of BMI in processing long-term data. Chapter 4 proposed an automated BMI to address these issues. A stabilization diagram is firstly built and automatically interpreted by modal validation criteria and clustering strategy to obtain the initial frequency. A series of effective bandwidth factors within a predefined factor range is then determined for the selection of frequency bandwidth. The proposed automation method is verified by a numerical example and then applied to the Z24 benchmark bridge for long-term data analysis. Results show that the automation method can accurately identify modal parameters with minimum human intervention, even for closely spaced and weakly excited modes. Overall, both initial frequency and frequency bandwidth in BMI are automatically determined, requiring minimal human interference to achieve sufficient accuracy. With the proposed method, a large number of measurements can be automatically treated without any loss of physical modes of interest. This makes the method suitable and promising for real applications, e.g., long-term health monitoring.

The basic principles of SSI and BMI are different. SSI uses state-space models to extract system state and output realized mathematical matrices from measured vibration data; modal parameters are identified by interpreting the matrices. While, BMI constructs a model representing the difference between analytical and measured response, directly converts measurements to FFT data; the physical meaning is strictly obeyed. The modal parameters are then identified as the most probable values based on Bayes’ theorem. As
for uncertainty quantification, SSI employs the propagation of first-order perturbation from measured data to modal parameters, the covariance matrix is determined. But BMI naturally provides uncertainties of modal parameters, as the posterior distribution is obtained. It is also worth mentioning that SSI gives a larger uncertainty estimation than BMI because more sources of uncertainty are considered (see Section 3.3.1.2); BMI estimates the uncertainty induced by only modeling error and measurement noise.

**Phase two: Bayesian model updating using modal parameters acquired from phase one**

The conventional Bayesian model updating approach (BMUA) is mainly used to update stiffness with the assumption that structural mass is well known and invariable due to damage. Because simultaneously updating stiffness and mass lead to unidentifiable case or coupling effect of stiffness and mass, this assumption in conventional BMUA is questionable to update stiffness when the mass has significantly changed. Chapter 5 proposes a new updating framework based on two structural systems: original and modified systems. A modified system is created by adding known mass or stiffness to the original system. Different from the conventional BMUA, two sets of measured vibration data are used to address the coupling effect. The new eigen-equations are derived by incorporating added mass or stiffness, yielding a new prior PDF. The objective functions are formulated by taking the posterior PDF’s negative logarithm. Finally, the analytical formulations of modal parameters (frequency and mode shape) and structural parameters (mass and stiffness) are derived using an asymptotic approximation method, and they were updated iteratively. In addition, the inverse of the Hessian matrix of the objective function determines the covariance matrix of uncertain parameters. Two numerical simulations (2D and 3D shear structures) are utilized to demonstrate the performance of the proposed
approach. The newly proposed BMUA successfully identifies mass and stiffness and address the coupling effect, which is considered as the main contribution of this research work.

The work in Chapter 6 is considered as an extension of the work in Chapter 5. Chapter 5 adopted an asymptotic optimization method to circumvent high-dimensional integrals involved in the posterior PDF for Bayesian inference. The analytical formulations of optimal model parameters are derived by the linear optimization method. However, the asymptotic approximation method assumes that parameters have unimodal and Gaussian distribution, which does not necessarily guarantee an actual physical model, especially for multi-modal and non-Gaussian posterior. Also, an insufficient amount of data and complex model class may lead to an unidentifiable problem. To this end, Chapter 6 proposed a new BMUA, which intrinsically addressed the coupling effect of mass and stiffness by two sets of data from the original and modified system with added mass/stiffness. The new characteristic equations are constructed. The posterior PDF is also reformulated. Differential Evolution Adaptive Metropolis (DREAM) is then employed to generate samples for approximation of the posterior PDF. The proposed BMUA simultaneously identifies the mass and stiffness; their uncertainties are also straightforward provided by the estimated PDF. A numerical study on a ten-story shear building and an experimental study on a three-story aluminum frame small-scale model is used at intact and damaged structural states to verify the accuracy and feasibility of the proposed method. It is also found that BMUA with added mass showed more reliable updating results than BMUA with added stiffness. Mainly because it is more convenient and accurate to measure the magnitude of added mass compared to that of added stiffness. In other words, the
calculation of added stiffness involves more uncertainties, which impairs the accuracy of the BMUA framework with added stiffness.

Although Bayesian model updating with DREAM is promising, it is computationally demanding because many FE model evaluations are required. As a result, it becomes impractical for complex and large-scale engineering structures. Chapter 7 proposed two time-saving strategies, including variance-based GSA for dropping insignificant parameters to reduce model dimensionality, and Kriging modeling to substitute FE model and further improve computational efficiency. Finally, Bayesian model updating with DREAM algorithm is implemented to update structural parameters using vibrational data. A real-world application of a cable-stayed pedestrian bridge demonstrated that the proposed updating framework gives satisfactory results with the much-reduced time cost.

8.2 Recommendations of future work

Although the proposed research work in this dissertation has been demonstrated to have a satisfactory performance in numerical study, laboratory tests, and real-world application, there are some aspects for potential future work to enhance the current work. Recommendations for future work are mentioned as follows:

- In the proposed automated SSI and BMI, the validation examples, e.g., steel frame pedestrian bridge and highway concrete bridge, have a relatively wider frequency range (e.g., 0-15Hz), including few weakly-excited modes and closed spaced modes. In contrast, long-span or suspension bridges exhibit low frequency range (e.g., 0-1Hz) and multiple extremely closed-spaced modes. Further verification of the proposed automated modal identification methods for structures with low-frequency range is needed.
Optimized sensor location should be further developed to acquire sufficient and important measurement information about structural dynamics with a small network of sensors that is practically available.

In the proposed automated SSI, a clustering technique with an adaptive threshold is used to group modes with similar features, which can be defined as hard clustering. It would be worth developing a soft clustering method to assign each mode with a probability to be part of a specific class, which may be more reasonable and accurate to identify modal parameters.

The proposed Bayesian model updating framework identifies mass and stiffness using global information, e.g., natural frequency and mode shape. However, global information may not be able to reflect local damage, such as holes and cracks. Modal damping is sensitive to local damage. Hence, the development of Bayesian model updating incorporating damping information as well as frequency and mode shape would be helpful to advance vibration-based damage detection.

In the current BMUA with added mass, the modified system is created by adding stationary masses to the original structure, which may not always be practical in real-world settings. Therefore, moving mass, e.g., vehicles on bridges or elevators in buildings, can be considered to create modified systems.

It is found that in the proposed BMUA with added stiffness, the accuracy of the estimation of stiffness addition dominates the updating performance. However, it is challenging to precisely calculate the magnitude of stiffness addition.
Therefore, it is necessary to propose a method to update stiffness addition and structural parameters together.

- Although the proposed Bayesian model updating with variance-based global sensitivity analysis and Kriging model is efficient, it still requires thousands of model evaluations to achieve an accurate approximation of the posterior PDF. In the future study, a more efficient Bayesian model updating framework will be investigated, such as the Gaussian mixture model and Bayesian variational inference, which need fewer iterations to estimate posterior distribution with multi-modality and non-Gaussian.

