8-2023

Stability of Cauchy's equation on $\boldsymbol{\Delta}$+.

Holden Wells
University of Louisville

STABILITY OF CAUCHY'S EQUATION ON Δ^+

By

Holden Wells
B.S., University of Louisville, 2018
M.A., University of Louisville, 2019

A Dissertation
Submitted to the Faculty of the
College of Arts and Sciences of the University of Louisville
in Partial Fulfillment of the Requirements
for the Degree of

Doctor of Philosophy
in
Applied and Industrial Mathematics

Department of Mathematics
University of Louisville
Louisville, Kentucky

August 2023

STABILITY OF CAUCHY'S EQUATION ON Δ^+

Submitted by

Holden Wells

A Dissertation Approved on

August 1, 2023

by the Following Dissertation Committee:

Dr. Thomas Riedel,
Dissertation Director

Dr. Robert Powers

Dr. Lee Larson

Dr. Paul Himes

DEDICATION

For Flannery and Emmy

ACKNOWLEDGEMENTS

I want to thank my advisor, Dr. Riedel, for the time he has taken to guide me on this process, and his willingness to listen to me state a claim three times before saying what I mean. I truly appreciate the time spent discussing mathematics and what it is to be a working mathematician.

I also want to express my thanks and appreciation for the members of my committee who have not only been generous at every step of the way with their schedules, but also have taken formative roles in my education.

I am indebted to the countless (but countable) collection of people required to support someone to this point. Thank you to the friends, teachers, coaches, teammates, and colleagues who supported and formed me. I especially want to thank my parents: My father for talking about physics on the ride home from school, and my mother giving me the emotional skills to navigate difficulty.

ABSTRACT

STABILITY OF CAUCHY'S EQUATION ON Δ^+

Holden Wells

August 1, 2023

The most famous functional equation $f(x + y) = f(x) + f(y)$ known as Cauchy's equation due to its appearance in the seminal analysis text Cours d'Analyse [9], was used to understand fundamental aspects of the real numbers and the importance of regularity assumptions in mathematical analysis. Since then, the equation has been abstracted and examined in many contexts. One such examination, introduced by Stanislaw Ulam and furthered by Donald Hyers, was that of stability[20]. Hyers demonstrated that Cauchy's equation exhibited stability over Banach Spaces in the following sense: functions that approximately satisfy Cauchy's equation are approximated with the same level of error by functions that are solutions of Cauchy's equation, namely linear maps. Here we pose the question of the stability of Cauchy's equation for functions defined on the monoid known as Δ^+, the space of cumulative distribution functions. We present stability results analogous to those of Hyer's and Ulam and results involving a new perspective on stability. We furnish a connection between the two perspectives and examples of the need for some regularity assumptions.

TABLE OF CONTENTS

CHAPTER 1
INTRODUCTION

In this dissertation, we set out to examine what properties must be true of approximate solutions to an important functional equation, Cauchy's Equation, in a novel setting, the space of cumulative distribution functions on non-negative random variables, or Δ^+. Of particular interest will be examining the relationship between these approximate solutions and the true solutions. This relationship is called the stability of the functional equation. In the process of examining the stability of Cauchy's equation in Δ^+, we will discuss what Cauchy's equation means in Δ^+, and we will explore how to meaningfully and naturally generalize the notion of approximation in a space which is not strictly numerical. Alongside this process, we will observe what this means about the construction of Δ^+, and why examining Cauchy's equation allows us to naturally explore this construction.

In the introduction chapter, we provide a brief history of the functional equations and their stability as a means of developing motivation and understanding for results presented in Chapter 4 as well as a means of providing a sense of place for these results in the broader mathematical context.

In the following two chapters, we present results relevant to Δ^+. Chapter 2 provides history, motivation, and structural understanding of the space, and Chapter 3 gives the necessary understanding of Cauchy's equation in the context of Δ^+ required to appreciate stability.

Finally, in Chapter 4 we present new results concerning stability.

History of Equation Solving

Functional Equations is the field of mathematics in which equations are solved for unknown functions. By solving we specifically mean identify the possible functions (or combinations of functions) which make a given equation true.

