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A new SIR model with mobility.

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A NEW SIR MODEL WITH MOBILITY

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DEDICATION

To my mom, Laura Applegate

To my brother, Aaron Applegate

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ABSTRACT

A NEW SIR MODEL WITH MOBILITY

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In this paper, a mobility-based SIR model is built to understand the spread of the pandemic. A traditional SIR model used in epidemiology describes the transition of particles among states, such as susceptible, infected, and recovered states. However, the traditional model has no movement of particles. There are many variations of SIR models when it comes to the factor of mobility, the majority of studies use mobility intensity or population density as a measure of mobility. In this paper, a new dynamical SIR model, including the spatial motion of three-type particles, is constructed and the long-time behavior of the first and second moments of this dynamical system are studied. The intermittency and Lyapunov exponents are derived and analyzed as well.

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CHAPTER 1
LITERATURE REVIEW

The susceptible-infectious-recovered (SIR) model is a type of compartmental model used in epidemiology, and it is one way to model infectious diseases. The SIR model was developed by Ronald Ross, William Hamer, and others in the early twentieth century [17]. The sizes of these sub-populations at time t are denoted by $S(t), I(t)$ and $R(t)$. The traditional SIR model represents the number of particles in each compartment at a particular time t. Wiess [17] proposed a simple use of ordinary differential equations, instead of partial differential equations or agent based models, where $\dfrac{dS}{dt} = -\beta SI, \dfrac{dI}{dt} = \beta SI - \nu I$ and $\dfrac{dR}{dt} = \nu I$. The bi-linear incidence term βSI for the number of new infected individuals per unit time corresponds to homogeneous mixing of the infected and susceptible classes [17]. The model supposes each infected individual has κ contacts, then $\kappa S/N$ of these contacts are susceptible individuals, and if τ is the transmittability of the infected disease, then each infected individual infects $\kappa\tau S/N$ susceptible individuals per unit of time [17]. The paper concludes that it is easy to prove that the disease always dies out- $I(\infty) = 0$ for all initial conditions- without having a formula for $I(t)$ [17]. This model is good, but somewhat basic. Not all diseases die out, under some conditions the number of infected can go to infinity.

Linda Allen studies the continuous version of the SIR model, which behaves in the same manner as the discrete model. Given the total population N remains constant, and initial conditions $S(0), I(0), R(0)$ for the susceptible, infected, and recovered groups respectively. For $N = S(0) + I(0) + R(0)$, and transmission rate α going from the susceptible group to the infected group, and transmission rate γ going from the infected group to the recovered group. The reproductive rate in the continuous case is $\mathcal{R} = \dfrac{S(0)\alpha}{N\gamma}$. The value of $\mathcal{R}$ determines the global behavior of the discrete SIR model [1]. If $\mathcal{R} \leq 1$, there is no epidemic, but if $\mathcal{R} > 1$, there is an epidemic [1]. The conclusion is that the infective class eventually decreases and approaches zero. The model proposed is a model with K sub-populations, but determining whether an epidemic occurs within a sub-population of the multi-population SIR model is not as straightforward as in the single population case. This model is effective, but is more fundamental and does not take into account migration.

Allen studies the differences between deterministic and stochastic epidemic models, and one of the most important differences is the asymptotic dynamics [2] [1]. Mathematical models of population dynamics, analysis of which is an essential part of modern mathematical biology, have the important feature of birth and death mechanisms. The main feature is that the parameters of mathematical models- rate of jumps, birth and death rates, etc.- are constants and there is a local equilibrium between the rate of production of new particles and their annihilation [8].

The famous stochastic epidemic model is proposed by Liggett, who studies a contact process that is thought of as a model on d dimensional integer lattice $\mathbb{Z}^d$ for the spread of infection [11]. The collection of susceptible individuals that may be infected at any given time is the set of vertices of a connected, undirected graph S.

The *contact process* on S with infection parameter $\lambda \geq 0$ is a continuous time Markov process η_t on $\{0,1\}^S$. This model differs from ours because it has one particle at one site and it only allows change to the nearest neighbors, meaning healthy individuals become infected at a rate that is proportional to the number of infected neighbors [11]. Additionally, Liggett's model does not have migration of the particles.

The model proposed by Kondratiev et. al. [8] introduced a contact model with underlying random walk generated by a Markov process generator $(LF)(\gamma) = \sum_{x\in\gamma} d(x,\gamma)(D_x^- F)(\gamma) + \int_{\mathbb{R}^d} b(x,\gamma)(D_x^+ F)(\gamma)dx$. where $(D_x^- F)(\gamma) = F(\gamma\backslash x) - F(\gamma)$, and $(D_x^+ F)(\gamma) = F(\gamma \cup x) - F(\gamma)$ [9], where γ is the configuration space, $d(x,\gamma)$ is the rate at which particle x of the configuration γ dies, $b(x,\gamma)$ is the rate at which a new particle is born at x [8]. The main result proposed is that there is no limiting distribution for the population dynamics models in a time independent random environment. This model uses a generator $\mathcal{L}$ of the underlying Markov process $X(s), s \geq 0$, and $V(x, w_m)$ is a random potential. This model is similar to the generator in our model, and the differential equation utilized for density is similar to our general inhomogeneous equation. Kondratiev introduces a limit theorem for the branching process with the random space evolution in the supercritical regime. However, this model only allows one particle at each site. This model is built on $\mathbb{R}^d$ instead of $\mathbb{Z}^d$, thus the techniques in this model are not applicable to discrete $\mathbb{Z}^d$ space, and only the offspring particles have mobility, the parent particles do not move.

Molchanov et. al groups study a continuous time branching random walks generated with non-local Laplacian operator on multidimensional lattice $\mathbb{Z}^d$. The model uses a matrix $A = (a(x,y))_{x,y\in\mathbb{Z}^d}$ of transition intensities of random walk, $a(x,y) \geq 0$ for $x \neq y$. There is the assumption that the branching mechanism in the sources is independent of the walk and defined by an infinitesimal generating function [19]. The Green function, $G_\lambda(x,y)$, of the generator $\mathcal{A}$ of the symmetrical random walk is the Laplace transform of the transition probability $p(t,x,y)$, $G_\lambda(x,y) = \int_0^\infty e^{-\lambda t} p(t,x,y)dt$. This is used in our model, as the Green function has a sense of the mean number of hits of the particle at the point y for the process starting from the point x, as $t \to \infty$ [19]. The analysis of branching random walks depends on whether $G_0(0,0)$ is finite or infinite. The concept of weakly supercritical branching random walks is defined and the asymptotic behavior of the Green function is analyzed and the asymptotic behavior of the eigenvalue $\lambda_0(\beta)$ for the evolutionary operator for $\beta \downarrow \beta_c$ [19]. This paper relates to our model with spatial migration because the general solution derived for inhomogeneous equations depends on the transition probability $p(t-s, 0, x-z)$, which relates to the Green function. However, they only analyzed the single type branching random walk and there are no categories of particles.

CHAPTER 2
INTRODUCTION

The susceptible-infectious-recovered (SIR) model is a type of compartmental model used in epidemiology, and it is one way to model infectious diseases. The SIR model was developed by Ronald Ross, William Hamer, and others in the early twentieth century [1] [17]. They introduced and analyzed the most basic transmission model for infectious diseases caused by bacteria, viruses, or fungi [2] [17]. The traditional SIR model only considers the transitions between susceptible, infected, and recovered population groups. The traditional SIR model represents the number of particles in each compartment at a particular time t, and the particles move between the groups with transition rates β and γ (See Fig. 1 below). The traditional SIR Model also has the assumption that once a particle is in the recovered group, it is immune to the disease [2] [17].

Figure 2.1: Traditional SIR Model

Mathematical models of population dynamics, which are an essential part of mathematical biology, have important features- such as birth and death mechanisms [6] [12] [8] [21]. Mathematical models can be considered to be part of the theory of branching processes with random spatial dynamics [8] [15] [21]. Some previous SIR models have looked at mobility intensity, which describes the area intensity or the distribution of intensity [2] [6]. The new SIR model in this paper will introduce spatial mobility within the model on $\mathbb{Z}^d$ space.

This means that the new model with migration has the assumption that all of the particles can have spatial motion, within the susceptible, infected, and recovered groups (in addition to the inter-compartmental motion). We assume the total population is fixed, i.e- $N(t) = S(t) + I(t) + R(t)$ but $N(t, x)$, the total population at position x at time t, is varying. We define κ as the probability that a particle will migrate. We define β as the transition rate from the susceptible group S to the infected group I, and γ is the transition rate from infected group I to recovered group R. Regarding the migration direction, it is determined by $a(z)$ where $a(z)$ is the probability kernel of the Poisson process. Define the probability kernel as $a(z)$, where $a(z) = a(-z)$. The movement of one particle moving from location x to location $x + z$, $(x \to x + z)$, has probability $a(z)dt$ during the infinitesimal time period $(t, t + dt)$. The movement of one particle going from location $x + z$ to x, $(x + z \longrightarrow x)$, has probability $a(-z)dt$. There is the assumption that $\sum\limits_{z \in \mathbb{Z}^d} a(z) = 0$ and $\sum\limits_{z \neq 0} a(z) = 1$, which implies that $a(0) = -1$. Additionally, we assume that the spatial motion of healthy particles is the same as the spatial motion of an infected particle, and that the only one type of movement can happen at a time, meaning a particle can jump to another location or they can jump states. The possible events are:

① $S : x \longrightarrow x + z$ in S with probability $\kappa a(z)dt, \forall x, z \in \mathbb{Z}^d$.

This is the event that in a short time period $(t, t + dt)$, a particle at location x moves to location $x + z$ within the susceptible group.

② $I : x \longrightarrow x + z$ in I with probability $\kappa a(z)dt, \forall x, z \in \mathbb{Z}^d$.

This is the event that in a short time period $(t, t + dt)$, a particle at location x moves to location $x + z$ within the infected group.

③ $R : x \longrightarrow x + z$ in R with probability $\kappa a(z)dt, \forall x, z \in \mathbb{Z}^d$.

This is the event that in a short time period $(t, t + dt)$, a particle at location x moves to location $x + z$ within the recovered group.

④ $S : x \longrightarrow x$ in I with probability $\beta dt, \forall x, z \in \mathbb{Z}^d$.

This is the event that a particle at location x in the susceptible group transitions to the infected group.

⑤ $I : x \longrightarrow x$ in R with probability $\gamma dt, \forall x, z \in \mathbb{Z}^d$.

This is the event that a particle at location x in the infected group transitions to the recovered group.

CHAPTER 3
DERIVING THE DIFFERENTIAL EQUATIONS FOR THE FIRST MOMENTS

One of the goals of this paper is to find the first and second moments of S, I and R and analyze the stability of the moments. The first step in doing this is to derive the differential equations for the generating functions and find the solutions. We derive the differential equations using the Kolmogorov Forward Equations.

Theorem 1 *The differential equations for the first moment of the susceptible, infected, and recovered groups are*

$$\begin{cases} \dfrac{\partial E[S(t,x)]}{\partial t} = \kappa \mathscr{L} E[S(t,x)] - \beta E[I(t,x)] & (1a) \\ \dfrac{\partial E[I(t,x)]}{\partial t} = \kappa \mathscr{L} E[I(t,x)] + (\beta - \gamma) E[I(t,x)] & (1b) \\ \dfrac{\partial E[R(t,x)]}{\partial t} = \kappa \mathscr{L} E[R(t,x)] + \gamma E[I(t,x)] & (1c) \end{cases} \quad (1)$$

Note that the discrete Laplace operator is defined to be $\mathscr{L} f(t,x) = \sum_{z \neq 0} a(z)[f(t,x+z) - f(t,x)]$.

Proof of Theorem 1: For the susceptible group: $S(t+dt,x) = S(t,x) + \xi(dt)$ where

$$\xi(dt) = \begin{cases} 1 & w.p & \sum_{z\neq 0} S(t,x+z)\kappa a(z)dt & (1) \\ -1 & w.p & \sum_{z\neq 0} S(t,x)\kappa a(z)dt + I(t,x)\beta dt & (2) \\ 0 & w.p & 1 - (1) - (2) \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x + z$ in the susceptible group moves to location x, meaning location x gains a particle, and thus the event has probability $\sum_{z \neq 0} S(t, x + z)\kappa a(z)dt$. The case that $\xi(dt) = -1$ is the event that either a particle at location x moves to location $x + z$ (meaning that location x loses a particle - which has probability $\sum_{z \neq 0} S(t, x)\kappa a(z)dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t, x)\beta dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the susceptible group, so it has probability $1 - \sum_{z \neq 0} S(t, x + z)\kappa a(z)dt - \sum_{z \neq 0} S(t, x)\kappa a(z)dt - I(t, x)\beta dt$.

$$E[S(t + dt, x)] = E[E[S(t + dt, x)|\mathscr{F}_t]]$$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we have that

$$E[S(t + dt, x)] = E\Big[(S(t, x) + 1)[\sum_{z \neq 0} S(t, x + z)\kappa a(z)dt] + (S(t, x) - 1)\cdot [\sum_{z \neq 0} S(t, x)\kappa a(z)dt + I(t, x)\beta dt] + S(t, x)[1 - \sum_{z \neq 0} S(t, x + z)\kappa a(z)dt - (\sum_{z \neq 0} S(t, x)\kappa a(z)dt + I(t, x)\beta dt)]\Big]$$

Recall that we defined our Laplace operator as $\mathscr{L} f(t, x) = \sum_{z \neq 0} a(z)[f(t, x + z) - f(t, x)]$. We can also distribute the expectation and divide both sides by dt and we have that, as $dt \to 0$

$$\frac{\partial E[S(t,x)]}{\partial t} = \kappa \mathscr{L} E[S(t,x)] - \beta E[I(t,x)]$$

For the infected group: $I(t+dt,x) = I(t,x) + \xi(dt)$ where

$$\xi(dt) = \begin{cases} 1 & w.p \quad I(t,x)\beta dt + \sum_{z\neq 0} I(t,x+z)\kappa a(-z)dt \quad ① \\ -1 & w.p \quad I(t,x)\gamma dt + \sum_{z\neq 0} I(t,x)\kappa a(z)dt \quad ② \\ 0 & w.p \quad 1 - ① - ② \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x+z$ in the infected group moves to location x, meaning location x gains a particle (which has probability $\sum_{z\neq 0} I(t,x+z)\kappa a(z)dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t,x)\beta dt$). The case that $\xi(dt) = -1$ is the event that either a particle at location x in the infected group moves to location $x+z$ (meaning that location x loses a particle - which has probability $\sum_{z\neq 0} I(t,x)\kappa a(z)dt$), or a particle at location x moves from the infected group to the recovered group (which has probability $I(t,x)\gamma dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the infected group, so it has probability $1 - I(t,x)\beta dt - \sum_{z\neq 0} I(t,x+z)\kappa a(-z)dt - I(t,x)\gamma dt - \sum_{z\neq 0} I(t,x)\kappa a(z)dt$.

$$E[I(t+dt,x)] = E[E[I(t+dt,x)|\mathscr{F}_t]]$$

Similarly to the susceptible group, we use the Kolmogorov Forward Equations, and derive the following equation, as $dt \to 0$

$$\frac{\partial E[I(t,x)]}{\partial t} = \kappa \mathscr{L} E[I(t,x)] + (\beta - \gamma)E[I(t,x)]$$

For the recovered group: $R(t+dt,x) = R(t,x) + \xi(dt)$ where

$$\xi(dt) = \begin{cases} 1 & w.p \quad I(t,x)\gamma dt + \sum_{z\neq 0} R(t,x+z)\kappa a(-z)dt \quad ① \\ -1 & w.p \quad \sum_{z\neq 0} R(t,x)\kappa a(z)dt \quad ② \\ 0 & w.p \quad 1 - ① - ② \end{cases}$$

The case that $\xi(dt) = 1$ is the event that either a particle at location $x+z$ in the recovered group moves to location x (meaning location x gains a particle (which has probability $\sum_{z\neq 0} R(t,x+z)\kappa a(z)dt$), or a particle at location x in the infected group becomes recovered (which has probability $I(t,x)\gamma dt$). The case that $\xi(dt) = -1$ is the event that a particle at location x in the recovered group moves to location $x+z$ (meaning that location x loses a particle - which has probability $\sum_{z\neq 0} R(t,x)\kappa a(z)dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the recovered group, so it has probability $1 - I(t,x)\gamma dt - \sum_{z\neq 0} R(t,x+z)\kappa a(-z)dt - \sum_{z\neq 0} R(t,x)\kappa a(z)dt$.

$$E[R(t+dt,x)] = E[E[R(t+dt,x)|\mathscr{F}_t]]$$

By the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we get that

$$\frac{\partial E[R(t,x)]}{\partial t} = \kappa \mathscr{L} E[R(t,x)] + \gamma E[I(t,x)]$$

CHAPTER 4

FIRST MOMENTS OF THE SIR MODEL

4.1 First Moments in Homogeneous Space

To solve the differential equations there are 2 cases- homogeneous space and inhomogeneous space. The first is homogeneous space: Assume the space is homogeneous, then the spaces are equivalent, meaning x and $x+z$ are the same. Thus the Laplace operator $\mathscr{L}f(t,x) = \sum_{z \neq 0} a(z)[f(t,x+z) - f(t,x)] = 0$. Then the differential equations from Chapter 3 Equation (1) now no longer have the Laplace operator and the equations become:

$$\begin{cases} \dfrac{\partial E[S(t,x)]}{\partial t} = -\beta E[I(t,x)] \\ \dfrac{\partial E[I(t,x)]}{\partial t} = (\beta - \gamma) E[I(t,x)] \\ \dfrac{\partial E[R(t,x)]}{\partial t} = \gamma E[I(t,x)] \end{cases} \tag{2}$$

Theorem 2 *In the homogeneous space, as $t \longrightarrow \infty$, with initial conditions $S(0) = \rho_0 > 0, I(0) = 1, I(0,x) = \begin{cases} 1 \ \ if \ \ x = y \\ \\ 0 \ \ if \ \ x \neq y \end{cases}$, $R(0) = 0$, if $\beta \neq \gamma$, the steady states $E[S(t,x)]$, $E[I(t,x)]$, and $E[R(t,x)]$ exist and the solutions are*

$$\begin{cases} E[S(t,x)] = \dfrac{-\beta e^{(\beta-\gamma)t} + \beta + \rho_0(\beta-\gamma)}{\beta-\gamma} \\ E[I(t,x)] = e^{(\beta-\gamma)t} \\ E[R(t,x)] = \dfrac{\gamma e^{(\beta-\gamma)t} - \gamma}{(\beta-\gamma)} \end{cases} \quad (3) \quad .$$

If $\beta = \gamma$, the solutions are $\begin{cases} E[S(t,x)] = \rho_0 - \beta C t \\ E[I(t,x)] = C \in \mathbb{R} \\ E[R(t,x)] = 1 + \gamma C t \end{cases}$

Proof of Theorem 2: Solving the ODE system (2) using regular ODE methods, we get the solutions for the first moments of the SIR model.

4.2 First Moments in Inhomogeneous Space

Now we need to solve the differential equations given in Chapter 3 Equation (1) in the inhomogeneous space. Assume the space is inhomogeneous, then the spaces x and $x+z$ are not equivalent and $\mathscr{L} f(t,x) = \sum_{z \neq 0} a(z)[f(t,x+z) - f(t,x)] \neq 0$. To solve the in-homogeneous equations, we need a few new definitions and theorems:

Definition 1 (Fourier Transform) $\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} f(x) e^{ikx}$

Definition 2 (Inverse Fourier Transform) *For the Fourier transform* $\hat{f}(k) = \sum_{x \in \mathbb{Z}^d} e^{itx} f(x)$, *the inverse Fourier transform is* $f(x) = \frac{1}{(2\pi)^d} \int_{T^d} \hat{f}(k) e^{-ikx} dk$ *where* $T^d = [-\pi, \pi]^d$

Definition 3 (Intensity of Mobility Effect) *Define* $\kappa \hat{a}(k)$, *where*

$\hat{a}(k) = \sum_{z \in \mathbb{Z}^d} a(z)e^{ikz}$, *as a measure of the intensity of the dynamical movement of particles/mobility effect.*

Lemma 1 (Fourier Transform of function $\mathscr{L}f(k)$) *Define $\hat{\mathscr{L}}(k)$ as the fourier symbol of the operator $\mathscr{L}$, then $\hat{\mathscr{L}}(k) = \hat{a}(k) \leq 0$ and $\widehat{\mathscr{L}f}(k) = \hat{f}(k)\hat{\mathscr{L}}(k)$*

Proof of Lemma 1: Recall that $\mathscr{L}f(t,x) = \sum_{z \neq 0} a(z)[f(t,x+z) - f(t,x)]$

Applying the Fourier transform from Definition 1 to the Laplace Operator, we get:

$$\widehat{\mathscr{L}f}(k) = \sum_{x \in \mathbb{Z}^d} e^{ikx} \sum_{z \neq 0} a(z)[f(t,x+z) - f(t,x)]$$

$$\widehat{\mathscr{L}f}(k) = \sum_{z \neq 0} a(z) \left[e^{-ikz} \sum_{x \in \mathbb{Z}^d} e^{ik(x+z)} f(x+z) - \sum_{x \in \mathbb{Z}^d} e^{ikx} f(x) \right]$$

$$\widehat{\mathscr{L}f}(k) = \hat{f}(k) \sum_{z \neq 0} a(z) \left[cos(kz) - 1 + isin(-kz) \right]$$

Since it is on a symmetric $\mathbb{Z}^d$ space, $isin(-kz) = 0$ thus,

$$\widehat{\mathscr{L}f}(k) = \hat{f}(k) \sum_{z \neq 0} a(z)(cos(kz) - 1)$$

$$\hat{a}(k) = \sum_{z \in \mathbb{Z}^d} a(z)(cos(kz) + isin(kz)) = \sum_{z \in \mathbb{Z}^d} a(z)cos(kz)$$

$$\hat{a}(k) = \sum_{z \neq 0} a(z)cos(kz) + \sum_{z = 0} a(z)cos(kz)$$

Since $a(z) = a(-z)$ and $\sum_{z \neq 0} a(z) = 1$, we know $a(0) = -1$ and then

$$\hat{a}(k) = \sum_{z \neq 0} a(z)cos(kz) + a(0)cos(0) = \sum_{z \neq 0} a(z)cos(kz) - 1$$

$$\hat{a}(k) + 1 = \sum_{z \neq 0} a(z)cos(kz) \text{ and } \hat{a}(k) = \sum_{z \neq 0} a(z)(cos(kz) - 1) \leq 0$$

$$\widehat{\mathscr{L}f}(k) = \hat{f}(k) \sum_{z \neq 0} a(z)(cos(kz) - 1) = \hat{f}(k)\Big[\sum_{z \neq 0} a(z)cos(kz) - \sum_{z \neq 0} a(z)\Big]$$

$$\widehat{\mathscr{L}f}(k) = \hat{f}(k)((\hat{a}(k) + 1) - 1) = \hat{f}(k)\hat{a}(k) \implies \hat{\mathscr{L}}(k) = \hat{a}(k) \leq 0$$

Thus, $\widehat{\mathscr{L}f}(k) = \hat{f}(k)\hat{\mathscr{L}}(k)$.

Now we can use Definitions 1 and 2 to solve for the first moment of $I(t, x)$ in the inhomogeneous space. For the differential equation of $E[I(t, x)]$ from Chapter 3 Equation ($1b$), we can apply the Fourier transform from Definition 1 to both sides of the equation and get:

$$\frac{\partial \hat{E}[I(t, k)]}{\partial t} = \kappa \hat{\mathscr{L}}(k)\hat{E}[I(t, k)] + (\beta - \gamma)\hat{E}[I(t, k)]$$

Assume $E[I(0, x)] = \begin{cases} 1 \ \ if \ \ x = 0 \\ 0 \ \ if \ \ x \neq 0 \end{cases}$,

then $\hat{E}[I(0, k)] = \sum_{x \in \mathbb{Z}^d} E[I(0, x)]e^{ikx} = 1$

$$\hat{E}[I(t, k)] = \hat{E}[I(0, k)]e^{\left[\kappa \hat{\mathscr{L}}(k) + (\beta - \gamma)\right]t} = e^{\left[\kappa \hat{\mathscr{L}}(k) + (\beta - \gamma)\right]t}$$

Using the Inverse Fourier formula from Definition 2, we have that

$$E[I(t,x)] = \frac{1}{(2\pi)^d}\int_{T^d} \hat{E}[I(t,k)]e^{-ikx}dk =$$

$$E[I(t,x)] = \frac{1}{(2\pi)^d}\int_{T^d} e^{[\kappa\hat{\mathscr{L}}(k) + (\beta-\gamma)]t}e^{-ikx}dk$$

Now to solve for the first moments of the susceptible and recovered groups in the in-homogeneous space we need a more general solution.

Lemma 2 *The transition probability of the particles $p(t,x,y)$ is the fundamental solution to the following equation*

$$\begin{cases} \dfrac{\partial p(t,x,y)}{\partial t} = \kappa\mathscr{L}p(t,x,y) \\ p(0,x,y) = \delta(x-y) = \begin{cases} 1 \;\; if \;\; x = y \\ 0 \;\; if \;\; x \neq y \end{cases} \end{cases}$$

and $p(t,x,y) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{ik(x-y)}e^{\kappa\hat{\mathscr{L}}(k)t}dk$

Proof of Lemma 2: Then the Fourier transform of $\dfrac{\partial p(t,x,k)}{\partial t}$ becomes

$$\begin{cases} \dfrac{\partial \hat{p}(t,x,k)}{\partial t} = \kappa\hat{\mathscr{L}}(k)\hat{p}(t,x,k) \\ \hat{p}(0,x,k) = \sum\limits_{y\in\mathbb{Z}^d}\delta(x-y)e^{iky} = e^{ikx} \end{cases} \implies \hat{p}(t,x,k) = \hat{p}(0,x,k)e^{\kappa\hat{\mathscr{L}}(k)t}$$

Thus we have $\hat{p}(t,x,k) = e^{ikx}e^{\kappa\hat{\mathscr{L}}(k)t}$

Using the inverse Fourier formula from Definition 2, we have that

$$p(t,x,y) = \frac{1}{(2\pi)^d}\int_{T^d}\left[e^{ikx}e^{\kappa\hat{\mathscr{L}}(k)t}\right]e^{-iky}dk = \frac{1}{(2\pi)^d}\int_{T^d} e^{ik(x-y)}e^{\kappa\hat{\mathscr{L}}(k)t}dk$$

Meaning that the transition probability depends on the distance between location x and location y. Then plugging $x = 0$ and $t = t - s$ into $p(t, x, y)$ we have that

$$p(t-s, 0, x-z) = \frac{1}{(2\pi)^d} \int_{T^d} (e^{ik(0)} e^{\kappa \hat{\mathscr{L}}(k)(t-s)}) e^{-ik(x-z)} dk$$

$$p(t-s, 0, x-z) = \frac{1}{(2\pi)^d} \int_{T^d} e^{ik(z-x)} e^{\kappa \hat{\mathscr{L}}(k)(t-s)} dk \square$$

The transition probability is important because it also relates to the Green function.

$$G_\lambda(x, y) = \int_0^\infty e^{-\lambda t} p(t, x, y) dt \text{ for } \lambda \geq 0$$

$$G_\lambda(x, y) = \frac{1}{(2\pi)^d} \int_{T^d} \frac{e^{ik(x-y)}}{\lambda - \kappa \hat{\mathscr{L}}(k)} dk \text{ then } G_0(x, y) = \frac{1}{(2\pi)^d} \int_{T^d} \frac{e^{ik(x-y)}}{-\kappa \hat{\mathscr{L}}(k)} dk$$

Where $G_0(x, y)$ is the expected value of the number of visits of susceptible, infected, or recovered population to location y if the original location is x. The random walk on $\mathbb{Z}^d$ is on a symmetric space so each location is equivalent. Therefore either all states are transient or all states are recurrent (either positive recurrent or non-recurrent). $G_0(0, 0) = \frac{1}{(2\pi)^d} \int_{T^d} \frac{1}{-\kappa \hat{\mathscr{L}}(k)} dk$ and if $G_0(0, 0) < \infty$ then the random walk is transient (there is a positive probability to never return to the original location) and if $G_0(0, 0) = \infty$ then the random walk is recurrent- meaning the random walk will return to the original location infinitely many times.

Theorem 3 (General Solution for Inhomogeneous Equation) *The general solution to the inhomogeneous equation*

$$\begin{cases} \dfrac{\partial U(t,x)}{\partial t} = \kappa \mathscr{L} U(t,x) + V(t)U(t,x) + f(t,x) \\ U(0,x) = \rho_0 > 0 \end{cases}$$

is $U(t,x) = \rho_0 E_x\left[e^{\displaystyle\int_0^t V(X_s)ds}\right] + e^{\displaystyle\int_0^t V(X_s)ds} \displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\tilde{f}(s,z)ds$ *where* X_s *is the stochastic process random walk driven by* $\kappa\mathscr{L}f = \kappa \sum\limits_{z\in\mathbb{Z}^d} a(z)[f(x+z) - f(x)]$ *and* $V(X_s)$ *is the potential of the equation.*

Proof of Theorem 3: By Duhamel's principle, we define the solution of

$$\begin{cases} \dfrac{\partial U(t,x)}{\partial t} = \kappa \mathscr{L} U(t,x) + V(t)U(t,x) + f(t,x) \\ U(0,x) = \rho_0 > 0 \end{cases}$$

to be $U(t,x) = U^h(t,x) + w(t,x)$, where $U^h(t,x)$ is the corresponding homogeneous solution and $w(t,x)$ is a particular solution to the in-homogeneous equation, and we will solve for $U^h(t,x)$ and $w(t,x)$ separately.

For $U^h[(t,x)]$, we have $\begin{cases} \dfrac{\partial U^h(t,x)}{\partial t} = \kappa \mathscr{L} U^h(t,x) + V(t)U^h(t,x) \\ U^h(0,x) = \rho_0 > 0 \end{cases}$

and the solution to this differential equation is $U^h(t,x) = \rho_0 E_x\left[e^{\displaystyle\int_0^t V(X_s)ds}\right]$ by the Kac-Feyman Formula, where X_s is the stochastic process random walk driven by $\kappa\mathscr{L}f = \kappa \sum\limits_{z\in\mathbb{Z}^d} a(z)[f(x+z) - f(x)]$.

For $w(t,x)$, we have $\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = \kappa \mathscr{L} w(t,x) + V(t)w(t,x) + f(t,x) \\ w(0,x) = 0 \end{cases}$ and the solution to this differential equation is $w(t,x) = e^{\displaystyle\int_0^t V(X_s)ds} \tilde{w}(t,x)$, so now we need to solve for $\tilde{w}(t,x)$

$$\frac{\partial w(t,x)}{\partial t} = \frac{\partial}{\partial t}\left[e^{\displaystyle\int_0^s V(s)ds}\right]\tilde{w}(t,x) + e^{\displaystyle\int_0^s V(s)ds}\left[\frac{\partial \tilde{w}(t,x)}{\partial t}\right]$$

$$\frac{\partial w(t,x)}{\partial t} = \Big(\int_0^t V(s)ds\Big) e^{\displaystyle\int_0^t V(s)ds} \tilde{w}(t,x) + e^{\displaystyle\int_0^s V(s)ds}\left[\frac{\partial \tilde{w}(t,x)}{\partial t}\right]$$

$$\kappa \mathscr{L} w(t,x) + V(t)w(t,x) + f(t,x) = V(t)e^{\int_0^t V(s)ds}\tilde{w}(t,x) + e^{\int_0^s V(s)ds}\left[\frac{\partial \tilde{w}(t,x)}{\partial t}\right]$$

$$e^{\displaystyle\int_0^t V(s)ds}\left[\frac{\partial \tilde{w}(t,x)}{\partial t}\right] = e^{\displaystyle\int_0^t V(s)ds} \kappa \mathscr{L} \tilde{w}(t,x) + f(t,x)$$

Then for $\tilde{w}(t,x)$ we have $\begin{cases} \dfrac{\partial \tilde{w}(t,x)}{\partial t} = \kappa \mathscr{L} \tilde{w}(t,x) + f(t,x) e^{-\displaystyle\int_0^t V(s)ds} \\ \tilde{w}(0,x) = 0 \end{cases}$

$\begin{cases} \dfrac{\partial \tilde{w}(t,x)}{\partial t} = \kappa \mathscr{L} \tilde{w}(t,x) + \tilde{f}(t,x) \\ \tilde{w}(0,x) = 0 \end{cases}$ and to solve this we need to use the Fourier Transform and transition probability.

The Fourier transform of $\dfrac{\partial \tilde{w}(t,x)}{\partial t}$ is $\dfrac{\partial \hat{\tilde{w}}(t,k)}{\partial t} = \kappa \hat{\mathscr{L}}(k)\hat{\tilde{w}}(t,k) + \hat{\tilde{f}}(t,k)$ and solving this differential equation we have that

$$\hat{\tilde{w}}(t,k) = C(t)e^{\kappa\hat{\mathscr{L}}(k)t} \implies C'(t)e^{\kappa\hat{\mathscr{L}}(k)t} = \hat{\tilde{f}}(t,k)$$

$$\implies C'(t) = \hat{\tilde{f}}(t,k)e^{-\kappa\hat{\mathscr{L}}(k)t}$$

Plugging this $C'(t)$ into $\hat{\tilde{w}}(t,k) = \int_0^t \hat{\tilde{f}}(t,k)e^{-\kappa\hat{\mathscr{L}}(k)s}ds(e^{\kappa\hat{\mathscr{L}}(k)t})$

$$\hat{\tilde{w}}(t,k) = \int_0^t \hat{\tilde{f}}(s,k)e^{\kappa\hat{\mathscr{L}}(k)(t-s)}ds$$

Now plugging in $\hat{p}(t-s,0,k) = e^{\kappa\hat{\mathscr{L}}(k)(t-s)}$ into the equation for $\hat{\tilde{w}}(t,k)$ we have that $\hat{\tilde{w}}(t,k) = \int_0^t \hat{\tilde{f}}(t,k)e^{\kappa\hat{\mathscr{L}}(k)(t-s)}ds$ we have that

$$\hat{\tilde{w}}(t,k) = \int_0^t \hat{\tilde{f}}(s,k)\hat{p}(t-s,0,k)ds = \int_0^t \sum_{z\in\mathbb{Z}^d} \tilde{f}(s,x)e^{ikx} \sum_{y\in\mathbb{Z}^d} p(t-s,0,y)e^{iky}$$

Now applying the Inverse Fourier Transform from Definition 2 we have that

$$\tilde{w}(t,x) = \int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\tilde{f}(s,z)ds$$

Plugging $\tilde{w}(t,x)$ into $w(t,x) = e^{\int_0^t V(X_s)ds}\tilde{w}(t,x)$ we get that

$$w(t,x) = e^{\int_0^t V(X_s)ds} \int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\tilde{f}(s,z)ds$$

Now we have the solution to the general differential equation to be $U(t,x) = U^h(t,x) + w(t,x)$

Plugging in what we have for $U^h(t,x)$ and $w(t,x)$ we get that

$$U(t,x) = \rho_0 E_x \left[e^{\displaystyle\int_0^t V(X_s)ds} \right] + e^{\displaystyle\int_0^t V(s)ds} \int_0^t \sum_{z \in \mathbb{Z}^d} p(t-s,0,x-z)\tilde{f}(s,z)ds \square$$

Now let us study the susceptible group, where Equation $(1a)$ gives the differential equation for $S(t,x)$ in the inhomogeneous space. To solve for $E[S(t,x)]$ in the inhomogeneous space we want to apply Theorem 3, where $f(t,x) = -\beta E[I(t,x)]$ but there is no potential $V(t)$ in these equations so then we have the solution to be $E[S(t,x)] = E^h[S(t,x)] + w(t,x)$ where

$$\begin{cases} \dfrac{\partial E^h[S(t,x)]}{\partial t} = \kappa \mathscr{L} E^h[S(t,x)] + V(t)E^h[S(t,x)] = \kappa \mathscr{L} E^h[S(t,x)] \\ E^h[S(0,x)] = \rho_0 \delta_0(x) \end{cases}$$
$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = \kappa \mathscr{L} w(t,x) + V(t)w(t,x) + f(t,x) = \kappa \mathscr{L} w(t,x) + f(t,x) \\ w(0,x) = 0 \end{cases}$$

To solve the differential equation of the homogeneous equation $E^h[S(t,x)]$ we will use Definitions 1 and 2:

$$\frac{\partial \hat{E}^h[S(t,k)]}{\partial t} = \kappa \hat{\mathscr{L}}(k) \hat{E}^h[S(t,k)] \implies \hat{E}^h[S(t,k)] = \hat{E}^h[S(0,k)] e^{\kappa \hat{\mathscr{L}}(k)t}$$

$$\hat{E}^h[S(t,k)] = \rho_0 e^{\kappa \hat{\mathscr{L}}(k)t} \implies E^h[S(t,x)] = \frac{1}{(2\pi)^d}\int_{T^d} \rho_0 e^{\kappa \hat{\mathscr{L}}(k)t} e^{-ikx} dk$$

To solve the differential equation of $w(t,x)$ we will use the Fourier transform and Inverse Fourier transform formula to get:

Note that since we have potential $V(u) = 0$, we have that

$f(t,x) = e^{\int_0^t V(s)ds} \tilde{f}(t,x) = \tilde{f}(t,x)$ and

$w(t,x) = e^{\int_0^t V(s)ds} \tilde{w}(t,x) = e^{\int_0^t 0 ds} \tilde{w}(t,x) = \tilde{w}(t,x)$

$\implies w(t,x) = \tilde{w}(t,x)$ and $f(t,x) = \tilde{f}(t,x)$

$$w(t,x) = \int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z) f(s,z) ds$$

$$w(t,x) = \int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)(-\beta) E[I(s,z)] ds$$

Then for $E[S(t,x)] = E^h[S(t,x)] + w(t,x)$ we have

$$E[S(t,x)] = \tfrac{1}{(2\pi)^d}\int_{T^d} \rho_0 e^{\kappa \hat{\mathscr{L}}(k)t} e^{-ikx} dk - \beta \int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z) E[I(s,z)] ds$$

$$E[S(t,x)] = \frac{1}{(2\pi)^d}\int_{T^d} \rho_0 e^{\kappa \hat{\mathscr{L}}(k)t} e^{-ikx} dk -$$
$$\frac{\beta}{(2\pi)^d}\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z) \left(\int_{T^d} e^{[\kappa \hat{\mathscr{L}}(k) + (\beta-\gamma)]s} e^{-ikz} dk\right) ds$$

Recall that the differential equation for $R(t,x)$ in the inhomogeneous space is Equation (1c). To solve for $E[R(t,x)]$ in the inhomogeneous space, we want to again apply Theorem 3, where $f(t,x)=\gamma E[I(t,x)]$ but there is no potential $V(t)$ in these equations so we have the solution to be $E[R(t,x)]=E^h[R(t,x)]+w(t,x)$ where

$$\begin{cases}\dfrac{\partial E^h[R(t,x)]}{\partial t}=\kappa\mathscr{L}E^h[R(t,x)]+V(t)E^h[R(t,x)]=\kappa\mathscr{L}E^h[R(t,x)]\\ E^h[R(0,x)]=0\end{cases}\quad \text{and}$$

$$\begin{cases}\dfrac{\partial w(t,x)}{\partial t}=\kappa\mathscr{L}w(t,x)+V(t)w(t,x)+f(t,x)=\kappa\mathscr{L}w(t,x)+f(t,x)\\ w(0,x)=0\end{cases}$$

To solve the differential equation of the homogeneous equation $E^h[R(t,x)]$ using Definitions 1 and 2:

$$\frac{\partial \hat{E}^h[R(t,k)]}{\partial t}=\kappa\hat{\mathscr{L}}(k)\hat{E}^h[R(t,k)] \implies \hat{E}^h[R(t,k)]=\hat{E}^h[R(0,k)]e^{\kappa\hat{\mathscr{L}}(k)t}=0$$

$$\hat{E}^h[R(t,x)]=\frac{1}{(2\pi)^d}\int_{T^d}\hat{E}^h[R(t,k)]e^{-ikx}dk \implies \hat{E}^h[R(t,x)]=0$$

Similarly to the susceptible group, to solve the differential equation of $w(t,x)$ we will use the Fourier transform, Inverse Fourier transform formula, and the transition probability to get:

Note that since we have potential $V(u)=0$, we have that $\tilde{f}(t,x)=f(t,x)$, $w(t,x)=\tilde{w}(t,x)$, and $w(t,x)=\displaystyle\int_0^t\sum_{z\in\mathbb{Z}^d}p(t-s,0,x-z)f(s,z)ds$

$$w(t,x) = \gamma \int_0^t \sum_{z \in \mathbb{Z}^d} p(t-s,0,x-z) E[I(s,z)] ds$$

Then for $E[R(t,x)] = E^h[R(t,x)] + w(t,x)$ we have

$$E[R(t,x)] = 0 + \gamma \int_0^t \sum_{z \in \mathbb{Z}^d} p(t-s,0,x-z) E[I(s,z)] ds$$

$$E[R(t,x)] = \gamma \tfrac{1}{(2\pi)^d} \int_0^t \sum_{z \in \mathbb{Z}^d} p(t-s,0,x-z) \left(\int_{T^d} e^{[\kappa \hat{\mathscr{L}}(k) + (\beta-\gamma)]s} e^{-ikz} dk \right) ds$$

Therefore the final solutions for the first moments in inhomogeneous space are:

$$\begin{cases} E[I(t,x)] = \dfrac{1}{(2\pi)^d} \displaystyle\int_{T^d} e^{[\kappa \hat{\mathscr{L}}(k) + (\beta-\gamma)]t} e^{-ikx} dk & (4a) \\ E[S(t,x)] = \dfrac{1}{(2\pi)^d} \left[\displaystyle\int_{T^d} \rho_0 e^{\kappa \hat{\mathscr{L}}(k)t} e^{-ikx} dk - \right. & (4b) \\ \left. \beta \displaystyle\int_0^t \sum_{z \in \mathbb{Z}^d} p(t-s,0,x-z) \left(\int_{T^d} e^{[\kappa \hat{\mathscr{L}}(k) + (\beta-\gamma)]s} e^{-ikz} dk \right) \right] ds & \\ E[R(t,x)] = \dfrac{\gamma}{(2\pi)^d} \displaystyle\int_0^t \sum_{z \in \mathbb{Z}^d} p(t-s,0,x-z) \cdot & (4c) \\ \left(\displaystyle\int_{T^d} e^{[\kappa \hat{\mathscr{L}}(k) + (\beta-\gamma)]s} e^{-ikz} dk \right) ds & \end{cases} \quad (4)$$

4.3 First Moments in Inhomogeneous Space using matrices

The homogeneous space equations uses simple ODE methods to solve for the first moments, but the inhomogeneous space solutions are more complicated and the first moments written in solution (4) are not as straightforward, making it difficult to analyze the asymptotics. Thus to make the solutions more clear we are going to

use the matrix format. Let $U(t,x) = \begin{bmatrix} S(t,x) \\ I(t,x) \\ R(t,x) \end{bmatrix}$ and then $m_1^U(t,x) = \begin{bmatrix} m_1^S(t,x) \\ m_1^I(t,x) \\ m_1^R(t,x) \end{bmatrix}$.
The Fourier transform of equation (1) from Chapter 3 is $\dfrac{\partial \hat{m}_1^U(t,k)}{\partial t} = \hat{A}_1 \hat{m}_1^U(t,k)$, where the matrix $\hat{A}_1 = \begin{bmatrix} \kappa\hat{a}(k) & -\beta & 0 \\ 0 & \kappa\hat{a}(k)+\beta-\gamma & 0 \\ 0 & \gamma & \kappa\hat{a}(k) \end{bmatrix}$ has eigenvalues are $\lambda_1 = \lambda_2 = \kappa\hat{a}(k)$ and $\lambda_3 = \kappa\hat{a}(k)+\beta-\gamma$.