- The proposed vibration-based SHM in this dissertation only works on the basis of linear and time-invariant model assumption; another vibration-based SHM might be developed to detect damages in the case of non-linear and time-varying structural behaviors.
REFERENCES


ASCE, 2009. 2009 report card for America’s infrastructure.*American Society of Civil Engineers.*


https://doi.org/10.1016/j.marstruc.2015.10.003


https://doi.org/10.1016/j.ymssp.2010.08.002

https://doi:10.1061/(ASCE)EM.1943-7889.0000213

https://10.1016/j.ymssp.2011.06.017

https://10.1016/j.ymssp.2011.06.019


https://doi.org/10.1016/j.ymssp.2016.03.025


https://doi.org/10.1002/stc.1571


Brincker, R., Zhang, L., and Andersen, P. 2001. “Modal identification of output-only systems using frequency domain decomposition”. *SmMaS*, 10 (3), 441


FEMA 2006. “Federal emergency management agency: techniques for the seismic rehabilitation of existing buildings.”, FEMA


http://10.1214/ss/1177011136


https://doi.org/10.1016/j.ymssp.2009.10.006

https://doi.org/10.1016/j.ymssp.2014.07.010

https://doi.org/10.1016/j.ymssp.2016.07.012


https://doi.org/10.1016/j.engstruct.2021.111988


Huang, Q., Gardoni, P., and Hurlebaus, S. 2012. “A probabilistic damage detection approach using vibration-based nondestructive testing”. Structural safety, 38, 11-21


at-shaped building using a parallel technique”. *Engineering Structures*, 193, 12-27. [https://doi.org/10.1016/j.engstruct.2019.05.023](https://doi.org/10.1016/j.engstruct.2019.05.023)

Lam, H.-F., Yang, J., and Au, S.-K. 2015. “Bayesian model updating of a coupled-slab system using field test data utilizing an enhanced Markov Chain Monte Carlo simulation algorithm”. *Engineering Structures*, 102, 144-155. [https://doi.org/10.1016/j.engstruct.2015.08.005](https://doi.org/10.1016/j.engstruct.2015.08.005)


https://doi.org/10.1007/s12205-020-0983-4


Los Alamos National Laboratory, USA, 1

Earthquake Engineering & Structural Dynamics, 29 (8), 1131-1152. http://dx.doi.org/10.1002/1096-9845(200008)29:8

AIAA Journal, 46 (11), 2955-2965. http://dx.doi.org/10.2514/1.38115

Mechanical Systems and Signal Processing, 143, 106837

Multidimensional Systems and Signal Processing, 32 (1), 335-357


Tian, Y., Zhang, J., and Han, Y. 2019a. “Structural scaling factor identification from output-only data by a moving mass technique”. 

Tian, Z., Howden, S., Ma, Z., Xiao, W., and Yu, L. 2019b. “Pulsed laser-scanning laser doppler vibrometer (PL-SLDV) phased arrays for damage detection in

278
aluminum plates”. *Mechanical Systems and Signal Processing*, 121, 158-170


Yan, W.-J., and Kataygiotis, L. S. 2015. “A novel Bayesian approach for structural model updating utilizing statistical modal information from multiple setups”. *Structural Safety*, 52, 260-271. [https://doi.org/10.1016/j.strusafe.2014.06.004](https://doi.org/10.1016/j.strusafe.2014.06.004)


APPENDIX A

NOMENCLATURE

Chapter 3

\( A, B \)  
System state and output matrices

\( x_k \)  
Discrete-time state vector

\( y_k \)  
Measured response vector

\( \omega_k \)  
Process white noise vector

\( v_k \)  
Measurement white noise vector

\( H \)  
Block Hankel matrix

\( T \)  
Block Toeplitz matrix

\( \hat{R} \)  
Output correlation

\( O_i \)  
Observability matrix

\( \Gamma_i \)  
Controllability matrix

\( \lambda_i \)  
The \( i \)-th eigenvalue

\( \varphi_i \)  
The \( i \)-th eigenvector

\( f_i \)  
The \( i \)-th frequency (Hz)

\( \zeta_i \)  
The \( i \)-th damping ratio

\( \phi_i \)  
The \( i \)-th mode shape

\( f_s \)  
The sampling frequency

\( i \)  
Time lag

\( T_i \)  
Fundamental period, (unit: second)

\( t \)  
Sampling interval

\( \text{Re}(\cdot), \text{Im}(\cdot) \)  
Real and imaginary part

\( \theta \)  
Phase angle in degree
\[ \Sigma_{H^{cov}} \] Covariance of Hankel matrix
\[ J_{O,H} \] Sensitivity of \( O \) with respect to Hankel matrix
\[ J_{A,O}, J_{C,O} \] Sensitivity of \( A \) and \( C \) with respect to \( O \)
\[ \Sigma_{A,C} \] Covariance of \( A \) and \( C \)
\[ J_{f_i,A}, J_{\xi_i,A}, J_{\phi_i,A} \] Sensitivity of \( f_i, \xi_i, \) and \( \phi_i \) with respect to \( A \)
\[ F_i \] The \( i \)-th frequency with two standard derivations
\[ \Phi_i \] The \( i \)-th mode shape with two standard derivations
\[ \omega \] Weighting factor in clustering threshold
\[ c \] Weighting factor in clustering distance
\[ \sigma_{f_i}, \sigma_{\phi_i} \] Standard derivation of the \( i \)-th frequency and mode shape
\[ V \] Minimum distance vector
\[ \bar{\mu}, \bar{\sigma} \] Mean and standard derivation of \( V \)
\[ \hat{\mu}_{MCD}, \hat{\Sigma}_{MCD} \] Mean and covariance of MCD
\[ \sigma_z \] Standard derivation of the \( z \)-th clusters

Chapter 4

\[ C_k \] Covariance matrix of FFT data
\[ \text{c.o.v} \] Coefficient of variation
\[ e \] Difference between model response and measured data
\[ \theta \] Modal parameters
\[ \mathbb{E}(\cdot) \] The expectation of the item in parenthesis
\[ \mathbf{E}_k \] Theoretical PSD matrix
\[ \zeta_0 \] Initial damping ratio
\[ T_d \] Data duration
\[ f \] Natural frequency
\[ f_0 \] Initial frequency
\[ f_i \] The \( i \)-th difference between initial and identified frequency
\[ f \] Frequency difference vector
\[ F_k \] The FFT of measured data, \( k = 1, \ldots, N_q \), \( N_q \) is the Nyquist frequency
\( H_k \)  
Theoretical spectral density matrix of the modal acceleration

\( I_{2n} \)  
\( 2n \times 2n \) identity matrix

\( k \)  
Stiffness

\( m \)  
Mass

\( n \)  
The number of DOFs

\( N \)  
The number of sampling points per channel

\( N_f \)  
The number of FFT points in the selected frequency band

\( S \)  
Spectral density of modal excitation

\( S_e \)  
Spectral density of the prediction error

\( \Delta t \)  
Sampling interval

\( \ddot{x} \)  
Theoretical acceleration

\( \hat{x} \)  
Measured acceleration

\( Z_k \)  
Vector of the real and imaginary part of \( F_k \), \( Z_k = (Re F_k; Im F_k) \)

\( \beta_{ik} \)  
Frequency ratio, \( \beta_{ik} = f_i/f_k \); \( f_i \) and \( f_k \) are the \( i \)th modal frequency and the FFT frequency abscissa

\( P(\{Z_k\}|\theta) \)  
Likelihood function of observed data \( Z_k \)

\( P(\theta|\{Z_k\}) \)  
The posterior probability density function of \( \theta \)