Generally, one motivation for solving equations is to discover the answer to a practical problem. In the book, الكتاب المختصر في حساب الجبر والمقابلة (Romanized: al-Kitab al Mukhtasar fi Hisab al-Jabr wal-Muqabalah). The author, Muhammad ibn Musa al-Khwarizimi, addresses a complex inheritance situation that has to adhere to a set of rules observed in Baghdad circa 830:

"A man dies, leaving two sons behind him, and bequeathing one third of his capital to a stranger. He leaves ten dirhems [currency] of property and a claim of ten dirhems upon one of the sons."[3]

The observed rules specify that the debts forgiven by the estate are considered part of the size of the estate. Since both the share of the stranger and the share of debt forgiveness to the indebted son are dependent on the size of the estate, a near circular issue appears. To resolve the problem, al-Khwārizmī pulls the clever trick of assuming the situation has already been resolved and that the resolution is by choosing the amount of debt forgiven to be "thing" (or in our case x). Then concludes that the estate has been distributed correctly x would satisfy the equation $\frac{2}{3}(10+x) = 2x$. Through a series of repeatable steps for which al-Khwārizmī became the namesake of the word algorithm, it is established that the unique value of x that solves the equation is 5.

However, the utility of equations and their solutions extends beyond solving every day problems, as is often the case in functional equations, solutions can define properties. While the equation $\frac{2}{3}(10+x) = 2x$ has a unique solution, there are many choices of a, b, c which make the equation $a^2+b^2 = c^2$ true, and characterizing these solutions is synonymous with characterizing right triangles.

For a functional equations context, we examine how to parse 2^π. Given that exponentiation, was first a tool for abbreviating iterated multiplication, the phrase 2^π might seem rather odd. What would it mean to multiply something by itself π times? To begin to get an idea, we observe for natural numbers m and n we have

$$2^{m+n} = \underbrace{2 \cdot 2 \cdot \ldots \cdot 2 \cdot 2}_{m+n} = \underbrace{(2 \cdot \ldots \cdot 2)}_{m} \cdot \underbrace{(2 \cdot \ldots \cdot 2)}_{n} = 2^m \cdot 2^n$$

and in fact this property holds regardless of the base. If we create a function $\epsilon(x)$ that captures this property and applies it to all real numbers x and y we have

$$\epsilon(x+y) = \epsilon(x)\epsilon(y)$$

Since multiplication is uninteresting if all of the factors are one, and negative numbers introduce unnecessary complication for the introduction, we will also require $\epsilon(1) \neq 1$ and $\epsilon(1) > 0$. If we assign a reference value to $\epsilon(1)$, say b, we see that this function behaves like basic exponentiation would on natural numbers since by induction we have for a natural number n

$$\epsilon(n) = \epsilon(1 + \cdots + 1) = \epsilon(1) \cdot \cdots \cdot \epsilon(1) = \epsilon(1)^n = b^n.$$

Furthermore for natural numbers m and n

$$b^{m+n} = \epsilon(m+n) = \epsilon(n) + \epsilon(m) = b^m b^n.$$

From these facts, other properties emerge,

$$\epsilon(1) = \epsilon(1+0) = \epsilon(1)\epsilon(0)$$

meaning $b^0 = 1$, and

$$1 = \epsilon(0) = \epsilon(1-1) = \epsilon(1)\epsilon(-1)$$

which can be interpreted as meaning $b^{-1} = \frac{1}{\epsilon(1)} = \frac{1}{b}$. We also see that

$$\epsilon(x) = \epsilon\left(\frac{x}{n} + \cdots + \frac{x}{n}\right) = \epsilon\left(\frac{x}{n}\right)^n$$

which admits the interpretation that $b^{x/n} = \sqrt[n]{b^x}$. Therefore, we can meaningfully parse $b^{x/n}$ so long as we can evaluate b^x. In particular this means that we can evaluate all rational powers of b. If we add the assumption that ϵ is a continuous function (since for any $a \in \mathbb{Q}$ ϵ restricted to $\mathbb{Q}$ is bounded and monotone on $(-\infty, a]$, we have the existence a continuous extension of the restricted ϵ), we immediately have the ability to parse all real exponents. Since b was arbitrarily chosen, we in fact have a way to meaningfully discuss 2^π as the limit of the sequence $2^3, 2^{3.1}, 2^{3.14}, \ldots$.