Case 1: $\beta = \gamma$

Then $\lambda_1 = \lambda_2 = \lambda_3 = \kappa\hat{a}(k)$ multiplicity 3 and $(\hat{A}_1 - \kappa\hat{a}(k)I)^3 = 0$.

$e^{\hat{A}_1 t} = \begin{bmatrix} e^{\kappa\hat{a}(k)t} & -\beta t e^{\kappa\hat{a}(k)t} & 0 \\ 0 & e^{\kappa\hat{a}(k)t} & 0 \\ 0 & \gamma t e^{\kappa\hat{a}(k)t} & e^{\kappa\hat{a}(k)t} \end{bmatrix}$ with initial conditions $x_0 = \begin{bmatrix} \rho_0 & 1 & 0 \end{bmatrix}^T$

$$\hat{m}_1^U(t,k) = e^{\hat{A}_1 t} x_0 = \begin{bmatrix} \rho_0 e^{\kappa\hat{a}(k)t} - \beta t e^{\kappa\hat{a}(k)t} \\ e^{\kappa\hat{a}(k)t} \\ \gamma t e^{\kappa\hat{a}(k)t} \end{bmatrix} = \begin{bmatrix} \hat{m}_1^S(t,k) \\ \hat{m}_1^I(t,k) \\ \hat{m}_1^R(t,k) \end{bmatrix} \implies$$

$$m_1^U(t,x) = \begin{bmatrix} m_1^S(t,x) \\ m_1^I(t,x) \\ m_1^R(t,x) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(\rho_0 e^{\kappa\hat{a}(k)t} - \beta t e^{\kappa\hat{a}(k)t}\right) e^{-ikx} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(e^{\kappa\hat{a}(k)t}\right) e^{-ikx} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(\gamma t e^{\kappa\hat{a}(k)t}\right) e^{-ikx} dk \end{bmatrix} \tag{5}$$

Case 2: $\beta \neq \gamma$

$\lambda_1 = \lambda_2 = \kappa\hat{a}(k)$ has multiplicity 2 and $\lambda_3 = \kappa\hat{a}(k) + \beta - \gamma$ has multiplicity 1. Then $X(t) = \begin{bmatrix} e^{\kappa\hat{a}(k)t} & 0 & \left(\frac{\beta}{\gamma-\beta}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t} \\ 0 & 0 & e^{(\kappa\hat{a}(k)+\beta-\gamma)t} \\ 0 & e^{\kappa\hat{a}(k)t} & \left(\frac{\gamma}{\beta-\gamma}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t} \end{bmatrix}$

and $X(0) = \begin{bmatrix} 1 & 0 & \left(\frac{\beta}{\gamma-\beta}\right) \\ 0 & 0 & 1 \\ 0 & 1 & \left(\frac{\gamma}{\beta-\gamma}\right) \end{bmatrix}$, $X^{-1}(0) = \begin{bmatrix} 1 & \left(\frac{\beta}{\beta-\gamma}\right) & 0 \\ 0 & \left(\frac{\gamma}{\gamma-\beta}\right) & 1 \\ 0 & 1 & 0 \end{bmatrix}$

$$e^{\hat{A}_1 t} = X(t)X^{-1}(0)$$

$e^{\hat{A}_1 t} = \begin{bmatrix} e^{\kappa\hat{a}(k)t} & \left(\frac{\beta}{\beta-\gamma}\right)e^{\kappa\hat{a}(k)t} + \left(\frac{\beta}{\gamma-\beta}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t} & 0 \\ 0 & e^{(\kappa\hat{a}(k)+\beta-\gamma)t} & 0 \\ 0 & \left(\frac{\gamma}{\gamma-\beta}\right)e^{\kappa\hat{a}(k)t} + \left(\frac{\gamma}{\beta-\gamma}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t} & e^{\kappa\hat{a}(k)t} \end{bmatrix}$ with initial conditions $x_0 = [\rho_0 \;\; 1 \;\; 0]^T$

$$\hat{m}_1^U(t,k) = \begin{bmatrix} \rho_0 e^{\kappa\hat{a}(k)t} + \left(\dfrac{\beta}{\beta-\gamma}\right)e^{\kappa\hat{a}(k)t} - \left(\dfrac{\beta}{\beta-\gamma}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t} \\ e^{(\kappa\hat{a}(k)+\beta-\gamma)t} \\ \left(\dfrac{\gamma}{\gamma-\beta}\right)e^{\kappa\hat{a}(k)t} - \left(\dfrac{\gamma}{\gamma-\beta}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t} \end{bmatrix}$$

$$m_1^U(t,x) = \begin{bmatrix} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left[\rho_0 e^{\kappa\hat{a}(k)t} + \left(\frac{\beta}{\beta-\gamma}\right)e^{\kappa\hat{a}(k)t} - \left(\frac{\beta}{\beta-\gamma}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t}\right]e^{-ikx}dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(e^{(\kappa\hat{a}(k)+\beta-\gamma)t}\right)e^{-ikx}dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left[\left(\frac{\gamma}{\gamma-\beta}\right)e^{\kappa\hat{a}(k)t} - \left(\frac{\gamma}{\gamma-\beta}\right)e^{(\kappa\hat{a}(k)+\beta-\gamma)t}\right]e^{-ikx}dk \end{bmatrix} \quad (6)$$

The representation here in equation (6) is equivalent to the solutions in (4) on page 21 and we will show the proof for the recovered group as an example, since the proofs are similar for the susceptible, infected, and recovered groups. The solution

$(4c)$ can be rewritten as

$$m_1^R(t,x) = \frac{\gamma}{(2\pi)^d} \sum_{z\in\mathbb{Z}^d} \int_{T^d} \int_0^t p(t-s,0,x-z) e^{(\kappa\hat{a}(k)+\beta-\gamma)s} e^{-ikz} ds dk$$

$$p(t-s,0,x-z) = \frac{1}{(2\pi)^d} \int_{T^d} e^{ik(z-x)} e^{\kappa\hat{a}(k)(t-s)} dk$$

$$m_1^R(t,x) = \frac{\gamma}{(2\pi)^d} \sum_{z\in\mathbb{Z}^d} \int_{T^d} \int_0^t \frac{1}{(2\pi)^d} \int_{T^d} e^{-ikx} e^{\kappa\hat{a}(k)t+(\beta-\gamma)s} dk ds dk$$

$$= \frac{\gamma}{(2\pi)^d} \sum_{z\in\mathbb{Z}^d} \int_{T^d} \frac{1}{(2\pi)^d} \int_{T^d} e^{-ikx} e^{\kappa\hat{a}(k)t} \int_0^t e^{(\beta-\gamma)s} ds dk dk$$

$$= \frac{\gamma}{(2\pi)^d} \sum_{z\in\mathbb{Z}^d} \int_{T^d} \Big[\frac{1}{(2\pi)^d} \int_{T^d} e^{-ikx} e^{\kappa\hat{a}(k)t} dk\Big] \Big[\Big(\frac{1}{\gamma-\beta}\Big)\Big(1-e^{(\beta-\gamma)t}\Big)\Big] dk$$

$$= \frac{\gamma}{(2\pi)^d} \int_{T^d} \sum_{z\in\mathbb{Z}^d} p(t,0,x) \Big[\Big(\frac{1}{\gamma-\beta}\Big)\Big(1-e^{(\beta-\gamma)t}\Big)\Big] dk$$

$$= \frac{1}{(2\pi)^d} \int_{T^d} \Big(e^{\kappa\hat{a}(k)t} e^{-ikx}\Big) \Big[\Big(\frac{\gamma}{\gamma-\beta}\Big)\Big(1-e^{(\beta-\gamma)t}\Big)\Big] dk$$

$$= \frac{1}{(2\pi)^d} \int_{T^d} \Big[\Big(\frac{\gamma}{\gamma-\beta}\Big)\Big(e^{\kappa\hat{a}(k)t} - e^{(\kappa\hat{a}(k)+\beta-\gamma)t}\Big)\Big] e^{-ikx} dk,$$ which is equivalent to row 3 in (6).

CHAPTER 5
ANALYZING THE BEHAVIOR OF THE FIRST MOMENTS OF THE SIR MODEL

5.1 Analyzing the behavior of the first moments in Homogeneous Space

As $t \to \infty$	$m_1^S(t,x) \to$	$m_1^I(t,x) \to$	$m_1^R(t,x) \to$
Subcritical: $\beta < \gamma$	$\to \frac{\beta+\rho_0(\beta-\gamma)}{\beta-\gamma}$	$\to 0$	$\to \frac{-\gamma}{\beta-\gamma}$
Critical: $\beta = \gamma$	$\to 0$	$\to C \in \mathbb{R}$	$\to \infty$
Supercritical: $\beta > \gamma$	$\to 0$	$\to \infty$	$\to \infty$

Table 5.1: Asymptotic Behavior of the First Moments in Homogeneous Space

We can see in Figure 5.1 that while both $E[(I(t,x)]$ and $E[R(t,x)]$ go to infinity, the graph shows that $E[I(t,x)]$ (the blue line) goes to infinity at a faster rate than $E[R(t,x)]$ (the green line).

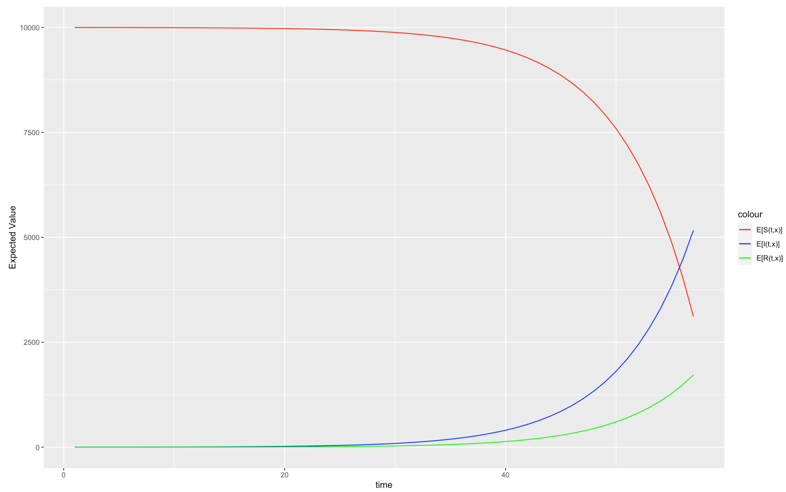

Figure 5.1: Expected Value when $\beta > \gamma$

5.2 Analyzing the behavior of the first moments in Inhomogeneous Space

A summary of the asymptotic behavior of the first moments of the susceptible, infected, and recovered groups as $t \longrightarrow \infty$ can be found in Table 7.1. Let $\theta = \kappa\hat{a}(k) + \beta - \gamma$, $\alpha = \kappa\hat{a}(k)$, $C_1 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{-ikx} dk$, $C_2 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \frac{\beta e^{-ikx}}{\beta - \gamma} dk$, and $C_3 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \frac{\gamma e^{-ikx}}{\beta - \gamma} dk$.

As $t \to \infty$	$m_1^S(t,x) \to$	$m_1^I(t,x) \to$	$m_1^R(t,x) \to$
$\beta = \gamma$, $\alpha < 0$, $\theta < 0$,	$\to 0$	$\to 0$	$\to 0$
$\beta = \gamma$, $\alpha = 0$, $\theta = 0$,	$\to 0$	$\to C_1$	$\to \infty$
$\beta < \gamma$, $\alpha < 0$, $\theta < 0$,	$\to 0$	$\to 0$	$\to 0$
$\beta < \gamma$, $\alpha = 0$, $\theta < 0$,	$\to C_2$	$\to 0$	$\to C_3$
$\beta > \gamma$, $\alpha < 0$, $\theta < 0$	$\to 0$	$\to 0$	$\to 0$
$\beta > \gamma$, $\alpha < 0$, $\theta > 0$,	$\to \infty$	$\to \infty$	$\to \infty$
$\beta > \gamma$, $\alpha = 0$, $\theta > 0$,	$\to 0$	$\to \infty$	$\to 0$

Table 5.2: Asymptotic Behavior of the First Moments in Inhomogeneous Space

When $\beta > \gamma, \alpha < 0$ and $\theta = \kappa\hat{a}(k) + \beta - \gamma < 0$, we have the event that the infection rate is higher than the recovery rate, but the infection rate minus the recovery rate $(\beta - \gamma)$ is smaller than the mobility effect $\kappa\hat{a}(k)$, meaning the mobility effect is stronger. The result is that the expected value of the susceptible, infected, and recovered populations goes to 0 as time t goes to infinity. If the first moment decreases to negative infinity simply from the mathematical expression view, as t goes to infinity, in this case, once the first moment hits the state 0, it will stay in state 0 forever. Another noteworthy event is when $\beta = \gamma$, meaning the infection rate is equal to the recovery rate, and the mobility effect $\kappa\hat{a}(k) < 0$, then the expected value of the infected population at location x goes to 0 as time t goes to infinity. The event where $\beta = \gamma$ and the mobility effect $\kappa\hat{a}(k) = 0$, we have that the expected value of the infected population at location x goes to a finite constant C_1 as t goes to infinity. This means that the expected value of the infected population goes to a steady state, rather than going to 0. This makes our model different from the classical SIR model because the infected population does not always go to 0 when the infection rate is equal to the recovery rate because we have active movement to and from outside location x.

CHAPTER 6
DERIVING THE DIFFERENTIAL EQUATIONS FOR THE SECOND MOMENTS

Now we want to find the differential equations for the second moments for the $S(t,x), I(t,x), R(t,x), S(t,x)I(t,x)$, and $R(t,x)I(t,x)$ groups, where each group has 2 cases: when the locations $x = y$ and when the locations $x \neq y$. The differential equations for the second moments will then be solved to find the second moments, which will be used in determining the variance and the long term behavior of the susceptible, infected, and recovered groups. Note a few operator definitions:

$$\begin{aligned}
\mathscr{L} f(t,x) &= \sum_{z\neq 0} a(z)(f(t,x+z) - f(t,x)) \\
\mathscr{L}_x f(t,x,y) &= \sum_{z\neq 0} a(z)(f(t,x+z,y) - f(t,x,y)) \\
\mathscr{L}_y f(t,x,y) &= \sum_{z\neq 0} a(z)(f(t,x,y+z) - f(t,x,y)) \\
\mathscr{L}_{S_x} E[S(t,x)I(t,x)] &= \sum_{z\neq 0} a(z)E[I(t,x)S(t,x+z) - I(t,x)S(t,x)] \\
\mathscr{L}_{I_x} E[S(t,x)I(t,x)] &= \sum_{z\neq 0} a(z)E[S(t,x)I(t,x+z) - S(t,x)I(t,x)] \\
&\text{and let } v = ||y - x||
\end{aligned}$$

Theorem 4 *The differential equations for the second moment of the susceptible group are*

$$
\begin{cases}
\text{Susceptible when } x = y: \\
\dfrac{\partial m_2^S(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_x m_2^S(t,x,x) + \kappa \mathscr{L} m_1^S(t,x) - 2\beta m_2^{SI}(t,x,x) + 2\kappa m_1^S(t,x) \\
+\beta m_1^I(t,x) \qquad (7a) \\
\text{Susceptible when } x \neq y: \\
\dfrac{\partial m_2^S(t,v)}{\partial t} = \kappa \mathscr{L}_x m_2^S(t,v) + \kappa \mathscr{L}_y m_2^S(t,v) - 2\beta m_2^{SI}(t,v) - \kappa a(v) m_1^S(t,x) \\
-\kappa a(v) m_1^S(t,y) \qquad (7b)
\end{cases}
$$

Theorem 5 *The differential equations for second moment of the infected group are*

$$
\begin{cases}
\text{Infected when } x = y: \\
\dfrac{\partial m_2^I(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_x m_2^I(t,x,x) + \kappa \mathscr{L} m_1^I(t,x) + 2(\beta - \gamma) m_2^I(t,x,x) + 2\kappa m_1^I(t,x) \\
+(\beta + \gamma) m_1^I(t,x) \qquad (8a) \\
\text{Infected when } x \neq y: \\
\dfrac{\partial m_2^I(t,v)}{\partial t} = \kappa \mathscr{L}_x m_2^I(t,v) + \kappa \mathscr{L}_y m_2^I(t,v) + 2(\beta - \gamma) m_2^I(t,v) - \kappa a(v) m_1^I(t,x) \\
-\kappa a(v) m_1^I(t,y) \qquad (8b)
\end{cases}
$$

Theorem 6 *The differential equations for the second moment of the recovered group are*

$$
\begin{cases}
\text{Recovered when } x = y: \\
\dfrac{\partial m_2^R(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_x m_2^R(t,x,x) + \kappa \mathscr{L} m_1^R(t,x) + 2\gamma m_2^{RI}(t,x,x) + 2\kappa m_1^R(t,x) \\
+\gamma m_1^I(t,x) \qquad (9a) \\
\text{Recovered when } x \neq y: \\
\dfrac{\partial m_2^R(t,v)}{\partial t} = \kappa \mathscr{L}_x m_2^R(t,v) + \kappa \mathscr{L}_y m_2^R(t,v) + 2\gamma m_2^{RI}(t,v) - \kappa a(v) m_1^R(t,x) \\
-\kappa a(v) m_1^R(t,y) \qquad (9b)
\end{cases}
$$

Theorem 7 *The differential equations for the second moment of the susceptible-infected groups are*

$$
\begin{cases}
\text{Susceptible-Infected when } x = y \\
\dfrac{\partial m_2^{SI}(t,x,x)}{\partial t} = \kappa \mathscr{L}_{S_x} m_2^{SI}(t,x,x) + \kappa \mathscr{L}_{I_x} m_2^{SI}(t,x,x) + (\beta - \gamma) m_2^{SI}(t,x,x) \\
-\beta m_2^I(t,x,x) - \beta m_1^I(t,x) \qquad (10a) \\
\text{Susceptible-Infected when } x \neq y \\
\dfrac{\partial m_2^{SI}(t,v)}{\partial t} = \kappa \mathscr{L}_x m_2^{SI}(t,v) + \kappa \mathscr{L}_y m_2^{SI}(t,v) + (\beta - \gamma) m_2^{SI}(t,v) - \beta m_2^I(t,v) (10b)
\end{cases}
$$

Note that $\dfrac{\partial m_2^{IS}(t,v)}{\partial t} = \dfrac{\partial m_2^{SI}(t,v)}{\partial t}$ and $\dfrac{\partial m_2^{IS}(t,x,x)}{\partial t} = \dfrac{\partial m_2^{SI}(t,x,x)}{\partial t}$

Theorem 8 *The differential equations for the second moment of the recovered-infected groups are*

$$\begin{cases} \text{Recovered-Infected when } x = y \\ \dfrac{\partial m_2^{RI}(t,x,x)}{\partial t} = \kappa \mathscr{L}_{R_x} m_2^{RI}(t,x,x) + \kappa \mathscr{L}_{I_x} m_2^{RI}(t,x,x) + (\beta - \gamma) m_2^{RI}(t,x,x) \\ + \gamma m_2^I(t,x,x) - \gamma m_1^I(t,x) \qquad (11a) \\ \text{Recovered-Infected when } x \neq y \\ \dfrac{\partial m_2^{RI}(t,v)}{\partial t} = \kappa \mathscr{L}_x m_2^{RI}(t,v) + \kappa \mathscr{L}_y m_2^{RI}(t,v) + (\beta - \gamma) m_2^{RI}(t,v) + \gamma m_2^I(t,v) \quad (11b) \end{cases}$$

Note that $\dfrac{\partial m_2^{RI}(t,v)}{\partial t} = \dfrac{\partial m_2^{IR}(t,v)}{\partial t}$ and $\dfrac{\partial m_2^{RI}(t,x,x)}{\partial t} = \dfrac{\partial m_2^{IR}(t,x,x)}{\partial t}$

Proof of Theorem 4: For the susceptible group, when deriving the differential equations for the second moment, we are going to use a similar method to the one used for the first moment- the Kolmogorov Forward Equations. There are 2 cases: when the locations are equivalent and $x = y$, and when the locations are different and $x \neq y$:

Case 1: $S(t+dt,x)$ when $x = y$, then $m_2(t+dt,x,y) = E[S^2(t+dt,x,x)]$

For the second moment when $x = y$ we have that $E[S^2(t+dt,x)] = E[(S(t,x) + \xi(dt))^2]$ where

$$\xi(dt) = \begin{cases} 1 & w.p & \sum\limits_{z \neq 0} S(t,x+z)\kappa a(z)dt & (1) \\ -1 & w.p & \sum\limits_{z \neq 0} S(t,x)\kappa a(z)dt + I(t,x)\beta dt & (2) \\ 0 & w.p & 1 - (1) - (2) \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x + z$ in the susceptible group moves to location x, meaning location x gains a particle, and thus the event has probability $\sum_{z\neq 0} S(t,x+z)\kappa a(z)dt$. The case that $\xi(dt) = -1$ is the event that either a particle at location x moves to location $x+z$ (meaning that location x loses a particle - which has probability $\sum_{z\neq 0} S(t,x)\kappa a(z)dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t,x)\beta dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the susceptible group, so it has probability $1 - \sum_{z\neq 0} S(t,x+z)\kappa a(z)dt - \sum_{z\neq 0} S(t,x)\kappa a(z)dt - I(t,x)\beta dt$.

$$E[S^2(t+dt,x)] = E[E[S^2(t+dt,x)|\mathscr{F}(t)]] = E[S^2(t,x)] + E[2S(t,x)\xi(dt)] + E[\xi^2(dt)]$$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we have that

$$E[S^2(t+dt,x)] = E[S^2(t,x)] + 2E\big[S(t,x)(1)(\sum_{z\neq 0} S(t,x+z)\kappa a(z)dt) + S(t,x)(-1)(\sum_{z\neq 0} S(t,x)\kappa a(z)dt + I(t,x)\beta dt) + S(t,x)(0)[1 - \sum_{z\neq 0} S(t,x+z)\kappa a(z)dt - (\sum_{z\neq 0} S(t,x)\kappa a(z)dt + I(t,x)\beta dt)]\big] + E\big[(1)^2(\sum_{z\neq 0} S(t,x+z)\kappa a(z)dt) + (-1)^2(\sum_{z\neq 0} S(t,x)\cdot \kappa a(z)dt + I(t,x)\beta dt) + (0)^2[1 - \sum_{z\neq 0} S(t,x+z)\kappa a(z)dt - (\sum_{z\neq 0} S(t,x)\kappa a(z)dt + I(t,x)\beta dt)]\big]$$

When we multiply out the quantities and cancel terms, re-write sums in terms of our Laplace operators defined in the beginning of the chapter, distribute the expectation and divide both sides by dt, we get that

$$\frac{\partial m_2^S(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_x m_2^S(t,x,x) + \kappa \mathscr{L} m_1^S(t,x) - 2\beta m_2^{SI}(t,x,x) + 2\kappa m_1^S(t,x) +$$
$\beta m_1^I(t,x)$

Case 2: $S(t{+}dt,x)$ when $x \neq y$, then $m_2(t{+}dt,x,y) = E[S(t{+}dt,x)S(t{+}dt,y)]$
For the second moment when $x \neq y$ we have that,

$$E[S(t+dt,x)S(t+dt,y)] = E[(S(t,x)+\xi(dt,x))(S(t,y)+\xi(dt,y))]$$

Recall that during $(t,t+dt)$ only one event can happen, either a particle can move or it can jump states, therefore there are several combinations for x and y. For example, $(x=1,y=-1)$, $(x=1,y=0)$, so on and so forth. The probabilities for the various combinations of $\xi(dt,x)$ and $\xi(dt,y)$ are:

The event that a particle in the susceptible group goes from y to x:

$P(\xi(dt,x)=1,\xi(dt,y)=-1) = \kappa a(x-y)S(t,y)dt$

The event that a particle in the susceptible group goes from $x+z$ to x but not to y:

$P(\xi(dt,x)=1,\xi(dt,y)=0) = \kappa \sum\limits_{z\neq 0} a(z)S(t,x+z)dt - \kappa a(x-y)S(t,y)dt$

The event that a particle in the susceptible group goes from x to y:

$P(\xi(dt,x)=-1,\xi(dt,y)=1) = \kappa a(x-y)S(t,x)dt$

The event that a particle in the susceptible group goes from x to $x+z$ but not to y or a particle at location x transitions $S \to I$:

$P(\xi(dt,x)=-1,\xi(dt,y)=0) = \kappa \sum\limits_{z\neq 0} a(z)S(t,x)dt - \kappa a(x-y)S(t,x)dt + \beta I(t,x)dt$

The event that a particle in the susceptible group goes from $y+z$ to y but not to x:

$P(\xi(dt,x)=0,\xi(dt,y)=1) = \kappa \sum\limits_{z\neq 0} a(z)S(t,y+z)dt - \kappa a(x-y)S(t,x)dt$

The event that a particle in the susceptible group goes from y to $y+z$ but not to x or a particle at location y transitions $S \to I$:

$P(\xi(dt,x)=0,\xi(dt,y)=-1)=\kappa\sum_{z\neq 0}a(-z)S(t,y)dt-\kappa a(x-y)S(t,y)dt+\beta I(t,y)dt$
The event that no particle moves in the susceptible group:
$P(\xi(dt,x)=0,\xi(dt,y)=0)=1-\kappa\sum_{z\neq 0}a(z)S(t,y+z)dt-\kappa\sum_{z\neq 0}a(z)S(t,y)dt-\kappa\sum_{z\neq 0}a(z)S(t,x+z)dt-\kappa\sum_{z\neq 0}a(z)S(t,x)dt+\kappa a(x-y)S(t,y)dt+\kappa a(x-y)S(t,x)dt-\beta I(t,x)dt-\beta I(t,y)dt$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event for all combinations of $\xi(dt,x)$ and $\xi(dt,y)$, and we have that

$$E[S(t+dt,x)S(t+dt,y)]=E[(S(t,x)+\xi(dt,x))(S(t,y)+\xi(dt,y))]$$

When we multiply out the quantities and cancel terms, re-write sums in terms of our Laplace operators defined in the beginning of the chapter, distribute the expectation and divide both sides by dt, and let $v=||y-x||$ we get that

$$\frac{\partial m_2^S(t,v)}{\partial t}=\kappa\mathscr{L}_x m_2^S(t,v)+\kappa\mathscr{L}_y m_2^S(t,v)-2\beta m_2^{SI}(t,v)-\kappa a(v)m_1^S(t,x)-\kappa a(v)m_1^S(t,y)$$

Proof of Theorem 5: For the infected group, when deriving the differential equations of the second moment, it is a very similar process to the one used for the susceptible group. There are 2 cases: when $x=y$ and when $x\neq y$:
Case 1: $I(t+dt,x)$ when $x=y$, then $m_2(t+dt,x,y)=E[I^2(t+dt,x,x)]$

For the second moment when $x = y$ we have that,

$E[I^2(t+dt,x)] = E[(I(t,x)+\xi(dt))^2]$ where

$$\xi(dt) = \begin{cases} 1 & w.p \quad \beta I(t,x)dt + \sum_{z\neq 0} I(t,x+z)\kappa a(-z)dt \quad (1) \\ -1 & w.p \quad \gamma I(t,x)dt + \sum_{z\neq 0} I(t,x)\kappa a(z)dt \quad (2) \\ 0 & w.p \quad 1 - (1) - (2) \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x+z$ in the infected group moves to location x, meaning location x gains a particle (which has probability $\sum_{z\neq 0} I(t,x+z)\kappa a(z)dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t,x)\beta dt$). The case that $\xi(dt) = -1$ is the event that either a particle at location x in the infected group moves to location $x+z$ (meaning that location x loses a particle - which has probability $\sum_{z\neq 0} I(t,x)\kappa a(z)dt$), or a particle at location x moves from the infected group to the recovered group (which has probability $I(t,x)\gamma dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the infected group, so it has probability $1 - I(t,x)\beta dt - \sum_{z\neq 0} I(t,x+z)\kappa a(-z)dt - I(t,x)\gamma dt - \sum_{z\neq 0} I(t,x)\kappa a(z)dt$.