\( L(\theta) \)  
Negative log-likelihood function

\( \kappa \)  
Bandwidth factor

\( \zeta \)  
Damping ratios

\( \Phi \)  
Mode shapes

\( \Phi(\cdot) \)  
Standard Gaussian cumulative distribution function

\( P_{dam}^{\ell}(d) \)  
Probability of damage occurrence with damage extent \( d \)

\( \theta_{\ell}, \sigma_{\ell} \)  
The \( \ell \)-th modal frequency estimate and its standard derivation

Chapter 5

\( M \)  
System mass matrix

\( K \)  
System stiffness matrix

\( \lambda \)  
Eigenvalues

\( \phi \)  
Eigenvectors
\( \Delta m \)  
Added mass  
\( \Delta k \)  
Added stiffness  
\( \Omega \)  
Vector of uncertainty parameters  
\( C \)  
Structural model class  
\( D \)  
Measured data  
\( p(\Omega|C) \)  
Prior probability density function  
\( p(D|\Omega,C) \)  
Likelihood function of observed data \( D \)  
\( p(\Omega|D,C) \)  
Posterior probability density function of parameters \( \Omega \)  
\( p(D|C) \)  
Normalizing constant (also denoted as \( c_0 \))  
\( \hat{\lambda} \)  
Measured eigenvalues  
\( \hat{\psi} \)  
Measured mode shapes  
\( N_d \)  
The number of degree of freedom  
\( \theta \)  
Stiffness parameters vector  
\( \beta \)  
Mass parameters vector  
\( K_l \)  
The \( l \)th elemental stiffness matrix  
\( M_l \)  
The \( l \)th elemental mass matrix  
\( K_0 \)  
Constant stiffness matrix (set as zero)  
\( M_0 \)  
Constant mass matrix (set as zero)  
\( d \)  
Fractional damage level  
\( p_l^{dam}(d) \)  
Probability of damage at damage extent \( d \)  
\( \Phi(\cdot) \)  
The cumulative distribution function  
\( \sigma_l \)  
The standard derivation  
\( \lambda' \)  
Eigenvalues in modified system  
\( \phi' \)  
Eigenvectors in modified system  
\( ME_m \)  
Eigen-equation error when updating mass  
\( ME_k \)  
Eigen-equation error when updating stiffness  
\( N_m \)  
The number of measured modes  
\( \sigma_{eq}^2 \)  
Eigen-equation error variance  
\( \Sigma_{eq} \)  
Prior covariance matrix,  
\( I \)  
Identity matrix
\( \text{exp} \) \quad \text{Exponential function}

\( \beta \) \quad \text{Nominal mass parameters}

\( \Sigma_\beta \) \quad \text{Covariance matrix of } \beta

\( \sigma_\beta \) \quad \text{Standard derivation of } \beta

\( \theta \) \quad \text{Nominal stiffness parameters}

\( \Sigma_\theta \) \quad \text{Covariance matrix of } \theta

\( \sigma_\theta \) \quad \text{Standard derivation of } \theta

\( \varepsilon \) \quad \text{Measurement error}

\( \Sigma_e \) \quad \text{Covariance matrix}

\( L_0 \) \quad \text{Selection matrix of ‘1s’ or ‘0s’}

\( J_m \) \quad \text{Objective function of updating mass}

\( J_k \) \quad \text{Objective function of updating stiffness}

\((\cdot)^*\) \quad \text{Updated parameters of } (\cdot)

\( \Gamma \) \quad \text{Covariance matrix of objective function}

\( K_e \) \quad \text{Analytical stiffness}

\( M_e \) \quad \text{Analytical mass}

\( E \) \quad \text{Young’s modulus}

\( I \) \quad \text{The moment of inertial}

\( \Omega \) \quad \text{Uncertainty parameter}

\( P(D|\Omega, C) \) \quad \text{Likelihood function of measured data } D \text{ in model class } C

\( P(\Omega|D) \) \quad \text{Noninformative prior probability density function}

\( P(D|C) \) \quad \text{Normalizing constant}

\( P(\Omega|D, C) \) \quad \text{Posterior probability density function}

\( c_0 \) \quad \text{Constant value reflecting } P(D|C) \text{ and } P(\Omega|D)

\( \tilde{f}_m \) \quad \text{The } m\text{th measured frequency}

\( f_m(\Omega) \) \quad \text{The } m\text{th calculated frequency given a set of } \Omega.

\( \varepsilon_{f,m} \) \quad \text{The } m\text{th frequency error between } \tilde{f}_m \text{ and } f_m(\Omega)

\( \Phi_m \) \quad \text{The } m\text{th measured mode shape}
The $m$th calculated mode shape given a set of $\Omega$.

$\epsilon_{ms,m}$ The $m$th mode shape error between $\Phi_m$ and $\phi_m(\Omega)$

$J(\Omega)$ Objective function with respect to $\Omega$

$\kappa$ Variance of measured data

$\sigma_{f,m}$ Standard derivation of the $m$th measured frequency

$\sigma_{ms,m}$ Standard derivation of the $m$th measured mode shape

$\Delta m$ Added mass

$M$ System mass matrix

$K$ System stiffness matrix

$\lambda$ Eigenvalue before modification

$\phi$ Mode shape before modification

$\lambda'$ Eigenvalue after modification

$\phi'$ Mode shape after modification

$\Delta f$ Frequency change after adding mass

$\psi$ Unscaled mode shape in the original system

$M^*$ Analytical mass

$N$ Problem dimension

$P$ The number of Markov chains

$I_{max}$ The maximum iteration

$\gamma$ Individual samples at each Markov chain

$\Omega_t^l$ Samples at the $l$th iteration

$\alpha$ The acceptance rate

$u$ Samples from a uniform distribution $U(0, 1)$

$Q_1$ Lower quartile

$Q_3$ Upper quartile

$R_{stat}$ Scale reduction factor

$Z$ Mean of the variance of total $P$ Markov chains

$B/\gamma$ Variance of the mean of $P$ parallel Markov chains

$\theta$ Stiffness coefficient

$\beta$ Mass coefficient
$\Delta k$ Added stiffness

Chapter 7

$\tilde{f}_m$ The $m$th measured frequency

$f_m(\Omega)$ The $m$th calculated frequency given a set of $\Omega$.

$\varepsilon_{f,m}$ The $m$th frequency fractional error between $\tilde{f}_m$ and $f_m(\Omega)$

$\tilde{\phi}_m$ The $m$th measured mode shape

$\phi_m(\Omega)$ The $m$th calculated mode shape given a set of $\Omega$.