This example has both historical value and investigative value. On the historical, the inverse function of ϵ, the logarithm is argued to be the first function to be defined by a functional equation [1]. Burgi defined and constructed logarithms by using the properties of exponents established above [8], Napier constructed them via primitive differential equation [38], and Briggs constructed them explicitly by the functional equation

$$\lambda(xy) = \lambda(x) + \lambda(y)$$

(although this was via description and not in modern function notation)[6]. In all three cases, the logarithm was defined in a manner that can be expressed via a functional equation.

On the investigative side, the construction of the exponential function as above raises the question of how necessary it is to assume continuity. Is it perhaps possible that there is enough structure imposed in the defining equation itself to do away with continuity (or at least make a weaker assumption) and still get the same result?

Cauchy's Equation

To gain some traction, we look to one of the foundational works of modern analysis, Cauchy's Cours D'Analyse. In his book, Cauchy introduced the four functional equations to emphasize the use of regularity assumptions like continuity and the necessity of being explicit in assuming them [9]. Below are the three thematically important equations

$$f(x+y) = f(x) + f(y) \tag{1.1}$$

$$\epsilon(x+y) = \epsilon(x)\epsilon(y) \tag{1.2}$$

$$\lambda(xy) = \lambda(x) + \lambda(y) \tag{1.3}$$

the first of which is referred to as Cauchy's equation due in part to its role in defining solutions to all three equations. This role is highlighted in the following theorem:

Theorem 1 *Let $f : \mathbb{R} \to \mathbb{R}$ be a solution of Cauchy's equation, then ϵ is a solution to Cauchy's exponential equation, and λ is a solution to Cauchy's logarithmic equation. That is,*

$$\epsilon(x+y) = \epsilon(x)\epsilon(y)$$

$$\lambda(xy) = \lambda(x) + \lambda(y)$$

if and only if there exists $a, b \in (0, \infty)$ such that $\epsilon(x) = a^{f(x)}$ and $\lambda(x) = f(\log_b(x))$ for all $x \in \mathbb{R}$.

We therefore have as an immediate corollary that the existence of discontinuous solutions to Cauchy's equation is equivalent to discontinuous solutions of the Cauchy's logarithmic and exponential equations. If any such solutions exist, then the assumption of continuity would be necessary.

While Cauchy was unable to characterize any discontinuous solutions, he did completely characterize the continuous solutions in a method similar to our examination of the exponential equation:

Theorem 2 *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Then for all $x, y \in \mathbb{R}$*

$$f(x + y) = f(x) + f(y)$$

if and only if there exists $c \in \mathbb{R}$ such that $f(x) = cx$ for all $x \in \mathbb{R}$. In particular, if f need only satisfy the equation for $x, y \in \mathbb{Q}$ the assumption of continuity may be eliminated.

Several mathematicians including Banach, Sierpinski, and Steinhaus worked to reduce the strength of the regularity assumptions required for Cauchy's equation to necessarily produce linear solutions. One such relaxation is the following

Theorem 3 *Let $f : \mathbb{R} \to \mathbb{R}$ be a solution to Cauchy's equation. There exists $c \in \mathbb{R}$ such that $f(x) = cx$ if and only if f can be bounded on a set of positive measure.*

At the same time, particular attention was being paid to the logical foundations of mathematics. One topic of interest was the axiom of choice which was being investigated by Zermelo. With the goal of demonstrating the existence of a discontinuous solution of Cauchy's equation, Hamel used the well ordering principle which was newly derived from Zermelo's work to construct what is now known as a Hamel basis [19]. In doing so, Hamel showed the necessity of some regularity assumptions to yield linear solutions.