$E[I^2(t+dt,x)] = E[E[I^2(t+dt,x)|\mathscr{F}(t)]] = E[I^2(t,x)] + E[2I(t,x)\xi(dt)] + E[\xi^2(dt)]$

$\dfrac{\partial m_2^I(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_x m_2^I(t,x,x) + \kappa \mathscr{L} m_1^I(t,x) + 2(\beta-\gamma)m_2^I(t,x,x) + (\beta+\gamma)m_1^I(t,x) + 2\kappa m_1^I(t,x)$

Case 2: $I(t{+}dt,x)$ when $x \neq y$, then $m_2(t{+}dt,x,y) = E[I(t{+}dt,x)I(t{+}dt,y)]$ For the second moment when $x \neq y$ we have that,

$$E[I(t+dt,x)I(t+dt,y)] = E[(I(t,x)+\xi(dt,x))(I(t,y)+\xi(dt,y))]$$

Recall that during $(t, t+dt)$ only one event can happen, either a particle can move or it can transition states, therefore the probabilities for the various combinations of $\xi(dt,x)$ and $\xi(dt,y)$ are:

The event that a particle in the infected group goes from y to x:

$P(\xi(dt,x) = 1, \xi(dt,y) = -1) = \kappa a(x-y)I(t,y)dt$

The event that a particle in the infected group goes from $x+z$ to x but not from y or a particle at location x transitions $S \to I$:

$P(\xi(dt,x) = 1, \xi(dt,y) = 0) = \kappa \sum_{z\neq 0} a(z)I(t,x+z)dt - \kappa a(x-y)I(t,y)dt + \beta I(t,x)dt$

The event that a particle in the infected group goes from x to y:

$P(\xi(dt,x) = -1, \xi(dt,y) = 1) = \kappa a(x-y)I(t,x)dt$

The event that a particle in the infected group goes from x to $x+z$ but not to y or a particle at location x transitions $I \to R$:

$P(\xi(dt,x) = -1, \xi(dt,y) = 0) = \kappa \sum_{z\neq 0} a(z)I(t,x)dt - \kappa a(x-y)I(t,x)dt + \gamma I(t,x)dt$

The event that a particle in the infected group goes from $y+z$ to y but not from x or a particle at location y transitions $S \to I$:

$P(\xi(dt,x) = 0, \xi(dt,y) = 1) = \kappa \sum_{z\neq 0} a(z)I(t,y+z)dt - \kappa a(x-y)I(t,x)dt + \beta I(t,y)dt$

The event that a particle in the infected group goes from y to $y+z$ but not to x or a particle at location y transitions $I \to R$:

$P(\xi(dt,x) = 0, \xi(dt,y) = -1) = \kappa \sum_{z\neq 0} a(-z)I(t,y)dt - \kappa a(x-y)I(t,y)dt + \gamma I(t,y)dt$

The event that no particle moves in the infected group:

$P(\xi(dt,x) = 0, \xi(dt,y) = 0) = 1 - \kappa \sum_{z\neq 0} a(z)I(t,y+z)dt - \kappa \sum_{z\neq 0} a(z)I(t,y)dt - \kappa \sum_{z\neq 0} a(z)I(t,x+z)dt - \kappa \sum_{z\neq 0} a(z)I(t,x)dt + \kappa a(x-y)I(t,y)dt + \kappa a(x-y)I(t,x)dt - \beta I(t,y)dt - \beta I(t,x)dt - \gamma I(t,y)dt - \gamma I(t,x)dt$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we have that

$E[I(t+dt,x)I(t+dt,y)] = E[(I(t,x)+\xi(dt,x))(I(t,y)+\xi(dt,y))]$

$$\frac{\partial m_2^I(t,v)}{\partial t} = \kappa \mathscr{L}_x m_2^I(t,v) + \kappa \mathscr{L}_y m_2^I(t,v) + 2(\beta-\gamma)m_2^I(t,v) - \kappa a(v)m_1^I(t,x) - \kappa a(v)m_1^I(t,y)$$

Proof of Theorem 6: For the recovered group, when deriving the differential equations of the second moment, it is a very similar process to the one used for the susceptible and infected groups and there are 2 cases- when $x = y$ and when $x \neq y$:

Case 1: $R(t+dt,x)$ when $x = y$, then $m_2(t+dt,x,y) = E[R^2(t+dt,x,x)]$

For the second moment when $x = y$ we have that,

$E[R^2(t+dt,x)] = E[(R(t,x)+\xi(dt))^2]$ where

$$\xi(dt) = \begin{cases} 1 & w.p \quad I(t,x)\gamma dt + \sum_{z\neq 0} R(t,x+z)\kappa a(-z)dt \quad ① \\ -1 & w.p \quad \sum_{z\neq 0} R(t,x)\kappa a(z)dt \quad ② \\ 0 & w.p \quad 1 - ① - ② \end{cases}$$

The case that $\xi(dt) = 1$ is the event that either a particle at location $x + z$

in the recovered group moves to location x (meaning location x gains a particle (which has probability $\sum_{z \neq 0} R(t, x+z)\kappa a(z)dt$), or a particle at location x in the infected group becomes recovered (which has probability $I(t,x)\gamma dt$). The case that $\xi(dt) = -1$ is the event that a particle at location x in the recovered group moves to location $x+z$ (meaning that location x loses a particle - which has probability $\sum_{z \neq 0} R(t,x)\kappa a(z)dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the recovered group, so it has probability $1 - I(t,x)\gamma dt - \sum_{z \neq 0} R(t, x+z)\kappa a(-z)dt - \sum_{z \neq 0} R(t,x)\kappa a(z)dt.$

$$E[R^2(t+dt, x)] = E[R^2(t,x)] + 2E[R(t,x)\xi(dt)] + E[\xi^2(dt)]$$

$\dfrac{\partial m_2^R(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_x m_2^R(t,x,x) + \kappa \mathscr{L} m_1^R(t,x) + 2\gamma m_2^{RI}(t,x,x) + \gamma m_1^I(t,x) + 2\kappa m_1^R(t,x)$

Case 2: $R(t+dt, x)$ when $x \neq y$, then $m_2(t+dt, x, y) = E[R(t+dt,x)R(t+dt,y)]$

For the second moment when $x \neq y$ we have that,

$$E[R(t+dt,x)R(t+dt,y)] = E[(R(t,x) + \xi(dt,x))(R(t,y) + \xi(dt,y))]$$

Recall that during $(t, t+dt)$ only one event can happen, either a particle can move or it can jump states, therefore the probabilities for the various combinations of x and y for $\xi(dt,x)$ and $\xi(dt,y)$ are:

The event that a particle in the recovered group goes from y to x:

$P(\xi(dt,x) = 1, \xi(dt,y) = -1) = \kappa a(x-y)R(t,y)dt$

The event that a particle in the recovered group goes from $x+z$ to x but not to y or a particle at location x transitions $I \to R$:

$P(\xi(dt,x)=1,\xi(dt,y)=0) = \kappa\sum_{z\neq 0} a(z)R(t,x+z)dt - \kappa a(x-y)R(t,y)dt + \gamma I(t,x)dt$

The event that a particle in the recovered group goes from x to y:

$P(\xi(dt,x)=-1,\xi(dt,y)=1) = \kappa a(x-y)R(t,x)dt$

The event that a particle in the recovered group goes from x to $x+z$ but not to y:

$P(\xi(dt,x)=-1,\xi(dt,y)=0) = \kappa\sum_{z\neq 0} a(z)R(t,x)dt - \kappa a(x-y)R(t,x)dt$

The event that a particle in the recovered group goes from $y+z$ to y but not to x or a particle at location y transitions $I \to R$:

$P(\xi(dt,x)=0,\xi(dt,y)=1) = \kappa\sum_{z\neq 0} a(z)R(t,y+z)dt - \kappa a(x-y)R(t,x)dt + \gamma I(t,y)dt$

The event that a particle in the recovered group goes from y to $y+z$ but not to x:

$P(\xi(dt,x)=0,\xi(dt,y)=-1) = \kappa\sum_{z\neq 0} a(-z)R(t,y)dt - \kappa a(x-y)R(t,y)dt$

The event that no particle moves in the recovered group:

$P(\xi(dt,x)=0,\xi(dt,y)=0) = 1 - \kappa\sum_{z\neq 0} a(z)R(t,y+z)dt - \kappa\sum_{z\neq 0} a(z)R(t,y)dt - \kappa\sum_{z\neq 0} a(z)R(t,x+z)dt - \kappa\sum_{z\neq 0} a(z)R(t,x)dt + \kappa a(x-y)R(t,y)dt + \kappa a(x-y)R(t,x)dt - \gamma I(t,y)dt - \gamma I(t,x)dt$

$$E[R(t+dt,x)R(t+dt,y)] = E[(R(t,x)+\xi(dt,x))(R(t,y)+\xi(dt,y))]$$

$$\frac{\partial m_2^R(t,v)}{\partial t} = \kappa\mathscr{L}_x m_2^R(t,v) + \kappa\mathscr{L}_y m_2^R(t,v) + 2\gamma m_2^{RI}(t,v) - \kappa a(v)m_1^R(t,x) - \kappa a(v)m_1^R(t,y)$$

Proof of Theorem 7: For the susceptible-infected group, there are 2 cases

($x = y$ and $x \neq y$) and there are several combinations for each case:

Case 1: when $x = y$, then we have

$E[S(t+dt, x)I(t+dt, y)] = E[S(t+dt, x)I(t+dt, x)] = E[E[S(t+dt, x)I(t+dt, x)|\mathscr{F}_t]]$

$E[S(t + dt, x)I(t + dt, x)] = E[(S(t, x) + \xi_S(dt, x))(I(t, x) + \xi_I(dt, x))]$

Recall that during $(t, t + dt)$ only one event can happen, either a particle can move or it can jump states, therefore the probabilities for $\xi_S(dt, x)$ and $\xi_I(dt, x)$ are:

The event that a particle goes from $x + z$ to x in S but x doesn't move within I:

$P(\xi_S(dt, x) = 1, \xi_I(dt, x) = 0) = \kappa \sum_{z \neq 0} a(z)S(t, x + z)dt$

The event that a particle at location x transitions $S \to I$:

$P(\xi_S(dt, x) = -1, \xi_I(dt, x) = 1) = \beta I(t, x)dt$

The event that a particle goes from x to $x + z$ in S but x doesn't move within I:

$P(\xi_S(dt, x) = -1, \xi_I(dt, x) = 0) = \kappa \sum_{z \neq 0} a(z)S(t, x)dt$

The event that a particle doesn't move in S but $x + z$ goes to x in I:

$P(\xi_S(dt, x) = 0, \xi_I(dt, x) = 1) = \kappa \sum_{z \neq 0} a(z)I(t, x + z)dt$

The event that a particle doesn't move in S but x goes to $x + z$ in I or a particle at x transitions from $I \to R$:

$P(\xi_S(dt, x) = 0, \xi_I(dt, x) = -1) = \kappa \sum_{z \neq 0} a(z)I(t, x)dt + \gamma I(t, x)dt$

The event that a particle doesn't move:

$P(\xi_S(dt, x) = 0, \xi_I(dt, x) = 0) = 1 - \kappa \sum_{z \neq 0} a(z)S(t, x + z)dt - \kappa \sum_{z \neq 0} a(z)S(t, x)dt - \kappa \sum_{z \neq 0} a(z)I(t, x + z)dt - \kappa \sum_{z \neq 0} a(z)I(t, x)dt - \beta I(t, x)dt - \gamma I(t, x)dt$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we have that

$$E[S(t+dt,x)I(t+dt,x)] = E[(S(t,x)+\xi_S(dt,x))(I(t,x)+\xi_I(dt,x))]$$

$$\frac{\partial m_2^{SI}(t,x,x)}{\partial t} = \kappa \mathscr{L}_{S_x} m_2^{SI}(t,x,x) + \kappa \mathscr{L}_{I_x} m_2^{SI}(t,x,x) + (\beta-\gamma) m_2^{SI}(t,x,x) - \beta m_2^I(t,x,x) - \beta m_1^I(t,x)$$

Case 2: when $x \neq y$, we have that:

$E[S(t+dt,x)I(t+dt,y)] = E[E[S(t+dt,x)I(t+dt,y)|\mathscr{F}_t]]$

$E[S(t+dt,x)I(t+dt,y)] = E[(S(t,x)+\xi_S(dt,x))(I(t,y)+\xi_I(dt,y))]$

Recall that during $(t,t+dt)$ only one event can happen, either a particle can move or it can jump states, therefore the probabilities for $\xi_S(dt,x)$ and $\xi_I(dt,y)$ are:

The event that a particle goes from $x+z$ to x in S but y doesn't move within I:

$P(\xi_S(dt,x)=1, \xi_I(dt,y)=0) = \kappa \sum_{z\neq 0} a(z) S(t,x+z)dt$

The event that a particle goes from x to $x+z$ in S or a particle at location x transitions $S \to I$:

$P(\xi_S(dt,x)=-1, \xi_I(dt,y)=0) = \kappa \sum_{z\neq 0} a(z) S(t,x)dt + \beta I(t,x)dt$

The event that a particle doesn't move in S but $y+z$ goes to y in I or a particle at location y transitions $S \to I$:

$P(\xi_S(dt,x)=0, \xi_I(dt,y)=1) = \kappa \sum_{z\neq 0} a(z) I(t,y+z)dt + \beta I(t,y)dt$

The event that a particle at location x doesn't move in S but y goes to $y+z$ in I or $I \to R$ at y:

$P(\xi_S(dt,x)=0, \xi_I(dt,y)=-1) = \kappa \sum_{z\neq 0} a(z) I(t,y)dt + \gamma I(t,y)dt$

The event that a particle doesn't move:

$P(\xi_S(dt,x)=0,\xi_I(dt,y)=0) = 1-\kappa\sum_{z\neq 0} a(z)S(t,x+z)dt - \kappa\sum_{z\neq 0} a(z)S(t,x)dt -$
$\kappa\sum_{z\neq 0} a(z)I(t,y+z)dt - \kappa\sum_{z\neq 0} a(z)I(t,y)dt - \beta I(t,x)dt - \beta I(t,y)dt - \gamma I(t,x)dt$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we have that

$$E[S(t+dt,x)I(t+dt,y)] = E[(S(t,x)+\xi_S(dt,x))(I(t,y)+\xi_I(dt,x))]$$

$$\frac{\partial m_2^{SI}(t,v)}{\partial t} = \kappa\mathscr{L}_x m_2^{SI}(t,v) + \kappa\mathscr{L}_y m_2^{SI}(t,v) + (\beta-\gamma)m_2^{SI}(t,v) - \beta m_2^I(t,v)$$

Proof of Theorem 8: For the recovered-infected group, there are 2 cases ($x=y$ and $x\neq y$) and there are several combinations for each case:

Case 1: when $x=y$, then we have

$E[R(t+dt,x)I(t+dt,y)] = E[R(t+dt,x)I(t+dt,x)] = E[E[R(t+dt,x)I(t+dt,x)|\mathscr{F}_t]]$

$E[R(t+dt,x)I(t+dt,x)] = E[(R(t,x)+\xi_R(dt,x))(I(t,x)+\xi_I(dt,x))]$

Recall that during $(t,t+dt)$ only one event can happen, either a particle can move or it can jump states, therefore the probabilities for $\xi_R(dt,x)$ and $\xi_I(dt,x)$ are:

The event that a particle at location x goes from $I\longrightarrow R$:

$P(\xi_R(dt,x)=1,\xi_I(dt,x)=-1) = \gamma I(t,x)dt$

The event that a particle goes from $x+z$ to x in R but x doesn't move within I:

$P(\xi_R(dt,x)=1,\xi_I(dt,x)=0) = \kappa\sum_{z\neq 0} a(z)R(t,x+z)dt$

The event that a particle goes from x to $x+z$ in R but x doesn't move within I:

$P(\xi_R(dt,x)=-1,\xi_I(dt,x)=0) = \kappa\sum_{z\neq 0} a(z)R(t,x)dt$

The event that a particle doesn't move in R but $x+z$ goes to x in I or a particle at location x transitions $S \longrightarrow I$:

$P(\xi_R(dt,x)=0, \xi_I(dt,x)=1) = \kappa \sum_{z\neq 0} a(z)I(t,x+z)dt + \beta I(t,x)dt$

The event that a particle doesn't move in R but x goes to $x+z$ in I:

$P(\xi_R(dt,x)=0, \xi_I(dt,x)=-1) = \kappa \sum_{z\neq 0} a(z)I(t,x)dt$

The event that a particle doesn't move:

$P(\xi_R(dt,x)=0, \xi_I(dt,x)=0) = 1 - \gamma I(t,x)dt - \kappa \sum_{z\neq 0} a(z)R(t,x+z)dt$
$- \kappa \sum_{z\neq 0} a(z)R(t,x)dt - \kappa \sum_{z\neq 0} a(z)I(t,x+z)dt - \beta I(t,x)dt - \kappa \sum_{z\neq 0} a(z)I(t,x)dt$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we have that

$$E[R(t+dt,x)I(t+dt,x)] = E[(R(t,x)+\xi_R(dt,x))(I(t,x)+\xi_I(dt,x))]$$

$$\frac{\partial m_2^{RI}(t,x,x)}{\partial t} = \kappa \mathscr{L}_{R_x} m_2^{RI}(t,x,x) + \kappa \mathscr{L}_{I_x} m_2^{RI}(t,x,x) + (\beta-\gamma) m_2^{RI}(t,x,x) + \gamma m_2^{I}(t,x,x) - \gamma m_1^{I}(t,x)$$

Case 2: when $x \neq y$, we have that:

$E[R(t+dt,x)I(t+dt,y)] = E[E[R(t+dt,x)I(t+dt,y)|\mathscr{F}_t]]$

$E[R(t+dt,x)I(t+dt,y)] = E[(R(t,x)+\xi_R(dt,x))(I(t,y)+\xi_I(dt,y))]$

Recall that during $(t,t+dt)$ only one event can happen, either a particle can move or it can jump states, therefore the probabilities for $\xi_R(dt,x)$ and $\xi_I(dt,y)$ are:

The event that a particle goes from $x+z$ to x in R or a particle at location x transitions $I \longrightarrow R$ but y doesn't move within I:

$P(\xi_R(dt,x)=1,\xi_I(dt,y)=0)=\kappa\sum_{z\neq 0}a(z)R(t,x+z)dt+\gamma I(t,x)dt$

The event that a particle goes from x to $x+z$ in R but y doesn't move within I:

$P(\xi_R(dt,x)=-1,\xi_I(dt,y)=0)=\kappa\sum_{z\neq 0}a(z)R(t,x)dt$

The event that a particle at x doesn't move in R but $y+z$ goes to y in I or a particle at location y transitions $S \to I$:

$P(\xi_R(dt,x)=0,\xi_I(dt,y)=1)=\kappa\sum_{z\neq 0}a(z)I(t,y+z)dt+\beta I(t,y)dt$

The event that a particle at location x doesn't move in R but a particle at y goes to $y+z$ in I or $I \to R$ at y:

$P(\xi_R(dt,x)=0,\xi_I(dt,y)=-1)=\kappa\sum_{z\neq 0}a(z)I(t,y)dt+\gamma I(t,y)dt$

The event that a particle doesn't move:

$P(\xi_R(dt,x)=0,\xi_I(dt,y)=0)=1-\kappa\sum_{z\neq 0}a(z)R(t,x+z)dt-\kappa\sum_{z\neq 0}a(z)R(t,x)dt-$
$\kappa\sum_{z\neq 0}a(z)I(t,y+z)dt-\kappa\sum_{z\neq 0}a(z)I(t,y)dt-\beta I(t,y)dt-\gamma I(t,x)dt-\gamma I(t,y)dt$

Using the Kolmogorov Forward Equations, we take the expected value of the event multiplied by the probability of the event, and we have that

$$E[R(t+dt,x)I(t+dt,y)]=E[(R(t,x)+\xi_R(dt,x))(I(t,y)+\xi_I(dt,x))]$$

$$\frac{\partial m_2^{RI}(t,v)}{\partial t}=\kappa\mathscr{L}_x m_2^{RI}(t,v)+\kappa\mathscr{L}_y m_2^{RI}(t,v)+(\beta-\gamma)m_2^{RI}(t,v)+\gamma m_2^{I}(t,v)$$

CHAPTER 7
SECOND MOMENTS OF THE SIR MODEL

Now we want to solve the differential equations for the second moments for the $S(t,x), I(t,x), R(t,x), S(t,x)I(t,x)$, and $R(t,x)I(t,x)$ groups, where each group has 2 cases: when the locations $x = y$ and when the locations $x \neq y$ and each case has 2 subcases: homogeneous space and inhomogeneous space.

Theorem 9 *The second moments for the infected groups when $x = y$ and $x \neq y$ are:*

$$
\begin{cases}
\text{Homogeneous space:} \\
m_2^I(t,x,x) = \rho_0 e^{2(\beta-\gamma)t} + \Big(\dfrac{\beta+\gamma+2\kappa}{-\beta+\gamma}\Big)\Big[e^{(\beta-\gamma)t} - e^{2(\beta-\gamma)t}\Big] & (12a) \\
\text{Inhomogeneous space:} \\
m_2^I(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[2\kappa\hat{\mathscr{L}}_x(k) + 2(\beta-\gamma)]t} e^{-ikx} dk + \\
e^{2(\beta-\gamma)t}\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\Big[\kappa\mathscr{L} m_1^I(s,z) + (\beta+\gamma+2\kappa) m_1^I(s,z)\Big]\cdot & (12b) \\
\Big(e^{2(\beta-\gamma)s}\Big) ds \\
\text{Homogeneous space:} \\
m_2^I(t,v) = \rho_0 e^{2(\beta-\gamma)t} + \Big(\dfrac{2\kappa a(v)}{(\beta-\gamma)}\Big)\Big[e^{(\beta-\gamma)t} - e^{2(\beta-\gamma)t}\Big] & (12c) \\
\text{Inhomogeneous space:} \\
m_2^I(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_x(k) + \kappa\hat{\mathscr{L}}_y(k) + 2(\beta-\gamma)]t} e^{-ikv} dk + & (12d) \\
e^{2(\beta-\gamma)t}\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\Big[-2\kappa a(v) m_1^I(s,z)\Big]\Big(e^{2(\beta-\gamma)s}\Big) ds
\end{cases}
$$

Theorem 10 *The second moments of the susceptible-infected group when $x = y$ and $x \neq y$ are:*

$$
\begin{cases}
\text{Homogeneous space:} \\
m_2^{SI}(t,x,x) = \rho_0 e^{(\beta-\gamma)t} + \Big(\dfrac{-\beta\rho_0(\gamma-\beta)+\beta(\beta+\gamma+2\kappa)}{(\beta-\gamma)(\gamma-\beta)}\Big)\Big[e^{2(\beta-\gamma)t} \\
-e^{(\beta-\gamma)t}\Big] + \Big(\dfrac{\beta(\beta+\gamma+2\kappa)-\beta(\beta-\gamma)}{\beta-\gamma}\Big)te^{(\beta-\gamma)t} \\
\text{Inhomogeneous space:} \\
m_2^{SI}(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d}\rho_0 e^{[\kappa\hat{\mathscr{L}}_{S_x}(k)+\kappa\hat{\mathscr{L}}_{I_x}(k)+(\beta-\gamma)]t}e^{-ikx}dk- \\
\beta e^{(\beta-\gamma)t}\displaystyle\int_0^t\sum_{z\in\mathbb{Z}^d}p(t-s,0,x-z)\Big[m_2^I(s,z,z)+m_1^I(s,z)\Big]\Big(e^{(\beta-\gamma)s}\Big)ds \\
\text{Homogeneous space:} \\
m_2^{SI}(t,v) = \rho_0 e^{(\beta-\gamma)t} + \Big(\dfrac{2\beta\kappa a(v)-\rho_0\beta(\beta-\gamma)}{(\beta-\gamma)^2}\Big)\Big[e^{2(\beta-\gamma)t}-e^{(\beta-\gamma)t}\Big]+ \\
\Big(\dfrac{-2\beta\kappa a(v)}{(\beta-\gamma)}\Big)\Big[te^{(\beta-\gamma)t}\Big] \\
\text{Inhomogeneous space:} \\
m_2^{SI}(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d}\rho_0 e^{[\kappa\hat{\mathscr{L}}_x(k)+\kappa\hat{\mathscr{L}}_y(k)+(\beta-\gamma)]t}e^{-ikv}dk \\
-\beta e^{(\beta-\gamma)t}\displaystyle\int_0^t\sum_{z\in\mathbb{Z}^d}p(t-s,0,x-z)e^{(\beta-\gamma)s}\Big(m_2^I(s,z)\Big)ds
\end{cases}
$$

Theorem 11 *The second moments of the susceptible group when $x = y$ and $x \neq y$ are:*

$$\begin{cases}
\text{Homogeneous space:}\\
m_2^S(t,x,x) = \rho_0 + \Big(\dfrac{-2\beta^2(\beta+\gamma+2\kappa) + 2\beta^2(\beta-\gamma)}{(\beta-\gamma)}\Big)\Big[\dfrac{(\beta t - \gamma t - 1)e^{(\beta-\gamma)t} + 1}{(\beta-\gamma)^2}\Big]\\
+\dfrac{2\kappa[\beta + (\beta-\gamma)(N-1)]t}{(\beta-\gamma)} + \Big(\dfrac{2\beta\rho_0(\beta-\gamma) + 2\beta^2(\beta+\gamma+2\kappa)}{2(\beta-\gamma)^3}\Big)\Big[e^{2(\beta-\gamma)t} - 1\Big] +\\
\Big(\dfrac{-2\beta\rho_0(\beta-\gamma)^2 - 2\beta^2\rho_0(\beta-\gamma) - 2\beta^2(\beta+\gamma+2\kappa) + \beta(\beta-\gamma)^2 - 2\kappa\beta(\beta-\gamma)}{(\beta-\gamma)^3}\Big)\cdot\\
\Big[e^{(\beta-\gamma)t} - 1\Big]\\
\text{Inhomogeneous space:}\\
m_2^S(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{2\kappa\hat{\mathscr{L}}_x(k)t} e^{-ikx} dk +\\
\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\Big[\kappa\mathscr{L} m_1^S(s,z) - 2\beta m_2^{SI}(s,z,z) + \beta m_1^I(s,z) + 2\kappa m_1^S(s,z)\Big] ds\\
\text{Homogeneous space:}\\
m_2^S(t,v) = \rho_0 + \Big(\dfrac{4\beta^2\kappa a(v)}{(\beta-\gamma)}\Big)\Big[\dfrac{(\beta t - \gamma t - 1)e^{(\beta-\gamma)t} + 1}{(\beta-\gamma)^2}\Big] -\\
\dfrac{2\kappa a(v)[\beta + (\beta-\gamma)(N-1)]t}{(\beta-\gamma)} + \Big(\dfrac{2\beta^2\rho_0(\beta-\gamma) - 4\beta^2\kappa a(v)}{2(\beta-\gamma)^3}\Big)\Big[e^{2(\beta-\gamma)t} - 1\Big] +\\
\Big(\dfrac{-2\beta\rho_0(\beta-\gamma)^2 + 4\beta^2\kappa a(v) - 2\beta^2\rho_0(\beta-\gamma) + 2\beta\kappa a(v)(\beta-\gamma)^2}{(\beta-\gamma)^3}\Big)\Big[e^{(\beta-\gamma)t} - 1\Big]\\
\text{Inhomogeneous space:}\\
m_2^S(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_x(k) + \kappa\hat{\mathscr{L}}_y(k)]t} e^{-ikv} dk +\\
\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\Big[-2\beta m_2^{SI}(s,z,z) - 2\kappa a(v) m_1^S(s,z)\Big] ds
\end{cases}$$

Theorem 12 *The second moments of the recovered-infected group when $x = y$ and $x \neq y$ are:*

$$
\begin{cases}
\text{Homogeneous space:} \\
m_2^{RI}(t,x,x) = \rho_0 e^{(\beta-\gamma)t} + \Big(\dfrac{\gamma\rho_0(\gamma-\beta) - \gamma(\beta+\gamma+2\kappa)}{(\beta-\gamma)(\gamma-\beta)}\Big)\Big[e^{2(\beta-\gamma)t} - \\
e^{(\beta-\gamma)t}\Big] + \Big(\dfrac{\gamma(\beta+\gamma+2\kappa) - \gamma(\gamma-\beta)}{\gamma-\beta}\Big) t e^{(\beta-\gamma)t} \\
\text{Inhomogeneous space:} \\
m_2^{RI}(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_{R_x}(k) + \kappa\hat{\mathscr{L}}_{I_x}(k) + (\beta-\gamma)]t} e^{-ikx} dk + \\
\gamma e^{(\beta-\gamma)t}\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\Big[m_2^I(s,z,z) - m_1^I(s,z)\Big]\Big(e^{(\beta-\gamma)s}\Big) ds \\
\text{Homogeneous space:} \\
m_2^{RI}(t,v) = \rho_0 e^{(\beta-\gamma)t} + \Big(\dfrac{\gamma\rho_0(\beta-\gamma) - 2\gamma\kappa a(v)}{(\beta-\gamma)^2}\Big)\Big[e^{2(\beta-\gamma)t} - e^{(\beta-\gamma)t}\Big] + \\
\Big(\dfrac{2\gamma\kappa a(v)}{(\beta-\gamma)}\Big)\Big[t e^{(\beta-\gamma)t}\Big] \\
\text{Inhomogeneous space:} \\
m_2^{RI}(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_x(k) + \kappa\hat{\mathscr{L}}_y(k) + (\beta-\gamma)]t} e^{-ikv} dk \\
+\gamma e^{(\beta-\gamma)t}\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z) e^{(\beta-\gamma)s}\Big(m_2^I(s,z)\Big) ds
\end{cases}
$$

Theorem 13 *The second moments of the recovered group are:*

$$\begin{cases} \text{Homogeneous space:} \\ m_2^R(t,x,x) = \Big(\dfrac{-2\gamma\kappa}{\beta-\gamma}\Big)t + \Big(\dfrac{2\gamma^2(\beta+\gamma+2\kappa) - 2\gamma^2(\gamma-\beta)}{(\gamma-\beta)}\Big)\cdot \\ \Big[\dfrac{(\beta t - \gamma t - 1)e^{(\beta-\gamma)t} + 1}{(\beta-\gamma)^2}\Big] + \Big(\dfrac{\gamma(\gamma-\beta)(\beta-\gamma) + 2\kappa\gamma(\gamma-\beta)}{(\gamma-\beta)(\beta-\gamma)^2} + \\ \dfrac{2\gamma^2(\gamma-\beta) - 2\gamma^2(\beta+\gamma+2\kappa) + 2\gamma\rho_0(\gamma-\beta)(\beta-\gamma)}{(\gamma-\beta)(\beta-\gamma)^2}\Big)\cdot \\ \Big[e^{(\beta-\gamma)t} - 1\Big] + \Big(\dfrac{2\gamma^2(\gamma-\beta) - 2\gamma^2(\beta+\gamma+2\kappa)}{2(\gamma-\beta)(\beta-\gamma)^2}\Big)\Big[e^{2(\beta-\gamma)t} - 1\Big] \\ \text{Inhomogeneous space:} \\ m_2^R(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{2\kappa\hat{\mathscr{L}}_x(k)t} e^{-ikx} dk + \\ \displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\Big[\kappa\mathscr{L} m_1^R(s,z) + 2\gamma m_2^{RI}(s,z,z) + \gamma m_1^I(s,z) \\ +2\kappa m_1^R(s,z)\Big] ds \\ \text{Homogeneous space:} \\ m_2^R(t,v) = \Big(\dfrac{2\gamma\kappa a(v)}{\beta-\gamma}\Big)t + + \Big(\dfrac{2\gamma^2\rho_0(\beta-\gamma) - 4\gamma^2\kappa a(v)}{2(\beta-\gamma)^3}\Big)\Big[e^{2(\beta-\gamma)t} - 1\Big] + \\ \Big(\dfrac{2\gamma\rho_0(\gamma-\beta)^2 - 2\gamma\kappa a(v)(\beta-\gamma) + 2\gamma^2\rho_0(\beta-\gamma) - 4\gamma^2\kappa a(v)}{(\beta-\gamma)^3}\Big)\Big[e^{(\beta-\gamma)t} - 1\Big] + \\ \Big(\dfrac{4\gamma^2\kappa a(v)}{(\beta-\gamma)}\Big)\Big[\dfrac{(\beta t - \gamma t - 1)e^{(\beta-\gamma)t} + 1}{(\beta-\gamma)^2}\Big] \\ \text{Inhomogeneous space:} \\ m_2^R(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_x(k) + \kappa\hat{\mathscr{L}}_y(k)]t} e^{-ikv} dk + \\ \displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z)\Big[2\gamma m_2^{RI}(s,z) - 2\kappa a(v) m_1^R(s,z)\Big] ds \end{cases}$$

7.1 Second Moments in Homogeneous Space

Proof of Theorem 9: Recall from Chapter 6 Equations ($8a$) and ($8b$) and for each equation $x = y$ and $x \neq y$ there are 2 cases: homogeneous space and inhomogeneous space

Case 1: Homogeneous space when $x = y$ (then the spaces x and $x + z$ are equivalent and $\mathscr{L}_x m_2^I(t, x, x) = 0 = \mathscr{L} m_1^I(t, x)$)

Then $\dfrac{\partial m_2^I(t, x, x)}{\partial t} = 2(\beta - \gamma) m_2^I(t, x, x) + (\beta + \gamma + 2\kappa) e^{(\beta - \gamma)t}$ with initial condition $m_2^I(0, x, x) = \rho_0 > 0$

Thus, $m_2^I(t, x, x) = \rho_0 e^{2(\beta - \gamma)t} + \Big(\dfrac{\beta + \gamma + 2\kappa}{-\beta + \gamma}\Big)\Big[e^{(\beta - \gamma)t} - e^{2(\beta - \gamma)t}\Big]$.

Case 2: Homogeneous space when $x \neq y$ (then $\mathscr{L}_x m_2^I(t, v) = 0$ and $m_1^I(t, x) = m_1^I(t, y) = e^{(\beta - \gamma)t}$)

Then $\dfrac{\partial m_2^I(t, v)}{\partial t} = 2(\beta - \gamma) m_2^I(t, v) - 2\kappa a(v) e^{(\beta - \gamma)t}$ with initial condition $m_2^I(0, x, x) = \rho_0 > 0$

By solving this ODE, we get $m_2^I(t, v) = \rho_0 e^{2(\beta - \gamma)t} + \Big(\dfrac{2\kappa a(v)}{(\beta - \gamma)}\Big)\Big[e^{(\beta - \gamma)t} - e^{2(\beta - \gamma)t}\Big]$

7.2 Second Moments in Inhomogeneous Space

Case 1: In-homogeneous space when $x = y$ (then the spaces x and $x + z$ are not equivalent and $\mathscr{L}_x m_2^I(t, x, x) \neq 0$)

$$\frac{\partial m_2^I(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_x m_2^I(t,x,x) + \kappa \mathscr{L} m_1^I(t,x) + 2(\beta-\gamma) m_2^I(t,x,x) + (\beta+\gamma+2\kappa) m_1^I(t,x)$$ with initial condition $m_2^I(0,x,x) = \rho_0 > 0$

Utilizing the general solution for inhomogeneous equations from Theorem 3 we have that $m_2^I(t,x,x) = m_2^{I^h}(t,x,x) + w(t,x,x)$

$$m_2^I(t,x,x) = \frac{1}{(2\pi)^d} \int_{T^d} \rho_0 e^{[2\kappa \hat{\mathscr{L}_x}(k) + 2(\beta-\gamma)]t} e^{-ikx} dk + e^{2(\beta-\gamma)t} \int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z) \Big[\kappa \mathscr{L} m_1^I(s,z) + (\beta+\gamma+2\kappa) m_1^I(s,z)\Big] \Big(e^{2(\beta-\gamma)s}\Big) ds$$

Case 2: Inhomogeneous space when $x \neq y$ (then $\mathscr{L}_x m_2^I(t,v) \neq 0$)

$$\frac{\partial m_2^I(t,v)}{\partial t} = \kappa \mathscr{L}_x m_2^I(t,v) + \kappa \mathscr{L}_y m_2^I(t,v) + 2(\beta-\gamma) m_2^I(t,v) - \kappa a(v) m_1^I(t,x) - \kappa a(v) m_1^I(t,y)$$ with initial condition $m_2^I(0,x,x) = \rho_0 > 0$

$$m_2^I(t,v) = \frac{1}{(2\pi)^d} \int_{T^d} \rho_0 e^{[\kappa \hat{\mathscr{L}_x}(k) + \kappa \hat{\mathscr{L}_y}(k) + 2(\beta-\gamma)]t} e^{-ikv} dk + e^{2(\beta-\gamma)t} \int_0^t \sum_{z\in\mathbb{Z}^d} p(t-s,0,x-z) \Big[-2\kappa a(v) m_1^I(s,z)\Big] \Big(e^{2(\beta-\gamma)s}\Big) ds$$

Note that solving the differential equations for the second moments of the susceptible, recovered, susceptible-infected, and recovered-infected groups follows the same procedure as solving for the second moment of the infected group.

7.3 Second Moments in Inhomogeneous Space using matrices

When $x = y$:

For the second moments of the susceptible, infected, recovered, susceptible-infected, and recovered-infected groups we are going to let

$$D(t,x,x) = \begin{bmatrix} S(t,x,x) \\ I(t,x,x) \\ R(t,x,x) \\ S(t,x)I(t,x) \\ R(t,x)I(t,x) \end{bmatrix} \text{ and then } m_2^D(t,x,x) = \begin{bmatrix} m_2^S(t,x,x) \\ m_2^I(t,x,x) \\ m_2^R(t,x,x) \\ m_2^{SI}(t,x,x) \\ m_2^{RI}(t,x,x) \end{bmatrix}$$

The differential equations for the second moments of the S, I, R, SI and RI groups are given in Chapter 6 Equations $(7a) - (11a)$. For the matrix format we have that $\dfrac{\partial \hat{m}_2^D(t,x,k)}{\partial t} = \hat{A}_2 \hat{m}_2^D(t,x,k) + \hat{B}_2 \hat{m}_1(t,k)$ where

$$\hat{A}_2 = \begin{bmatrix} 2\kappa\hat{a}(k) & 0 & 0 & -2\beta & 0 \\ 0 & 2\kappa\hat{a}(k) + 2(\beta-\gamma) & 0 & 0 & 0 \\ 0 & 0 & 2\kappa\hat{a}(k) & 0 & 2\gamma \\ 0 & -\beta & 0 & 2\kappa\hat{a}(k) + \beta - \gamma & 0 \\ 0 & \gamma & 0 & 0 & 2\kappa\hat{a}(k) + \beta - \gamma \end{bmatrix}$$

and $\hat{B}_2 = \begin{bmatrix} \kappa\hat{a}(k)+2\kappa & \beta & 0 \\ 0 & \kappa\hat{a}(k)+2\kappa+\beta+\gamma & 0 \\ 0 & \gamma & \kappa\hat{a}(k)+2\kappa \\ 0 & -\beta & 0 \\ 0 & -\gamma & 0 \end{bmatrix}$

For matrix $\hat{A}_2$ we have eigenvalues $\lambda_1 = 2\kappa\hat{a}(k), \lambda_2 = 2\kappa\hat{a}(k), \lambda_3 = 2(\kappa\hat{a}(k)+\beta-\gamma)$, $\lambda_4 = 2\kappa\hat{a}(k)+\beta-\gamma, \lambda_5 = 2\kappa\hat{a}(k)+\beta-\gamma$ and the eigenvectors are

$v_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$,

$v_3 = \begin{bmatrix} \frac{\beta^2}{(\beta-\gamma)\gamma} & \frac{\beta-\gamma}{\gamma} & \frac{\gamma}{\beta-\gamma} & \frac{-\beta}{\gamma} & 1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 0 & 0 & \frac{2\gamma}{\beta-\gamma} & 0 & 1 \end{bmatrix}$,

$v_5 = \begin{bmatrix} \frac{2\beta}{\gamma-\beta} & 0 & 0 & 1 & 0 \end{bmatrix}$

Case 1: $\beta = \gamma$

Then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 2\kappa\hat{a}(k)$ multiplicity 5 and $(\hat{A}_2 - 2\kappa\hat{a}(k)I)^5 = 0$.

$e^{\hat{A}_2 t} = \begin{bmatrix} e^{2\kappa\hat{a}(k)t} & \beta^2 t^2 e^{2\kappa\hat{a}(k)t} & 0 & -2\beta t e^{2\kappa\hat{a}(k)t} & 0 \\ 0 & e^{2\kappa\hat{a}(k)t} & 0 & 0 & 0 \\ 0 & \gamma^2 t^2 e^{2\kappa\hat{a}(k)t} & e^{2\kappa\hat{a}(k)t} & 0 & 2\gamma t e^{2\kappa\hat{a}(k)t} \\ 0 & -\beta t e^{2\kappa\hat{a}(k)t} & 0 & e^{2\kappa\hat{a}(k)t} & 0 \\ 0 & \gamma t e^{2\kappa\hat{a}(k)t} & 0 & 0 & e^{2\kappa\hat{a}(k)t} \end{bmatrix}$ with initial conditions $x_0 = \begin{bmatrix} \rho_0^2 & 1 & 0 & \rho_0 & 0 \end{bmatrix}^T$.