$\varepsilon_{ms,m}$ The $m$th mode shape fractional error between $\tilde{\phi}_m$ and $\phi_m(\Omega)$

$P(\Omega|D, C)$ Posterior probability density function of parameters $\Omega$

$J(\Omega)$ Objective function with respect to $\Omega$

$\kappa$ Variance of measured data

$\sigma_{f,m}$ Standard derivation of the $m$th measured frequency

$\sigma_{ms,m}$ Standard derivation of the $m$th measured mode shape

$x$ Structural parameter vector

$Y$ Model response vector

$F(x)$ Polynomial regression function

$\beta$ Regression coefficient vector

$Z(x)$ Stationary Gaussian process error with zero mean and variance

$R$ Correlation matrix

$\alpha_k$ The $k$-th correlation coefficient

$m_s$ The number of training samples

$\hat{y}(x)$ Predicted response

$H(x)$ Correlation vector between training samples and prediction points

$\hat{\sigma}_y(x)$ Prediction error of $\hat{y}(x)$

$x$ Sample point

$P$ The number of sample point

$\pi$ Stratification of sequence $(0, 1, \ldots, P - 1)$

$U$ Random value from a uniform distribution $(0, 1)$

MSE Mean square error
<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>$R^2$</td>
<td>Coefficient of determination</td>
</tr>
<tr>
<td>$V(\bullet)$</td>
<td>Variance operator</td>
</tr>
<tr>
<td>$E(\bullet)$</td>
<td>Expectation operator</td>
</tr>
<tr>
<td>$S_i$</td>
<td>First-order sensitivity index</td>
</tr>
<tr>
<td>$S_{Ti}$</td>
<td>Total sensitivity index</td>
</tr>
<tr>
<td>$X_{selected}$</td>
<td>Selected significant parameters</td>
</tr>
<tr>
<td>$X_{total}$</td>
<td>All possible parameter candidates</td>
</tr>
</tbody>
</table>
APPENDIX B

COMPARATIVE STUDY BETWEEN CONTACT AND NON-CONTACT SENSOR

This appendix presents the comparative study between contact (accelerometer) and non-contact (high-speed camera) sensor to acquire vibration measurement for a laboratory-scale three-story shear frame in Section 6.4.2. The automated SSI in Chapter 3 and automated BMI in Chapter 4 are utilized to identify modal parameters using accelerations and displacements measured by accelerometers and camera, respectively. The test setup with three accelerometers is the same as in Section 6.4.2; the test setup with a high-speed camera is shown in Figure B.1.

![Test setup of shear frame: (a) displacement measurement; (b) high-speed camera](image)

The measurement using high-speed camera was conducted with the help of Dr. Jeffrey Hay, a CEO of RDI Technologies. The principles and technical introduction of high-speed camera can be found in Dr. Jeffrey’ dissertation (Hay, 2011). The measurement system
was also patented in 2014 (Kielkopf and Hay, 2014), which allows to measure dynamic characteristics for civil infrastructures. The specification of non-contact high-speed camera and associated data processing software packages in this test are listed as below:

- FLIR Grasshopper 3 GS3-U3-23S6M-C with a Sony IMX174 mono sensor:
  Resolution: 1920 × 1200
- USB3 cable
- RDI BridgeView software
- Microsoft Surface Book

After hitting the top floor by a rubber hammer, the displacements and accelerations at from top to bottom were recorded using a high-speed camera and accelerometers at a sampling frequency of 120.2 Hz and 2000 Hz, respectively. The data duration was 10 seconds. For a fair comparison, acceleration data was down sampled to 125 Hz. It is worth mentioning that the high-speed camera was used to capture vibration displacements without any artificial target marks. Instead, the camera automatically traces the motion of edge points of shear frame.

A total three different tests were considered, including a healthy case and two damage cases, which are the same as in Section 6.4.2. In three cases, the modal parameter identification for original and modified system with added mass was performed by the automated SSI and BMI. The results are shown in figures and tables in Sections B.1-B.3 in which natural frequencies and mode shapes are included. The stabilization diagrams with singular value spectrum for each case are also presented. In each table, the Acc and Cam denote accelerometer and camera, respectively; S/N denotes signal-to-noise ratio.
identified by BMI. The higher S/N values is, the lower noise level is during the measurement.

B.1 Healthy case

![Stabilization Diagram for Original System](image1)

**Figure B.2.** The stabilization diagram of original system in healthy case: (a) acceleration measurement; (b) displacement measurement

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>SSI Acc</th>
<th>SSI Cam</th>
<th>BMI Acc</th>
<th>BMI Cam</th>
<th>Error (%)</th>
<th>S/N Acc</th>
<th>S/N Cam</th>
<th>S/N Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.95</td>
<td>8.32</td>
<td>7.93</td>
<td>8.30</td>
<td>4.60</td>
<td>27993</td>
<td>39077</td>
<td>4.86</td>
</tr>
<tr>
<td>2</td>
<td>23.6</td>
<td>24.27</td>
<td>23.76</td>
<td>24.36</td>
<td>2.85</td>
<td>1712</td>
<td>1329</td>
<td>2.49</td>
</tr>
<tr>
<td>3</td>
<td>35.18</td>
<td>36.22</td>
<td>35.30</td>
<td>36.29</td>
<td>2.95</td>
<td>4568</td>
<td>509</td>
<td>2.82</td>
</tr>
</tbody>
</table>

![Stabilization Diagram for Modified System](image2)

**Figure B.3.** The stabilization diagram of modified system in healthy case: (a) acceleration measurement; (b) displacement measurement
Table B.2. Measured frequency for modified system in healthy case (Hz)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>SSI Acc</th>
<th>SSI Cam</th>
<th>Error (%)</th>
<th>BMI Acc</th>
<th>BMI S/N</th>
<th>BMI Cam</th>
<th>BMI S/N</th>
<th>Error (%)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>7.82</td>
<td>8.03</td>
<td>2.69</td>
<td>7.91</td>
<td>70068</td>
<td>8.09</td>
<td>143117</td>
<td>2.15</td>
</tr>
<tr>
<td>2</td>
<td>22.81</td>
<td>23.38</td>
<td>2.47</td>
<td>22.90</td>
<td>70981</td>
<td>23.45</td>
<td>4121</td>
<td>2.42</td>
</tr>
<tr>
<td>3</td>
<td>34.02</td>
<td>34.86</td>
<td>2.48</td>
<td>34.09</td>
<td>3373</td>
<td>34.92</td>
<td>132</td>
<td>2.44</td>
</tr>
</tbody>
</table>

Figure B.4. Measured mode shapes in healthy case: (a) original system; (b) modified system

B.2 Damage case 1

Figure B.5. The stabilization diagram of original system in damage case 1: (a) acceleration measurement; (b) displacement measurement
Table B.3. Measured frequency for original system in damage case 1 (Hz)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>SSI</th>
<th>BMI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Acc</td>
<td>S/N</td>
</tr>
<tr>
<td>1</td>
<td>7.55</td>
<td>7.65</td>
</tr>
<tr>
<td>2</td>
<td>21.24</td>
<td>21.31</td>
</tr>
<tr>
<td>3</td>
<td>34.01</td>
<td>34.09</td>
</tr>
</tbody>
</table>

Figure B.6. The stabilization diagram of modified system in damage case 1: (a) acceleration measurement; (b) displacement measurement

Table B.4. Measured frequency for modified system in damage case 1 (Hz)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>SSI</th>
<th>BMI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Acc</td>
<td>S/N</td>
</tr>
<tr>
<td>1</td>
<td>7.28</td>
<td>7.30</td>
</tr>
<tr>
<td>2</td>
<td>20.41</td>
<td>20.51</td>
</tr>
<tr>
<td>3</td>
<td>32.71</td>
<td>32.78</td>
</tr>
</tbody>
</table>

Figure B.7. Measured mode shapes in damage case 1: (a) original system; (b) modified system
B.3 Damage case 2

Figure B.8. The stabilization diagram of original system in damage case 2: (a) acceleration measurement; (b) displacement measurement