Theorem 4 *Let $\mathscr{H}$ be a Hamel basis for $\mathbb{R}$ over $\mathbb{Q}$, and let $g : \mathscr{H} \to \mathbb{R}$. Then for some set of scalars $\{c_{x,h} | h \in \mathscr{H}\}$, finitely many of which are nonzero,*

$$x = \sum_{h \in \mathscr{H}} c_{x,h} h.$$

A function f satisfies

$$f(x) = \sum_{h \in \mathscr{H}} c_{x,h} g(h)$$

if and only if f is a solutions of Cauchy's equation. Furthermore, if g is such that there exists $h_1, h_2 \in \mathscr{H}$ such that $\frac{g(h_1)}{h_1} \neq \frac{g(h_2)}{h_2}$, then f is not continuous.

Corollary 1 *There exists discontinuous solutions of equation 1.2 and equation 1.3*

As was Cauchy's intent, we see that by examining the solutions of equation 1.1 deep analytic questions are raised which in turn develop analytic intuition. Furthermore, Cauchy's equation often is useful in solving and understanding other functional equations. As a result, Cauchy's equation has been cast in settings other than $\mathbb{R}$. One natural setting for which we will have use later is that of the positive half interval for which Aczel and Erdos attained the following results [2]

Theorem 5 *If $g : [0, \infty) \to \mathbb{R}$ is a solution to Cauchy's Equation, then there exists a solution to Cauchy's Equation $f : \mathbb{R} \to \mathbb{R}$ for which $g(x) = f(x)$ for all $x \in [0, \infty)$.*

Corollary 2 *If g is as above and g maps onto $[0, \infty)$, then there exists $c \in [0, \infty)$ such that $g(x) = cx$*

More generally, we observe that Cauchy's equation (along with the logarithmic and exponential equations) is really about defining automorphisms and homomorphisms, so solutions are also important algebraically. As a result of the algebraic and analytic properties of solutions of Cauchy's equation, oftentimes solutions of other functional equations will have some relationship to solutions of Cauchy's equation making it a natural jumping off point for investigating functional equations in various spaces. It is for this reason that we have the goal of understanding Cauchy's equation on Δ^+. We will return to viewing Cauchy's equation in this more general light once we have developed other aspects important to our goals.

Stability

With some understanding of Cauchy's equation, we now turn our attention to the notion of stability. Ulam in a lecture given at the University of Wisconsin raised the question of what approximate solutions of Cauchy's equation would look like. In particular, if there was some margin of error ε such that a function $f(x+y)$ was ε close to $f(x)+f(y)$ for all $x, y \in \mathbb{R}$, would f necessarily have any relationship to a function which satisfied Cauchy's equation? Shortly after, Hyers [20] answered in the positive with the following theorem:

Theorem 6 *Let E and E' be Banach spaces and let $\varepsilon > 0$. If for all $x, y \in E$ the function $f : E \to E'$ satisfies*

$$\|f(x+y) - (f(x) + f(y))\| < \varepsilon$$

then $L(x) = \lim_{n\to\infty} \frac{f(2^n x)}{2^n}$ exists for all $x \in E$ and is a linear transformation of E into E'. Furthermore, $L(x)$ is the unique linear transformation satisfying $\|f(x) - L(x)\| \leq \varepsilon$

This relationship between functions that are nearly solutions a functional equation and the functions that are solutions of the equation is the stability of the functional equation. Given that there are several ways of defining what it means to "nearly" have a property, there are several notions of stability.

In fact, there is a way to examine a more general notion of stability than the above theorem. The following result due to Rassias [43] relaxes the notion of uniform error to error dependent on the size of x and y.

Theorem 7 *Let E and E' be Banach spaces, $\varepsilon \geq 0$, and $p \in [0,1)$. If $f : E \to E'$ is a mapping such that $f(tx)$ is continuous in t for each fixed x and*

$$\frac{\|f(x+y)-(f(x)+f(y))\|}{\|x\|^p+\|y\|^p} \leq \varepsilon \text{ for any } x,y \in E$$

then there exists a unique linear transformation $T : E \to E'$ such that

$$\frac{\|f(x)-T(x)\|}{\|x\|^p} \leq \frac{2\theta}{2-2^p} \text{ for any } x \in E$$

There are several generalizations of the result by Rassias. Forti [13] provided one such generalization. Below we state a simpler version of Forti's result with relaxed hypothesis that we will need for later results.