We have the solution $\hat{m}_2^D(t,x,k) = e^{\hat{A}_2 t}x_0 + e^{\hat{A}_2 t}\int_0^t e^{-\hat{A}_2 s}\hat{B}_2\hat{m}_1^U(s,k)ds$ where

$$\hat{B}_2 = \begin{bmatrix} \kappa\hat{a}(k)+2\kappa & \beta & 0 \\ 0 & \kappa\hat{a}(k)+2\kappa+\beta+\gamma & 0 \\ 0 & \gamma & \kappa\hat{a}(k)+2\kappa \\ 0 & -\beta & 0 \\ 0 & -\gamma & 0 \end{bmatrix},$$

$$\hat{m}_1^U(t,k) = \begin{bmatrix} \rho_0 e^{\kappa\hat{a}(k)t} - \beta t e^{\kappa\hat{a}(k)t} \\ e^{\kappa\hat{a}(k)t} \\ \gamma t e^{\kappa\hat{a}(k)t} \end{bmatrix}$$

$$e^{-\hat{A}_2 s} = \begin{bmatrix} e^{-2\kappa\hat{a}(k)s} & \beta^2 s^2 e^{-2\kappa\hat{a}(k)s} & 0 & 2\beta s e^{-2\kappa\hat{a}(k)s} & 0 \\ 0 & e^{-2\kappa\hat{a}(k)s} & 0 & 0 & 0 \\ 0 & \gamma^2 s^2 e^{-2\kappa\hat{a}(k)s} & e^{-2\kappa\hat{a}(k)s} & 0 & -2\gamma s e^{2\kappa\hat{a}(k)s} \\ 0 & \beta s e^{-2\kappa\hat{a}(k)s} & 0 & e^{-2\kappa\hat{a}(k)s} & 0 \\ 0 & -\gamma s e^{-2\kappa\hat{a}(k)s} & 0 & 0 & e^{-2\kappa\hat{a}(k)s} \end{bmatrix}$$

We have that $e^{\hat{A}_2 t}x_0 = \begin{bmatrix} \rho_0^2 e^{2\kappa\hat{a}(k)t} + \beta^2 t^2 e^{2\kappa\hat{a}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{a}(k)t} \\ e^{2\kappa\hat{a}(k)t} \\ \gamma^2 t^2 e^{2\kappa\hat{a}(k)t} \\ -\beta t e^{2\kappa\hat{a}(k)t} + \rho_0 e^{2\kappa\hat{a}(k)t} \\ \gamma t e^{2\kappa\hat{a}(k)t} \end{bmatrix}$

$$\hat{m}_2^D(t,x,k) = e^{\hat{A}_2 t}x_0 + e^{\hat{A}_2 t}\int_0^t e^{-\hat{A}_2 s}\hat{B}_2\hat{m}_1^U(s,k)ds$$

$$
\begin{bmatrix}
\rho_0^2 e^{2\kappa\hat{a}(k)t} + \beta^2 t^2 e^{2\kappa\hat{a}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{a}(k)t} \\
-\Big(\frac{\rho_0(\kappa\hat{a}(k)+2\kappa)}{\kappa\hat{a}(k)}\Big)\Big[e^{\kappa\hat{a}(k)} - e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa)+2\beta^2}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{2\kappa\hat{a}(k)t} \\
-e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] + \Big(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{\kappa\hat{a}(k)t}\Big(-(\kappa\hat{a}(k))^2 \\
-2\kappa\hat{a}(k)t-2\Big) + 2e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{\kappa\hat{a}(k)}\Big)\cdot \\
\Big[t^2e^{\kappa\hat{a}(k)t} - t^2e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[te^{2\kappa\hat{a}(k)t} - \\
te^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] - \Big(\frac{2\beta^2}{\kappa\hat{a}(k)}\Big)\Big[te^{\kappa\hat{a}(k)t} - te^{2\kappa\hat{a}(k)t}\Big] \\
e^{2\kappa\hat{a}(k)t} - \Big(\frac{\kappa\hat{a}(k)+2\kappa+\beta+\gamma}{\kappa\hat{a}(k)}\Big)\Big[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\Big] \\
\gamma^2t^2e^{2\kappa\hat{a}(k)t} - \Big(\frac{\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{\kappa\hat{a}(k)}\Big)\Big[t^2e^{\kappa\hat{a}(k)t} - t^2e^{2\kappa\hat{a}(k)t}\Big] \\
-\Big(\frac{\gamma+\kappa\hat{a}(k)\gamma+2\kappa\gamma}{\kappa\hat{a}(k)}\Big)\Big[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{2\gamma^2}{(\kappa\hat{a}(k))^2}\Big)\cdot \\
\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] + \Big(\frac{\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{\kappa\hat{a}(k)t}\cdot \\
(-(\kappa\hat{a}(k)t)^2 - 2\kappa\hat{a}(k)t - 2) + 2e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{2\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\cdot \\
\Big[te^{2\kappa\hat{a}(k)t} - te^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] + \Big(\frac{2\gamma^2}{\kappa\hat{a}(k)}\Big)\Big[te^{\kappa\hat{a}(k)t} - te^{2\kappa\hat{a}(k)t}\Big] \\
-\beta te^{2\kappa\hat{a}(k)t} + \rho_0e^{2\kappa\hat{a}(k)t} + \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)+\beta}{\kappa\hat{a}(k)}\Big)\Big[e^{\kappa\hat{a}(k)t} - \\
e^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] \\
\gamma te^{2\kappa\hat{a}(k)t} - \Big(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)+\gamma}{\kappa\hat{a}(k)}\Big)\Big[te^{\kappa\hat{a}(k)t} - te^{2\kappa\hat{a}(k)t}\Big] - \\
\Big(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big]
\end{bmatrix}
$$

When $\beta = \gamma$, $m_2^D(t,x,x) = \begin{bmatrix} m_2^S(t,x,x) \\ m_2^I(t,x,x) \\ m_2^R(t,x,x) \\ m_2^{SI}(t,x,x) \\ m_2^{RI}(t,x,x) \end{bmatrix} =$

$$
\begin{bmatrix}
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{ \rho_0^2 e^{2\kappa\hat{a}(k)t} + \beta^2 t^2 e^{2\kappa\hat{a}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{a}(k)t} \\
-\Big(\frac{\rho_0(\kappa\hat{a}(k)+2\kappa)}{\kappa\hat{a}(k)}\Big)\Big[e^{\kappa\hat{a}(k)} - e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa)+2\beta^2}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{2\kappa\hat{a}(k)t} \\
-e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] + \Big(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{\kappa\hat{a}(k)t}\Big(- (\kappa\hat{a}(k))^2 - \\
2\kappa\hat{a}(k)t - 2\Big) + 2e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{\kappa\hat{a}(k)}\Big)\Big[t^2 e^{\kappa\hat{a}(k)t} - \\
t^2 e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[t e^{2\kappa\hat{a}(k)t} \\
-t e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] - \Big(\frac{2\beta^2}{\kappa\hat{a}(k)}\Big)\Big[t e^{\kappa\hat{a}(k)t} - t e^{2\kappa\hat{a}(k)t}\Big]\Big\} e^{-ikx} dk \\
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{ e^{2\kappa\hat{a}(k)t} - \Big(\frac{\kappa\hat{a}(k)+2\kappa+\beta+\gamma}{\kappa\hat{a}(k)}\Big)\Big[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\Big]\Big\} e^{-ikx} dk \\
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{ \gamma^2 t^2 e^{2\kappa\hat{a}(k)t} - \Big(\frac{\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{\kappa\hat{a}(k)}\Big)\Big[t^2 e^{\kappa\hat{a}(k)t} - t^2 e^{2\kappa\hat{a}(k)t}\Big] \\
-\Big(\frac{\gamma+\kappa\hat{a}(k)\gamma+2\kappa\gamma}{\kappa\hat{a}(k)}\Big)\Big[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{2\gamma^2}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{2\kappa\hat{a}(k)t} \\
-e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] + \Big(\frac{\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{\kappa\hat{a}(k)t}(-(\kappa\hat{a}(k)t)^2 - \\
2\kappa\hat{a}(k)t - 2) + 2e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{2\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[t e^{2\kappa\hat{a}(k)t} \\
-t e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] + \Big(\frac{2\gamma^2}{\kappa\hat{a}(k)}\Big)\Big[t e^{\kappa\hat{a}(k)t} - t e^{2\kappa\hat{a}(k)t}\Big]\Big\} e^{-ikx} dk \\
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{ -\beta t e^{2\kappa\hat{a}(k)t} + \rho_0 e^{2\kappa\hat{a}(k)t} + \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)+\beta}{\kappa\hat{a}(k)}\Big) \cdot \\
\Big[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{2\kappa\hat{a}(k)t} - \\
e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big]\Big\} e^{-ikx} dk \\
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{ \gamma t e^{2\kappa\hat{a}(k)t} - \Big(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)+\gamma}{\kappa\hat{a}(k)}\Big)\Big[t e^{\kappa\hat{a}(k)t} - t e^{2\kappa\hat{a}(k)t}\Big] \\
-\Big(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big]\Big\} e^{-ikx} dk
\end{bmatrix}
$$

Case 2: $\beta \neq \gamma$

For matrix $\hat{A}_2$ we have eigenvalues $\lambda_1 = 2\kappa\hat{a}(k), \lambda_2 = 2\kappa\hat{a}(k), \lambda_3 = 2(\kappa\hat{a}(k) + \beta - \gamma), \lambda_4 = 2\kappa\hat{a}(k) + \beta - \gamma, \lambda_5 = 2\kappa\hat{a}(k) + \beta - \gamma$ and the eigenvectors are
$v_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$,
$v_3 = \begin{bmatrix} \dfrac{\beta^2}{(\beta-\gamma)\gamma} & \dfrac{\beta-\gamma}{\gamma} & \dfrac{\gamma}{\beta-\gamma} & \dfrac{-\beta}{\gamma} & 1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 0 & 0 & \dfrac{2\gamma}{\beta-\gamma} & 0 & 1 \end{bmatrix}$,
$v_5 = \begin{bmatrix} \dfrac{2\beta}{\gamma-\beta} & 0 & 0 & 1 & 0 \end{bmatrix}$
and $X_1(t) = \begin{bmatrix} e^{2\kappa\hat{a}(k)t} & 0 & 0 & 0 & 0 \end{bmatrix}^T$, $X_2(t) = \begin{bmatrix} 0 & 0 & e^{2\kappa\hat{a}(k)t} & 0 & 0 \end{bmatrix}^T$,
$X_3(t) = \begin{bmatrix} \left(\frac{\beta^2}{(\beta-\gamma)\gamma}\right)C & \left(\frac{\beta-\gamma}{\gamma}\right)C & \left(\frac{\gamma}{\beta-\gamma}\right)C & \left(\frac{-\beta}{\gamma}\right)C & C \end{bmatrix}^T$
where $C = e^{2(\kappa\hat{a}(k) + \beta + \gamma)t}$,
$X_4(t) = \begin{bmatrix} \left(\dfrac{2\beta}{\gamma-\beta}\right)e^{(2\kappa\hat{a}(k)t + \beta + \gamma)t} & 0 & 0 & e^{(2\kappa\hat{a}(k)t + \beta + \gamma)t} & 0 \end{bmatrix}^T$,
$X_5(t) = \begin{bmatrix} 0 & 0 & \left(\dfrac{-2\gamma}{\gamma-\beta}\right)e^{(2\kappa\hat{a}(k)t + \beta + \gamma)t} & 0 & e^{(2\kappa\hat{a}(k)t + \beta + \gamma)t} \end{bmatrix}^T$
Aligning the $X_i(t)$ vectors we have

$$X(t) = \begin{bmatrix} e^{2\kappa\hat{a}(k)t} & 0 & \left(\frac{\beta^2}{(\beta-\gamma)\gamma}\right)e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & \left(\frac{2\beta}{\gamma-\beta}\right)C & 0 \\ 0 & 0 & \left(\frac{\beta-\gamma}{\gamma}\right)C & 0 & 0 \\ 0 & e^{2\kappa\hat{a}(k)t} & \left(\frac{\gamma}{\beta-\gamma}\right)C & 0 & \left(\frac{-2\gamma}{\gamma-\beta}\right)C \\ 0 & 0 & \left(\frac{-\beta}{\gamma}\right)C & C & 0 \\ 0 & 0 & C & 0 & C \end{bmatrix}$$

$$X(0)=\begin{bmatrix}1 & 0 & \left(\frac{\beta^2}{(\beta-\gamma)\gamma}\right) & \left(\frac{-2\beta}{\beta-\gamma}\right) & 0\\ 0 & 0 & \left(\frac{\beta-\gamma}{\gamma}\right) & 0 & 0\\ 0 & 1 & \left(\frac{\gamma}{\beta-\gamma}\right) & 0 & \left(\frac{-2\gamma}{\gamma-\beta}\right)\\ 0 & 0 & \left(\frac{-\beta}{\gamma}\right) & 1 & 0\\ 0 & 0 & 1 & 0 & 1\end{bmatrix}$$

$$X^{-1}(0)=\begin{bmatrix}1 & \left(\frac{\beta^2}{(\beta-\gamma)^2}\right) & 0 & \left(\frac{-2\beta}{\gamma-\beta}\right) & 0\\ 0 & \left(\frac{\gamma^2}{(\beta-\gamma)^2}\right) & 1 & 0 & \left(\frac{2\gamma}{\gamma-\beta}\right)\\ 0 & \left(\frac{\gamma}{\beta-\gamma}\right) & 0 & 0 & 0\\ 0 & \left(\frac{\beta}{\beta-\gamma}\right) & 0 & 1 & 0\\ 0 & \left(\frac{-\gamma}{\beta-\gamma}\right) & 0 & 0 & 1\end{bmatrix}$$

$$e^{\hat{A}_2 t}=X(t)X^{-1}(0)=$$

$$\begin{bmatrix}e^{2\kappa\hat{a}(k)t} & \left(\frac{\beta^2}{(\beta-\gamma)^2}\right)e^{2(\beta+\gamma)t}-\left(\frac{2\beta^2}{(\beta-\gamma)^2}\right)e^{(\beta+\gamma)t} & 0 & \left(\frac{-2\beta}{\beta-\gamma}\right)e^{(\beta+\gamma)t} & 0\\ 0 & e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & 0 & 0 & 0\\ 0 & \left(\frac{-\gamma^2}{(\beta-\gamma)^2}\right)e^{2(\beta+\gamma)t} & e^{2\kappa\hat{a}(k)t} & 0 & \left(\frac{2\gamma}{\beta-\gamma}\right)e^{(\beta+\gamma)t}\\ 0 & \left(\frac{-\beta}{\beta-\gamma}\right)e^{2(\beta+\gamma)t}+\left(\frac{\beta}{\beta-\gamma}\right)e^{(\beta+\gamma)t} & 0 & e^{(2\kappa\hat{a}(k)+\beta+\gamma)t} & 0\\ 0 & \left(\frac{\gamma}{\beta-\gamma}\right)e^{2(\beta+\gamma)t}-\left(\frac{\gamma}{\beta-\gamma}\right)e^{(\beta+\gamma)t} & 0 & 0 & e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\end{bmatrix}$$

with initial conditions $x_0=\begin{bmatrix}\rho_0^2 & 1 & 0 & \rho_0 & 0\end{bmatrix}^T$.

We have the solution $\hat{m}_2^D(t,x,k) = e^{\hat{A}_2 t}x_0 + e^{\hat{A}_2 t}\int_0^t e^{-\hat{A}_2 s}\hat{B}_2\hat{m}_1^U(s,k)ds$ where

$$\hat{B}_2 = \begin{bmatrix} \kappa\hat{a}(k) + 2\kappa & \beta & 0 \\ 0 & \kappa\hat{a}(k) + 2\kappa + \beta + \gamma & 0 \\ 0 & \gamma & \kappa\hat{a}(k) + 2\kappa \\ 0 & -\beta & 0 \\ 0 & -\gamma & 0 \end{bmatrix},$$

$$\hat{m}_1^U(t,k) = \begin{bmatrix} \rho_0 e^{\kappa\hat{a}(k)} - \beta t e^{\kappa\hat{a}(k)t} \\ e^{\kappa\hat{a}(k)t} \\ \gamma t e^{\kappa\hat{a}(k)t} \end{bmatrix}$$

$$\hat{m}_2^D(t,x,k) = e^{\hat{A}_2 t} x_0 + e^{\hat{A}_2 t} \int_0^t e^{-\hat{A}_2 s} \hat{B}_2 \hat{m}_1^U(s,k) ds =$$

$$
\begin{bmatrix}
\rho_0^2 e^{2\kappa\hat{a}(k)t} + \left(\frac{\beta^2}{(\beta-\gamma)^2}\right)e^{2(\beta+\gamma)t} - \left(\frac{2\beta^2}{(\beta-\gamma)^2}\right)e^{(\beta+\gamma)t} - \rho_0\left(\frac{2\beta}{\beta-\gamma}\right)e^{(\beta+\gamma)t} \\
\left(\frac{-\rho_0(\kappa\hat{a}(k)+2\kappa)-\beta}{\kappa\hat{a}(k)}\right)\left[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] + \left(\frac{\beta(\kappa\hat{a}(k)+2\kappa)}{(\kappa\hat{a}(k))^2}\right)\left[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\right] \\
+\left(\frac{2\beta^2(\beta-\gamma)+\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{3\kappa\hat{a}(k)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{5\kappa\hat{a}(k)(\beta-\gamma)^2}\right)\cdot \\
\left[e^{-3\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{2\kappa\hat{a}(k)t}\right] \\
+\left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(5\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t} - e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\cdot \\
\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] + \left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right]+ \\
\left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-3\kappa\hat{a}(k)t} - e^{(\beta+\gamma)t}\right] - \left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\cdot \\
\left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right] - \left(\frac{2\beta^2}{(\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{(\beta+\gamma)t}\right] \\
e^{2(\kappa\hat{a}(k)t+\beta+\gamma)t} - \left(\frac{\kappa\hat{a}(k)+2\kappa+\beta+\gamma}{\kappa\hat{a}(k)+2\beta+2\gamma}\right)\left[e^{\kappa\hat{a}(k)t} - e^{2(\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
\left(\frac{-\gamma^2}{(\beta-\gamma)^2}\right)e^{2(\beta+\gamma)t} + \left(\frac{\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] - \left(\frac{\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k))(\beta-\gamma)^2}\right)\cdot \\
\left[e^{-\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{2\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{5\kappa\hat{a}(k)(\beta-\gamma)^2}\right)\left[e^{-3\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{2\gamma^2}{3\kappa\hat{a}(k)(\beta-\gamma)}\right)\cdot \\
\left[e^{-\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] + \left(\frac{2\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(5\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t} - e^{2\kappa\hat{a}(k)t}\right] \\
-\left(\frac{\gamma}{\kappa\hat{a}(k)}\right)\left[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] + \left(\frac{\gamma(\kappa\hat{a}(k)+2\kappa)}{(\kappa\hat{a}(k))^2}\right)\left[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\right] \\
-\left(\frac{2\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-3\kappa\hat{a}(k)t} - e^{(\beta+\gamma)t}\right] + \left(\frac{2\gamma^2(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\cdot \\
\left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right] + \left(\frac{2\gamma^2}{(\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{(\beta+\gamma)t}\right] \\
\left(\frac{-\beta}{\beta-\gamma}\right)e^{2(\beta+\gamma)t} + \left(\frac{\beta}{\beta-\gamma}\right)e^{(\beta+\gamma)t} + \rho_0 e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}+ \\
\left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] - \left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\cdot \\
\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right] - \left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
+\left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
+\left(\frac{\beta}{(\kappa\hat{a}(k)+\beta+\gamma)}\right)\left[e^{\kappa\hat{a}(k)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
\left(\frac{\gamma}{\beta-\gamma}\right)e^{2(\beta+\gamma)t} - \left(\frac{\gamma}{\beta-\gamma}\right)e^{(\beta+\gamma)t} \\
-\left(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] + \left(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\cdot \\
\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right] - \left(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
+\left(\frac{\gamma(\kappa\hat{a}(k)+2\kappa+\beta+\gamma)}{(3\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
+\left(\frac{\gamma}{(\kappa\hat{a}(k)+\beta+\gamma)}\right)\left[e^{\kappa\hat{a}(k)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right]
\end{bmatrix}
$$

When $\beta \neq \gamma$, $m_2^D(t,x,x) = \left(\frac{1}{2\pi}\right)^d \int_{T^d} \hat{m}_2^D(t,x,k)e^{-ikx}dk =$

$$\begin{bmatrix} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left\{\hat{m}_2^S(t,x,k)\right\} e^{-ikx}dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left\{\hat{m}_2^I(t,x,k)\right\} e^{-ikx}dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left\{\hat{m}_2^R(t,x,k)\right\} e^{-ikx}dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left\{\hat{m}_2^{SI}(t,x,k)\right\} e^{-ikx}dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left\{\hat{m}_2^{RI}(t,x,k)\right\} e^{-ikx}dk \end{bmatrix}$$

When $x \neq y$:

The differential equations for the second moments of the S, I, R, SI and RI groups when $x \neq y$ are given in Chapter 6 Equations $(7b) - (11b)$. For the matrix format we have that $\dfrac{\partial \hat{m}_2^D(t,k)}{\partial t} = \hat{A}_3 \hat{m}_2^D(t,k) + \hat{B}_3 \vec{P}$ where

$$\hat{A}_3 = \begin{bmatrix} 2\kappa\hat{a}(k) & 0 & 0 & -2\beta & 0 \\ 0 & 2\kappa\hat{a}(k) + 2(\beta-\gamma) & 0 & 0 & 0 \\ 0 & 0 & 2\kappa\hat{a}(k) & 0 & 2\gamma \\ 0 & -\beta & 0 & 2\kappa\hat{a}(k)+\beta-\gamma & 0 \\ 0 & \gamma & 0 & 0 & 2\kappa\hat{a}(k)+\beta-\gamma \end{bmatrix} \text{ and}$$

$$\hat{B}_3 = \begin{bmatrix} -2\kappa\hat{a}(k) & 0 & 0 & 0 & 0 \\ 0 & -2\kappa\hat{a}(k) & 0 & 0 & 0 \\ 0 & 0 & -2\kappa\hat{a}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \vec{P} \text{ where } \vec{P} = \begin{bmatrix} \hat{m}_1^S(t,k) \\ \hat{m}_1^I(t,k) \\ \hat{m}_1^R(t,k) \\ 0 \\ 0 \end{bmatrix}$$

For matrix $\hat{A}_3$ we have eigenvalues $\lambda_1 = 2\kappa\hat{a}(k), \lambda_2 = 2\kappa\hat{a}(k), \lambda_3 = 2(\kappa\hat{a}(k) + \beta - \gamma), \lambda_4 = 2\kappa\hat{a}(k) + \beta - \gamma, \lambda_5 = 2\kappa\hat{a}(k) + \beta - \gamma$
and the eigenvectors are $v_1 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$,
$v_3 = \begin{bmatrix} \frac{\beta^2}{(\beta-\gamma)\gamma} & \frac{\beta-\gamma}{\gamma} & \frac{\gamma}{\beta-\gamma} & \frac{-\beta}{\gamma} & 1 \end{bmatrix}$, $v_4 = \begin{bmatrix} 0 & 0 & \frac{2\gamma}{\beta-\gamma} & 0 & 1 \end{bmatrix}$,
$v_5 = \begin{bmatrix} \frac{2\beta}{\gamma-\beta} & 0 & 0 & 1 & 0 \end{bmatrix}$

Case 1: $\beta = \gamma$

*Note that the $\hat{A}_2 = \hat{A}_3$ matrix for $m_2^D(t, v)$ is the same as for $m_2^D(t, x, x)$. Then $\lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = 2\kappa\hat{a}(k)$ multiplicity 5 and $(\hat{A}_3 - 2\kappa\hat{a}(k)I)^5 = 0$.

$$e^{\hat{A}_3 t} = \begin{bmatrix} e^{2\kappa\hat{a}(k)t} & \beta^2 t^2 e^{2\kappa\hat{a}(k)t} & 0 & -2\beta t e^{2\kappa\hat{a}(k)t} & 0 \\ 0 & e^{2\kappa\hat{a}(k)t} & 0 & 0 & 0 \\ 0 & \gamma^2 t^2 e^{2\kappa\hat{a}(k)t} & e^{2\kappa\hat{a}(k)t} & 0 & 2\gamma t e^{2\kappa\hat{a}(k)t} \\ 0 & -\beta t e^{2\kappa\hat{a}(k)t} & 0 & e^{2\kappa\hat{a}(k)t} & 0 \\ 0 & \gamma t e^{2\kappa\hat{a}(k)t} & 0 & 0 & e^{2\kappa\hat{a}(k)t} \end{bmatrix}$$ with initial conditions $x_0 = \begin{bmatrix} \rho_0^2 & 1 & 0 & \rho_0 & 0 \end{bmatrix}^T$.

We have the solution $\hat{m}_2^D(t, k) = e^{\hat{A}_3 t} x_0 + e^{\hat{A}_3 t} \int_0^t e^{-\hat{A}_3 s} \hat{B}_3 \vec{P} ds$ where

$$\hat{B}_3 = \begin{bmatrix} -2\kappa\hat{a}(k) & 0 & 0 & 0 & 0 \\ 0 & -2\kappa\hat{a}(k) & 0 & 0 & 0 \\ 0 & 0 & -2\kappa\hat{a}(k) & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \vec{P} = \begin{bmatrix} \rho_0 e^{\kappa\hat{a}(k)t} - \beta t e^{\kappa\hat{a}(k)t} \\ e^{\kappa\hat{a}(k)t} \\ \gamma t e^{\kappa\hat{a}(k)t} \\ 0 \\ 0 \end{bmatrix}$$

$$e^{-\hat{A}_3 s} =$$

$$\begin{bmatrix} e^{-2\kappa\hat{a}(k)s} & \beta^2 s^2 e^{-2\kappa\hat{a}(k)s} & 0 & 2\beta s e^{-2\kappa\hat{a}(k)s} & 0 \\ 0 & e^{-2\kappa\hat{a}(k)s} & 0 & 0 & 0 \\ 0 & \gamma^2 s^2 e^{-2\kappa\hat{a}(k)s} & e^{-2\kappa\hat{a}(k)s} & 0 & -2\gamma s e^{2\kappa\hat{a}(k)s} \\ 0 & \beta s e^{-2\kappa\hat{a}(k)s} & 0 & e^{-2\kappa\hat{a}(k)s} & 0 \\ 0 & -\gamma s e^{-2\kappa\hat{a}(k)s} & 0 & 0 & e^{-2\kappa\hat{a}(k)s} \end{bmatrix}$$

$$\hat{m}_2^D(t,k) = e^{\hat{A}_3 t}x_0 + e^{\hat{A}_3 t}\int_0^t e^{-\hat{A}_3 s}\hat{B}_3\vec{P}ds$$

$$\begin{bmatrix}
\rho_0^2 e^{2\kappa\hat{a}(k)t} + \beta^2 t^2 e^{2\kappa\hat{a}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{a}(k)t} + \\
\Big(2\rho_0\Big)\Big[e^{\kappa\hat{a}(k)} - e^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{2\beta}{\kappa\hat{a}(k)}\Big)\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] \\
+\Big(\frac{4\beta^2}{(\kappa\hat{a}(k))^2}\Big)\cdot\Big[e^{\kappa\hat{a}(k)t}\Big(-(\kappa\hat{a}(k))^2 - 2\kappa\hat{a}(k)t - 2\Big) + 2e^{2\kappa\hat{a}(k)t}\Big] + \Big(2\beta^2\Big)\cdot \\
\Big[t^2 e^{\kappa\hat{a}(k)t} - t^2 e^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{4\beta^2}{\kappa\hat{a}(k)}\Big)\Big[te^{2\kappa\hat{a}(k)t} - te^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] \\
e^{2\kappa\hat{a}(k)t} + 2\Big[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\Big] \\
\gamma^2 t^2 e^{2\kappa\hat{a}(k)t} + \Big(2\gamma^2\Big)\Big[t^2 e^{\kappa\hat{a}(k)t} - t^2 e^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{2\gamma^2}{\kappa\hat{a}(k)}\Big)\Big[te^{2\kappa\hat{a}(k)t} \\
-te^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] - \Big(\frac{2\gamma^2}{(\kappa\hat{a}(k))^2}\Big)\Big[e^{\kappa\hat{a}(k)t}(-(\kappa\hat{a}(k)t)^2 \\
-2\kappa\hat{a}(k)t - 2) + 2e^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{2\gamma}{\kappa\hat{a}(k)}\Big)\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] \\
-\beta t e^{2\kappa\hat{a}(k)t} + \rho_0 e^{2\kappa\hat{a}(k)t} - \\
\Big(2\beta\Big)\Big[te^{\kappa\hat{a}(k)t} - te^{2\kappa\hat{a}(k)t}\Big] - \Big(\frac{2\beta}{\kappa\hat{a}(k)}\Big)\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\Big] \\
\gamma t e^{2\kappa\hat{a}(k)t} + \Big(2\gamma\Big)\Big[te^{\kappa\hat{a}(k)t} - te^{2\kappa\hat{a}(k)t}\Big] + \Big(\frac{2\gamma}{\kappa\hat{a}(k)}\Big)\Big[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}\cdot \\
(\kappa\hat{a}(k)t+1)\Big]
\end{bmatrix}$$

When $\beta = \gamma$, $m_2^D(t,v) = \Big(\frac{1}{2\pi}\Big)^d \int_{T^d} \hat{m}_2^D(t,k)e^{-ikv}dk$

Case 2: $\beta \neq \gamma$

For matrix $\hat{A}_3$, we have eigenvalues $\lambda_1 = 2\kappa\hat{a}(k), \lambda_2 = 2\kappa\hat{a}(k), \lambda_3 = 2(\kappa\hat{a}(k) + \beta - \gamma), \lambda_4 = 2\kappa\hat{a}(k) + \beta - \gamma, \lambda_5 = 2\kappa\hat{a}(k) + \beta - \gamma$ and the eigenvectors are $v_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$, $v_2 = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \end{bmatrix}$, $v_3 = \begin{bmatrix} \frac{\beta^2}{(\beta-\gamma)\gamma} & \frac{\beta-\gamma}{\gamma} & \frac{\gamma}{\beta-\gamma} & \frac{-\beta}{\gamma} & 1 \end{bmatrix}$,

$$v_4 = \begin{bmatrix} 0 & 0 & \dfrac{2\gamma}{\beta-\gamma} & 0 & 1 \end{bmatrix},\ v_5 = \begin{bmatrix} \dfrac{2\beta}{\gamma-\beta} & 0 & 0 & 1 & 0 \end{bmatrix}$$

Aligning the $X_i(t)$ vectors we have $X(t) =$

$$\begin{bmatrix} e^{2\kappa\hat{a}(k)t} & 0 & \left(\frac{\beta^2}{(\beta-\gamma)\gamma}\right)e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & \left(\frac{2\beta}{\gamma-\beta}\right)e^{(2\kappa\hat{a}(k)+\beta+\gamma)t} & 0 \\ 0 & 0 & \left(\frac{\beta-\gamma}{\gamma}\right)e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & 0 & 0 \\ 0 & e^{2\kappa\hat{a}(k)t} & \left(\frac{\gamma}{\beta-\gamma}\right)e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & 0 & \left(\frac{-2\gamma}{\gamma-\beta}\right)e^{(2\kappa\hat{a}(k)t+\beta+\gamma)t} \\ 0 & 0 & \left(\frac{-\beta}{\gamma}\right)e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & e^{(2\kappa\hat{a}(k)+\beta-\gamma)t} & 0 \\ 0 & 0 & e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & 0 & e^{(2\kappa\hat{a}(k)+\beta-\gamma)t} \end{bmatrix}$$

$$X(0) = \begin{bmatrix} 1 & 0 & \left(\frac{\beta^2}{(\beta-\gamma)\gamma}\right) & \left(\frac{-2\beta}{\beta-\gamma}\right) & 0 \\ 0 & 0 & \left(\frac{\beta-\gamma}{\gamma}\right) & 0 & 0 \\ 0 & 1 & \left(\frac{\gamma}{\beta-\gamma}\right) & 0 & \left(\frac{-2\gamma}{\gamma-\beta}\right) \\ 0 & 0 & \left(\frac{-\beta}{\gamma}\right) & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{bmatrix}$$

$$X^{-1}(0) = \begin{bmatrix} 1 & \left(\frac{\beta^2}{(\beta-\gamma)^2}\right) & 0 & \left(\frac{-2\beta}{\gamma-\beta}\right) & 0 \\ 0 & \left(\frac{\gamma^2}{(\beta-\gamma)^2}\right) & 1 & 0 & \left(\frac{2\gamma}{\gamma-\beta}\right) \\ 0 & \left(\frac{\gamma}{\beta-\gamma}\right) & 0 & 0 & 0 \\ 0 & \left(\frac{\beta}{\beta-\gamma}\right) & 0 & 1 & 0 \\ 0 & \left(\frac{-\gamma}{\beta-\gamma}\right) & 0 & 0 & 1 \end{bmatrix}$$

$$e^{\hat{A}_3 t} = X(t)X^{-1}(0) =$$

$$
\begin{bmatrix}
e^{2\kappa\hat{a}(k)t} & \left(\frac{\beta^2}{(\beta-\gamma)^2}\right)e^{2(\beta+\gamma)t} - \left(\frac{2\beta^2}{(\beta-\gamma)^2}\right)e^{(\beta+\gamma)t} & 0 & \left(\frac{-2\beta}{\beta-\gamma}\right)e^{(\beta+\gamma)t} & 0 \\
0 & e^{2(\kappa\hat{a}(k)+\beta+\gamma)t} & 0 & 0 & 0 \\
0 & \left(\frac{-\gamma^2}{(\beta-\gamma)^2}\right)e^{2(\beta+\gamma)t} & e^{2\kappa\hat{a}(k)t} & 0 & \left(\frac{2\gamma}{\beta-\gamma}\right)e^{(\beta+\gamma)t} \\
0 & \left(\frac{-\beta}{\beta-\gamma}\right)e^{2(\beta+\gamma)t} + \left(\frac{\beta}{\beta-\gamma}\right)e^{(\beta+\gamma)t} & 0 & e^{(2\kappa\hat{a}(k)+\beta+\gamma)t} & 0 \\
0 & \left(\frac{\gamma}{\beta-\gamma}\right)e^{2(\beta+\gamma)t} - \left(\frac{\gamma}{\beta-\gamma}\right)e^{(\beta+\gamma)t} & 0 & 0 & e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}
\end{bmatrix}
$$

with initial conditions $x_0 = \begin{bmatrix} \rho_0^2 & 1 & 0 & \rho_0 & 0 \end{bmatrix}^T$.

$$\hat{m}_2^D(t,k) = e^{\hat{A}_3 t} x_0 + e^{\hat{A}_3 t} \int_0^t e^{-\hat{A}_3 s} \hat{B}_3 \vec{P} ds =$$

$$
\left[
\begin{array}{c}
\rho_0^2 e^{2\kappa\hat{a}(k)t} + \left(\frac{\beta^2}{(\beta-\gamma)^2}\right) e^{2(\beta+\gamma)t} - \left(\frac{2\beta^2}{(\beta-\gamma)^2}\right) e^{(\beta+\gamma)t} + \rho_0 \left(\frac{-2\beta}{\beta-\gamma}\right) e^{(\beta+\gamma)t} + \\
\left(2\rho_0\right)\left[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] + \left(2\beta\right)\left[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] \\
-\left(\frac{2\beta^2\kappa\hat{a}(k)}{3\kappa\hat{a}(k)(\beta-\gamma)^2}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] + \left(\frac{4\beta^2\kappa\hat{a}(k)}{5\kappa\hat{a}(k)(\beta-\gamma)^2}\right) \cdot \\
\left[e^{-3\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] + \left(\frac{4\beta^2\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t}\right. \\
\left. -e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{4\beta^2\kappa\hat{a}(k)}{(5\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right) \cdot \left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t} - e^{2\kappa\hat{a}(k)t}\right] \\
+\left(\frac{2\beta^2\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] - \left(\frac{4\beta^2\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right) \cdot \\
\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right] - \left(\frac{4\beta^2\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right) \cdot \\
\left[e^{-3\kappa\hat{a}(k)t} - e^{(\beta+\gamma)t}\right] - \left(\frac{4\beta^2\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t} - \right. \\
\left. e^{(\beta+\gamma)t}\right] \\
e^{2(\kappa\hat{a}(k)t+\beta+\gamma)t} + \left(\frac{2\kappa\hat{a}(k)}{\kappa\hat{a}(k)+2\beta+2\gamma}\right)\left[e^{\kappa\hat{a}(k)t} - e^{2(\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
\left(\frac{-\gamma^2}{(\beta-\gamma)^2}\right) e^{2(\beta+\gamma)t} + \left(\frac{-2\gamma^2\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)^2}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] \\
+\left(\frac{2\gamma^2\kappa\hat{a}(k)}{3\kappa\hat{a}(k)(\beta-\gamma)^2}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] + \left(\frac{4\gamma^2\kappa\hat{a}(k)}{5\kappa\hat{a}(k)(\beta-\gamma)^2}\right) \cdot \\
\left[e^{-3\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{4\gamma^2\kappa\hat{a}(k)}{(5\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t}\right. \\
\left. -e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{2\gamma\kappa\hat{a}(k)}{(\kappa\hat{a}(k))^2}\right)\left[e^{2\kappa\hat{a}(k)t} - e^{\kappa\hat{a}(k)t}(\kappa\hat{a}(k)t+1)\right] \\
+\left(\frac{4\gamma^2\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)^2}\right)\left[e^{-3\kappa\hat{a}(k)t} - e^{(\beta+\gamma)t}\right] \\
+\left(\frac{4\gamma^2\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+2(\beta+\gamma))(\beta-\gamma)^2}\right)\left[e^{-(3\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right] \\
\left(\frac{-\beta}{\beta-\gamma}\right) e^{2(\beta+\gamma)t} + \left(\frac{\beta}{\beta-\gamma}\right) e^{(\beta+\gamma)t} + \rho_0 e^{(2\kappa\hat{a}(k)+\beta+\gamma)t} + \\
\left(\frac{-2\beta\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] + \left(\frac{2\beta\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right) \cdot \\
\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(\beta+\gamma)t}\right] + \left(\frac{2\beta\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t}\right. \\
\left. -e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] + \left(\frac{2\beta\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t}\right. \\
\left. -e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
\left(\frac{\gamma}{\beta-\gamma}\right) e^{2(\beta+\gamma)t} - \left(\frac{\gamma}{\beta-\gamma}\right) e^{(\beta+\gamma)t} + \left(\frac{2\gamma\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right) \cdot \\
\left[e^{-\kappa\hat{a}(k)t} - e^{2(\beta+\gamma)t}\right] - \left(\frac{2\gamma\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - \right. \\
\left. e^{(\beta+\gamma)t}\right] + \left(\frac{2\gamma\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+\beta+\gamma)(\beta-\gamma)}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right] \\
+\left(\frac{2\gamma\kappa\hat{a}(k)}{(3\kappa\hat{a}(k)+2\beta+2\gamma)(\beta-\gamma)}\right)\left[e^{-(\kappa\hat{a}(k)+\beta+\gamma)t} - e^{(2\kappa\hat{a}(k)+\beta+\gamma)t}\right]
\end{array}
\right]
$$

CHAPTER 8
ANALYZING THE BEHAVIOR OF THE SECOND MOMENTS OF THE SIR MODEL

8.1 Analyzing the behavior of the second moments in Homogeneous Space

The behavior of the second moments of the susceptible, infected, recovered, susceptible-infected, recovered-infected groups in the homogeneous space as $t \longrightarrow \infty$ can be broken up in 3 cases: $\beta < \gamma, \beta = \gamma, \beta > \gamma$. The asymptotic behavior of $m_2^S(t,x,x)$, $m_2^I(t,x,x)$, $m_2^R(t,x,x)$, $m_2^{SI}(t,x,x)$, and $m_2^{RI}(t,x,x)$ in homogeneous space as $t \longrightarrow \infty$ is summarized in the Table 8.1 below.