Table B.5. Measured frequency for original system in damage case 2 (Hz)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>SSI</th>
<th>BMI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Acc</td>
<td>Cam</td>
</tr>
<tr>
<td>1</td>
<td>6.93</td>
<td>7.10</td>
</tr>
<tr>
<td>2</td>
<td>20.77</td>
<td>21.21</td>
</tr>
<tr>
<td>3</td>
<td>29.89</td>
<td>30.62</td>
</tr>
</tbody>
</table>

Figure B.9. The stabilization diagram of modified system in damage case 2: (a) acceleration measurement; (b) displacement measurement
Table B.6. Measured frequency for modified system in damage case 2 (Hz)

<table>
<thead>
<tr>
<th>Mode No.</th>
<th>SSI</th>
<th>BMI</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Acc</td>
<td>Cam</td>
</tr>
<tr>
<td>1</td>
<td>6.69</td>
<td>6.89</td>
</tr>
<tr>
<td>2</td>
<td>20.11</td>
<td>20.70</td>
</tr>
<tr>
<td>3</td>
<td>28.85</td>
<td>29.58</td>
</tr>
</tbody>
</table>

Figure B.10. Measured mode shapes in damage case 2: (a) original system; (b) modified system

It is found in Tables B.1-B.6 that the identified natural frequencies from accelerometers are coincident well with those from high-speed camera. The maximum error is less than 5%. The identified mode shapes for each case also have a good agreement using both sensors, as shown in Figures B.4, B.7 and B.10. However, more undesirable modes appeared in the stabilization diagrams obtained from displacement measurements, which may be attributed to harmonic excitation or represent tortional modes that cannot been visualized in planar view. Therefore, more efforts have to be made to distinguish spurious modes from physical modes when processing camera-recorded vibration data, such as checking mode shapes for each potential mode.

In addition, we found the singular value spectrum from accelerations is smoother than that from displacements, indicating the accelerations were well collected and have higher
quality than displacements. This is also reflected on S/N values identified by BMI. The first mode identified by both sensors has similar S/N with the same order of magnitude, but the second and third modes identified by high-speed camera have much smaller S/N compared to those by accelerometers, even S/N values in some cases are extremely small, such as the S/N values of 10 and 28 in Tables B.4 and B.5, illustrating the camera-based measurement has high levels of noise, the displacements acquired by high-speed camera are heavily noise-contaminated. The noise is even more noticeable when the field of view is zoomed out (Tomac and Slavič, 2022). This may be explained by that high-speed camera has lower dynamic range than accelerometers, the amplitude of displacement is typically very small (in the range of micrometer) and significantly below the camera’s pixel size (Beberniss and Ehrhardt, 2017, Javh et al., 2018, Bregar et al., 2021). Therefore, the displacements measured by high-speed camera need a careful and wise processing strategy in modal identification, otherwise it may make modal parameters unidentifiable.

Based on the comparative study in this appendix, it is concluded that although high-speed camera has advantages 1) producing dense, spatial, and full-field measurements; 2) avoiding sensor mounting and sensor mass attached to structures; 3) making non-contact and distant monitoring possible where traditional sensors have difficulties in accessing, currently it is still challenging to completely replace the traditional sensors because of relatively low measurement accuracy and high levels of noise.
APPENDIX C

PSEUDOCODE OF AUTOMATED SSI

The pseudocode of automated SSI in Chapter 3

Generate system pole matrix, $W$, by SSI-cov/ref. $W$ includes:
- Frequency and damping column vector: $F$ and $D$
- Mode shape matrix: $\Phi$

Stage 1 starts: initially eliminate spurious modes:
- conventional validation criteria: damping ratio check: remove $\zeta < 0$ and $\zeta > 10$
- modal complexity check: remove MPCR $< 0.3$ and MPD $> 0.7$
- uncertainty criterion: uncertainty check: remove COV of frequency $> 2$

Stage 2 starts: compute clustering threshold, $d_{max}$

initialize the cluster number $z$ as 1

For $i = 1$: $n_{max}$ (starting from pole matrix from the lowest to the highest model order)

For $j = 1$: the last pole (scan poles from the first to the last pole at each model order)

If pole at $W[i, j]$ is not assembled

Create new cluster $C_z$

Save pole $W[i, j]$ at the first component of this cluster

For $x = (i + 1)$: $n_{max}$ (Consecutively scan model order)

For $y = 1$: the last element

If pole at $W[x, y]$ is not assembled

Compute the weighted distance $d_{wx}$ in cluster $C_z$

End if

End for

If the weighted distance $d_{wx}$ of pole $m$ is lower than $d_{max}$

Collect $m$ to cluster $C_z$. If no pole is detected, put 0
(if detect more than one pole, select pole with the minimum distance:)

If cluster $C_z$ contains more than 2 poles

Update clustering threshold $d_{max} = \sqrt{\text{median}(d_{wx} - \text{median}(d_{wx}))}$

End if

End if

End for

Repeat procedures until achieving $n_{max}$

Delete cluster containing poles $< \frac{1}{2} n_{max}$

End if

End for

Detect outlying modes by robust outlier detection: function ‘robustcov’ in MATLAB

Calculate the average of all elements in each cluster as representative

Calculate uncertainty of each physical cluster

APPENDIX D

MATLAB PROGRAM OF SHEAR FRAME IN LAB TEST

The three-story shear frame in lab test in Chapter 6 was modeled as a planar structure with
3 DOFs, as shown in Figure D.1.

![Figure D.1. Modeling of shear frame](image)

Note: $m$, $k$, and $y$ are mass, stiffness, and response, respectively.

With the context of structural dynamics, the system stiffness matrix $K$ in this example can
be expressed as:

$$K = \begin{bmatrix}
k_1 + k_2 & -k_2 & 0 \\
-k_2 & k_2 + k_3 & -k_3 \\
0 & -k_3 & k_3
\end{bmatrix}
$$

where $k_i$ is the $i$th elemental stiffness, $i = 1, 2, 3$, which herein is the sum of stiffness of
four columns at each floor. The stiffness of each column is calculated as $\frac{12EI}{L^3}$, $E$, $I$, and $L$
are the young’s modulus, the cross-sectional moment of inertial with respect to vibration
direction, and the length of column, respectively.

The lumped mass matrix is used for dynamic analysis in this example. For the uniform
material, the mass matrix is simply a diagonal matrix in which the diagonal element is
equal to sum of half mass of two consecutive floors. The system mass matrix \( M \) can be expressed as:

\[
M = \begin{bmatrix}
\frac{m_1 + m_2}{2} & 0 & 0 \\
0 & \frac{m_2 + m_3}{2} & 0 \\
0 & 0 & \frac{m_3}{2}
\end{bmatrix}
\]  
(D.2)

where \( m_i \) is the \( i \)th elemental mass, \( i = 1, 2, 3 \), which is equal to mass of the \( i \)th floor, \( \rho V \), \( \rho \) and \( V \) are the mass density of material and the volume of each plate.

Therefore, the natural frequency and mode shape are calculated using the characteristic equation:

\[
(K - \lambda M) \phi = 0 \quad (D.2)
\]

where \( \lambda \) and \( \phi \) are the eigenvalue (square of natural frequency) and mode shape, respectively. The following screenshots show the MATLAB program for dynamic analysis.