Theorem 8 *Let $(S,+)$ be an abelian semi group and $(X,+,d)$ be a complete metric abelian semi group uniquely divisible by 2 for which $d(2x,2y) = cd(x,y)$ for some $c > 1$. Let $f : S \to X$,*

$$\varepsilon(x,y) := \sum_{i=1}^{\infty} c^{-i} d\Big(f(2^i(x+y), 2f(2^i y)\Big),$$

and

$$e_n(x,y) = d(f(2^n(x+y)), f(2^n x) + f(2^n y))$$

If for every $x,y \in S$ and the series $\varepsilon(x,x)$ is convergent and $e_n(x,y) = o(2^n)$, then

1. There exists a unique homomorphism $f_1 : S \to X$ satisfying $d(f_1(x), f(x)) = \varepsilon(x,x)$.

2. If for some pair $\tilde{x}, \tilde{y}$ in S we have $\liminf[2^{-n} d(f(2^n\tilde{x}), f(2^n\tilde{y}))] > 0$ then f_1 is not a constant function.

We furnish proof to justify the simplification

Proof:

Fix $x,y \in \mathbb{R}^+$. We show that $2^{-n}f(2^n x)$ is a Cauchy Sequence. Let $n > m$. Observe that

$$c^n d(2^{-n} f(2^n x), 2^{-m} f(2^m x)) = d(f(2^n x), 2^{n-m} f(2^m x))$$

Applying the triangle inequality $n-m-1$ times to the expression $d(f(2^n x), 2^{n-m}f(2^m x))$ we have that

$d(f(2^n x), 2^{n-m}f(2^m x)) \leq d(f(2^n x), 2f(2^{n-1}x) + \cdots + d(2^{n-m+1}f(2^{m+1}x), 2^{n-m}f(2^m x)).$

Recalling $e_{i-1}(x,x) = d(f(2^i x), 2f(2^{i-1}x))$ we obtain

$$
\begin{aligned}
d(2^{-n}f(2^n x), 2^{-m}f(2^m x)) &= c^{-n}d(f(2^n x), 2^{n-m}f(2^m x)) \\
&\leq c^{-n}\sum_{i=m+1}^{\infty} c^{n-i}e_{i-1}(x,x) \\
&= \sum_{i=m+1}^{\infty} c^{-i}e_{i-1}(x,x)
\end{aligned}
$$

Since the latter sum is the tail of a convergent series we conclude that $2^{-n}f(2^n x)$ is a Cauchy Sequence. Therefore, there is a function f_1 that is the point wise limit of $2^{-n}f(2^n x)$.

We now show that f_1 is an additive function. Observe that

$$\lim_{n\to\infty} d(c^{-n}f(2^n x + 2^n y), c^{-n}(f(2^n x) + f(2^n y))] = d[f_1(x+y), f_1(x) + f_1(y))$$

On the other hand because $d(f(2^n(x+y)), f(2^n x) + f(2^n y)) = o(2^n)$,

$$\lim_{n\to\infty} d[c^{-n}f(2^n x + 2^n y), c^{-n}(f(2^n x) + f(2^n y))] = 0$$

Thus, f_1 is an additive function.

Let $\tilde{x}$ and $\tilde{y}$ be as in our hypothesis.

$$
\begin{aligned}
d(f_1(\tilde{x}), f_1(\tilde{y})) &= \lim_{n\to\infty}[c^{-n}d(f(2^n\tilde{x}), f(2^n\tilde{y}))] \\
&= \liminf[c^{-n}d(f(2^n\tilde{x}), f(2^n\tilde{y}))] > 0
\end{aligned}
$$

So f_1 is not constant.