As $t \to \infty$	$m_2^S(t,x,x)$	$m_2^I(t,x,x)$	$m_2^R(t,x,x)$	$m_2^{SI}(t,x,x)$	$m_2^{RI}(t,x,x)$
$\beta > \gamma$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta < \gamma$	$\to \infty$	$\to 0$	$\to \infty$	$\to 0$	$\to 0$
$\beta = \gamma$	$\to \infty$	$\to \infty$	$\to \infty$	$\to 0$	$\to \infty$

Table 8.1: Asymptotic Behavior of the Second Moments in Homogeneous Space when $x = y$

The asymptotic behavior of the second moments in homogeneous space when $x \neq y$ and $v = ||y = x||$ as $t \longrightarrow \infty$ is summarized in Table 8.2

As $t \to \infty$	$m_2^S(t,v)$	$m_2^I(t,v)$	$m_2^R(t,v)$	$m_2^{SI}(t,v)$	$m_2^{RI}(t,v)$
$\beta > \gamma$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta < \gamma$	$\to \infty$	$\to 0$	$\to \infty$	$\to 0$	$\to 0$
$\beta = \gamma$	$\to 0$	$\to 0$	$\to \infty$	$\to 0$	$\to \infty$

Table 8.2: Asymptotic Behavior of the Second Moments in Homogeneous Space when $x \neq y$

8.2 Analyzing the behavior of the second moments in Inhomogeneous Space

The long term behavior of $m_2^S(t,x,x)$, $m_2^I(t,x,x)$, $m_2^R(t,x,x)$, $m_2^{SI}(t,x,x)$, and $m_2^{RI}(t,x,x)$ then follows from the long term behavior of $m_2^I(t,x,x)$. The asymptotic behavior of the second moments of the susceptible, infected, recovered, susceptible-infected, and recovered-infected groups in inhomogeneous space when $x = y$ as $t \longrightarrow \infty$ is summarized in Table 8.3 below. The asymptotic behavior of the second moments of the susceptible, infected, recovered, susceptible-infected, and recovered-infected groups in inhomogeneous space when $x \neq y$ as $t \longrightarrow \infty$ is summarized in Table 8.4 below.

Let $\alpha = \kappa\hat{a}(k)$, $\theta = \kappa\hat{a}(k)+\beta-\gamma$, $\mu = \kappa\hat{a}(k)+\beta+\gamma$, $C_1 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{-ikx}dk$,

$$C_2 = \frac{1}{(2\pi)^d}\int_{T^d}\Big(\frac{\kappa+\beta+\gamma}{2\beta+2\gamma}\Big)e^{-ikx}dk, \text{ and } C_3 = \frac{1}{(2\pi)^d}\int_{T^d}e^{-ikv}dk$$

When $\beta + \gamma > 0, \alpha < 0$ and $\mu = \kappa\hat{a}(k) + \beta + \gamma < 0$, we have the infection rate plus the recovery rate $(\beta + \gamma)$ is positive but smaller than the mobility effect $\kappa\hat{a}(k)$, meaning the mobility effect is stronger. The result is that the second moment of the infected population goes to 0 as time t goes to infinity. Another notable event is when $\beta = \gamma$, meaning the infection rate is equal to the recovery rate. When $\beta = \gamma$ and the mobility effect $\kappa\hat{a}(k) < 0$, the second moment of the infected population at location x goes to 0 as time t goes to infinity. The event where $\beta = \gamma$ and the mobility effect $\kappa\hat{a}(k) = 0$, we have that the second moment of the infected population at location x goes to a finite constant C_3 as t goes to infinity. This means that the second moment of the infected population goes to a steady state. This makes our model different from the classical SIR model because the infected population does not always go to 0 when the infection rate is equal to the recovery rate because we have active movement to and from outside location x.

As $t \to \infty$	$m_2^S(t,x,x)$	$m_2^I(t,x,x)$	$m_2^R(t,x,x)$	$m_2^{SI}(t,x,x)$	$m_2^{RI}(t,x,x)$
$\beta - \gamma = 0$, $\alpha < 0$, $\theta < 0$	$\to 0$	$\to 0$	$\to 0$	$\to 0$	$\to 0$
$\beta - \gamma = 0$, $\alpha = 0$, $\theta = 0$	$\to \infty$	$\to C_1$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma < 0$, $\alpha < 0$, $\mu < 0$	$\to \infty$	$\to 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma < 0$, $\alpha = 0$, $\mu < 0$	$\to \infty$	$\to C_2$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma > 0$, $\alpha < 0$, $\mu < 0$	$\to \infty$	$\to 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma > 0$, $\alpha < 0$, $\mu > 0$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma > 0$, $\alpha = 0$, $\mu > 0$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$

Table 8.3: Asymptotic Behavior of the Second Moments in Inhomogeneous Space when $x = y$

As $t \to \infty$	$m_2^S(t,v)$	$m_2^I(t,v)$	$m_2^R(t,v)$	$m_2^{SI}(t,v)$	$m_2^{RI}(t,v)$
$\beta - \gamma = 0$, $\alpha < 0$, $\theta < 0$	$\to 0$	$\to 0$	$\to 0$	$\to 0$	$\to 0$
$\beta - \gamma = 0$, $\alpha = 0$, $\theta = 0$	$\to \infty$	$\to C_3$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma < 0$, $\alpha < 0$, $\mu < 0$	$\to \infty$	$\to 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma < 0$, $\alpha = 0$, $\mu < 0$	$\to \infty$	$\to 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma > 0$, $\alpha < 0$, $\mu < 0$	$\to \infty$	$\to 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma > 0$, $\alpha < 0$, $\mu > 0$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta + \gamma > 0$, $\alpha = 0$, $\mu > 0$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$	$\to \infty$

Table 8.4: Asymptotic Behavior of the Second Moments in Inhomogeneous Space when $x \neq y$

CHAPTER 9
INTRODUCTION TO INTERMITTENCY AND LYAPUNOV EXPONENTS FOR THE SIR MODEL

Now we will study the intermittency effect in our model. A random field u has *intermittency effect* if $u(t, \cdot)$ develops high peaks on few, small, remote islands-the *intermittent islands*, which carry most of the total mass $U(t) = \sum_{z \in \mathbb{Z}} u(t, z)$ [10] [3] [5]. Molchanov defines intermittency as: a field $n(t, x), x \in \mathbb{Z}^d$ is intermittent as $t \to \infty$ on a non-decreasing family of sets $D(t)$ if $\lim \dfrac{En^2(t, x)}{(En(t, x))^2} = \infty$ uniformly in $x \in D(t)$ [5].

König [10] studies *intermittency* in terms of the Lyapunov exponents. They state that *intermittency* refers to unusually large fluctuations of the field, such that for large values of time t there exists spots (or clusters) where the concentration of particles is very high, the distances between these clusters being very high, and most of the mass of the particles is concentrated in these clusters [3]. The Lyapunov exponent is defined as $\gamma_p = \lim_{t \to \infty} \dfrac{1}{t} \log m_p(t)$, where $m_p(t)$ is the p^{th} moment of the solution. The Lyapunov exponents have the follow convexity property: $\dfrac{\gamma_p(\kappa)}{p} \leq \dfrac{\gamma_{p+1}(\kappa)}{p+1}, p \geq 0, \kappa > 0$. The convex property of the Lyapunov exponent influences the intermittency phenomena. Carmona and Molchanov [3] state that the family of homogeneous fields $\{u(t, x); x \in \mathbb{Z}^d\}$ is intermittent when $t \to \infty$ if one has

$\gamma_1 \leq \frac{\gamma_2}{2} \leq ... \leq \frac{\gamma_p}{p} \leq \frac{\gamma_{p+1}}{p+1} \leq ....$

For the epidemic model, it is interesting to study if the infected population will form clusters, or if the infected population will mix with the susceptible population. For our SIR model with mobility, intermittency in the infected group means that the model forms clusters of infected people. For our model, we define $m_2(t, x, y) = E[u(t, x)u(t, y)]$, and for the infected group we have $m_2^I(t, x, y) = E[I(t, x)I(t, y)]$. When $x = y$, if $\lim_{t\to\infty} \frac{m_2^I(t, x, x)}{m_1^I(t, x)^2} = \infty$, then it implies that the infected group will form clusters. When $x \neq y$, we have that if $\lim_{t\to\infty} \frac{m_2^I(t, x, y)}{m_1^I(t, x)m_1^I(t, y)} = \infty$, then it will have clusterization of the infected population. The Lyapunov exponent tells us how the population will increase or decrease. For Lyapunov exponent λ_1, we have $m_1^I(t, x) = e^{\lambda_1 t}$. If $\lambda_1 > 0$, the population will increase exponentially. If $\lambda_1 < 0$, the population will decrease exponentially. For the Lyapunov exponents of the first moments of our model, we have that $\lambda_1 = \lim_{t\to\infty} \frac{\ln(m_1^I(t, x))}{t}$. For the Lyapunov exponents of the second moments of our model, we have that $\lambda_{2,x,x} = \lim_{t\to\infty} \frac{\ln(m_2^I(t, x, x))}{t}$ and $\lambda_{2,x,y} = \lim_{t\to\infty} \frac{\ln(m_2^I(t, x, y))}{t}$.

CHAPTER 10
INTERMITTENCY ANALYSIS FOR THE SIR MODEL

When $x = y$ in Homogeneous Space:

$\lim\limits_{t\to\infty} \dfrac{m_2^I(t,x,x)}{m_1^I(t,x)^2}$ where $m_1^I(t,x) = e^{(\beta-\gamma)t}$ and

$$m_2^I(t,x,x) = \rho_0 e^{2(\beta-\gamma)t} + \left(\frac{\beta+\gamma+2\kappa}{-\beta+\gamma}\right)\left[e^{(\beta-\gamma)t} - e^{2(\beta-\gamma)t}\right]$$

$$\frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2} = \rho_0 + \left(\frac{\beta+\gamma+2\kappa}{-\beta+\gamma}\right)\left[e^{-(\beta-\gamma)t} - 1\right]$$

If $\beta < \gamma$, $\lim\limits_{t\to\infty} e^{-(\beta-\gamma)t} - 1 \to \infty$, thus $\lim\limits_{t\to\infty} \dfrac{m_2^I(t,x,x)}{(m_1^I(t,x))^2} \to \infty$

Let $C_1 = \rho_0 - \left(\dfrac{\beta+\gamma+2\kappa}{-\beta+\gamma}\right)$

If $\beta > \gamma$, $\lim\limits_{t\to\infty} e^{-(\beta-\gamma)t} - 1 \to 0$, thus $\lim\limits_{t\to\infty} \dfrac{m_2^I(t,x,x)}{(m_1^I(t,x))^2} \to C_1$

If $\beta = \gamma$ and $C \in \mathbb{R}$, $\lim\limits_{t\to\infty} \dfrac{\rho_0}{C^2} + \dfrac{(\beta+\gamma+2\kappa)t}{C} \to \infty$

As $t \to \infty$	$\lim_{t\to\infty} \frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2}$	Result
$\beta < \gamma$	$\to \infty$	Intermittency
$\beta > \gamma$	$\to C_1 < \infty$	No intermittency
$\beta = \gamma$	$\to \infty$	Intermittency

Table 10.1: Intermittency Analysis when $x = y$ in Homogeneous Space

When $x \neq y$ in Homogeneous Space:

$$\lim_{t\to\infty} \frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)} \text{ where } m_1^I(t,x) = m_1^I(t,y) = e^{(\beta-\gamma)t}$$

$$m_2^I(t,x,y) = \rho_0 e^{2(\beta-\gamma)t} + \left(\frac{2\kappa a(v)}{\beta-\gamma}\right)\left[e^{(\beta-\gamma)t} - e^{2(\beta-\gamma)t}\right]$$

$$\frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)} = \rho_0 + \left(\frac{2\kappa a(v)}{\beta-\gamma}\right)\left[e^{-(\beta-\gamma)t} - 1\right]$$

Let $C_2 = \rho_0 - \left(\frac{2\kappa a(v)}{\beta-\gamma}\right)$

As $t \to \infty$	$\lim_{t\to\infty} \frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)}$	Result
$\beta < \gamma$	$\to \infty$	Intermittency
$\beta > \gamma$	$\to C_2 < \infty$	No intermittency
$\beta = \gamma$	$\to -\infty$	Intermittency

Table 10.2: Intermittency Analysis when $x \neq y$ in Homogeneous Space

When $x = y$ in Inhomogeneous Space:

Since $m_1^I(t,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{[\kappa\hat{\mathscr{L}}(k) + (\beta-\gamma)]t} e^{-ikx} dk = e^{(\beta-\gamma)t} p(t,0,x)$ and plugging in $(12b)$ from Chapter 7, we get

$$\frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2} = \frac{\rho_0}{p(t,0,x)} + \Big(\frac{\kappa}{3(\beta-\gamma)(p(t,0,x))^2}\Big)\Big[e^{3(\beta-\gamma)t} - 1\Big] + \Big(\frac{\beta+\gamma+2\kappa}{3(\beta-\gamma)p(t,0,x)}\Big) \cdot \Big[e^{3(\beta-\gamma)t} - 1\Big]$$

Since $p(t,x,y) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{ik(x-y)} e^{\kappa\hat{\mathscr{L}}(k)t} = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} cos(x-y)\hat{p}(t,0,k)dk \leq \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d}\hat{p}(t,0,k)dk = p(t,0,0)$, we have $p(t,x,y) \leq p(t,0,0)$ and as $t \to \infty, p(t,0,0) = \dfrac{C}{t^{d/2}} + o(t^{-d/2})$ [13]. Thus $\lim\limits_{t\to\infty} \dfrac{C}{t^{d/2}} + o(t^{-d/2}) = 0$.

Thus $\frac{\rho_0}{p(t,0,x)} + \Big(\frac{\kappa}{3(\beta-\gamma)(p(t,0,x))^2}\Big)\Big[e^{3(\beta-\gamma)t} - 1\Big] + \Big(\frac{\beta+\gamma+2\kappa}{3(\beta-\gamma)p(t,0,x)}\Big) \cdot \Big[e^{3(\beta-\gamma)t} - 1\Big]$
$\geq \frac{\rho_0}{p(t,0,0)} + \Big(\frac{\kappa}{3(\beta-\gamma)(p(t,0,0))^2}\Big)\Big[e^{3(\beta-\gamma)t} - 1\Big] + \Big(\frac{\beta+\gamma+2\kappa}{3(\beta-\gamma)p(t,0,0)}\Big) \cdot \Big[e^{3(\beta-\gamma)t} - 1\Big]$

Case 1: $\beta < \gamma, C \neq 0$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} - \Big(\frac{\kappa}{3(\beta-\gamma)(p(t,0,0))^2}\Big) - \Big(\frac{\beta+\gamma+2\kappa}{3(\beta-\gamma)p(t,0,0)}\Big) = \infty$$

Case 2: $\beta > \gamma$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} + \Big(\frac{\kappa}{3(\beta-\gamma)(p(t,0,0))^2}\Big)\Big[e^{3(\beta-\gamma)t} - 1\Big] + \Big(\frac{\beta+\gamma+2\kappa}{3(\beta-\gamma)p(t,0,0)}\Big) \cdot \Big[e^{3(\beta-\gamma)t} - 1\Big] = \infty$$

Case 3: $\beta = \gamma$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} + \Big(\frac{\kappa t}{(p(t,0,0))^2}\Big) - \Big(\frac{(\beta+\gamma+2\kappa)t}{p(t,0,0)}\Big) = \infty$$

As $t \to \infty$	$\lim_{t\to\infty} \frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2}$	Result
$\beta < \gamma$	$\to \infty$	Intermittency
$\beta > \gamma$	$\to \infty$	Intermittency
$\beta = \gamma$	$\to \infty$	Intermittency

Table 10.3: Intermittency Analysis when $x = y$ in Inhomogeneous Space

When $x \neq y$ in Inhomogeneous Space:

Using the solutions $m_1^I(t,x) = e^{(\beta - \gamma)t} p(t,0,x)$ and $m_1^I(t,y) = e^{(\beta - \gamma)t} p(t,0,y)$ and $(12d)$ from Chapter 7, we have that

$$\frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)} = \frac{\rho_0}{p(t,0,y)} - \Big(\frac{2\kappa a(v)}{3(\beta-\gamma)p(t,0,y)}\Big)\Big[e^{3(\beta - \gamma)t} - 1\Big]$$

We have $p(t,x,y) \leq p(t,0,0)$ and as $t \to \infty, p(t,0,0) = \frac{C}{t^{d/2}} + o(t^{-d/2})$ [13] and $\lim_{t\to\infty} \frac{C}{t^{d/2}} + o(t^{-d/2}) = 0$.

Case 1: $\beta < \gamma, C \neq 0$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} - \Big(\frac{2\kappa a(v)}{3(\beta-\gamma)p(t,0,0)}\Big) = \infty$$

Case 2: $\beta > \gamma$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} + \Big(\frac{2\kappa a(v)}{3(\beta-\gamma)p(t,0,0)}\Big)\Big[e^{3(\beta-\gamma)t} - 1\Big] = \infty$$

Case 3: $\beta = \gamma$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} - \left(\frac{2\kappa a(v)t}{p(t,0,0)}\right) = \infty$$

As $t \to \infty$	$\lim\limits_{t\to\infty} \dfrac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)}$	Result
$\beta < \gamma$	$\to \infty$	Intermittency
$\beta > \gamma$	$\to \infty$	Intermittency
$\beta = \gamma$	$\to \infty$	Intermittency

Table 10.4: Intermittency Analysis when $x \neq y$ in Inhomogeneous Space

In homogeneous space, when $\beta > \gamma$, we have the event that the infection rate is higher than the recovery rate and there is no intermittency phenomenon. This means that the infection is so widespread that as $t \to \infty$ and it does not matter where the location is, everywhere will have the infection. This is compared to the case where $\beta < \gamma$, meaning the infection rate is less than the recovery rate, or when $\beta = \gamma$ and the infection rate is equal to the recovery rate, and the result is that there will be peaks/clusters with a higher concentration of infection population in some locations, and thus there is the intermittency phenomenon.

In inhomogeneous space, when $\beta < \gamma$ (meaning the infection rate is less than the recovery rate), $\beta > \gamma$ (meaning the infection rate is greater than the infection rate), and when $\beta = \gamma$ (meaning the infection rate is equal to the recovery rate), the result is that the infection will form spots/groups with a higher concentration of infection in some locations and the intermittency phenomenon appears in the configuration space.

CHAPTER 11
LYAPUNOV ANALYSIS FOR THE SIR MODEL

Homogeneous Space:

Theorem 14 *For the first moment of the infected group in homogeneous space, the Lyapunov Exponents are* $\lambda_{1,1} = \beta - \gamma$ *when* $\beta - \gamma < 0$, $\lambda_{1,2} = \beta - \gamma$ *when* $\beta - \gamma > 0$, *and* $\lambda_{1,3} = 0$ *when* $\beta - \gamma = 0$

Proof of Theorem 14:

Lyapunov Exponent $\lambda_{1,i} = \lim\limits_{t\to\infty} \dfrac{\ln(m_1^I(t,x))}{t}$ where $m_1^I(t,x) = e^{(\beta - \gamma)t}$

Case 1: $\beta - \gamma < 0$, $\lim\limits_{t\to\infty} \dfrac{\ln\left(e^{(\beta-\gamma)t}\right)}{t} = \beta - \gamma$

Case 2: $\beta - \gamma > 0$, $\lim\limits_{t\to\infty} \dfrac{\ln\left(e^{(\beta-\gamma)t}\right)}{t} = \beta - \gamma$

Case 3: $\beta - \gamma = 0$, $m_1^I(t,x) = C_1 \in \mathbb{R}$, $\lim\limits_{t\to\infty} \dfrac{\ln(C_1)}{t} = 0$

For case 1, $\beta - \gamma$ is negative, so the Lyapunov exponent $\lambda_{1,1}$ is negative, and $m_1^I(t,x)$ is decreasing. For case 2, $\lambda_{1,2}$ is positive, and thus $m_1^I(t,x)$ is increasing. For case 3, $\lambda_{1,3}$ is 0 and $m_1^I(t,x)$ is neither increasing nor decreasing.

Theorem 15 *For the second moment of the infected group in homogeneous space, when $x = y$, the Lyapunov Exponents are $\lambda_{2,1} = \beta - \gamma$ when $\beta - \gamma < 0$, $\lambda_{2,2} = \dfrac{2(\beta-\gamma)\rho_0 + 2(\beta+\gamma+2\kappa)}{\rho_0 + \left(\dfrac{\beta+\gamma+2\kappa}{\beta-\gamma}\right)}$ when $\beta - \gamma > 0$, and $\lambda_{2,3} = 0$ when $\beta - \gamma = 0$*

Proof of Theorem 15:

Lyapunov Exponent $\lambda_{2,i} = \lim\limits_{t\to\infty} \dfrac{\ln(m_2^I(t,x,x))}{t}$ where $m_2^I(t,x,x)$ is given by $(12a)$ in Chapter 7 Theorem 9

Case 1: $\beta - \gamma < 0$, using L'Hôpital's Rule we get that:

$$\lim_{t\to\infty} \frac{2(\beta-\gamma)\rho_0 e^{(\beta-\gamma)t} - (\beta+\gamma+2\kappa)\left[1 - 2e^{(\beta-\gamma)t}\right]}{\rho_0 e^{(\beta-\gamma)t} + \left(\dfrac{\beta+\gamma+2\kappa}{\gamma-\beta}\right)\left[1 - e^{(\beta-\gamma)t}\right]} = \beta - \gamma$$

Case 2: $\beta - \gamma > 0$, $\lim\limits_{t\to\infty} \dfrac{2(\beta-\gamma)\rho_0 - (\beta+\gamma+2\kappa)\left[e^{-(\beta-\gamma)t} - 2\right]}{\rho_0 + \left(\dfrac{\beta+\gamma+2\kappa}{\gamma-\beta}\right)\left[e^{-(\beta-\gamma)t} - 1\right]}$

$$\lambda_{2,2} = \frac{2(\beta-\gamma)\rho_0 + 2(\beta+\gamma+2\kappa)}{\rho_0 + \left(\dfrac{\beta+\gamma+2\kappa}{\beta-\gamma}\right)}$$

Case 3: $\beta - \gamma = 0$, $m_2^I(t,x,x) = \rho_0 + (\beta+\gamma+2\kappa)Ct$ where $C \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} = \lim_{t\to\infty} \frac{(\beta+\gamma+2\kappa)C}{\rho_0 + (\beta+\gamma+2\kappa)Ct} = 0$$

For case 1, $\beta - \gamma$ is negative, so the Lyapunov exponent $\lambda_{2,1} = \beta - \gamma$ is negative, and the second moment of the infected group when $x = y$ is decreasing. For case 2, $\lambda_{2,2}$ is positive, and thus $m_2^I(t,x,x)$ is increasing. For case 3, $\lambda_{2,3}$ is 0 and

$m_2^I(t,x,x)$ is neither increasing nor decreasing.

Theorem 16 *For the second moment of the infected group in homogeneous space, when $x \neq y$, the Lyapunov Exponents are $\lambda_{3,1} = \beta - \gamma$ when $\beta - \gamma < 0$, $\lambda_{3,2} = \dfrac{2(\beta-\gamma)\rho_0 - 4\kappa a(v)}{\rho_0 - \left(\dfrac{2\kappa a(v)}{\beta-\gamma}\right)}$ when $\beta - \gamma > 0$, and $\lambda_{3,3} = 0$ when $\beta - \gamma = 0$*

Proof of Theorem 16: Lyapunov Exponent $\lambda_{3,i} = \lim\limits_{t\to\infty} \dfrac{\ln(m_2^I(t,v))}{t}$ where $m_2^I(t,v)$ is given by $(12c)$ in Chapter 7 Theorem 9

Case 1: $\beta - \gamma < 0$, using L'Hôpital's Rule we get that:

$$\lim_{t\to\infty} \frac{2(\beta-\gamma)\rho_0 e^{(\beta-\gamma)t} + (2\kappa a(v))\left[1 - 2e^{(\beta-\gamma)t}\right]}{\rho_0 e^{(\beta-\gamma)t} + \left(\dfrac{2\kappa a(v)}{\beta-\gamma}\right)\left[1 - e^{(\beta-\gamma)t}\right]} = \beta - \gamma$$

Case 2: $\beta - \gamma > 0$, $\lim\limits_{t\to\infty} \dfrac{2(\beta-\gamma)\rho_0 + (2\kappa a(v))\left[e^{-(\beta-\gamma)t} - 2\right]}{\rho_0 + \left(\dfrac{2\kappa a(v)}{\beta-\gamma}\right)\left[e^{-(\beta-\gamma)t} - 1\right]}$

$$\lambda_{3,2} = \frac{2(\beta-\gamma)\rho_0 - 4\kappa a(v)}{\rho_0 - \left(\dfrac{2\kappa a(v)}{\beta-\gamma}\right)}$$

Case 3: $\beta - \gamma = 0$, $m_2^I(t,v) = \rho_0 - 2\kappa a(v)Ct$ where $C \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \lim_{t\to\infty} \frac{-2\kappa a(v)C}{\rho_0 - 2\kappa a(v)Ct} = 0$$

For case 1, $\beta - \gamma$ is negative, so the Lyapunov exponent $\lambda_{3,1} = \beta - \gamma$ is negative, and the second moment of the infected group when $x \neq y$ is decreasing. For case 2, if $\beta - \gamma > 2\kappa a(v)$, then $\lambda_{3,2}$ is positive, and thus $m_2^I(t,v)$ is increasing.

For case 2, if $\beta - \gamma < 2\kappa a(v)$, then $\lambda_{3,2}$ is negative and the second moment of the infected group is decreasing. For case 3, $\lambda_{3,3}$ is 0 and $m_2^I(t,v)$ is neither increasing nor decreasing.

As $t \to \infty$	$\lim_{t\to\infty} \frac{\ln m_1^I(t,x)}{t}$	$\lim_{t\to\infty} \frac{\ln m_2^I(t,x,x)}{t}$	$\lim_{t\to\infty} \frac{\ln m_2^I(t,v)}{t}$
$\beta < \gamma$	$< \infty$	$< \infty$	$< \infty$
$\beta > \gamma$	$< \infty$	$< \infty$	$< \infty$
$\beta = \gamma$	$< \infty$	$< \infty$	$< \infty$

Table 11.1: Lyapunov Exponents in Homogeneous Space

Inhomogeneous Space:

Theorem 17 *For the first moment of the infected group in inhomogeneous space, when $x = y$, the Lyapunov Exponents are $\lambda_{4,1} = \beta - \gamma + \kappa\hat{\mathscr{L}}(k)$ when $\beta - \gamma < 0$, $\lambda_{4,2} = \beta - \gamma + \kappa\hat{\mathscr{L}}(k)$, and $\lambda_{4,3} = \kappa\hat{\mathscr{L}}(k)$ when $\beta - \gamma = 0$*
Note that $\frac{d}{dt}p(t,0,x) = \kappa\hat{\mathscr{L}}(k)p(t,0,x)$

Proof of Theorem 17: Lyapunov Exponent $\lambda_{4,i} = \lim_{t\to\infty} \frac{\ln(m_1^I(t,x))}{t}$ where $m_1^I(t,x) = e^{(\beta - \gamma)t}p(t,0,x)$

Case 1: $\beta - \gamma < 0$, $\lim_{t\to\infty} \frac{(\beta-\gamma)t + \ln(p(t,0,x))}{t} = \beta - \gamma + \lim_{t\to\infty} \frac{\frac{d}{dt}p(t,0,x)}{p(t,0,x)}$

$\lambda_{4,1} = \beta - \gamma + \kappa\hat{\mathscr{L}}(k)$

Case 2: $\beta - \gamma > 0$, $\lim_{t\to\infty} \dfrac{\ln(m_1^I(t,x))}{t} = \beta - \gamma + \kappa\hat{\mathscr{L}}(k)$

Case 3: $\beta - \gamma = 0$, $m_1^I(t,x) = p(t,0,x)$, $\lim_{t\to\infty} \dfrac{\ln(m_1^I(t,x))}{t} = \kappa\hat{\mathscr{L}}(k)$

For case 1, $\beta - \gamma$ is negative and $\kappa\hat{\mathscr{L}}(k)$ is negative, thus $\lambda_{4,1} < 0$ and the first moment of the infected group is decreasing. For case 2, if $\beta - \gamma > \hat{\mathscr{L}}(k)$, then $\lambda_{4,2} > 0$ and the first moment of the infected group is increasing. For case 2, if $\beta - \gamma < \hat{\mathscr{L}}(k)$, then $\lambda_{4,2} < 0$ and then $m_1^I(t,x)$ is decreasing. For case 3, $\lambda_{4,3}$ is negative and $m_1^I(t,x)$ is decreasing.

Theorem 18 *For the second moment of the infected group in inhomogeneous space, when $x = y$, the Lyapunov Exponents are $\lambda_{5,1} = 2(\beta - \gamma)$ when $\beta - \gamma < 0$, $\lambda_{5,2} = 5(\beta - \gamma)$ when $\beta - \gamma > 0$, and $\lambda_{5,3} = 0$ when $\beta - \gamma = 0$*

Proof of Theorem 18: Lyapunov Exponent $\lambda_{5,i} = \lim_{t\to\infty} \dfrac{\ln(m_2^I(t,x,x))}{t}$ where $m_2^I(t,x,x)$ is given by $(12b)$ in Chapter 7 Theorem 9

Case 1: $\beta - \gamma < 0$, using L'Hôpital's Rule and note that $p(t,x,y) < p(t,0,0)$ and as $t \to \infty, p(t,0,0) = \dfrac{C}{t^{d/2}} + o(t^{-d/2}) < \infty$, $C_1 = \dfrac{C}{t^{d/2}} + o(t^{-d/2})$ and $\lim_{t\to\infty} C_1 = 0$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} =$$
$$\frac{\rho_0\kappa\hat{\mathscr{L}}(k)C_1 + 2(\beta-\gamma)\rho_0 C_1 - \Big(\dfrac{2\kappa}{3}\Big) - \Big(\dfrac{\beta+\gamma+2\kappa}{3(\beta-\gamma)}\Big)\kappa\hat{\mathscr{L}}(k)C_1 - \Big(\dfrac{2(\beta+\gamma+2\kappa)}{3}\Big)C_1}{\rho_0 C_1 - \Big(\dfrac{\kappa}{3(\beta-\gamma)}\Big) - \Big(\dfrac{\beta+\gamma+2\kappa}{3(\beta-\gamma)}\Big)C_1}$$

Thus $\lim \dfrac{\ln(m_2^I(t,x,x))}{t} = 2(\beta-\gamma)$

Case 2: $\beta-\gamma>0$

$$\lim_{t\to\infty}\frac{\ln(m_2^I(t,x,x))}{t} = \frac{\Big(\frac{5\kappa}{3}\Big) + \Big(\frac{\beta+\gamma+2\kappa}{3(\beta-\gamma)}\Big)\kappa\hat{\mathscr{L}}(k)C_1 + \Big(\frac{5(\beta+\gamma+2\kappa)}{3}\Big)C_1}{\Big(\frac{\kappa}{3(\beta-\gamma)}\Big) + \Big(\frac{\beta+\gamma+2\kappa}{3(\beta-\gamma)}\Big)C_1}$$

Thus $\lim \dfrac{\ln(m_2^I(t,x,x))}{t} = 5(\beta-\gamma)$

Case 3: $\beta-\gamma=0$, $m_2^I(t,x,x) = \rho_0 p(t,0,x) + \kappa t + (\beta+\gamma+2\kappa)tp(t,0,x)$

$$\lim_{t\to\infty}\frac{\ln(m_2^I(t,x,x))}{t} = \lim_{t\to\infty}\frac{(\beta+\gamma+2\kappa)\kappa\hat{\mathscr{L}}(k)p(t,0,x)}{\kappa+(\beta+\gamma+2\kappa)p(t,0,x)}$$

$$\lim_{t\to\infty}\frac{\ln(m_2^I(t,x,x))}{t} = \frac{(\beta+\gamma+2\kappa)\kappa\hat{\mathscr{L}}(k)\Big(\frac{C}{t^{d/2}}+o(t^{-d/2})\Big)}{\kappa+(\beta+\gamma+2\kappa)\Big(\frac{C}{t^{d/2}}+o(t^{-d/2})\Big)} = \frac{0}{\kappa} = 0$$

For case 1, $\beta-\gamma$ is negative and therefore the second moment of the infected group when $x=y$ is decreasing. For case 2, $\beta-\gamma$ is positive, then the second moment of the infected group is increasing. For case 3, $\lambda_{5,3}=0$ and the second moment of the infected group is neither increasing nor decreasing.

Theorem 19 *For the second moment of the infected group in inhomogeneous space, when $x \neq y$, the Lyapunov Exponents are*
$\lambda_{6,1} = \dfrac{\rho_0\kappa\hat{\mathscr{L}}(k) + 2(\beta-\gamma)\rho_0 + \Big(\frac{2\kappa a(v)}{3(\beta-\gamma)}\Big)\kappa\hat{\mathscr{L}}(k) + \Big(\frac{4\kappa a(v)}{3}\Big)}{\rho_0 + \frac{2\kappa a(v)}{3(\beta-\gamma)}}$ *when* $\beta-\gamma<0$,
$\lambda_{6,2} = \kappa\hat{\mathscr{L}}(k) + 5(\beta-\gamma)$ *when* $\beta-\gamma>0$, *and* $\lambda_{6,3} = \kappa\hat{\mathscr{L}}(k)$ *when* $\beta-\gamma=0$

Proof of Theorem 19: Lyapunov Exponent $\lambda_{6,i} = \lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t}$ where $m_2^I(t,v)$ is given by $(12d)$ in Chapter 7 Theorem 9

Case 1: $\beta - \gamma < 0$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \frac{\rho_0\kappa\hat{\mathscr{L}}(k) + 2(\beta-\gamma)\rho_0 + \left(\frac{2\kappa a(v)}{3(\beta-\gamma)}\right)\kappa\hat{\mathscr{L}}(k) + \left(\frac{4\kappa a(v)}{3}\right)}{\rho_0 + \frac{2\kappa a(v)}{3(\beta-\gamma)}}$$

Case 2: $\beta - \gamma > 0$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \lim_{t\to\infty} \frac{-\left(\frac{2\kappa a(v)}{3(\beta-\gamma)}\right)\kappa\hat{\mathscr{L}}(k)p(t,0,x) - \left(\frac{10\kappa a(v)}{3}\right)p(t,0,x)}{-\left(\frac{2\kappa a(v)}{3(\beta-\gamma)}\right)p(t,0,x)}$$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \kappa\hat{\mathscr{L}}(k) + 5(\beta-\gamma)$$

Case 3: $\beta - \gamma = 0$, $m_2^I(t,v) = \rho_0 p(t,0,x) - 2\kappa a(v)tp(t,0,x)$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \frac{\rho_0\kappa\hat{\mathscr{L}}(k)\frac{1}{t} - 2\kappa a(v)\kappa\hat{\mathscr{L}}(k) - 2\kappa a(v)\frac{1}{t}}{\rho_0\frac{1}{t} - 2\kappa a(v)} = \kappa\hat{\mathscr{L}}(k)$$

For case 1, $\beta-\gamma$ is negative and if $\rho_0(\beta-\gamma)+2a(v) < 0$, then $\lambda_{6,1} > 0$ and the second moment of the infected group when $x \neq y$ is increasing. For case 1, $\beta - \gamma$ if $\rho_0(\beta-\gamma) + 2a(v) > 0$, then $\lambda_{6,1}$ is negative and therefore $m_2^I(t,v)$ is decreasing. For case 2, $\beta - \gamma$ is positive and $\kappa\hat{\mathscr{L}}(k) < 0$, if $5(\beta-\gamma) > \kappa\hat{\mathscr{L}}(k)$, then $\lambda_{6,2} > 0$ and the second moment of the infected group is increasing. For case 2, if $5(\beta-\gamma) < \kappa\hat{\mathscr{L}}(k)$, then $\lambda_{6,2} < 0$ and thus $m_2^I(t,v)$ is decreasing. For case 3, $\lambda_{6,3} < 0$ and the second moment of the infected group is decreasing.

As $t \to \infty$	$\lim\limits_{t\to\infty} \dfrac{\ln m_1^I(t,x)}{t}$	$\lim\limits_{t\to\infty} \dfrac{\ln m_2^I(t,x,x)}{t}$	$\lim\limits_{t\to\infty} \dfrac{\ln m_2^I(t,v)}{t}$
$\beta < \gamma$	$< \infty$	$< \infty$	$< \infty$
$\beta > \gamma$	$< \infty$	$< \infty$	$< \infty$
$\beta = \gamma$	$< \infty$	$< \infty$	$< \infty$

Table 11.2: Lyapunov Exponents in Inhomogeneous Space

For our SIR model, the homogeneous space we have that the Lyapunov exponents for the first and second moments are finite for all three cases, when $\beta < \gamma$, $\beta > \gamma$, and $\beta = \gamma$. For the inhomogeneous space we have that the Lyapunov exponents for the first and second moments are finite for all three cases, when $\beta < \gamma$, $\beta > \gamma$, and $\beta = \gamma$. If $\lambda_{1,i}$ or $\lambda_{4,i}$ is positive, then the first moment of the infected group is increasing. If $\lambda_{1,i}$ or $\lambda_{4,i}$ is negative, then the first moment of the infected group is decreasing. If the Lyapunov exponent is 0, then the $m_1^I(t,x)$ is neither increasing nor decreasing.

CHAPTER 12
INTRODUCTION TO THE SI MODEL

The set up for the new susceptible-infected (SI) model with migration has the assumption that all of the particles can have spatial motion, within the susceptible and infected groups (in addition to the inter-compartmental motion). We are assuming the total population is fixed, i.e- $N(t) = S(t) + I(t)$ but $N(t, x)$, the total population at position x at time t, is varying. We are now assuming that the spatial motion of healthy particles is not the same as the spatial motion of an infected particle, and now there are two probability kernels- $a(z)$ and $b(z)$. Define κ as the probability of a jump during the time period $(t, t + dt)$ and β is the transition rate from S to I. Define $a(z)$ is the probability kernel of the Poisson process (or the process intensity) and it determines the direction of a jump in the infected group, $b(z)$ is the probability kernel of the susceptible group, where $a(z) = a(-z)$ and $b(z) = b(-z)$. The movement of one particle from location x to location $x + z$ has probability kernels $a(z)dt$ or $b(z)dt$. The movement from location $x + z$ to location x has probability $a(-z)dt$ or $b(-z)dt$. Assume that $\sum\limits_{z \in \mathbb{Z}^d} a(z) = 0$ and $\sum\limits_{z \neq 0} a(z) = 1$, which implies that $a(0) = -1$, and similarly for $b(z)$. However, we keep the assumption that the only one type of movement can happen at a time, meaning a particle can jump to another location or they can jump states. The possible events are:

① $S : x \longrightarrow x + z$ in S with probability $\kappa b(z)dt$.