**The elemental mass and stiffness**

```matlab
% Elemental mass
% P: kg/m3; A: area m2; L: m length of each story
% -----------------------------------------------
function m=shear_Mass(P,A,L)
    m=P*A*L;

% Elemental stiffness
% E: youngs modulus; I: moment of inertia; L: length of each story
% -----------------------------------------------
function k=shear_Stiffness(E,I,L)
k=E*I/(L*L*L)*12;

The system mass and stiffness (assemble local matrix)

% Assemble elemental mass for system mass matrix
% -----------------------------------------------
function M=shear_MAssembly(P,M,m,m_column,i)
    for i=1:2
        M(i,i)=m(i)+m_column(i);
    end
    for i=3
        M(i,i)=m(i)+m_column(i)/2;
    end
    M=M;
```
% Assemble elemental stiffness for system stiffness matrix
% >>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>>
function K=shear_KAssembly(K,k,j)
for i=1:j
 for ii=1:j
  if ii==i
   K(i,ii)=k(i)+k(i+1);
  end
  if ii==i+1
   K(i,ii)=-k(i);
  end
  if ii==i+1
   K(i,ii)=-k(ii);
 end
end
K=K;

The model of shear frame

%Shear frame in lab test
clear;clc;
global H E L P B T Wd
nn=3; % No. frequency
mm=3; % No. mode shape
npar=3; % no. story
H=0.006;B=0.025; % column unit: m
T=0.025;Wd=0.3; % plate
E0=6.9e10;L=0.75;P0=2700; % unit m, N young's modulus 6.9e10;n/m2; kg/m3

CK = diag(eye(3)); % K coefficient
CM = diag(eye(3)); % M coefficient
A=T*Wd; % plate area
I=4*B*H^2*H*1/12; % moment of inertial of 4 columns
LL=ones(1,npar);
LL(:,i)=L/npar; % length of each story
LLL=ones(1,npar);
LLL(:,i)=0.3; % stiffness and mass matrix
K=zeros(npar,npar);
k=zeros(npar+1,1);
M=zeros(npar,npar);
m=zeros(npar,1);
m_column=zeros(npar,1); % mass of 4 columns kg

for R=1:npar
 E=CK(R)*E0;
P=CM(R)*P0;
k(R)=shear_Stiffness(E,I,LL(R));
K=shear_KAssembly(K,k,R);
 m_column(R)=4*P*0.006*0.025*0.25;
m(R)=shear_Mass(P,A,LLL(R));
M=shear_MAssembly(P,M,m,m_column,R);
end
% find eigenvalue and eigenvector
[V,D]=eig(K,M);
F=diag(sqrt(D))./(2*pi);
% normalization
V1=[V(:,1)./V(1,1) V(:,2)./V(1,2) V(:,3)./V(1,3)];
APPENDIX E

ANSYS PROGRAM

This Appendix presents the Ansys Parametric Design Language (APDL) program for the cable-stayed pedestrian bridge in Chapter 7.

```
finish
/clear
!==================================================Parameterization
*DIM,Etm,,6 ! No. young's modulus
*DIM,Dens,,6 ! No. mass density
*DIM,Str,,8 ! No. cable strain

*SET,Etm(1),2.02E11
*SET,Etm(2),2.02E11
*SET,Etm(3),2.02E11
*SET,Etm(4),2.02E11
*SET,Etm(5),2.02E11
*SET,Etm(6),1.95E11

*SET,Dens(1),7900
*SET,Dens(2),7900
*SET,Dens(3),7900
*SET,Dens(4),7900
*SET,Dens(5),7900
*SET,Dens(6),7900

!*SET,Str(1),1.776325E-04
!*SET,Str(2),1.829705E-04
!*SET,Str(3),1.722468E-04
!*SET,Str(4),2.052759E-04
!*SET,Str(5),3.118935E-04
!*SET,Str(6),2.051329E-04
!*SET,Str(7),1.389794E-04
!*SET,Str(8),4.279957E-05

*SET,Str(1),8.803E-04
*SET,Str(2),8.803E-04
*SET,Str(3),8.803E-04
*SET,Str(4),9.612E-04
*SET,Str(5),9.612E-04
*SET,Str(6),9.612E-04
*SET,Str(7),9.612E-04
*SET,Str(8),9.612E-04
```
Preprocessing
/prep7
/nerr,0

Material property
ET,1,shell181
ET,2,link180
r,1,0.1
r,2,2.348e-3,1,1
!Cross-sectional area

MP,dens,1,Dens(1)
mp,ex,1,Et(1)
mp,prxy,1,0.3

MP,dens,2,Dens(2)
mp,ex,2,Et(2)
mp,prxy,2,0.3

MP,dens,3,Dens(3)
mp,ex,3,Et(3)
mp,prxy,3,0.3

MP,dens,4,Dens(4)
mp,ex,4,Et(4)
mp,prxy,4,0.3

MP,dens,5,Dens(5)
mp,ex,5,Et(5)
mp,prxy,5,0.3

MP,dens,6,Dens(6)
MP,prxy,6,0.3
MP,ex,6,Etm(6)
Equal Geometric property
! Section of tower baseline
k,1,-0.35,0,0.5
k,2,0.35,0,0.5
k,3,0.35,0,-0.5
k,4,-0.35,0,-0.5
k,5,0,0,0.5
k,6,0.35,0,0.16
k,7,0.35,0,-0.16
k,8,0,0,-0.5
k,9,-0.35,0,-0.16
k,10,-0.35,0,0.16
k,11,0,0,0.28
k,12,0.13,0,0.16
k,13,0.13,0,-0.16
k,14,0,0,-0.28
k,15,-0.13,0,-0.16
k,16,-0.13,0,0.16
k,17,-0.35,11.26,0.43
k,18,0.35,11.26,0.43
k,19,0.35,11.26,-0.43
k,20,-0.35,11.26,-0.43
k,21,0,11.26,0.43
k,22,0.35,11.26,0.09
k,23,0.35,11.26,-0.09
k,24,0,11.26,-0.43
k,25,-0.35,11.26,-0.09
k,26,-0.35,11.26,0.09
k,27,0,11.26,0.21
k,28,0.13,11.26,0.09
k,29,0.13,11.26,-0.09
k,30,0,11.26,-0.21
k,31,-0.13,11.26,-0.09
k,32,-0.13,11.26,0.09
k,33,-0.35,11.26,0
k,34,0.35,11.26,0
k,35,-0.13,11.26,0
k,36,0.13,11.26,0
k,37,-0.35,24.6,0.35
k,38,0.35,24.6,0.35
k,39,0.35,24.6,-0.35
k,40,-0.35,24.6,-0.35
*do,i,5,10,1  
a,i,i+6,i+22,i+16
*endo  
a,21,27,45,41  
a,24,30,47,43  
a,33,35,48,44  
a,34,36,46,42  
cmsel,u,tower_1  
cm,tower_2,area  
allsel,all  
cm,tower,area  
  ! Girder and beam  
k,49,0.35,4.4,-0.3  
k,50,0.35,4.4,0.3  
k,51,0.35,5.18,-0.3  
k,52,0.35,5.18,0.3  
k,53,2.3,4.8,-0.3  
k,54,2.3,4.8,0.3  
k,55,2.3,5.18,-0.3  
k,56,2.3,5.18,0.3  
k,57,0.35,4.4,0  
k,58,0.35,5.18,0  
k,59,2.3,5.18,0  
k,60,2.3,4.8,0  
a,49,50,52,51  
a,49,50,54,53  
a,51,52,56,55  
a,53,54,56,55  
a,49,51,55,53  
a,50,52,56,54  
a,57,58,59,60  
cmsel,u,tower  
cm,hengliang1,area  
!Mirror  
cmsel,s,hengliang1,area  
ar sym,x,hengliang1,,,,0  
cm,hengliang,area
!=================================Mesh
alls
cmseg,s,tower
esize,0.2
mshape,0$mshkey,0
aatt,1,1,1
amesh,all              ! Mesh girder
cm,tower,elem

alls
asel,s,loc,z,-0.5,0.5
cmseg,u,tower
esize,0.3
mshape,0$mshkey,0
aatt,2,1,1
amesh,all              ! Mesh beam
cm,hengliang,elem

! Assign material and geometric property to elements

cmseg,s,stay-cable
latt,6,2,2
lesize,stay-cable,,,1
lmesh,stay-cable
cm,stay-cable,elem

alls
cmseg,s,uxj
esize,0.2
mshape,0$mshkey,1
aatt,5,1,1
amesh,all
cm,uxj,elem

alls
cmseg,s,uxing
esize,0.2
mshape,0$mshkey,0
aatt,5,1,1
amesh,all
cm,uxing,elem

!alls
!cmseg,s,uxing
!lesize,0.5
!mshape,0$mshkey,0
!aatt,5,1,1
!amesh,all
!cm,uxing,elem
!Grid mesh

!Boolean calculation
!Tower
CSYS, 0
cmsel,s,tower
aptn,all
cmsel,s,tower
cm,tower_2,area

asel,s,loc,z,-0.5,0.5
cmsel,u,hengliang
cm,tower,area
cmsel,u,tower_2
cm,tower_1,area

!Straight span
cmsel,s,tower_1
cmsel,a,qiaomian
aptn,all
asel,s,loc,z,-0.5,0.5
cmsel,u,hengliang
cmsel,u,tower_2
cm,tower_1,area
cmsel,a,tower_2
cm,tower,area

!Beam and tower
cmsel,s,tower_1
cmsel,a,hengliang
aptn,all
alls
asel,s,loc,z,-0.5,0.5
!cysy,0
!asel,s,loc,x,-0.5,0.5
asel,r,loc,x,-0.35,0.35

cm,tower,area
asel,s,loc,z,-0.5,0.5
cmsel,u,tower
cm,hengliang,area
alls
cmsel,s,tower

wpoffs,,11.47
wprota,,90
asbw,all

wpoffs,,3.51
asbw,all
! Anchor location
*GET,KPAX,KP,0,NUM,MAX
K,k,1,11.47,0.42874063
K,k,2,14.98,0.40769115
K,k,3,18.49,0.38664168
K,k,4,22,0.36559220
K,k,5,11.47,-0.42874063
K,k,6,14.98,-0.40769115
K,k,7,18.49,-0.38664168
K,k,8,22,-0.36559220
K,k,9,5.9,10.43
K,k,10,5.9,16.93
K,k,11,5.9,23.43
K,k,12,5.9,29.93
K,k,13,5.0855,-7.43
K,k,14,4.3505,-12.33
K,k,15,3.6155,-17.23
K,k,16,2.903,-21.98
*DO,i,k,kmax+1,kmax+8,1
L,i,i+8
*ENDDO

LSEL,s,,583,590,1
CM,stay-cable,line
CMSEL,s,uxing
ADELE,110,,1
ADELE,117,,1
ADELE,143,144,1,1
CM,uxing,area

CSYS,11
K,1037,12.5,-180,0
K,1038,9.5,-180,0
KGEN,2,1037,1038,1,,30,,2
KGEN,2,1037,1038,1,,60,,4
KGEN,2,1037,1038,1,,90,,6
KGEN,2,1037,1038,1,,120,,8
KGEN,2,1037,1038,1,,150,,10
KGEN,2,1037,1038,1,,180,,12
CSYS,12
K,1051,6.5,30,0
*GET, KP MAX, KP, 0, NUM, MAX
  k, k max +1, 6, 60, 0

  l, 79, 159

  adrag, 96, 97, 98, 99, 102, 231

  !adrag, 95, 225
  !adrag, 100, 225

  cm sel, u, tower
  cm sel, u, hengliang
  cm sel, u, qiao mian
  cm sel, u, u xing
  cm, ulink1, area

  c sys, 0
  *GET, KP MAX, KP, 0, NUM, MAX
  k, k max +1, 9, 5, 9, 46
  k, k max +2, 9, 3, 3, 59
  k, k max +3, 9, 3, 3, 61
  k, k max +4, 9, 2, 45, 65, 25
  l, k max +1
  l, k max +2, k max +3
  l, k max +3, k max +4

  adrag, 184, 185, 186, 187, 188, 189, 203, 212, 213, 218
  adrag, 190, 191, 192, 193, 194, 195, 203, 212, 213, 218
  adrag, 196, 203, 212, 213, 218
  adrag, 197, 203, 212, 213, 218

  alls
  cm sel, u, tower
  cm sel, u, hengliang
  cm sel, u, qiao mian
  cm sel, u, u xing
  cm, u x j1, area
  ar sym, x, u x j1, 0
  cm, u x j, area
! Side span
! U-shape
local,11,1,0,5.9,45,0,-90,0
CSYS,11

*GET,KPMAK,KP,0,NBM,MAX
k,kpmax+1,13,0,0
k,kpmax+2,12.5,0,0
k,kpmax+3,12,0,0
k,kpmax+4,10,0,0
k,kpmax+5,9.5,0,0
k,kpmax+6,9,0,0
k,kpmax+7,9,0,-0.2
k,kpmax+8,9.75,0,-0.7
k,kpmax+9,10,0,-0.7
k,kpmax+10,12,0,-0.7
k,kpmax+11,12.25,0,-0.7
k,kpmax+12,13,0,-0.2
k,kpmax+13,13,-180,0

*do,i,kpmax+1,kpmax+11
lstr,i,i+1
*enddo
lstr,kpmax+12,kpmax+1
lstr,kpmax+4,kpmax+9
lstr,kpmax+3,kpmax+10

!larc,kpmax+1,kpmax+11
l,kpmax+1,kpmax+13

!l,kpmax+11,kpmax+12
adrag,184,185,186,187,188,189,198
adrag,190,191,192,193,194,195,198
adrag,196,197,198
adrag,197,198

alls
cmsel,u,tower
cmsel,u,hengliang
cmsel,u,qiaomian
cm,uxing,area
! Main span
k,73,1.5,5.2,28.55
k,74,2.75,5.2,28.55
k,75,3.5,5.7,28.55
k,76,3.5,5.9,28.55
k,77,1.5,5.9,28.55
k,78,-1.5,5.9,28.55
k,79,-3.5,5.9,28.55
k,80,-3.5,5.7,28.55
k,81,-2.75,5.2,28.55
k,82,-1.5,5.2,28.55
k,83,1.5,5.2,-2
k,84,1.5,2.203,-21.98
k,85,1.5,2.203,-23.98

*do,i,73,81
l,i,i+1
*endo
l,82,73
lset,s,loc,z,28,29
cm,ad,line
l,77,73
l,78,82
l,73,83
l,83,84
l,84,85

adrag,ad,103,104,105
adrag,101,,103,104,105
adrag,102,,103,104,105

alls
cmsel,u,tower
cmsel,u,hengliang
cm,qiaomian,area
!-----------------------------------------------=Impose boundary condition