This is the event that in a short time period $(t, t + dt)$, a particle at location x moves to location $x + z$ within the susceptible group.

② $I : x \longrightarrow x + z$ in I with probability $\kappa a(z)dt$.

This is the event that in a short time period $(t, t + dt)$, a particle at location x moves to location $x + z$ within the infected group.

③ $S : x \longrightarrow x$ in I with probability βdt.

This is the event that a particle at location x in the susceptible group transitions to the infected group.

CHAPTER 13

DERIVING THE DIFFERENTIAL EQUATIONS FOR THE SI MODEL

We first want to determine the stability of $E[S(t,x)]$ and $E[I(t,x)]$ as $t \longrightarrow \infty$. The first step in doing this is to derive the differential equations for the generating functions and find the solutions. We derive the differential equations using the Kolmogorov Forward Equations.

Theorem 20 (Differential equations of the first moments) *The differential equations for the first moment of the susceptible and infected groups are*

$$\begin{cases} \dfrac{\partial E[S(t,x)]}{\partial t} = \kappa \mathscr{L}_{b(z)} E[S(t,x)] - \beta E[I(t,x)] & (13a) \\ \dfrac{\partial E[I(t,x)]}{\partial t} = \kappa \mathscr{L}_{a(z)} E[I(t,x)] + \beta E[I(t,x)] & (13b) \end{cases}$$

Note that the discrete Laplace operator is defined to be $\mathscr{L}_{a(z)} f(t,x) = \sum_{z \neq 0} a(z)[f(t,x+z) - f(t,x)]$ *and* $\mathscr{L}_{b(z)} f(t,x) = \sum_{z \neq 0} b(z)[f(t,x+z) - f(t,x)]$.

Proof:

For the susceptible group: $S(t+dt,x) = S(t,x) + \xi(dt)$ where

$$\xi(dt) = \begin{cases} 1 & w.p & \sum_{z \neq 0} \kappa b(z) S(t,x+z) dt & ① \\ -1 & w.p & \sum_{z \neq 0} \kappa b(z) S(t,x) dt + \beta I(t,x) dt & ② \\ 0 & w.p & 1 - ① - ② \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x + z$ in the susceptible group moves to location x, meaning location x gains a particle, and thus the event has probability $\sum_{z\neq 0} S(t, x + z)\kappa b(z)dt$. The case that $\xi(dt) = -1$ is the event that either a particle at location x moves to location $x + z$ (meaning that location x loses a particle - which has probability $\sum_{z\neq 0} S(t, x)\kappa b(z)dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t, x)\beta dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the susceptible group, so it has probability $1 - \sum_{z\neq 0} S(t, x + z)\kappa b(z)dt - \sum_{z\neq 0} S(t, x)\kappa b(z)dt - I(t, x)\beta dt$.

$$E[S(t + dt, x)] = E[E[S(t + dt, x)|\mathscr{F}_t]]$$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and our Laplace operator defined as $\mathscr{L}_{b(z)}f(t, x) = \sum_{z\neq 0} b(z)[f(t, x + z) - f(t, x)]$ and note that $m_1^S(t, x) = E[S(t, x)]$ we have that

$$\text{As } dt \longrightarrow 0\ ,\ \frac{\partial E[S(t, x)]}{\partial t} = \frac{\partial m_1^S(t, x)}{\partial t} = \kappa \mathscr{L}_{b(z)} m_1^S(t, x) - \beta m_1^I(t, x).$$

For the infected group: $I(t + dt, x) = I(t, x) + \xi(dt)$ where

$$\xi(dt) = \begin{cases} 1 & w.p \quad \beta I(t, x)dt + \sum_{z\neq 0} \kappa a(-z)I(t, x + z)dt \qquad (1) \\ -1 & w.p \quad \sum_{z\neq 0} \kappa a(z)I(t, x)dt \qquad (2) \\ 0 & w.p \quad 1 - (1) - (2) \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x + z$ in the infected group moves to location x, meaning location x gains a particle (which has probability $\sum_{z \neq 0} I(t, x+z)\kappa a(z) dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t, x)\beta dt$). The case that $\xi(dt) = -1$ is the event that a particle at location x in the infected group moves to location $x + z$ (meaning that location x loses a particle - which has probability $\sum_{z \neq 0} I(t, x)\kappa a(z) dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the infected group, so it has probability $1 - I(t, x)\beta dt - \sum_{z \neq 0} I(t, x+z)\kappa a(-z) dt - \sum_{z \neq 0} I(t, x)\kappa a(z) dt$.

$$E[I(t + dt, x)] = E[E[I(t + dt, x)|\mathscr{F}_t]]$$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and we have that

$$\text{As } dt \longrightarrow 0 \text{ , } \frac{\partial m_1^I(t, x)}{\partial t} = \kappa \mathscr{L}_{a(z)} m_1^I(t, x) + \beta m_1^I(t, x)$$

$\square$

Now that we have the differential equations for the first moments of the susceptible and infected groups, we can solve for the first moments.

CHAPTER 14
FIRST MOMENTS OF THE SI MODEL

When solving for the first moments of the susceptible and infected groups there are two sub-cases: homogeneous space and inhomogeneous space. For the homogeneous space there is the assumption the the spaces are equivalent, meaning x and $x+z$ are the same. Thus the Laplace operator $\mathscr{L}_{a(z)}f(t,x) = \sum_{z\neq 0} a(z)[f(t,x+z) - f(t,x)] = 0$, $\mathscr{L}_{b(z)}f(t,x) = \sum_{z\neq 0} b(z)[f(t,x+z) - f(t,x)] = 0$ and $\mathscr{L}_{a(z)+b(z)}f(t,x) = \sum_{z\neq 0}(a(z)+b(z))[f(t,x+z) - f(t,x)]$. For the inhomogeneous space there is the assumption that the spaces are not equivalent, so the Laplace operator $\mathscr{L}_i f(t,x) \neq 0$ where $i = a, b$

14.1 First Moments in Homogeneous space

The Laplace operator $\mathscr{L}_a f(t,x) = \sum_{z\neq 0} a(z)[f(t,x+z) - f(t,x)] = 0$ and $\mathscr{L}_b f(t,x) = \sum_{z\neq 0} b(z)[f(t,x+z) - f(t,x)] = 0$. Then the differential equations from Chapter 13 Equation (13a) and (13b) no longer have the Laplace operator and we can rewrite the expectations as the first moments and so the equations become:

$$\begin{cases} \dfrac{\partial m_1^S(t,x)}{\partial t} = -\beta m_1^I(t,x) \\ \dfrac{\partial m_1^I(t,x)}{\partial t} = \beta m_1^I(t,x) \end{cases} \tag{14}$$

Theorem 21 *In the homogeneous space, as* t *increases,* $0 \leq t \leq \frac{\ln(\rho_0+1)}{\beta}$, *with initial conditions* $S(0) = \rho_0 > 0$, $I(0) = 1$, $I(0,x) = \begin{cases} 1 \;\; if \;\; x = y \\ 0 \;\; if \;\; x \neq y \end{cases}$, *the steady states* $E[S(t,x)]$ *and* $E[I(t,x)]$ *exist and, when* $\beta \neq 0$, $\begin{cases} E[S(t,x)] = \rho_0 + 1 - e^{\beta t} \\ E[I(t,x)] = e^{\beta t} \end{cases}$

Proof of Theorem 21: Solving the ODE system (14) using regular ODE methods, we get the solutions for the first moments of the SI model.

14.2 First Moments in Inhomogeneous space

The Laplace operator $\mathscr{L}_{a(z)} f(t,x) = \sum\limits_{z \neq 0} a(z)[f(t,x+z) - f(t,x)] \neq 0$ and $\mathscr{L}_{b(z)} f(t,x) = \sum\limits_{z \neq 0} b(z)[f(t,x+z) - f(t,x)] \neq 0$ so we have the differential equations $(13a)$ and $(13b)$. In order to solve the inhomogeneous equations, we are going to use the definitions, lemmas, and theorems from Section 4.2 in Chapter 4.

Applying the Fourier transform from Definition 2 to the differential equation of $E[I(t,x)]$ to solve for the first moment of the infected group, and following the same process we used for the SIR model, we get that:

$$m_1^I(t,x) = \frac{1}{(2\pi)^d} \int_{T^d} \hat{m}_1^I(t,k) e^{-ikx} dk = \frac{1}{(2\pi)^d} \int_{T^d} e^{[\kappa \hat{\mathscr{L}}_{a(z)}(k) + \beta]t} e^{-ikx} dk$$

Now to solve for the first moments of the susceptible group in the inhomogeneous space we use the general solution from Theorem 3, where $f(t,x) = -\beta E[I(t,x)]$ but there is no potential $V(t)$ in these equations so then we have the solution to be $E[S(t,x)] = E^h[S(t,x)] + w(t,x)$ where

$$\begin{cases} \dfrac{\partial m_1^{S^h}(t,x)}{\partial t} = \kappa \mathscr{L}_{b(z)} m_1^{S^h}(t,x) + V(t) m_1^{S^h}(t,x) = \kappa \mathscr{L}_{b(z)} m_1^{S^h}(t,x) \\ m_1^{S^h}(0,x) = \rho_0 \delta_0(x) \end{cases}$$

$$\begin{cases} \dfrac{\partial w(t,x)}{\partial t} = \kappa \mathscr{L}_{b(z)} w(t,x) + V(t) w(t,x) + f(t,x) = \kappa \mathscr{L}_{b(z)} w(t,x) + f(t,x) \\ w(0,x) = 0 \end{cases}$$

$$m_1^S(t,x) = \frac{1}{(2\pi)^d} \int_{T^d} \rho_0 e^{\kappa \hat{\mathscr{L}}_{b(z)}(k)t} e^{-ikx} dk$$

$$-\beta \int_0^t \sum_{z \in \mathbb{Z}^d} p_b(t-s, 0, x-z) m_1^I(s,z) ds$$

14.3 First Moments in Inhomogeneous Space using matrices

To write the first moments of the susceptible and infected groups in matrix format we are going to let $F(t,x) = \begin{bmatrix} S(t,x) \\ I(t,x) \end{bmatrix}$ and then $m_1^F(t,x) = \begin{bmatrix} m_1^S(t,x) \\ m_1^I(t,x) \end{bmatrix}$. The Fourier transform of the matrix format of Equation $(13a)$ and $(13b)$ from Chapter 13 Theorem 20, is $\dfrac{\partial \hat{m}_1^F(t,k)}{\partial t} = \hat{A}_4 \hat{m}_1^F(t,k)$, where the matrix $\hat{A}_4 = \begin{bmatrix} \kappa \hat{b}(k) & -\beta \\ 0 & \kappa \hat{a}(k) + \beta \end{bmatrix}$ the eigenvalues are $\lambda_1 = \kappa \hat{a}(k) + \beta$ and $\lambda_2 = \kappa \hat{b}(k)$.

Case 1: $\kappa \hat{a}(k) + \beta = \kappa \hat{b}(k)$

Then $\lambda_1 = \lambda_2 = \kappa\hat{b}(k)$ multiplicity 2 and $(\hat{A}_4 - \kappa\hat{a}(k)I)^2 = 0$.

$$e^{\hat{A}_4 t} = \begin{bmatrix} e^{\kappa\hat{b}(k)t} & -\beta t e^{\kappa\hat{b}(k)t} \\ 0 & e^{\kappa\hat{b}(k)t} \end{bmatrix} \text{ with initial conditions } x_0 = \begin{bmatrix} \rho_0 & 1 \end{bmatrix}^T$$

$$\hat{m}_1^F(t,k) = \begin{bmatrix} \hat{m}_1^S(t,k) \\ \hat{m}_1^I(t,k) \end{bmatrix} = e^{\hat{A}_4 t} x_o = \begin{bmatrix} \rho_0 e^{\kappa\hat{b}(k)t} - \beta t e^{\kappa\hat{b}(k)t} \\ e^{\kappa\hat{b}(k)t} \end{bmatrix} \implies$$

$$m_1^F(t,x) = \begin{bmatrix} m_1^S(t,x) \\ m_1^I(t,x) \end{bmatrix} = \begin{bmatrix} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(\rho_0 e^{\kappa\hat{b}(k)t} - \beta t e^{\kappa\hat{b}(k)t}\right) e^{-ikx} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(e^{\kappa\hat{b}(k)t}\right) e^{-ikx} dk \end{bmatrix}$$

Case 2: $\kappa\hat{a}(k) + \beta \neq \kappa\hat{b}(k)$

$\lambda_1 = \kappa\hat{b}(k)$ has multiplicity 1 and $\lambda_2 = \kappa\hat{a}(k) + \beta$ has multiplicity 1. Then

$$X(t) = \begin{bmatrix} e^{(\kappa\hat{a}(k) + \beta)t} & e^{\kappa\hat{b}(k)t} \\ \left(\frac{\kappa\hat{b}(k) - \kappa\hat{a}(k) - \beta}{\beta}\right) e^{(\kappa\hat{a}(k) + \beta)t} & 0 \end{bmatrix}$$

and $X(0) = \begin{bmatrix} 1 & 1 \\ \left(\frac{\kappa\hat{b}(k) - \kappa\hat{a}(k) - \beta}{\beta}\right) & 0 \end{bmatrix}$, $X^{-1}(0) = \begin{bmatrix} 0 & \left(\frac{-\beta}{\kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)}\right) \\ 1 & \left(\frac{\beta}{\kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)}\right) \end{bmatrix}$

$e^{\hat{A}_4 t} = X(t)X^{-1}(0) = \begin{bmatrix} e^{\kappa\hat{b}(k)t} & \frac{-\beta e^{(\kappa\hat{a}(k) + \beta)t}}{\kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)} + \frac{\beta e^{(\kappa\hat{b}(k))t}}{\kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)} \\ 0 & -e^{(\kappa\hat{a}(k) + \beta)t} \end{bmatrix}$ with initial conditions $x_0 = [\rho_0 \;\; 1 \;\; 0]^T$

$$\hat{m}_1^F(t,k) = \begin{bmatrix} \rho_0 e^{\kappa\hat{b}(k)t} + \left(\frac{-\beta}{\kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)}\right)\left[e^{(\kappa\hat{a}(k) + \beta)t} - e^{\kappa\hat{b}(k)t}\right] \\ -e^{(\kappa\hat{a}(k) + \beta)t} \end{bmatrix}$$

$$m_1^F(t,x) = \begin{bmatrix} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(\rho_0 e^{\kappa\hat{b}(k)t} + \left(\frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)\left[e^{(\kappa\hat{a}(k)+\beta)t} - e^{\kappa\hat{b}(k)t}\right]\right) e^{-ikx} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \left(-e^{(\kappa\hat{a}(k)+\beta)t}\right) e^{-ikx} dk \end{bmatrix}$$

CHAPTER 15
ANALYZING THE BEHAVIOR OF THE FIRST MOMENTS OF THE SI MODEL

Now that we have solved for the first moments of the susceptible and infected groups, we want to analyze the long term behavior of the first moments and there are 2 cases: homogeneous space and inhomogeneous space.

15.1 Analyzing the behavior of the first moments in Homogeneous Space

The behavior of the susceptible and infected groups in the homogeneous space as $t \longrightarrow \infty$ can be broken up in 2 cases: $\beta = 0, \beta > 0$

As $t \to \infty$	$m_1^S(t,x) \to$	$m_1^I(t,x) \to$
$\beta = 0$	$\to 0$	$\to C_1 \in \mathbb{R}$
$\beta > 0$	$\to 0$	$\to \infty$

Table 15.1: Asymptotic Behavior of the First Moments in Homogeneous Space

As $t \to \infty$	$m_1^S(t,x) \to$	$m_1^I(t,x) \to$
$\beta \geq 0,\ \alpha = 0,\ \phi < 0$	$\to 0$	$\to 0$
$\beta = 0,\ \alpha = 0,\ \phi = 0$	$\to C_2$	$\to C_2$
$\beta > 0,\ \alpha = 0,\ \phi = 0$	$\to 0$	$\to C_2$
$\alpha < 0,\ \theta < 0,\ \phi < 0$	$\to 0$	$\to 0$
$\alpha < 0,\ \theta < 0,\ \phi = 0$	$\to C_3$	$\to 0$
$\alpha < 0,\ \theta > 0,\ \phi < 0$	$\to 0$	$\to 0$
$\alpha > 0,\ \theta > 0,\ \phi = 0$	$\to 0$	$\to 0$
$\alpha > 0,\ \theta > 0,\ \phi < 0$	$\to 0$	$\to 0$

Table 15.2: Asymptotic Behavior of the First Moments in Inhomogeneous Space

15.2 Analyzing the behavior of the first moments in Inhomogeneous Space

Now we want to analyze the long term behavior as $t \longrightarrow \infty$, with $\kappa > 0, \hat{\mathscr{L}}(k) \leq 0$. Table 15.2 summarizes the asymptotic behavior of the first moments of the susceptible and infected groups as $t \longrightarrow \infty$. Let $\alpha = \kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)$, $\theta = \kappa\hat{a}(k) + \beta$, $\phi = \kappa\hat{b}(k)$, $C_2 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{-ikx}dk$ and
$C_3 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d}\Big(\rho_0 + \dfrac{\beta}{\kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)}\Big)e^{-ikx}dk$.

When $\beta \geq 0, \alpha = 0$ and $\phi = \kappa\hat{b}(k) < 0$, we have the event that the mobility effect for the infected group plus the infection rate is equal to the mobility effect of the susceptible group, but the mobility effect of the susceptible group is negative. The result is that the expected value of the susceptible and infected populations goes to 0 as time t goes to infinity. If the first moment decreases to negative infinity, as t goes to infinity, in this case, once the first moment hits the state 0, it will stay in state 0 forever. Another interesting event is when $\alpha > 0$ (meaning the mobility effect of the infected group plus the infection rate is greater than the mobility effect of the susceptible group) but $\theta > 0$ and $\phi = 0$ (meaning the mobility effect of the infected group plus the infection rate is positive, and the mobility effect of the susceptible group is 0), then the expected value of the infected population at location x goes to 0 as time t goes to infinity.

CHAPTER 16

DERIVING THE DIFFERENTIAL EQUATIONS OF THE SECOND MOMENTS OF THE SI MODEL

Now we want to find the differential equations for the second moments for the $S(t,x), I(t,x)$, and $S(t,x)I(t,x)$, groups, where each group has 2 cases: when the locations $x = y$ and when the locations $x \neq y$. The differential equations for the second moments will then be solved to find the second moments, which will be used in determining the variance and the long term behavior of the susceptible, and infected groups. Note a few operator definitions:

$$\begin{aligned}
\mathscr{L}_{a(z)} f(t,x) &= \sum_{z\neq 0} a(z)(f(t,x+z) - f(t,x)) \\
\mathscr{L}_{b(z)} f(t,x) &= \sum_{z\neq 0} b(z)(f(t,x+z) - f(t,x)) \\
\mathscr{L}_{a(z)+b(z)} f(t,x) &= \sum_{z\neq 0} (a(z)+b(z))(f(t,x+z) - f(t,x)) \\
\mathscr{L}_{a(z),x} f(t,x,y) &= \sum_{z\neq 0} a(z)(f(t,x+z,y) - f(t,x,y)) \\
\mathscr{L}_{b(z),x} f(t,x,y) &= \sum_{z\neq 0} b(z)(f(t,x+z,y) - f(t,x,y)) \\
\mathscr{L}_{a(z),y} f(t,x,y) &= \sum_{z\neq 0} a(z)(f(t,x,y+z) - f(t,x,y)) \\
\mathscr{L}_{b(z),y} f(t,x,y) &= \sum_{z\neq 0} b(z)(f(t,x,y+z) - f(t,x,y)) \\
\mathscr{L}_{a(z),S} E[S(t,x)I(t,x)] &= \sum_{z\neq 0} a(z)E[I(t,x)S(t,x+z) - I(t,x)S(t,x)] \\
\mathscr{L}_{b(z),S} E[S(t,x)I(t,x)] &= \sum_{z\neq 0} b(z)E[I(t,x)S(t,x+z) - I(t,x)S(t,x)] \\
\mathscr{L}_{a(z),I} E[S(t,x)I(t,x)] &= \sum_{z\neq 0} a(z)E[S(t,x)I(t,x+z) - S(t,x)I(t,x)]
\end{aligned}$$

$$\mathscr{L}_{b(z),I}E[S(t,x)I(t,x)] = \sum_{z\neq 0} b(z)E[S(t,x)I(t,x+z) - S(t,x)I(t,x)]$$
$$v = ||y-x||$$

Theorem 22 *The differential equations for the second moment of the susceptible group are*

$$\begin{cases} \text{Susceptible when } x = y: \\ \dfrac{\partial m_2^S(t,x,x)}{\partial t} = 2\kappa\mathscr{L}_{b(z),x}m_2^S(t,x,x) + \kappa\mathscr{L}m_1^S(t,x) - 2\beta m_2^{SI}(t,x,x) + 2\kappa m_1^S(t,x) \\ +\beta m_1^I(t,x) \qquad (15a) \\ \text{Susceptible when } x \neq y: \\ \dfrac{\partial m_2^S(t,v)}{\partial t} = \kappa\mathscr{L}_{b(z),x}m_2^S(t,v) + \kappa\mathscr{L}_{b(z),y}m_2^S(t,v) - 2\beta m_2^{SI}(t,v) \\ -\kappa b(v)m_1^S(t,x) - \kappa b(v)m_1^S(t,y) \qquad (15b) \end{cases}$$

Proof of Theorem 22: For the susceptible group, when deriving the differential equations for the second moment, we are going to use a similar method to the one used for the first moment- the Kolmogorov Forward Equations. There are 2 cases: when the locations are equivalent and $x = y$, and when the locations are different and $x \neq y$:

Case 1: $S(t+dt,x)$ when $x = y$, then $m_2(t+dt,x,y) = E[S^2(t+dt,x,x)]$
For the second moment when $x = y$ we have that $E[S^2(t+dt,x)] = E[(S(t,x) + \xi(dt))^2]$ where

$$\xi(dt) = \begin{cases} 1 & w.p \quad \sum_{z\neq 0} S(t,x+z)\kappa b(z)dt \quad ① \\ -1 & w.p \quad \sum_{z\neq 0} S(t,x)\kappa b(z)dt + I(t,x)\beta dt \quad ② \\ 0 & w.p \quad 1 - ① - ② \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x+z$ in the susceptible group moves to location x, meaning location x gains a particle, and thus the event has probability $\sum_{z\neq 0} S(t,x+z)\kappa b(z)dt$. The case that $\xi(dt) = -1$ is the event that either a particle at location x moves to location $x+z$ (meaning that location x loses a particle - which has probability $\sum_{z\neq 0} S(t,x)\kappa b(z)dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t,x)\beta dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the susceptible group, so it has probability $1 - \sum_{z\neq 0} S(t,x+z)\kappa b(z)dt - \sum_{z\neq 0} S(t,x)\kappa b(z)dt - I(t,x)\beta dt$.

$E[S^2(t+dt,x)] = E[E[S^2(t+dt,x)|\mathscr{F}(t)]] = E[S^2(t,x)] + E[2S(t,x)\xi(dt)] + E[\xi^2(dt)]$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and we have that

$\frac{\partial m_2^S(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_{b(z)} m_2^S(t,x,x) + \kappa \mathscr{L} m_1^S(t,x) - 2\beta m_2^{SI}(t,x,x) + 2\kappa m_1^S(t,x) + \beta m_1^I(t,x)$

Case 2: $S(t{+}dt, x)$ when $x \neq y$, then $m_2(t{+}dt, x, y) = E[S(t{+}dt, x)S(t{+}dt, y)]$

Recall that during $(t, t + dt)$ only one event can happen, either a particle can move or it can jump states, therefore there are seven combinations for x and y. For example, $(x = 1, y = -1)$, $(x = 1, y = 0)$, so on and so forth. The probabilities for the various combinations of $\xi(dt, x)$ and $\xi(dt, y)$ are:

The event that a particle in the susceptible group goes from y to x:

$P(\xi(dt, x) = 1, \xi(dt, y) = -1) = \kappa b(x - y)S(t, y)dt$

The event that a particle in the susceptible group goes from $x + z$ to x but not to y:

$P(\xi(dt, x) = 1, \xi(dt, y) = 0) = \kappa \sum\limits_{z \neq 0} b(z)S(t, x + z)dt - \kappa b(x - y)S(t, y)dt$

The event that a particle in the susceptible group goes from x to y:

$P(\xi(dt, x) = -1, \xi(dt, y) = 1) = \kappa b(x - y)S(t, x)dt$

The event that a particle in the susceptible group goes from x to $x + z$ but not to y or a particle at location x transitions $S \to I$:

$P(\xi(dt, x) = -1, \xi(dt, y) = 0) = \kappa \sum\limits_{z \neq 0} b(z)S(t, x)dt - \kappa b(x - y)S(t, x)dt + \beta I(t, x)dt$

The event that a particle in the susceptible group goes from $y + z$ to y but not to x:

$P(\xi(dt, x) = 0, \xi(dt, y) = 1) = \kappa \sum\limits_{z \neq 0} b(z)S(t, y + z)dt - \kappa b(x - y)S(t, x)dt$

The event that a particle in the susceptible group goes from y to $y + z$ but not to x or a particle at location y transitions $S \to I$:

$P(\xi(dt, x) = 0, \xi(dt, y) = -1) = \kappa \sum\limits_{z \neq 0} b(-z)S(t, y)dt - \kappa b(x - y)S(t, y)dt + \beta I(t, y)dt$

The event that no particle moves:

$P(\xi(dt, x) = 0, \xi(dt, y) = 0) = 1 - \kappa \sum\limits_{z \neq 0} b(z)S(t, y + z)dt - \kappa \sum\limits_{z \neq 0} b(z)S(t, y)dt - \kappa \sum\limits_{z \neq 0} b(z)S(t, x + z)dt - \kappa \sum\limits_{z \neq 0} b(z)S(t, x)dt + \kappa b(x - y)S(t, y)dt + \kappa b(x - y)S(t, x)dt -$

$\beta I(t,x)dt - \beta I(t,y)dt$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and we have that as $dt \longrightarrow 0$,

$$\frac{\partial m_2^S(t,v)}{\partial t} = \kappa \mathscr{L}_{b(z),x} m_2^S(t,v) + \kappa \mathscr{L}_{b(z),y} m_2^S(t,v) - 2\beta m_2^{SI}(t,v) - \kappa b(v) m_1^S(t,x) - \kappa b(v) m_1^S(t,y)$$

Theorem 23 *The differential equations for second moment of the infected group are*

$$\begin{cases} \text{Infected when } x = y : \\ \dfrac{\partial m_2^I(t,x,x)}{\partial t} = 2\kappa \mathscr{L}_{a(z)} m_2^I(t,x,x) + \kappa \mathscr{L} m_1^I(t,x) + 2\beta m_2^I(t,x,x) + 2\kappa m_1^I(t,x) \\ +\beta m_1^I(t,x) \qquad (16a) \\ \text{Infected when } x \neq y : \\ \dfrac{\partial m_2^I(t,v)}{\partial t} = \kappa \mathscr{L}_{a(z),x} m_2^I(t,v) + \kappa \mathscr{L}_{a(z),y} m_2^I(t,v) + 2\beta m_2^I(t,v) \\ -\kappa a(v) m_1^I(t,x) - \kappa a(v) m_1^I(t,y) \qquad (16b) \end{cases}$$

Proof of Theorem 23: For the infected group, when deriving the differential equations of the second moment, it is a very similar process to the one used for the susceptible group. There are 2 cases: when $x = y$ and when $x \neq y$:

Case 1: $I(t+dt,x)$ when $x = y$, then $m_2(t+dt,x,y) = E[I^2(t+dt,x,x)]$

For the second moment when $x = y$ we have that,

$E[I^2(t+dt,x)] = E[(I(t,x) + \xi(dt))^2]$ where

$$\xi(dt) = \begin{cases} 1 \qquad w.p \qquad \beta I(t,x)dt + \sum\limits_{z\neq 0} I(t,x+z)\kappa a(-z)dt \qquad ① \\ -1 \qquad w.p \qquad \sum\limits_{z\neq 0} I(t,x)\kappa a(z)dt \qquad ② \\ 0 \qquad w.p \qquad 1 - ① - ② \end{cases}$$

The case that $\xi(dt) = 1$ is the event that a particle at location $x + z$ in the infected group moves to location x, meaning location x gains a particle (which has probability $\sum\limits_{z\neq 0} I(t,x+z)\kappa a(z)dt$), or a particle at location x in the susceptible group becomes infected (which has probability $I(t,x)\beta dt$). The case that $\xi(dt) = -1$ is the event that either a particle at location x in the infected group moves to location $x + z$ (meaning that location x loses a particle - which has probability $\sum\limits_{z\neq 0} I(t,x)\kappa a(z)dt$). The case that $\xi(dt) = 0$ is the event that there is no particle moving to or away from location x in the infected group, so it has probability $1 - I(t,x)\beta dt - \sum\limits_{z\neq 0} I(t,x+z)\kappa a(-z)dt - \sum\limits_{z\neq 0} I(t,x)\kappa a(z)dt$.

$E[I^2(t+dt,x)] = E[E[I^2(t+dt,x)|\mathscr{F}(t)]] = E[I^2(t,x)] + E[2I(t,x)\xi(dt)] + E[\xi^2(dt)]$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and we have that

As $dt \longrightarrow 0$, $\dfrac{\partial m_2^I(t,x,x)}{\partial t} = 2\kappa\mathscr{L}_{a(z),x} m_2^I(t,x,x) + \kappa\mathscr{L} m_1^I(t,x) + 2\beta m_2^I(t,x,x) + \beta m_1^I(t,x) + 2\kappa m_1^I(t,x)$

Case 2: $I(t+dt,x)$ when $x \neq y$, then $m_2(t+dt,x,y) = E[I(t+dt,x)I(t+dt,y)]$

Recall that during $(t, t+dt)$ only one event can happen, either a particle can move or it can transition states, therefore the probabilities for the various combinations of $\xi(dt,x)$ and $\xi(dt,y)$ are:

The event that a particle in the infected group goes from y to x:

$P(\xi(dt,x) = 1, \xi(dt,y) = -1) = \kappa a(x-y)I(t,y)dt$

The event that a particle in the infected group goes from $x+z$ to x but not from y or a particle at location x transitions $S \to I$:

$P(\xi(dt,x) = 1, \xi(dt,y) = 0) = \kappa \sum\limits_{z\neq 0} a(z)I(t,x+z)dt - \kappa a(x-y)I(t,y)dt + \beta I(t,x)dt$

The event that a particle in the infected group goes from x to y:

$P(\xi(dt,x) = -1, \xi(dt,y) = 1) = \kappa a(x-y)I(t,x)dt$

The event that a particle in the infected group goes from x to $x+z$ but not to y:

$P(\xi(dt,x) = -1, \xi(dt,y) = 0) = \kappa \sum\limits_{z\neq 0} a(z)I(t,x)dt - \kappa a(x-y)I(t,x)dt$

The event that a particle in the infected group goes from $y+z$ to y but not from x or a particle at location y transitions $S \to I$:

$P(\xi(dt,x) = 0, \xi(dt,y) = 1) = \kappa \sum\limits_{z\neq 0} a(z)I(t,y+z)dt - \kappa a(x-y)I(t,x)dt + \beta I(t,y)dt$

The event that a particle in the infected group goes from y to $y+z$ but not to x:

$P(\xi(dt,x) = 0, \xi(dt,y) = -1) = \kappa \sum\limits_{z\neq 0} a(-z)I(t,y)dt - \kappa a(x-y)I(t,y)dt$

The event that no particle moves in the infected group:

$P(\xi(dt,x) = 0, \xi(dt,y) = 0) = 1 - \kappa \sum\limits_{z\neq 0} a(z)I(t,y+z)dt - \kappa \sum\limits_{z\neq 0} a(z)I(t,y)dt - \kappa \sum\limits_{z\neq 0} a(z)I(t,x+z)dt - \kappa \sum\limits_{z\neq 0} a(z)I(t,x)dt + \kappa a(x-y)I(t,y)dt + \kappa a(x-y)I(t,x)dt - \beta I(t,y)dt - \beta I(t,x)dt$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and we have that as $dt \longrightarrow 0$, $\dfrac{\partial m_2^I(t,v)}{\partial t} = \kappa \mathscr{L}_{a(z),x} m_2^I(t,v) + \kappa \mathscr{L}_{a(z),y} m_2^I(t,v) + 2\beta m_2^I(t,v) - \kappa a(v) m_1^I(t,x) - \kappa a(v) m_1^I(t,y)$

Theorem 24 *The differential equations for the second moment of the susceptible-infected groups are*

$$\begin{cases} \text{Susceptible-Infected when } x = y \\ \dfrac{\partial m_2^{SI}(t,x,x)}{\partial t} = \kappa \mathscr{L}_{a(z)+b(z)} m_2^{SI}(t,x,x) + \beta m_2^{SI}(t,x,x) - \beta m_2^I(t,x,x) \\ -\beta m_1^I(t,x) \qquad (17a) \\ \text{Susceptible-Infected when } x \neq y \\ \dfrac{\partial m_2^{SI}(t,v)}{\partial t} = \kappa \mathscr{L}_{a(z)+b(z)} m_2^{SI}(t,v) + \beta m_2^{SI}(t,v) - \beta m_2^I(t,v) \qquad (17b) \end{cases}$$

Note that $\dfrac{\partial m_2^{IS}(t,v)}{\partial t} = \dfrac{\partial m_2^{SI}(t,v)}{\partial t}$ and $\dfrac{\partial m_2^{IS}(t,x,x)}{\partial t} = \dfrac{\partial m_2^{SI}(t,x,x)}{\partial t}$

Proof of Theorem 24: For the susceptible-infected group, there are 2 cases ($x = y$ and $x \neq y$) and there are several combinations for each case:

Case 1: when $x = y$, then we have

$E[S(t+dt,x)I(t+dt,y)] = E[S(t+dt,x)I(t+dt,x)] = E[E[S(t+dt,x)I(t+dt,x)|\mathscr{F}_t]]$

$E[S(t+dt,x)I(t+dt,x)] = E[(S(t,x) + \xi_S(dt,x))(I(t,x) + \xi_I(dt,x))]$

Recall that during $(t, t+dt)$ only one event can happen, either a particle can move or it can jump states, therefore the probabilities for $\xi_S(dt,x)$ and $\xi_I(dt,x)$ are:

The event that a particle goes from $x+z$ to x in S but x doesn't move within I:

$P(\xi_S(dt,x)=1,\xi_I(dt,x)=0)=\kappa\sum_{z\neq 0} b(z)S(t,x+z)dt$

The event that a particle at location x transitions $S\to I$:

$P(\xi_S(dt,x)=-1,\xi_I(dt,x)=1)=\beta I(t,x)dt$

The event that a particle goes from x to $x+z$ in S but x doesn't move within I:

$P(\xi_S(dt,x)=-1,\xi_I(dt,x)=0)=\kappa\sum_{z\neq 0} b(z)S(t,x)dt$

The event that a particle doesn't move in S but $x+z$ goes to x in I:

$P(\xi_S(dt,x)=0,\xi_I(dt,x)=1)=\kappa\sum_{z\neq 0} a(z)I(t,x+z)dt$

The event that a particle doesn't move in S but x goes to $x+z$ in I:

$P(\xi_S(dt,x)=0,\xi_I(dt,x)=-1)=\kappa\sum_{z\neq 0} a(z)I(t,x)dt$

The event that a particle doesn't move:

$P(\xi_S(dt,x)=0,\xi_I(dt,x)=0)=1-\kappa\sum_{z\neq 0} b(z)S(t,x+z)dt-\kappa\sum_{z\neq 0} b(z)S(t,x)dt-$
$\kappa\sum_{z\neq 0} a(z)I(t,x+z)dt-\kappa\sum_{z\neq 0} a(z)I(t,x)dt-\beta I(t,x)dt$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and we have that as $dt\longrightarrow 0$,

$\dfrac{\partial m_2^{SI}(t,x,x)}{\partial t}=\kappa\mathscr{L}_{b(z)_{S_x}}m_2^{SI}(t,x,x)+\kappa\mathscr{L}_{a(z)_{I_x}}m_2^{SI}(t,x,x)+\beta m_2^{SI}(t,x,x)-$
$\beta m_2^{I}(t,x,x)-\beta m_1^{I}(t,x)$

Case 2: when $x\neq y$, we have that:

$E[S(t+dt,x)I(t+dt,y)]=E[E[S(t+dt,x)I(t+dt,y)|\mathscr{F}_t]]$

$E[S(t+dt,x)I(t+dt,y)]=E[(S(t,x)+\xi_S(dt,x))(I(t,y)+\xi_I(dt,y))]$

Recall that during $(t,t+dt)$ only one event can happen, either a particle can move or it can jump states, therefore the probabilities for $\xi_S(dt,x)$ and $\xi_I(dt,y)$ are:

The event that a particle goes from $x+z$ to x in S but y doesn't move within I:

$P(\xi_S(dt,x)=1,\xi_I(dt,y)=0)=\kappa\sum_{z\neq 0}b(z)S(t,x+z)dt$

The event that a particle goes from x to $x+z$ in S or a particle at location x transitions $S\to I$:

$P(\xi_S(dt,x)=-1,\xi_I(dt,y)=0)=\kappa\sum_{z\neq 0}b(z)S(t,x)dt+\beta I(t,x)dt$

The event that a particle x doesn't move in S but $y+z$ goes to y in I or a particle at location y transitions $S\to I$:

$P(\xi_S(dt,x)=0,\xi_I(dt,y)=1)=\kappa\sum_{z\neq 0}a(z)I(t,y+z)dt+\beta I(t,y)dt$

The event that a particle at location x doesn't move in S but y goes to $y+z$ in I:

$P(\xi_S(dt,x)=0,\xi_I(dt,y)=-1)=\kappa\sum_{z\neq 0}a(z)I(t,y)dtdt$

The event that a particle doesn't move:

$P(\xi_S(dt,x)=0,\xi_I(dt,y)=0)=1-\kappa\sum_{z\neq 0}b(z)S(t,x+z)dt-\kappa\sum_{z\neq 0}b(z)S(t,x)dt-\kappa\sum_{z\neq 0}a(z)I(t,y+z)dt-\kappa\sum_{z\neq 0}a(z)I(t,y)dt-\beta I(t,x)dt-\beta I(t,y)dt$

Using the Kolmogorov Forward Equations, following the same process used for the SIR model, and we have that as $dt\longrightarrow 0$,

$$\frac{\partial m_2^{SI}(t,v)}{\partial t}=\kappa\mathscr{L}_{b(z)_{S_x}}m_2^{SI}(t,v)+\kappa\mathscr{L}_{a(z)_{I_y}}m_2^{SI}(t,v)+\beta m_2^{SI}(t,v)-\beta m_2^{I}(t,v)$$

CHAPTER 17
SECOND MOMENTS OF THE SI MODEL

Now we want to solve the differential equations for the second moments for the $S(t,x), I(t,x)$, and $S(t,x)I(t,x)$ groups, where each group has 2 cases: when the locations $x = y$ and when the locations $x \neq y$ and each case has 2 subcases: homogeneous space and inhomogeneous space.