! U-shape
nsel,s,loc,z,65.1,65.25  
d,all,all
  d,all,uy,,,,,ux

nsel,s,loc,z,64.75,65.1
nsel,r,loc,y,1.75,1.85  
  !d,all,ux,,,,,uy
  !d,all,uy,,,,,rotx,roty,rotz
  d,all,uy,,,,,ux
  !d,all,all

! Tower
nsel,s,loc,z,44.5,45.5
nsel,r,loc,y,5.1,5.3
nsel,r,loc,x,10.5,11.5  
  !D,all,ux,,,,,uy
  d,all,uy

nsel,s,loc,z,44.5,45.5
nsel,r,loc,y,5.1,5.3
nsel,r,loc,x,-10.5,-11.5  
  !D,all,ux,,,,,uy
  d,all,uy

! Connection between tower and pier
nsel,s,loc,y,-0.01,0.01
  d,all,all

! All fixed
nsel,s,loc,z,-24,-21.98  
  !nsel,r,loc,y,2.16,2.3
  !D,all,ux,,,,,uy
  d,all,all

! Beam and deck
nsel,s,loc,z,-0.4,0.4
nsel,r,loc,y,5.15,5.25

  CP,1,UY,4098,36325
  CP,2,UY,4100,36324
  CP,3,UY,4101,36323
  CP,4,UY,4170,36321
  CP,5,UY,4177,36320
  CP,6,UY,4176,36319
  CP,7,UY,4134,37021
  CP,8,UY,4141,37020
  CP,9,UY,4140,37019
!==================================
Solve

alls
/solu
acel,,9.8

inistate, set, dtyp, epel !Initial cable strain

inistate, defi, 3805,,,, Stra(1)
inistate, defi, 3806,,,, Stra(2)
inistate, defi, 3807,,,, Stra(3)
inistate, defi, 3808,,,, Stra(4)
inistate, defi, 3809,,,, Stra(5)
inistate, defi, 3810,,,, Stra(6)
inistate, defi, 3811,,,, Stra(7)
inistate, defi, 3812,,,, Stra(8)
alls
antype, 2
modopt, subsp, 20,,,, 1
mxpand, 20,,,, yes
lumpm, off
solve
/post1
set, list
Jice Zeng

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EDUCATION

University of Louisville, Louisville, KY, USA  
Ph.D. in Structural Engineering  
Dec. 2021  
Advisor: Young Hoon Kim  
GPA 4.0/4.0

Chongqing Jiaotong University, Chongqing, China  
M.Sc. in Civil Engineering  
GPA 85/100

Nottingham Trent University, Nottingham, United Kingdom  
International Student Exchange Program  
Feb-May 2015

Xihua University, Chengdu, China  
B.S. in Civil Engineering  
Jun. 2013  
GPA 3.5/4.0

PUBLICATIONS AND AWARDS

Publications

- Published:
• **Under review:**

• **In-progress:**
  1. Automated Operational Modal Analysis Using Gaussian Mixture Models and Variational Inference
  2. Probabilistic Structural Identification Using Variational Bayesian Model Updating and Gaussian Process Regression
  3. Multi-parameter Identification at Element Level Using Bayesian Model Updating with Incomplete Measurement
  4. The Comparative Study Using Different Markov Chain Monte Carlo Algorithms for Bayesian Inference

• **Conference publication:**

**Awards**
- 2021, Doctoral Dissertation Completion Award at University of Louisville, Louisville, KY
- 2019, International Student Tuition Support Award at University of Louisville, Louisville, KY
- 2018, Graduate Student Council Travel Funds at University of Louisville, Louisville, KY
- 2017~2018, International student fellowship, University of Louisville, Louisville, KY
- 2013~2015, University Scholarship, Chongqing University, Chongqing, China

**PRESENTATIONS**

**International Workshop on Structural Health Monitoring (IWSHM) Dec. 2021**
Will present research: ‘Bayesian Model Updating for with Differential Evolution Adaptive Metropolis (DREAM) Sampling Method and Kriging Model’, (Accepted), *Stanford University, CA*

**Engineering Mechanic Institute Conference (EMI) May. 2018**
Presented research: ‘Damage Identification and Damage Quantification Using Time-Variant Visual Images’, *Massachusetts Institute of Technology, Boston*
Graduate Student Regional Research Conference (GSRRC) March. 2018
Presented research: ‘Applicability of Static Condensation to Estimate Stiffness Loss Using Non-contact Based Sensors’, University of Louisville, KY

RESEARCH EXPERIENCE

Structural health monitoring (SHM) Sept. 2017-Now
Research Assistant University of Louisville, KY
- Detected damage and quantified damage of steel members using non-contact optical sensor (high-speed camera)
- Developed the automated operational modal analysis strategies for in-service steel pedestrian bridge and a concrete highway bridge
- Presented an algorithm of automated Bayesian modal analysis for parameter estimation and uncertainty quantification
- Proposed a novel Bayesian model updating framework to simultaneously identify mass and stiffness and further implement probabilistic damage detection
- Enhance Bayesian approach with DREAM sampling method and Kriging model to advance model updating performance and computational efficiency

Construction analysis and management Jan. 2016-May 2016
Research Assistant Chongqing Jiaotong University, China
- Investigated construction stages of city viaduct casting-in-place and examined stability of scaffold
- Discussed and optimized theoretically the way to pour concrete stiff skeleton arch bridge by means of AutoCAD (for drawings) and Midas/civil (for 3D Finite Element Model)

Research Assistant Chongqing Jiaotong University, China
- Evaluated fatigue and mechanical performance of bamboo bridge under varied conditions such as heat and moisture
- Assessed effect of fiber material (glass-steel plate) on strength of concrete

TEACHING

University of Louisville, Department of Civil and Environmental Engineering May. 2019-July. 2021
Teaching assistant:
CEE 322-Structural Analysis Summer, 2019
CEE 470-Surface Water Hydrology Fall, 2019
CEE 421-Concrete Design Spring, 2020
CEE 322-Structural Analysis Summer, 2020
PROFESSIONAL EXPERIENCE

The Eleventh Metallurgical Construction Group Co. LTD

**Working as construction manager**

Sichuan, China

Aug. 2016-Jul 2017

- Managed construction stage and supervised for Resettlement Housing Project worth $28.8M, including construction flow, risk, and safety assessment
- Cooperated with contractors, landscape architects and structural engineers
- Conducted inspection to confirm each construction item as per design drawing and relevant Chinese standards
- Resolve technical issues during project execution to meet design requirement

Sichuan Provincial Transport Department Highway Planning, Survey, Design, and Research Institute

**Civil Engineering Intern**

Sichuan, China

Oct. 2015-Dec. 2015

- Joined a team of 10 personnel which directed professional and high-quality service, including:
  1. Initiated design for bridge, retaining wall and culvert, specification, and initial cost estimates
  2. Examined existing structure and provided efficient and economical maintenance plan
  3. Reported on above programs, condition assessment and progress evaluation
- Coordinated site meetings with contractors, construction agencies

Chongqing Communications Planning Survey and Design Institute

**Civil Engineering Intern**

Chongqing, China

Jun. 2015-Sept. 2015

- Provided input to design, drawings using AutoCAD, wrote specification, created cost estimates and project presentation using Excel, Word, PowerPoint for municipal bridges projects
- Gathered information on project using Total Station device and GPS