Theorem 25 *The second moments for the infected groups when $x = y$ and $x \neq y$ are:*

$$
\begin{cases}
\text{Homogeneous space:} \\
m_2^I(t,x,x) = \rho_0 e^{2\beta t} + \left(\dfrac{\beta + 2\kappa}{-\beta}\right)\left[e^{\beta t} - e^{2\beta t}\right] & (18a) \\
\text{Inhomogeneous space:} \\
m_2^I(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[2\kappa\hat{\mathscr{L}}_{a(z)}(k) + 2\beta]t} e^{-ikx} dk + e^{2\beta t}. \\
\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p_a(t-s,0,x-z)\left[\kappa\mathscr{L}_{a(z)} m_1^I(s,z) + (\beta+2\kappa) m_1^I(s,z)\right]\left(e^{2\beta s}\right) ds & (18b) \\
\text{Homogeneous space:} \\
m_2^I(t,v) = \rho_0 e^{2\beta t} + \left(\dfrac{2\kappa a(v)}{\beta}\right)\left[e^{\beta t} - e^{2\beta t}\right] & (18c) \\
\text{Inhomogeneous space:} \\
m_2^I(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_{a(z)_x}(k) + \kappa\hat{\mathscr{L}}_{a(z)_y}(k) + 2\beta]t} e^{-ikv} dk + \\
e^{2\beta t}\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p_a(t-s,0,x-z)\left[-2\kappa a(v) m_1^I(s,z)\right]\left(e^{2\beta s}\right) ds & (18d)
\end{cases}
$$

Theorem 26 *The second moments of the susceptible-infected group when $x = y$ and $x \neq y$ are:*

$$\begin{cases} \text{Homogeneous space:} \\ m_2^{SI}(t,x,x) = (\rho_0+1)e^{\beta t} - e^{2\beta t} + 2\kappa t e^{\beta t} - \Big(\dfrac{\beta+2\kappa}{\beta}\Big)\Big[e^{2\beta t} - e^{\beta t}\Big] & (19a) \\ \text{Inhomogeneous space:} \\ m_2^{SI}(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_{a(z)+b(z)}(k)+\beta]t} e^{-ikx} dk - \\ \beta e^{\beta t}\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p_{(a+b)}(t-s,0,x-z)\Big[m_2^I(s,z,z) + m_1^I(s,z)\Big]\Big(e^{\beta s}\Big)ds & (19b) \\ \text{Homogeneous space:} \\ m_2^{SI}(t,v) = \rho_0 e^{\beta t} - 2\kappa a(v) t e^{\beta t} + \Big(\dfrac{2\kappa a(v) - \beta\rho_0}{\beta}\Big)\Big[e^{2\beta t} - e^{\beta t}\Big] & (19c) \\ \text{Inhomogeneous space:} \\ m_2^{SI}(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_{a(z)+b(z)}(k)+\beta]t} e^{-ikv} dk \\ -\beta e^{\beta t}\displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p_{a+b}(t-s,0,x-z) e^{\beta s}\Big(m_2^I(s,z,z)\Big)ds & (19d) \end{cases}$$

Theorem 27 *The second moments of the susceptible group are:*

$$\begin{cases} \text{Homogeneous space:} \\ m_2^S(t,x,x) = \rho_0 - \Big(\dfrac{4\kappa}{\beta}\Big)\Big[e^{\beta t}(\beta t - 1) + 1\Big] + 2\kappa N t + \Big(\dfrac{-2\beta\rho_0 - 3\beta - 6\kappa}{\beta}\Big) \cdot \\ \Big[e^{\beta t} - 1\Big] + \Big(\dfrac{2\beta + 2\kappa}{\beta}\Big)\Big[e^{2\beta t} - 1\Big] \\ \text{Inhomogeneous space:} \\ m_2^S(t,x,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{2\kappa \hat{\mathscr{L}}_{b(z)}(k) t} e^{-ikx} dk + \\ \displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p_b(t-s,0,x-z)\Big[-2\beta m_2^{SI}(s,z,z) + \beta m_1^I(s,z) + \kappa \mathscr{L} m_1^S(s,z) + \\ 2\kappa m_1^S(s,z)\Big] ds \\ \text{Homogeneous space:} \\ m_2^S(t,v) = \rho_0 + \Big(\dfrac{4\kappa a(v)}{\beta}\Big)\Big[e^{\beta t}(\beta t - 1) + 1\Big] - 2b(v)\rho_0 t + \Big(\dfrac{-2\kappa a(v) + \beta\rho_0}{\beta}\Big) \cdot \\ \Big[e^{2\beta t} - 1\Big] + \Big(\dfrac{4\kappa a(v) + 2b(v) - 4\beta\rho_0}{\beta}\Big)\Big[e^{\beta t} - 1\Big] \\ \text{Inhomogeneous space:} \\ m_2^S(t,v) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} \rho_0 e^{[\kappa \hat{\mathscr{L}}_{b(z)_x}(k) + \kappa \hat{\mathscr{L}}_{b(z)_y}(k)] t} e^{-ikv} dk + \\ \displaystyle\int_0^t \sum_{z\in\mathbb{Z}^d} p_b(t-s,0,x-z)\Big[-2\beta m_2^{SI}(s,z,z) - 2\kappa b(v) m_1^S(s,z)\Big] ds \end{cases}$$

17.1 Second Moments in Homogeneous space

Proof of Theorem 25: Recall from Chapter 16 Equation ($16a$) and ($16b$) and for each equation $x = y$ and $x \neq y$ there are 2 cases: homeogeneous and inhomogeneous space:

Case 1: Homogeneous space when $x = y$ (then the spaces x and $x + z$ are equivalent and $\mathscr{L}_x m_2^I(t, x, x) = 0 = \mathscr{L} m_1^I(t, x)$)

Then $\dfrac{\partial m_2^I(t, x, x)}{\partial t} = 2\beta m_2^I(t, x, x) + (\beta + 2\kappa)e^{\beta t}$ with initial conditions $m_2^I(0, x, x) = \rho_0 > 0$, and following the same process used for the SIR model.

$$m_2^I(t, x, x) = \rho_0 e^{2\beta t} + \Big(\frac{\beta + 2\kappa}{-\beta}\Big)\Big[e^{\beta t} - e^{2\beta t}\Big]$$

Case 2: Homogeneous space when $x \neq y$ (then $\mathscr{L}_x m_2^I(t, v) = 0$ and $m_1^I(t, x) = m_1^I(t, y) = e^{\beta t}$)

Then $\dfrac{\partial m_2^I(t, v)}{\partial t} = 2\beta m_2^I(t, v) - 2\kappa a(v)e^{\beta t}$ with initial conditions $m_2^I(0, v) = \rho_0 > 0$, and following the same procedure used for the SIR model.

$$m_2^I(t, v) = \rho_0 e^{2\beta t} + \Big(\frac{2\kappa a(v)}{\beta}\Big)\Big[e^{\beta t} - e^{2\beta t}\Big]$$

17.2 Second Moments in Inhomogeneous space

Case 1: In-homogeneous space when $x = y$ (then the spaces x and $x + z$ are not equivalent and $\mathscr{L}_x m_2^I(t, x, x) \neq 0$)

$\dfrac{\partial m_2^I(t, x, x)}{\partial t} = 2\kappa \mathscr{L}_{a(z)} m_2^I(t, x, x) + \kappa \mathscr{L}_{a(z)} m_1^I(t, x) + 2\beta m_2^I(t, x, x) + (2\kappa +$

$\beta)m_1^I(t,x)$ with initial conditions $m_2^I(0,x,x) = \rho_0 > 0$

Utilizing the general solution for inhomogeneous equations from Theorem 3 and following the same procedure used for the SIR model, we have that $m_2^I(t,x,x) = m_2^{I^h}(t,x,x) + w(t,x,x)$

$$m_2^I(t,x,x) = \frac{1}{(2\pi)^d}\int_{T^d} \rho_0 e^{[2\kappa\hat{\mathscr{L}}_{a(z)_x}(k) + 2\beta]t} e^{-ikx} dk + e^{2\beta t}\int_0^t \sum_{z\in\mathbb{Z}^d} p_a(t - s, 0, x-z)\Big[\kappa\mathscr{L}m_1^I(s,z) + (\beta+2\kappa)m_1^I(s,z)\Big]\Big(e^{2\beta s}\Big)ds$$

Case 2: Inhomogeneous space when $x \neq y$ (then $\mathscr{L}_{a(z)_x}m_2^I(t,v) \neq 0$)

$$\frac{\partial m_2^I(t,v)}{\partial t} = \kappa\mathscr{L}_{a(z)_x}m_2^I(t,v) + \kappa\mathscr{L}_{a(z)_y}m_2^I(t,v) + 2\beta m_2^I(t,v) - \kappa a(v)m_1^I(t,x) - \kappa a(v)m_1^I(t,y)$$ with initial conditions $m_2^I(0,v) = \rho_0 > 0$

Thus following the same procedure used for the SIR model, we have that $m_2^I(t,v) = m_2^{I^h}(t,v) + w(t,v)$

$$m_2^I(t,v) = \frac{1}{(2\pi)^d}\int_{T^d} \rho_0 e^{[\kappa\hat{\mathscr{L}}_{a(z)}(k) + \kappa\hat{\mathscr{L}}_{a(z)}(k) + 2\beta]t} e^{-ikv} dk + e^{2\beta t}\int_0^t \sum_{z\in\mathbb{Z}^d} p_a(t-s,0,x-z)\Big[-2\kappa a(v)m_1^I(s,z)\Big]\Big(e^{2\beta s}\Big)ds$$

Note that solving the differential equations for the second moments of the susceptible and susceptible-infected groups follows the same procedure as for the second moment of the infected group.

17.3 Second Moments in Inhomogeneous space using matrices

When $x = y$:

For the second moments of the susceptible, infected, and susceptible-infected groups we are going to let $G(t,x,x) = \begin{bmatrix} S(t,x,x) \\ I(t,x,x) \\ S(t,x)I(t,x) \end{bmatrix}$ and then $m_2^G(t,x,x) = \begin{bmatrix} m_2^S(t,x,x) \\ m_2^I(t,x,x) \\ m_2^{SI}(t,x,x) \end{bmatrix}$

Because our differential equations for the second moments of the S, I and SI groups are given in Chapter 16 Equations $(15a), (16a), (17a)$. The Fourier transform of the matrix format is
$\dfrac{\partial \hat{m}_2^G(t,x,k)}{\partial t} = \hat{A}_5 \hat{m}_2^G(t,x,k) + \hat{B}_5 \vec{Q}$ where

$$\hat{A}_5 = \begin{bmatrix} 2\kappa\hat{b}(k) & 0 & -2\beta \\ 0 & 2\kappa\hat{a}(k) + 2\beta & 0 \\ 0 & -\beta & \kappa\hat{a}(k) + \kappa\hat{b}(k) + \beta \end{bmatrix},$$

$\hat{B}_5 = \begin{bmatrix} \kappa\hat{b}(k) + 2\kappa & \beta & 0 \\ 0 & \kappa\hat{a}(k) + 2\kappa + \beta & 0 \\ 0 & -\beta & 0 \end{bmatrix}$ and $\vec{Q} = \begin{bmatrix} \hat{m}_1^S(t,x) & \hat{m}_1^I(t,x) & 0 \end{bmatrix}$

For matrix $\hat{A}_5$ we have eigenvalues $\lambda_1 = 2(\kappa\hat{a}(k) + \beta), \lambda_2 = 2\kappa\hat{b}(k), \lambda_3 = \kappa\hat{a}(k) + \beta + \kappa\hat{b}(k)$ and the eigenvectors are

$$v_1 = \begin{bmatrix} \dfrac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & -\dfrac{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}{\beta} & 1 \end{bmatrix},\ v_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$
$$v_3 = \begin{bmatrix} \dfrac{-2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & 0 & 1 \end{bmatrix}$$

Case 1: $\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k) = 0$

Then $\lambda_1 = \lambda_2 = \lambda_3 = 2\kappa\hat{b}(k)$ multiplicity 3 and $(\hat{A}_5 - 2\kappa\hat{b}(k)I)^3 = 0$.
$$e^{\hat{A}_5 t} = \begin{bmatrix} e^{2\kappa\hat{b}(k)t} & \beta^2 t^2 e^{2\kappa\hat{b}(k)t} & -2\beta t e^{2\kappa\hat{b}(k)t} \\ 0 & e^{2\kappa\hat{b}(k)t} & 0 \\ 0 & -\beta t e^{2\kappa\hat{b}(k)t} & e^{2\kappa\hat{b}(k)t} \end{bmatrix}$$
with initial conditions $x_0 = \begin{bmatrix} \rho_0^2 & 1 & \rho_0 \end{bmatrix}^T$.

We have the solution $\hat{m}_2^G(t,x,k) = e^{\hat{A}_5 t}x_0 + e^{\hat{A}_5 t}\int_0^t e^{-\hat{A}_5 s}\hat{B}_5\vec{Q}ds$ where
$$\hat{B}_5 = \begin{bmatrix} \kappa\hat{b}(k)+2\kappa & \beta & 0 \\ 0 & \kappa\hat{a}(k)+2\kappa+\beta & 0 \\ 0 & -\beta & 0 \end{bmatrix},\ \vec{Q} = \begin{bmatrix} \rho_0 e^{\kappa\hat{b}(k)t} - \beta t e^{\kappa\hat{b}(k)t} \\ e^{\kappa\hat{b}(k)t} \\ 0 \end{bmatrix}$$
and
$$e^{-\hat{A}_5 s} = \begin{bmatrix} e^{-2\kappa\hat{b}(k)s} & \beta^2 s^2 e^{-2\kappa\hat{b}(k)s} & 2\beta s e^{-2\kappa\hat{b}(k)s} \\ 0 & e^{-2\kappa\hat{b}(k)s} & 0 \\ 0 & \beta s e^{-2\kappa\hat{b}(k)s} & e^{-2\kappa\hat{b}(k)s} \end{bmatrix}$$
$$\hat{m}_2^G(t,x,k) = e^{\hat{A}_5 t}x_0 + e^{\hat{A}_5 t}\int_0^t e^{-\hat{A}_5 s}\hat{B}_5\vec{Q}ds$$

$$
\begin{bmatrix}
\rho_0^2 e^{2\kappa\hat{b}(k)t} + \beta^2 t^2 e^{2\kappa\hat{b}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{b}(k)t} \\
\left(\frac{-\rho_0(\kappa\hat{b}(k)+2\kappa)-\beta}{\kappa\hat{b}(k)}\right)\left[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\right] - \left(\frac{\beta(\kappa\hat{b}(k)+2\kappa)-2\beta^2}{(\kappa\hat{b}(k))^2}\right)\cdot \\
\left[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\right] + \left(\frac{\kappa\hat{a}(k)\beta^2+2\beta^2+\beta^3}{(\kappa\hat{b}(k))^3}\right)\cdot \\
\left[e^{\kappa\hat{b}(k)t}(-(\kappa\hat{b}(k)t)^2 - 2\kappa\hat{b}(k)t - 2) + 2e^{2\kappa\hat{b}(k)t}\right] - \left(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta)}{\kappa\hat{b}(k)}\right)\cdot \\
\left[t^2 e^{\kappa\hat{b}(k)t} - t^2 e^{2\kappa\hat{b}(k)t}\right] - \left(\frac{2\beta(\kappa\hat{a}(k)+2\kappa\beta+\beta^2)}{(\kappa\hat{b}(k))^2}\right)\left[te^{2\kappa\hat{b}(k)t} - \right. \\
\left. te^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\right] - \left(\frac{2\beta^2}{\kappa\hat{b}(k)}\right)\left[te^{\kappa\hat{b}(k)t} - te^{2\kappa\hat{b}(k)t}\right] \\
e^{2\kappa\hat{a}(k)t} - \left(\frac{\kappa\hat{a}(k)+2\kappa+\beta}{\kappa\hat{b}(k)}\right)\left[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\right] \\
-\beta te^{2\kappa\hat{b}(k)t} + \rho_0 e^{2\kappa\hat{b}(k)t} + \left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta)}{\kappa\hat{b}(k)}\right)\left[te^{\kappa\hat{b}(k)t} - \right. \\
\left. te^{2\kappa\hat{b}(k)t}\right] + \left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta)}{(\kappa\hat{b}(k))^2}\right)\left[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\right] + \\
\left(\frac{\beta}{\kappa\hat{b}(k)}\right)\left[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\right]
\end{bmatrix}
$$

When $\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)=0$, $m_2^G(t,x,x)=\begin{bmatrix} m_2^S(t,x,x) \\ m_2^I(t,x,x) \\ m_2^{SI}(t,x,x) \end{bmatrix}=$

$$\begin{bmatrix} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{\rho_0^2 e^{2\kappa\hat{b}(k)t} + \beta^2 t^2 e^{2\kappa\hat{b}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{b}(k)t} \\ \Big(\frac{-\rho_0(\kappa\hat{b}(k)+2\kappa)-\beta}{\kappa\hat{b}(k)}\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big] - \Big(\frac{\beta(\kappa\hat{b}(k)+2\kappa)-2\beta^2}{(\kappa\hat{b}(k))^2}\Big)\cdot \\ \Big[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big] + \Big(\frac{\kappa\hat{a}(k)\beta^2+2\beta^2+\beta^3}{(\kappa\hat{b}(k))^3}\Big)\cdot \\ \Big[e^{\kappa\hat{b}(k)t}(-(\kappa\hat{b}(k)t)^2 - 2\kappa\hat{b}(k)t - 2) + 2e^{2\kappa\hat{b}(k)t}\Big] - \Big(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta)}{\kappa\hat{b}(k)}\Big)\cdot \\ \Big[t^2 e^{\kappa\hat{b}(k)t} - t^2 e^{2\kappa\hat{b}(k)t}\Big] - \Big(\frac{2\beta(\kappa\hat{a}(k)+2\kappa\beta+\beta^2)}{(\kappa\hat{b}(k))^2}\Big)\Big[te^{2\kappa\hat{b}(k)t} - \\ te^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big] - \Big(\frac{2\beta^2}{\kappa\hat{b}(k)}\Big)\Big[te^{\kappa\hat{b}(k)t} - te^{2\kappa\hat{b}(k)t}\Big]\Big\} e^{-ikx} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{e^{2\kappa\hat{a}(k)t} - \Big(\frac{\kappa\hat{a}(k)+2\kappa+\beta}{\kappa\hat{b}(k)}\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big]\Big\} e^{-ikx} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{-\beta t e^{2\kappa\hat{b}(k)t} + \rho_0 e^{2\kappa\hat{b}(k)t} + \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta)}{\kappa\hat{b}(k)}\Big)\Big[te^{\kappa\hat{b}(k)t} \\ -te^{2\kappa\hat{b}(k)t}\Big] + \Big(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta)}{(\kappa\hat{b}(k))^2}\Big)\Big[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big] + \\ \Big(\frac{\beta}{\kappa\hat{b}(k)}\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big]\Big\} e^{-ikx} dk \end{bmatrix}$$

Case 2: $\kappa\hat{a}(k)+\beta \neq \kappa\hat{b}(k)$

For matrix $\hat{A}_5$ the eigenvectors are

$v_1 = \begin{bmatrix} \frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & -\frac{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}{\beta} & 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$,
$v_3 = \begin{bmatrix} \frac{-2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & 0 & 1 \end{bmatrix}$

We have $X(t) =$

$$\begin{bmatrix} \left(\frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)e^{2(\kappa\hat{a}(k)+\beta)t} & e^{2\kappa\hat{b}(k)t} & \left(\frac{-2\beta}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t} \\ \left(\frac{-(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))}{\beta}\right)e^{2(\kappa\hat{a}(k)+\beta)t} & 0 & 0 \\ e^{2(\kappa\hat{a}(k)+\beta)t} & 0 & e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t} \end{bmatrix}$$

$$X(0) = \begin{bmatrix} \left(\frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) & 1 & \left(\frac{-2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) \\ \left(\frac{-(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))}{\beta}\right) & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and}$$

$$X^{-1}(0) = \begin{bmatrix} 0 & \left(\frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) & 0 \\ 1 & \left(\frac{\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)^2 & \left(\frac{2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) \\ 0 & \left(\frac{\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) & 1 \end{bmatrix}$$

$e^{\hat{A}_5 t} = X(t)X^{-1}(0) =$

$$\begin{bmatrix} e^{2\kappa\hat{b}(k)t} & \frac{\beta^2 e^{2(\kappa\hat{a}(k)+\beta)t}+\beta^2 e^{2\kappa\hat{b}(k)t}-2\beta^2 e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2} & \frac{2\beta e^{2\kappa\hat{b}(k)t}-2\beta e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} \\ 0 & e^{2(\kappa\hat{a}(k)+\beta)t} & 0 \\ 0 & \frac{-\beta e^{2(\kappa\hat{a}(k)+\beta)t}+\beta e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t} \end{bmatrix}$$

with initial conditions $x_0 = \begin{bmatrix} \rho_0^2 & 1 & \rho_0 \end{bmatrix}^T$.

We have the solution $\hat{m}_2^G(t,x,k) = e^{\hat{A}_5 t}x_0 + e^{\hat{A}_5 t}\int_0^t e^{-\hat{A}_5 s}\hat{B}_5\vec{Q}ds$ where

$$\hat{B}_5 = \begin{bmatrix} \kappa\hat{b}(k)+2\kappa & \beta & 0 \\ 0 & \kappa\hat{a}(k)+2\kappa+\beta & 0 \\ 0 & -\beta & 0 \end{bmatrix}, \vec{Q} = \begin{bmatrix} \rho_0 e^{\kappa\hat{b}(k)t} - \beta t e^{\kappa\hat{b}(k)t} \\ e^{\kappa\hat{b}(k)t} \\ 0 \end{bmatrix}$$

$$\hat{m}_2^G(t,x,k) = e^{\hat{A}_5 t}x_0 + e^{\hat{A}_5 t}\int_0^t e^{-\hat{A}_5 s}\hat{B}_5\vec{Q}ds =$$

$$
\begin{bmatrix}
\rho_0^2 e^{2\kappa\hat{b}(k)t} + \left(\frac{\beta^2}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2}\right)\left[e^{2(\kappa\hat{a}(k)+\beta)t} + e^{2\kappa\hat{b}(k)t} - \right. \\
\left. 2e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\right] + \left(\frac{\rho_0\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)\left[e^{2\kappa\hat{b}(k)t} - \right. \\
\left. e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\right] + \left(\frac{-\rho_0(\kappa\hat{b}(k)+2\kappa)-\beta}{\kappa\hat{b}(k)}\right)\left[e^{\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] - \\
\left(\frac{\beta}{(\kappa\hat{b}(k))^2}\right)\left[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t-1)\right] + \left(\frac{2\beta^2}{\kappa\hat{b}(k)(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))}\right)\left[e^{\kappa\hat{a}(k)t}\right. \\
\left. -e^{2\kappa\hat{a}(k)t}\right] - \left(\frac{\beta^2}{\kappa\hat{b}(k)(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2}\right)\left[e^{-\kappa\hat{a}(k)t} - e^{2\kappa\hat{a}(k)t}\right] \\
-\left(\frac{\beta^2(\kappa\hat{a}(k)+2\kappa+\beta)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))}\right)\left[e^{\kappa\hat{b}(k)t} - e^{2(\kappa\hat{a}(k)+\beta)t}\right] \\
+\left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))}\right)\left[e^{-(2\kappa\hat{a}(k)+2\beta-3\kappa\hat{b}(k))t} - e^{2\kappa\hat{b}(k)t}\right] \\
+\left(\frac{2\beta^2(\kappa\hat{a}(k)+2\kappa+\beta)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2(\kappa\hat{a}(k)+\beta)}\right)\left[e^{\kappa\hat{b}(k))t} - e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\right] \\
-\left(\frac{2\beta^2}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))(\kappa\hat{a}(k)+\beta)}\right)\left[e^{\kappa\hat{b}(k))t} - e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\right] \\
e^{2(\kappa\hat{a}(k)+\beta)t} - \left(\frac{\kappa\hat{a}(k)+2\kappa+\beta}{2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k)}\right)\left[e^{\kappa\hat{b}(k)t} - e^{2(\kappa\hat{a}(k)+\beta)t}\right] \\
\frac{-\beta e^{2(\kappa\hat{a}(k)+\beta)t}+\beta e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} + \rho_0 e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t} + \\
\left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))}\right)\left[e^{\kappa\hat{b}(k)t} - e^{2(\kappa\hat{a}(k)+\beta)t}\right] \\
-\left(\frac{\beta(\kappa\hat{a}(k)+2\kappa+\beta)}{(\kappa\hat{a}(k)+\beta)(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))}\right)\left[e^{\kappa\hat{b}(k)t} - e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\right] \\
+\left(\frac{\beta}{\kappa\hat{a}(k)+\beta}\right)\left[e^{\kappa\hat{b}(k)t} - e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\right]
\end{bmatrix}
$$

When $x \neq y$:

The differential equations are given in Chapter 16 Equations $(15b)$, $(16b)$, $(17b)$. The Fourier transform of the matrix format is $\dfrac{\partial \hat{m}_2^G(t,k)}{\partial t} = \hat{A}_6 \hat{m}_2^G(t,k) + \hat{B}_6 \vec{Q}$ where

$$
\hat{A}_6 = \begin{bmatrix} 2\kappa\hat{b}(k) & 0 & -2\beta \\ 0 & 2\kappa\hat{a}(k)+2\beta & 0 \\ 0 & -\beta & \kappa\hat{a}(k)+\kappa\hat{b}(k)+\beta \end{bmatrix},
$$

$$\hat{B}_6 = \begin{bmatrix} -2\kappa\hat{b}(k) & 0 & 0 \\ 0 & -2\kappa\hat{a}(k) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \text{ and } \vec{Q} = \begin{bmatrix} \rho_0 e^{\kappa\hat{b}(k)t} - \beta t e^{\kappa\hat{b}(k)t} \\ e^{\kappa\hat{b}(k)t} \\ 0 \end{bmatrix}$$

For matrix $\hat{A}_6$ the eigenvectors are

$$v_1 = \begin{bmatrix} \frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & -\frac{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}{\beta} & 1 \end{bmatrix}, \; v_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix},$$

$$v_3 = \begin{bmatrix} \frac{-2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & 0 & 1 \end{bmatrix}$$

Case 1: $\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k) = 0$

Then $\lambda_1 = \lambda_2 = \lambda_3 = 2\kappa\hat{b}(k)$ multiplicity 3 and $(\hat{A}_6 - 2\kappa\hat{b}(k)I)^3 = 0$.

$$e^{\hat{A}_6 t} = \begin{bmatrix} e^{2\kappa\hat{b}(k)t} & \beta^2 t^2 e^{2\kappa\hat{b}(k)t} & -2\beta t e^{2\kappa\hat{b}(k)t} \\ 0 & e^{2\kappa\hat{b}(k)t} & 0 \\ 0 & -\beta t e^{2\kappa\hat{b}(k)t} & e^{2\kappa\hat{b}(k)t} \end{bmatrix}$$

with initial conditions $x_0 = \begin{bmatrix} \rho_0^2 & 1 & \rho_0 \end{bmatrix}^T$.

We have the solution $\hat{m}_2^G(t,k) = e^{\hat{A}_6 t}x_0 + e^{\hat{A}_6 t}\int_0^t e^{-\hat{A}_6 s}\hat{B}_6\vec{Q}ds$ where

$$\hat{B}_6 = \begin{bmatrix} -2\kappa\hat{b}(k) & 0 & 0 \\ 0 & -2\kappa\hat{a}(k) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \; \vec{Q} = \begin{bmatrix} \rho_0 e^{\kappa\hat{b}(k)t} - \beta t e^{\kappa\hat{b}(k)t} \\ e^{\kappa\hat{b}(k)t} \\ 0 \end{bmatrix}$$

and $$e^{-\hat{A}_6 s} = \begin{bmatrix} e^{-2\kappa\hat{b}(k)s} & \beta^2 s^2 e^{-2\kappa\hat{b}(k)s} & 2\beta s e^{-2\kappa\hat{b}(k)s} \\ 0 & e^{-2\kappa\hat{b}(k)s} & 0 \\ 0 & \beta s e^{-2\kappa\hat{b}(k)s} & e^{-2\kappa\hat{b}(k)s} \end{bmatrix}$$

$$\hat{m}_2^G(t,k) = e^{\hat{A}_6 t}x_0 + e^{\hat{A}_6 t}\int_0^t e^{-\hat{A}_6 s}\hat{B}_6\vec{Q}ds$$

$$\begin{bmatrix} \rho_0^2 e^{2\kappa\hat{b}(k)t} + \beta^2 t^2 e^{2\kappa\hat{b}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{b}(k)t} + \\ \Big(2\rho_0\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big] + \Big(\frac{2\beta}{\kappa\hat{b}(k)}\Big)\Big[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big] \\ -\Big(\frac{2\kappa\hat{a}(k)\beta^2}{(\kappa\hat{b}(k))^3}\Big)\Big[e^{\kappa\hat{b}(k)t}(-(\kappa\hat{b}(k)t)^2 - 2\kappa\hat{b}(k)t - 2) + 2e^{2\kappa\hat{b}(k)t}\Big] \\ +\Big(\frac{2\beta^2\kappa\hat{a}(k)}{\kappa\hat{b}(k)}\Big)\Big[t^2 e^{\kappa\hat{b}(k)t} - t^2 e^{2\kappa\hat{b}(k)t}\Big] \\ +\Big(\frac{4\beta^2\kappa\hat{a}(k)}{(\kappa\hat{b}(k))^2}\Big)\Big[te^{2\kappa\hat{b}(k)t} - te^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big] \\ e^{2\kappa\hat{b}(k)t} + \Big(\frac{2\kappa\hat{a}(k)}{\kappa\hat{b}(k)}\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big] \\ -\beta t e^{2\kappa\hat{b}(k)t} + \rho_0 e^{2\kappa\hat{b}(k)t} - \Big(\frac{2\beta\kappa\hat{a}(k)}{\kappa\hat{b}(k)}\Big)\Big[te^{\kappa\hat{b}(k)t} - te^{2\kappa\hat{b}(k)t}\Big] \\ -\Big(\frac{2\beta\kappa\hat{a}(k)}{(\kappa\hat{b}(k))^2}\Big)\Big[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big] \end{bmatrix}$$

When $\kappa\hat{a}(k) + \beta - \kappa\hat{b}(k) = 0$, $m_2^G(t,v) = \begin{bmatrix} m_2^S(t,v) \\ m_2^I(t,v) \\ m_2^{SI}(t,v) \end{bmatrix} =$

$$
\begin{bmatrix}
\begin{gathered}
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{ \rho_0^2 e^{2\kappa\hat{b}(k)t} + \beta^2 t^2 e^{2\kappa\hat{b}(k)t} - 2\rho_0\beta t e^{2\kappa\hat{b}(k)t} + \\
\Big(2\rho_0\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big] + \Big(\frac{2\beta}{\kappa\hat{b}(k))}\Big)\Big[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big] \\
-\Big(\frac{2\kappa\hat{a}(k)\beta^2}{(\kappa\hat{b}(k))^3}\Big)\Big[e^{\kappa\hat{b}(k)t}(-(\kappa\hat{b}(k)t)^2 - 2\kappa\hat{b}(k)t - 2) + 2e^{2\kappa\hat{b}(k)t}\Big] \\
+\Big(\frac{2\beta^2\kappa\hat{a}(k)}{\kappa\hat{b}(k)}\Big)\Big[t^2 e^{\kappa\hat{b}(k)t} - t^2 e^{2\kappa\hat{b}(k)t}\Big] \\
+\Big(\frac{4\beta^2\kappa\hat{a}(k)}{(\kappa\hat{b}(k))^2}\Big)\Big[te^{2\kappa\hat{b}(k)t} - te^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big]\Big\} e^{-ikv} dk
\end{gathered} \\
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{e^{2\kappa\hat{a}(k)t} + \Big(\frac{2\kappa\hat{a}(k)}{\kappa\hat{b}(k)}\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big]\Big\} e^{-ikv} dk \\
\begin{gathered}
\left(\frac{1}{2\pi}\right)^d \int_{T^d} \Big\{ -\beta t e^{2\kappa\hat{b}(k)t} + \rho_0 e^{2\kappa\hat{b}(k)t} - \Big(\frac{2\beta\kappa\hat{a}(k)}{\kappa\hat{b}(k)}\Big)\Big[te^{\kappa\hat{b}(k)t} - te^{2\kappa\hat{b}(k)t}\Big] \\
-\Big(\frac{2\beta\kappa\hat{a}(k)}{(\kappa\hat{b}(k))^2}\Big)\Big[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t+1)\Big]\Big\} e^{-ikv} dk
\end{gathered}
\end{bmatrix}
$$

Case 2: $\kappa\hat{a}(k) + \beta \neq \kappa\hat{b}(k)$

For matrix $\hat{A}_6$ the eigenvectors are

$v_1 = \begin{bmatrix} \dfrac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & -\dfrac{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}{\beta} & 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$,

$v_3 = \begin{bmatrix} \dfrac{-2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & 0 & 1 \end{bmatrix}$

We have $X(t) =$

$$
\begin{bmatrix}
\Big(\frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\Big)e^{2(\kappa\hat{a}(k)+\beta)t} & e^{2\kappa\hat{b}(k)t} & \Big(\frac{-2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\Big)e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k)t} \\
\Big(\frac{-(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))}{\beta}\Big)e^{2(\kappa\hat{a}(k)+\beta)t} & 0 & 0 \\
e^{2(\kappa\hat{a}(k)+\beta)t} & 0 & e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k)t}
\end{bmatrix}
$$

$$X(0) = \begin{bmatrix} \left(\frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) & 1 & \left(\frac{-2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) \\ \left(\frac{-(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))}{\beta}\right) & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \text{ and}$$

$$X^{-1}(0) = \begin{bmatrix} 0 & \left(\frac{-\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) & 0 \\ 1 & \left(\frac{\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)^2 & \left(\frac{2\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) \\ 0 & \left(\frac{\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right) & 1 \end{bmatrix}$$

$$e^{\hat{A}_6 t} = X(t)X^{-1}(0) =$$

$$\begin{bmatrix} e^{2\kappa\hat{b}(k)t} & \frac{\beta^2 e^{2(\kappa\hat{a}(k)+\beta)t}+\beta^2 e^{2\kappa\hat{b}(k)t}-2\beta^2 e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2} & \frac{2\beta e^{2\kappa\hat{b}(k)t}-2\beta e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} \\ 0 & e^{2(\kappa\hat{a}(k)+\beta)t} & 0 \\ 0 & \frac{-\beta e^{2(\kappa\hat{a}(k)+\beta)t}+\beta e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)} & e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t} \end{bmatrix}$$

with initial conditions $x_0 = \begin{bmatrix} \rho_0^2 & 1 & \rho_0 \end{bmatrix}^T$.

We have the solution $\hat{m}_2^G(t,k) = e^{\hat{A}_6 t}x_0 + e^{\hat{A}_6 t}\int_0^t e^{-\hat{A}_6 s}\hat{B}_6\vec{Q}ds$ where

$$\hat{B}_6 = \begin{bmatrix} -2\kappa\hat{b}(k) & 0 & 0 \\ 0 & -2\kappa\hat{a}(k) & 0 \\ 0 & 0 & 0 \end{bmatrix}, \vec{Q} = \begin{bmatrix} \rho_0 e^{\kappa\hat{b}(k)t} - \beta t e^{\kappa\hat{b}(k)t} \\ e^{\kappa\hat{b}(k)t} \\ 0 \end{bmatrix}$$

$$\hat{m}_2^G(t,k) = e^{\hat{A}_6 t}x_0 + e^{\hat{A}_6 t}\int_0^t e^{-\hat{A}_6 s}\hat{B}_6\vec{Q}ds =$$

$$\left[\begin{array}{c}
\rho_0^2 e^{2\kappa\hat{b}(k)t} + \left(\frac{\beta^2}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2}\right)\Big[e^{2(\kappa\hat{a}(k)+\beta)t} + e^{2\kappa\hat{b}(k)t} \\
-2e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] + \left(\frac{2\rho_0\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)\Big[e^{2\kappa\hat{b}(k)t} - \\
e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] + \Big(2\rho_0\Big)\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big] + \left(\frac{2\beta}{(\kappa\hat{b}(k))}\right)\cdot \\
\Big[e^{2\kappa\hat{b}(k)t} - e^{\kappa\hat{b}(k)t}(\kappa\hat{b}(k)t-1)\Big] + \left(\frac{2\beta\kappa\hat{a}(k)}{(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2}\right)\cdot \\
\Big[e^{-(2\kappa\hat{a}(k)+2\beta-3\kappa\hat{b}(k))t} - e^{2\kappa\hat{b}(k)t}\Big] + \left(\frac{2\kappa\hat{a}(k)\beta}{\kappa\hat{b}(k)(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2}\right)\cdot \\
\Big[e^{\kappa\hat{b}(k)t} - e^{2\kappa\hat{b}(k)t}\Big] + \left(\frac{2\kappa\hat{a}(k)\beta}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2(\kappa\hat{a}(k)+\beta)}\right)\cdot \\
\Big[e^{-(\kappa\hat{a}(k)+\beta-2\kappa\hat{b}(k))t} - e^{2\kappa\hat{b}(k)t}\Big] + \left(\frac{2\beta^2\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))}\right)\cdot \\
\Big[e^{\kappa\hat{b}(k)t} + e^{-(2\kappa\hat{a}(k)+2\beta-3\kappa\hat{b}(k))t} - 2e^{-(\kappa\hat{a}(k)+\beta+2\kappa\hat{b}(k))t} \\
-e^{2(\kappa\hat{a}(k)+\beta)t} - e^{2\kappa\hat{b}(k)t} + 2e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] \\
+\left(\frac{4\beta^2\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2(\kappa\hat{a}(k)+\beta)^2}\right)\Big[e^{-(\kappa\hat{a}(k)+\beta-2\kappa\hat{b}(k))t} - e^{\kappa\hat{b}(k))t} \\
-e^{2\kappa\hat{b}(k)t} + e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] - \left(\frac{4\kappa\hat{a}(k)\beta^2}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))^2(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))}\right)\cdot \\
\Big[e^{-(2\kappa\hat{a}(k)+2\beta-3\kappa\hat{b}(k))t} - e^{-(\kappa\hat{a}(k)+\beta-2\kappa\hat{b}(k))t} \\
-e^{2\kappa\hat{b}(k))t} + e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] \\
e^{2(\kappa\hat{a}(k)+\beta)t} + \left(\frac{2\kappa\hat{a}(k)}{2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k)}\right)\Big[e^{\kappa\hat{b}(k)t} - e^{2(\kappa\hat{a}(k)+\beta)t}\Big] \\
\left(\frac{\beta}{\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k)}\right)\Big[-e^{2(\kappa\hat{a}(k)+\beta)t} + e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] \\
+\rho_0 e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t} + \left(\frac{2\beta\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))}\right)\cdot \\
\Big[-e^{\kappa\hat{b}(k)t} + e^{-(\kappa\hat{a}(k)+\beta+2\kappa\hat{b}(k))t} + e^{2(\kappa\hat{a}(k)+\beta)t} \\
-e^{\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] + \left(\frac{2\beta\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+\beta)(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))}\right)\cdot \\
\Big[e^{\kappa\hat{b}(k)t} - e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big] - \left(\frac{2\beta\kappa\hat{a}(k)}{(\kappa\hat{a}(k)+\beta-\kappa\hat{b}(k))(2\kappa\hat{a}(k)+2\beta-\kappa\hat{b}(k))}\right)\cdot \\
\Big[e^{-(\kappa\hat{b}(k)+\beta-2\kappa\hat{b}(k))t} - e^{(\kappa\hat{a}(k)+\beta+\kappa\hat{b}(k))t}\Big]
\end{array}\right]$$

When $\kappa\hat{a}(k)+\beta \neq \hat{b}(k)$, $m_2^G(t,v) = \left(\frac{1}{2\pi}\right)^d \int_{T^d}\hat{m}_2^G(t,k)e^{-ikv}dk =$

$$\begin{bmatrix} \left(\frac{1}{2\pi}\right)^d \int_{T^d} \hat{m}_2^S(t,k) e^{-ikv} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \hat{m}_2^I(t,k) e^{-ikv} dk \\ \left(\frac{1}{2\pi}\right)^d \int_{T^d} \hat{m}_2^{SI}(t,k) e^{-ikv} dk \end{bmatrix}$$

CHAPTER 18
ANALYZING THE BEHAVIOR OF THE SECOND MOMENTS OF THE SI MODEL

18.1 Analyzing the behavior of the second moments in homogeneous space

The long term behavior of the second moments of the infected, susceptible-infected, and susceptible groups in the homogeneous space as $t \longrightarrow \infty$ can be broken up in 2 cases: $\beta > 0$ and $\beta = 0$. Tables 18.1 and 18.2 below summarizes the asymptotic behavior of the second moments of the susceptible, infected, susceptible-infected groups in homogeneous space as $t \longrightarrow \infty$.

As $t \to \infty$	$m_2^S(t,x,x)$	$m_2^I(t,x,x)$	$m_2^{SI}(t,x,x)$
$\beta > 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta = 0$	$\to \infty$	$\to \infty$	$\to C \in \mathbb{R}$

Table 18.1: Asymptotic Behavior of the Second Moments in Homogeneous Space when $x = y$

As $t \to \infty$	$m_2^S(t,v)$	$m_2^I(t,v)$	$m_2^{SI}(t,v)$
$\beta > 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\beta = 0$	$\to \infty$	$\to \infty$	$\to C \in \mathbb{R}$

Table 18.2: Asymptotic Behavior of the Second Moments in Homogeneous Space when $x \neq y$

18.2 Analyzing the behavior of the second moments in inhomogeneous space

The behavior of the second moments of the infected, susceptible-infected, and susceptible groups in the inhomogeneous space as $t \longrightarrow \infty$ can be broken up into 8 cases for the susceptible, infected and the susceptible-infected groups and is summarized in Tables 18.3 and 18.4.

Let $\alpha = \kappa\hat{a}(k) + \beta - \kappa\hat{b}(k)$, $\theta = \hat{a}(k) + \beta$, $\phi = \kappa\hat{b}(k)$, $C_4 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{-ikx}dk$, $C_5 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{-ikv}dk$, $C_6 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d}\Big(\frac{\kappa\hat{a}(k) + 2\kappa + \beta}{2\kappa\hat{a}(k) + 2\beta - \kappa\hat{b}(k)}\Big)e^{-ikx}dk$, and $C_7 = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d}\Big(\frac{2\kappa\hat{a}(k)}{2\kappa\hat{a}(k) + 2\beta - \kappa\hat{b}(k)}\Big)e^{-ikv}dk$.

As $t \to \infty$	$m_2^S(t,x,x) \to$	$m_2^I(t,x,x) \to$	$m_2^{SI}(t,x,x) \to$
$\alpha = 0,\ \phi < 0,\ \beta \geq 0$	$\to 0$	$\to 0$	$\to 0$
$\alpha = 0,\ \phi = 0,\ \beta \geq 0$	$\to \infty$	$\to C_4$	$\to \infty$
$\alpha < 0,\ \theta < 0,\ \phi < 0$	$\to \infty$	$\to 0$	$\to \infty$
$\alpha < 0,\ \theta < 0,\ \phi = 0$	$\to \infty$	$\to C_6$	$\to \infty$
$\alpha < 0,\ \theta > 0,\ \phi < 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\alpha > 0,\ \theta > 0,\ \phi = 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\alpha > 0,\ \theta > 0,\ \phi < 0$	$\to \infty$	$\to \infty$	$\to \infty$

Table 18.3: Asymptotic Behavior of the Second Moments in Inhomogeneous Space when $x = y$

When $\beta \geq 0, \alpha = 0$ and $\phi = \kappa\hat{b}(k) < 0$, we have the event that the mobility effect for the infected group plus the infection rate is equal to the mobility effect of the susceptible group, but the mobility effect of the susceptible group is negative. The result is that the second moment of the susceptible and infected populations goes to 0 as time t goes to infinity. The event that $\alpha > 0$ means that the infection rate plus the mobility effect of the infected group is larger than the mobility effect of the susceptible group, and for $\phi \leq 0$, the second moment of the susceptible group goes to infinity. This means that $E[S^2(t,x)] = var[S(t,x)] + E[S(t,x)]^2 \to \infty$, and if $E[S(t,x)]$ is finite then the variance of the susceptible group goes to infinity and the field will form the clusters/high peaks.

As $t \to \infty$	$m_2^S(t,v) \to$	$m_2^I(t,v) \to$	$m_2^{SI}(t,v) \to$
$\alpha = 0,\ \phi < 0,\ \beta \geq 0$	$\to 0$	$\to 0$	$\to 0$
$\alpha = 0,\ \phi = 0,\ \beta \geq 0$	$\to \infty$	$\to C_5$	$\to \infty$
$\alpha < 0,\ \theta < 0,\ \phi < 0$	$\to \infty$	$\to 0$	$\to \infty$
$\alpha < 0,\ \theta < 0,\ \phi = 0$	$\to \infty$	$\to C_7$	$\to \infty$
$\alpha < 0,\ \theta > 0,\ \phi < 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\alpha > 0,\ \theta > 0,\ \phi = 0$	$\to \infty$	$\to \infty$	$\to \infty$
$\alpha > 0,\ \theta > 0,\ \phi < 0$	$\to \infty$	$\to \infty$	$\to \infty$

Table 18.4: Asymptotic Behavior of the Second Moments in Inhomogeneous Space when $x \neq y$

Another significant event is when $\alpha < 0$ (meaning the mobility effect of the infected group plus the infection rate is less than the mobility effect of the susceptible group) but $\theta < 0$ and $\phi = 0$ (meaning the mobility effect of the infected group plus the infection rate is negative, and the mobility effect of the susceptible group is 0) then the second moment of the infected population at location x goes to a steady state C_7 as time t goes to infinity. The event that $\alpha < 0$ (meaning the mobility effect of the infected group plus the infection rate is less than the mobility effect of the susceptible group) but $\theta < 0$ and $\phi < 0$ (meaning the mobility effect of the infected group plus the infection rate is negative, and the mobility effect of the susceptible group is negative) then the second moment of the infected population at location x goes to 0 as time t goes to infinity, instead of a steady state. Comparing these two events shows that the mobility effect of the susceptible group being negative, rather than 0, causes the second moment of the infected population to go to 0, rather than to a steady state.

CHAPTER 19

INTERMITTENCY ANALYSIS FOR THE SI MODEL

When $x = y$ in Homogeneous Space:

$\lim_{t\to\infty} \frac{m_2^I(t,x,x)}{m_1^I(t,x)^2}$ where $m_1^I(t,x) = e^{\beta t}$ and

$$m_2^I(t,x,x) = \rho_0 e^{2\beta t} + \left(\frac{\beta + 2\kappa}{-\beta}\right)\left[e^{\beta t} - e^{2\beta t}\right]$$

$$\frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2} = \rho_0 - \left(\frac{\beta + 2\kappa}{\beta}\right)\left[e^{-\beta t} - 1\right]$$

Let $C_1 = \rho_0 + \left(\frac{\beta + 2\kappa}{\beta}\right)$

If $\beta > 0$, $\lim_{t\to\infty} e^{-\beta t} - 1 \to 0$, thus $\lim_{t\to\infty} \frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2} \to C_1$

If $\beta = 0$ and $C \in \mathbb{R}$, $\lim_{t\to\infty} \frac{\rho_0}{C^2} + \frac{2\kappa t}{C} \to \infty$

As $t \to \infty$	$\lim_{t\to\infty} \frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2}$	Result
$\beta > 0$	$\to C_1 < \infty$	No intermittency
$\beta = 0$	$\to \infty$	Intermittency

Table 19.1: Intermittency Analysis when $x = y$ in Homogeneous Space

When $x \neq y$ in Homogeneous Space:

$\lim_{t\to\infty} \frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)}$ where $m_1^I(t,x) = m_1^I(t,y) = e^{\beta t}$ and

$$m_2^I(t,x,y) = \rho_0 e^{2\beta t} + \Big(\frac{2\kappa a(v)}{\beta}\Big)\Big[e^{\beta t} - e^{2\beta t}\Big]$$

$$\frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)} = \rho_0 + \Big(\frac{2\kappa a(v)}{\beta}\Big)\Big[e^{-\beta t} - 1\Big]$$

Let $C_2 = \rho_0 - \Big(\frac{2\kappa a(v)}{\beta}\Big)$

As $t \to \infty$	$\lim_{t\to\infty} \frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)}$	Result
$\beta > 0$	$\to C_2 < \infty$	No intermittency
$\beta = 0$	$\to -\infty$	Intermittency

Table 19.2: Intermittency Analysis when $x \neq y$ in Homogeneous Space

When $x = y$ in Inhomogeneous Space:

Since $m_1^I(t,x) = \frac{1}{(2\pi)^d}\int_{T^d} e^{[\kappa\hat{\mathscr{L}}(k) + \beta]t} e^{-ikx} dk = e^{\beta t} p(t,0,x)$ and plugging in (18b) from Chapter 17, we get

$$\frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2} = \frac{\rho_0}{p(t,0,x)} + \Big(\frac{\kappa}{3\beta(p(t,0,x))^2}\Big)\Big[e^{3\beta t} - 1\Big] + \Big(\frac{\beta+2\kappa}{3\beta p(t,0,x)}\Big)\Big[e^{3\beta t} - 1\Big]$$

Since $p(t,x,y) = \frac{1}{(2\pi)^d}\int_{T^d} e^{ik(x-y)}e^{\kappa\hat{\mathscr{L}}(k)t} = \frac{1}{(2\pi)^d}\int_{T^d} cos(x-y)\hat{p}(t,0,k)dk \leq \frac{1}{(2\pi)^d}\int_{T^d}\hat{p}(t,0,k)dk = p(t,0,0)$, we have $p(t,x,y) \leq p(t,0,0)$ and as $t \to \infty$, $p(t,0,0) = \frac{C}{t^{d/2}} + o(t^{-d/2})$ [13]. Thus $\lim_{t\to\infty}\frac{C}{t^{d/2}} + o(t^{-d/2}) = 0^+$.

Thus $\frac{\rho_0}{p(t,0,x)} + \Big(\frac{\kappa}{3\beta(p(t,0,x))^2}\Big)\Big[e^{3\beta t} - 1\Big] + \Big(\frac{\beta+2\kappa}{3\beta p(t,0,x)}\Big)\Big[e^{3\beta t} - 1\Big] \geq \frac{\rho_0}{p(t,0,0)} + \Big(\frac{\kappa}{3\beta(p(t,0,0))^2}\Big)\Big[e^{3\beta t} - 1\Big] + \Big(\frac{\beta+2\kappa}{3\beta p(t,0,0)}\Big)\Big[e^{3\beta t} - 1\Big]$

Case 1: $\beta > 0$

$$\lim_{t\to\infty}\frac{\rho_0}{p(t,0,0)} + \Big(\frac{\kappa}{3\beta(p(t,0,0))^2}\Big)\Big[e^{3\beta t} - 1\Big] + \Big(\frac{\beta+2\kappa}{3\beta p(t,0,0)}\Big)\Big[e^{3\beta t} - 1\Big] = \frac{\infty}{0^+} = \infty$$

Case 2: $\beta = 0$

$$\lim_{t\to\infty}\frac{\rho_0}{p(t,0,0)} + \Big(\frac{\kappa t}{(p(t,0,0))^2}\Big) - \Big(\frac{2\kappa t}{p(t,0,0)}\Big) = \frac{\infty}{0^+} = \infty$$

As $t \to \infty$	$\lim_{t\to\infty}\frac{m_2^I(t,x,x)}{(m_1^I(t,x))^2}$	Result
$\beta > 0$	$\to \infty$	Intermittency
$\beta = 0$	$\to \infty$	Intermittency

Table 19.3: Intermittency Analysis when $x = y$ in Inhomogeneous Space

When $x \neq y$ in Inhomogeneous Space:

Using the solutions $m_1^I(t,x) = e^{\beta t}p(t,0,x)$, $m_1^I(t,y) = e^{\beta t}p(t,0,y)$ and (18d) from Chapter 17, we have that

$$\frac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)} = \frac{\rho_0}{p(t,0,y)} - \left(\frac{2\kappa a(v)}{3\beta p(t,0,y)}\right)\left[e^{3\beta t} - 1\right]$$

We have $p(t,x,y) \leq p(t,0,0)$ and as $t \to \infty$, $p(t,0,0) = \dfrac{C}{t^{d/2}} + o(t^{-d/2})$ [13]. Thus $\lim\limits_{t\to\infty} \dfrac{C}{t^{d/2}} + o(t^{-d/2}) = 0^+$.

Case 1: $\beta > 0$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} + \left(\frac{2\kappa a(v)}{3\beta p(t,0,0)}\right)\left[e^{3\beta t} - 1\right] = \frac{\infty}{0^+} = \infty$$

Case 2: $\beta = 0$

$$\lim_{t\to\infty} \frac{\rho_0}{p(t,0,0)} + \left(\frac{2\kappa a(v)t}{p(t,0,0)}\right) = \frac{\infty}{0^+} = \infty$$

As $t \to \infty$	$\lim\limits_{t\to\infty} \dfrac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)}$	Result
$\beta > 0$	$\to \infty$	Intermittency
$\beta = 0$	$\to \infty$	Intermittency

Table 19.4: Intermittency Analysis when $x \neq y$ in Inhomogeneous Space

In homogeneous space, when $\beta > 0$, we have the event that the infection rate is positive and there is no intermittency phenomenon. This means that the infection is so widespread that as $t \to \infty$ and it does not matter where the location is, everywhere will have the infection. This is compared to the case where $\beta =$

0, meaning the infection rate is equal to 0, and the result is that there will be peaks/clusters with a higher concentration of infection in some locations, and thus there is the intermittency phenomenon.

In inhomogeneous space, we have the intermittency phenomenon for both cases. This means that, when the infection rate β is greater than and equal to 0, the infection will form clusters with a higher concentration of infection in some locations and the intermittency phenomenon appears in the field.

CHAPTER 20
LYAPUNOV ANALYSIS FOR THE SI MODEL

Homogeneous Space:

Theorem 28 *For the first moment of the infected group in homogeneous space, the Lyapunov Exponents are* $\lambda_{7,1} = \beta$ *when* $\beta > 0$*, and* $\lambda_{7,2} = 0$ *when* $\beta = 0$

Proof of Theorem 28: Lyapunov Exponent $\lambda_{7,i} = \lim_{t\to\infty} \dfrac{\ln(m_1^I(t,x))}{t}$ where $m_1^I(t,x) = e^{\beta t}$

Case 1: $\beta > 0$, $\displaystyle\lim_{t\to\infty} \frac{\ln(m_1^I(t,x))}{t} = \lim_{t\to\infty} \frac{\ln\left(e^{\beta t}\right)}{t} = \beta$

Case 2: $\beta = 0$, $m_1^I(t,x) = C_1 \in \mathbb{R}$, $\displaystyle\lim_{t\to\infty} \frac{\ln(C_1)}{t} = 0$

For case 1, $\beta > 0$, the Lyapunov exponent is positive and therefore $m_1^I(t,x)$ is increasing. For case 2, the Lyapunov exponent is 0 and the first moment of the infected group is neither increasing nor decreasing.

Theorem 29 *For the second moment of the infected group in homogeneous space, when* $x = y$*, the Lyapunov Exponents are* $\lambda_{8,1} = \dfrac{2\beta\rho_0 + 2(\beta+2\kappa)}{\rho_0 + \left(\dfrac{\beta+2\kappa}{\beta}\right)}$ *when* $\beta > 0$*, and* $\lambda_{8,2} = 0$ *when* $\beta = 0$

Proof of Theorem 29: Lyapunov Exponent $\lambda_{8,i} = \lim_{t\to\infty} \dfrac{\ln(m_2^I(t,x,x))}{t}$ where $m_2^I(t,x,x)$ is given by $(18a)$ in Chapter 17 Theorem 25

Case 1: $\beta > 0$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} = \lim_{t\to\infty} \frac{2\beta\rho_0 - (\beta+2\kappa)\left[e^{-\beta t} - 2\right]}{\rho_0 e^{\beta t} + \left(\dfrac{\beta+2\kappa}{-\beta}\right)\left[e^{-\beta t} - 1\right]}$$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} = \frac{2\beta\rho_0 + 2(\beta+2\kappa)}{\rho_0 + \left(\dfrac{\beta+2\kappa}{\beta}\right)}$$

Case 2: $\beta = 0$, $m_2^I(t,x,x) = \rho_0 + (\beta+2\kappa)Ct$ where $C \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} = \lim_{t\to\infty} \frac{(\beta+2\kappa)C}{\rho_0 + (\beta+2\kappa)Ct} = 0$$

For case 1, $\beta > 0$ and $\lambda_{8,2} > 0$ and therefore $m_2^I(t,x,x)$ is increasing. For case 2, the Lyapunov exponent is 0 and the second moment of the infected group is neither increasing nor decreasing.

Theorem 30 *For the second moment of the infected group in homogeneous space, when $x \neq y$, the Lyapunov Exponents are* $\lambda_{9,1} = \dfrac{2\beta\rho_0 - 4\kappa a(v)}{\rho_0 - \left(\dfrac{2\kappa a(v)}{\beta}\right)}$ *when $\beta > 0$, and $\lambda_{9,2} = 0$ when $\beta = 0$*

Proof of Theorem 30: Lyapunov Exponent $\lambda_{9,i} = \lim_{t\to\infty} \dfrac{\ln(m_2^I(t,v))}{t}$ where $m_2^I(t,v)$ is given by $(18c)$ in Chapter 17 Theorem 25

Case 1: $\beta > 0$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \lim_{t\to\infty} \frac{2\beta\rho_0 + (2\kappa a(v))\left[e^{-\beta t} - 2\right]}{\rho_0 + \left(\frac{2\kappa a(v)}{\beta}\right)\left[e^{-\beta t} - 1\right]}$$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \frac{2\beta\rho_0 - 4\kappa a(v)}{\rho_0 - \left(\frac{2\kappa a(v)}{\beta}\right)}$$

Case 2: $\beta = 0$, $m_2^I(t,v) = \rho_0 + 2\kappa a(v)Ct$ where $C \in \mathbb{R}$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,v))}{t} = \lim_{t\to\infty} \frac{2\kappa a(v)C}{\rho_0 + 2\kappa a(v)Ct} = 0$$

For case 1, $\beta > 0$, if $2\beta\rho_0 > 4\kappa a(v)$, then $\lambda_{9,1} > 0$ and $m_2^I(t,v)$ is increasing. For case 1, if $2\beta\rho_0 < 4\kappa a(v)$, then $\lambda_{9,1} < 0$ and $m_2^I(t,v)$ is decreasing. For case 2, the Lyapunov exponent is 0 and the second moment of the infected group is neither increasing nor decreasing.

As $t \to \infty$	$\lim_{t\to\infty} \frac{\ln m_1^I(t,x)}{t}$	$\lim_{t\to\infty} \frac{\ln m_2^I(t,x,x)}{t}$	$\lim_{t\to\infty} \frac{\ln m_2^I(t,v)}{t}$
$\beta > 0$	$< \infty$	$< \infty$	$< \infty$
$\beta = 0$	$< \infty$	$< \infty$	$< \infty$

Table 20.1: Lyapunov Exponents in Homogeneous Space

Inhomogeneous Space:

Theorem 31 *For the first moment of the infected group in inhomogeneous space, the Lyapunov Exponents are* $\lambda_{10,1} = \beta + \kappa\hat{\mathscr{L}}_a(k)$ *when* $\beta > 0$*, and* $\lambda_{10,2} = \kappa\hat{\mathscr{L}}_a(k)$

when $\beta = 0$

Proof of Theorem 31: Lyapunov Exponent $\lambda_{10,i} = \lim_{t\to\infty} \dfrac{\ln(m_1^I(t,x))}{t}$ where $m_1^I(t,x) = \dfrac{1}{(2\pi)^d}\displaystyle\int_{T^d} e^{[\kappa\hat{\mathscr{L}}(k)+\beta]t} e^{-ikx} dk = e^{\beta t} p(t,0,x)$

Case 1: $\beta > 0$, $\displaystyle\lim_{t\to\infty} \frac{\ln(m_1^I(t,x))}{t} = \lim_{t\to\infty} \frac{\ln\left(e^{\beta t}p(t,0,x)\right)}{t}$

$$\lim_{t\to\infty} \frac{\ln(m_1^I(t,x))}{t} = \beta + \kappa\hat{\mathscr{L}}_a(k)$$

Case 2: $\beta = 0$, $m_1^I(t,x) = p(t,0,x)$

$$\lim_{t\to\infty} \frac{\ln(m_1^I(t,x))}{t} = \frac{\frac{d}{dt}p(t,0,x)}{p(t,0,x)} = \kappa\hat{\mathscr{L}}_a(k)$$

For case 1, $\beta > 0, \hat{\mathscr{L}}(k) < 0$ and if $\beta + \kappa\hat{\mathscr{L}}(k) > 0$, then $\lambda_{10,1} > 0$ and the first moment of the infected group is increasing. For case 1, if $\beta + \kappa\hat{\mathscr{L}}(k) < 0$, then $\lambda_{10,1} < 0$ and $m_1^I(t,x)$ is decreasing. For case 2, the Lyapunov exponent is negative and the first moment of the infected group is decreasing.

Theorem 32 *For the second moment of the infected group in inhomogeneous space, when* $x = y$*, the Lyapunov Exponents are* $\lambda_{11,1} = 5\beta$ *when* $\beta > 0$*, and* $\lambda_{11,2} = 0$ *when* $\beta = 0$

Proof of Theorem 32: Lyapunov Exponent $\lambda_{11,i} = \lim_{t\to\infty} \dfrac{\ln(m_2^I(t,x,x))}{t}$ where $m_2^I(t,x,x)$ is given by $(18b)$ in Chapter 17 Theorem 25

Case 1: $\beta > 0$, using L'Hôpital's Rule we get that and note that $p(t,x,y) < p(t,0,0)$ and as $t \to \infty, p(t,0,0) = \dfrac{C}{t^{d/2}} + o(t^{-d/2}) < \infty$. Let $C_1 = \dfrac{C}{t^{d/2}} + o(t^{-d/2})$, then $\lim\limits_{t\to\infty} C_1 = 0$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} = \frac{\Big(\frac{5\kappa}{3}\Big) + \Big(\frac{\beta+2\kappa}{3\beta}\Big)\kappa\hat{\mathscr{L}}_a(k)C_1 + \Big(\frac{5(\beta+2\kappa)}{3}\Big)C_1}{\Big(\frac{\kappa}{3\beta}\Big) + \Big(\frac{\beta+2\kappa}{3\beta}\Big)C_1}$$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} = 5\beta$$

Case 2: $\beta = 0$, $m_2^I(t,x,x) = \rho_0 p(t,0,x) + \kappa t + (+2\kappa)tp(t,0,x)$

$$\lim_{t\to\infty} \frac{\ln(m_2^I(t,x,x))}{t} = \frac{(2\kappa)\kappa\hat{\mathscr{L}}_a(k)\Big(\frac{C}{t^{d/2}} + o(t^{-d/2})\Big)}{\kappa + 2\kappa\Big(\frac{C}{t^{d/2}} + o(t^{-d/2})\Big)} = \frac{0}{\kappa} = 0$$

For case 1, $\beta > 0$, and the second moment of the infected group is increasing. For case 2, the Lyapunov exponent is 0 and $m_2^I(t,x,x)$ is neither increasing or decreasing.

Theorem 33 *For the second moment of the infected group in inhomogeneous space, when $x \neq y$, the Lyapunov Exponents are $\lambda_{12,1} = \kappa\hat{\mathscr{L}}_a(k) + 5\beta$ when $\beta > 0$, and $\lambda_{12,2} = \kappa\hat{\mathscr{L}}_a(k)$ when $\beta = 0$*

Proof of Theorem 33: Lyapunov Exponent $\lambda_{12,i} = \lim\limits_{t\to\infty} \dfrac{\ln(m_2^I(t,v)}{t}$ where $m_2^I(t,v)$ is given by $(18d)$ in Chapter 17 Theorem 25

Case 1: $\beta > 0$

$$\lim_{t\to\infty}\frac{\ln(m_2^I(t,v))}{t} = \lim_{t\to\infty}\frac{-\Big(\dfrac{2\kappa a(v)}{3\beta}\Big)\kappa\hat{\mathscr{L}}_a(k)p(t,0,x) - \Big(\dfrac{10\kappa a(v)}{3}\Big)p(t,0,x)}{-\Big(\dfrac{2\kappa a(v)}{3\beta}\Big)p(t,0,x)}$$

$$\lim_{t\to\infty}\frac{\ln(m_2^I(t,v))}{t} = \kappa\hat{\mathscr{L}}_a(k) + 5\beta$$

Case 2: $\beta = 0$, $m_2^I(t,v) = \rho_0 p(t,0,x) - 2\kappa a(v)tp(t,0,x)$

$$\lim_{t\to\infty}\frac{\ln(m_2^I(t,v))}{t} = \lim_{t\to\infty}\frac{\rho_0\kappa\hat{\mathscr{L}}_a(k)\frac{1}{t} - 2\kappa a(v)\hat{\mathscr{L}}(k) - 2\kappa a(v)\frac{1}{t}}{\rho_0\frac{1}{t} - 2\kappa a(v)} = \kappa\hat{\mathscr{L}}_a(k)$$

For case 1, β is positive and $\kappa\hat{\mathscr{L}}(k) < 0$, if $5\beta > \kappa\hat{\mathscr{L}}(k)$, then $\lambda_{12,1} > 0$ and the second moment of the infected group is increasing. For case 1, if $5\beta < \kappa\hat{\mathscr{L}}(k)$, then $\lambda_{12,1} < 0$ and thus $m_2^I(t,v)$ is decreasing. For case 2, $\lambda_{12,2} < 0$ and the second moment of the infected group is decreasing.

As $t\to\infty$	$\lim_{t\to\infty}\frac{\ln m_1^I(t,x)}{t}$	$\lim_{t\to\infty}\frac{\ln m_2^I(t,x,x)}{t}$	$\lim_{t\to\infty}\frac{\ln m_2^I(t,v)}{t}$
$\beta > 0$	$< \infty$	$< \infty$	$< \infty$
$\beta = 0$	$< \infty$	$< \infty$	$< \infty$

Table 20.2: Lyapunov Exponents in Inhomogeneous Space

For our SI model, the homogeneous space we have that the Lyapunov exponents for the first and second moments are finite for both cases, when $\beta > 0$, and $\beta = 0$. For the inhomogeneous space we have that the Lyapunov exponents for the first and second moments are finite for both cases, when $\beta > 0$, and $\beta = 0$. If $\lambda_{7,i}$ or $\lambda_{10,i}$ is positive, then the first moment of the infected group is increasing. If $\lambda_{7,i}$ or $\lambda_{10,i}$ is negative, then the first moment of the infected group is decreasing. If the Lyapunov exponent is 0, then the $m_1^I(t,x)$ is neither increasing nor decreasing.

CHAPTER 21
CONCLUSIONS

In this dissertation, a new SIR model with mobility is developed. The new model with migration has the assumption that all of the particles can have spatial motion, within the susceptible, infected, and recovered groups (in addition to the inter-compartmental motion). Additionally, for the SIR model, we assume that the spatial motion of healthy particles is the same as the spatial motion of an infected particle, and that the only one type of movement can happen at a time, meaning a particle can jump to another location or they can jump states.

In Chapter 3, we derived the differential equations for the first moments of the susceptible, infected and recovered groups using the Kolmogorov Forward Equations. In Chapter 4, we solved for the first moments of the susceptible, infected, and recovered groups in the homogeneous space, the inhomogeneous space, and in inhomogeneous space using matrices. In the homogeneous space we used regular ODE methods to solve for the first moments of S, I and R. In the inhomogeneous space, we used Fourier transforms, inverse Fourier transforms, and transition probabilities to derive a general solutions to the inhomogeneous equations based on the Kac-Feyman formula and the Duhamel's principle.

In Chapter 5, we analyzed the long term behavior of the first moments as

$t \to \infty$.When $\beta > \gamma, \alpha < 0$ and $\theta = \kappa\hat{a}(k) + \beta - \gamma < 0$, we have the event that the infection rate is higher than the recovery rate, but the infection rate minus the recovery rate $(\beta - \gamma)$ is smaller than the mobility effect $\kappa\hat{a}(k)$, meaning the mobility effect is stronger. The result is that the expected value of the susceptible, infected, and recovered populations goes to 0 as time t goes to infinity. Another noteworthy event is when $\beta = \gamma$, meaning the infection rate is equal to the recovery rate. When $\beta = \gamma$ and the mobility effect $\kappa\hat{a}(k) < 0$, the expected value of the infected population at location x goes to 0 as time t goes to infinity. The event where $\beta = \gamma$ and the mobility effect $\kappa\hat{a}(k) = 0$, we have that the expected value of the infected population at location x goes to a finite constant C_1 as t goes to infinity. This means that the expected value of the infected population goes to a steady state, rather than going to 0. This makes our model different from the classical SIR model because the infected population does not always go to 0 when the infection rate is equal to the recovery rate because we have active movement to and from outside location x.

In Chapters 6, 7 and 8 we used the same procedure to derive, solve for, and analyze the second moments of the SIR model, where the second moments $E[S(t,x,x)], E[I(t,x,x)], E[R(t,x,x)], E[S(t,x)I(t,x)]$, and $E[R(t,x)I(t,x)]$ groups, where each group has 2 cases: when the locations $x = y$ and when the locations $x \neq y$. In Chapters 9, 10 and 11, we analyzed the intermittency and Lyapunov exponents of the infected group of the SIR model. For the SIR model with mobility, intermittency in the infected group means that the model forms clusters of infected people. For our model, we define $m_2(t,x,y) = E[u(t,x)u(t,y)]$, and for the infected group we have $m_2^I(t,x,y) = E[I(t,x)I(t,y)]$. When $x = y$, we have that if $\lim_{t\to\infty} \dfrac{m_2^I(t,x,x)}{m_1^I(t,x)^2} = \infty$, then it has the intermittency phenomenon. When $x \neq y$, we have that if $\lim_{t\to\infty} \dfrac{m_2^I(t,x,y)}{m_1^I(t,x)m_1^I(t,y)} = \infty$, then it has the intermittency phenomenon. For the Lyapunov exponents of the first moments of our model, we have that $\lambda_1 = \lim_{t\to\infty} \dfrac{\ln(m_1^I(t,x))}{t}$. For the Lyapunov exponents of the second moments of our model, we have that $\lambda_{2,x,x} = \lim_{t\to\infty} \dfrac{\ln(m_2^I(t,x,x))}{t}$ and $\lambda_{2,x,y} = \lim_{t\to\infty} \dfrac{\ln(m_2^I(t,x,y))}{t}$.

In Chapters 12 through 20, we introduced the SI model where the susceptible and infected groups have different probability kernels. The set up for the new SI model with migration has the assumption that all of the particles can have spatial motion, within the susceptible and infected groups (in addition to the inter-compartmental motion). We are assuming the total population is fixed, i.e- $N(t) = S(t) + I(t)$ but $N(t,x)$, the total population at position x at time t, is varying. We are now assuming that the spatial motion of healthy particles is not the same as the spatial motion of an infected particle, and now there are two probability kernels- $a(z)$ and $b(z)$. Following the same procedure that we used for the SIR model, we derived the differential equations for the first and second moments, solved for the first and second moments, and analyzed the first and second moments of the SI model. Lastly, we analyzed the intermittency and Lyapunov exponents of the infected group of the SI model.

Now let us compare SIR model and SI model. The SIR model has the assumption that the mobility $a(z)$ of the susceptible and the infected groups are equal. The SI model has assumption that the susceptible and infected groups have different probability kernels, $b(z)$ and $a(z)$ respectively. In the SIR model, there is the event that $\beta - \gamma > 0$, $\alpha < 0$, meaning the mobility effect of the infected group is not 0, and $\theta = \kappa\hat{a}(k) + \beta - \gamma > 0$, meaning κ times the Fourier transform of mobility kernel $a(z)$ plus the infection rate β is more than the recovery rate γ, and we have that the first moment of the infected group goes to infinity and the steady state does not exist. The comparable event in the SI model, is the event that $\alpha > 0$ (meaning the mobility effect of the infected group plus the infection rate is greater than the mobility effect of the susceptible group) but $\theta > 0$ and $\phi = 0$ (meaning the mobility effect of the infected group plus the infection rate is positive, and the mobility effect of the susceptible group is 0) then the expected value of the infected population at location x goes to 0 as time t goes to infinity. This shows that the mobility rates for the susceptible and infected groups being different does effect the expected value of the infected group.

REFERENCES

[1] Linda JS Allen, *Some discrete-time si, sir, and sis epidemic models*, Mathematical biosciences **124** (1994), no. 1, 83–105.

[2] ———, *An introduction to stochastic epidemic models*, Mathematical epidemiology, Springer, 2008, pp. 81–130.

[3] René Carmona, René A Carmona, and Stanislav A Molchanov, *Parabolic anderson problem and intermittency*, vol. 518, American Mathematical Soc., 1994.

[4] René A Carmona and Stanislav A Molchanov, *Stationary parabolic anderson model and intermittency*, Probability theory and related fields **102** (1995), no. 4, 433–453.

[5] A Getan, S Molchanov, and B Vainberg, *Intermittency for branching walks with heavy tails*, Stochastics and Dynamics **17** (2017), no. 06, 1750044.

[6] Dan Han, Yulia Makarova, Stanislav Molchanov, and Elena Yarovaya, *Branching random walks with immigration*, International Conference on Analytical and Computational Methods in Probability Theory, Springer, 2017, pp. 401–408.

[7] Dan Han, Stanislav Molchanov, and Joseph Whitmeyer, *Population processes with immigration*, International Conference on Modern Problems of Stochastic Analysis and Statistics, Springer, 2016, pp. 411–434.

[8] Yuri Kondratiev, Oleksandr Kutoviy, and Stanislav Molchanov, *On the population dynamics in the stationary random environment*, U. of Bielefeld preprint (2011).

[9] Yuri Kondratiev and Anatoli Skorokhod, *On contact processes in continuum*, Infinite Dimensional Analysis, Quantum Probability and Related Topics **9** (2006), no. 02, 187–198.

[10] Wolfgang König, *The parabolic anderson model: Random walk in random potential*, Birkhäuser, 2016.

[11] Thomas M Liggett et al., *Stochastic interacting systems: contact, voter and exclusion processes*, vol. 324, springer science & Business Media, 1999.

[12] Yu Makarova, D Han, S Molchanov, and E Yarovaya, *Branching random walks with immigration. lyapunov stability*, arXiv preprint arXiv:2002.03999 (2020).

[13] Stanislav Alekseevich Molchanov and Elena Borisovna Yarovaya, *Large deviations for a symmetric branching random walk on a multidimensional lattice*, Proceedings of the Steklov Institute of Mathematics **282** (2013), no. 1, 186–201.

[14] Vladimir Panov, *Modern problems of stochastic analysis and statistics: Selected contributions in honor of valentin konakov*, vol. 208, Springer, 2017.

[15] Zhan Shi, *Branching random walks*, Springer, 2015.

[16] Kôhei Uchiyama, *Green's functions for random walks on n*, Proceedings of the London Mathematical Society **77** (1998), no. 1, 215–240.

[17] Howard Howie Weiss, *The sir model and the foundations of public health*, Materials matematics (2013), 0001–17.

[18] E Yarovaya, *Positive discrete spectrum of the evolutionary operator of supercritical branching walks with heavy tails*, Methodology and Computing in Applied Probability **19** (2017), no. 4, 1151–1167.

[19] ______, *Operator equations of branching random walks*, Methodology and Computing in Applied Probability **21** (2019), no. 3, 1007–1021.

[20] Elena Borisovna Yarovaya, *Spectral properties of evolutionary operators in branching random walk models*, Mathematical Notes **92** (2012), no. 1, 115–131.

[21] ______, *Spectral asymptotics of a supercritical branching random walk*, Probability theory and its applications **62** (2017), no. 3, 518–541.

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