All that remains to be shown is $d(f_1(x), f(x)) \leq \varepsilon(x,x)$ and that f_1 is unique. From earlier work we see that

$$
\begin{aligned}
d(2^{-n}f(2^n x), f(x)) &= c^{-n}d(f(2^n x), 2^{n-0}f(2^0 x)) \\
&\leq \sum_{i=1}^{n} 2^{-i}e_i(x)
\end{aligned}
$$

Letting n go to infinity on both sides we achieve the desired proximity between f and f_1. Suppose now that g is a different additive function with the same proximity to f. Then for some z, $d(g(z), f_1(z)) > 0$. Since both are additive functions, we further conclude

$$\lim_{n\to\infty} d(g(2^n z), f_1(2^n z)) = \lim_{n\to\infty} c^n d(g(z), f_1(z)) = \infty$$

On the other hand

$$\lim_{n\to\infty} d(g(2^n z), f_1(2^n z)) \leq \lim_{n\to\infty} d(g(2^n z), f(2^n z)) + \lim_{n\to\infty} d(f(2^n z), f_1(2^n z)) \leq 2\varepsilon(z, z)$$

The last piece is finite by hypothesis which contradicts our assumption of g being different from f_1 ■

We also want to emphasize that the Forti result is a generalization of the Rassias result in every way. Not only do we have that we have relaxed assumption around structure (metric groups vs. Banach Spaces), the following theorem shows that the margin of error is more permissive as well.

Theorem 9 *Let E and E' be Banach spaces, $\delta \geq 0$, and $p \in [0, 1)$, and*

$$\varepsilon(x, y) := \sum_{i=1}^{\infty} 2^{-i} \| f(2^{i+1}x) - 2f(2^i y) \|$$

If $f : E \to E'$ is a mapping such that $f(tx)$ is continuous in t for each fixed x and

$$\frac{\| f(x + y) - (f(x) + f(y)) \|}{\|x\|^p + \|y\|^p} \leq \delta \text{ for any } x, y \in E$$

then for every $x, y \in E$, the series $\varepsilon(x, x)$ converges and $\| f(2^n(x + y)) - (f(2^n x) + f(2^n y)) \|) = o(2^n)$

Proof:

Let $x \in E$, then

$$\begin{aligned}
\epsilon(x,x) &= \sum_{i=1}^{\infty} 2^{-i} \|f(2^{i+1}x) - 2f(2^i x)\| \\
&\leq \sum_{i=1}^{\infty} 2^{-i} \delta(\|2^i x\|^p + \|2^i x\|^p) \\
&= 2 \sum_{i=1}^{\infty} 2^{(p-1)i} \delta \|x\|^p \\
&< \infty
\end{aligned}$$

and thus for $x, y \in E$

$$\begin{aligned}
\lim_{n\to\infty} 2^{-n} \|f(2^n x + 2^n y) - (f(2^n x) + f(2^n y))\| &\leq \lim_{n\to\infty} 2^{-n} (\|2^n x\|^p + \|2^n y\|^p)\delta \\
&\leq \lim_{n\to\infty} 2^{p-n} (\|x\|^p + \|y^p\|)\delta \\
&= 0
\end{aligned}$$

■

From these results, we take a moment to solidify what we mean by stability in the sense of Hyers, Ulam, and Rassias with a sequence of definitions. To keep things reasonably self contained, we present definitions whose scope is limited to Cauchy's equation.

Definition 1 *Let S be a set and (X, d) a metric space. Let $\sigma : S^2 \to S$ and $\xi : X^2 \to X$ be functions. If for all $s, t \in S$ there exists $\delta(s,t) \in [0, \infty)$ such that the function $f : S \to X$ satisfies*

$$d(f(\sigma(s,t)), \xi(f(s), f(t)) \leq \delta(s,t)$$

then f is metric quasi additive with error $\delta(s,t)$

Definition 2 *Let S be a set and (X, d) a metric space. If for all $s \in S$ there exists a solution of Cauchy's Equation, f', and $\varepsilon(s) \in [0, \infty)$ such that the function $f : S \to X$ satisfies*

$$d(f(s), f'(s)) \leq \varepsilon(s)$$

then f is a metric additive approximator with error $\varepsilon(s)$

When we refer to Hyers Ulam Rassias stability of Cauchy's equation, we are specifically referring to stating relationships between metric quasi additive functions and metric additive approximators. In the specific case δ and ε are constant functions, the relationship between the two objects is the Hyers-Ulam stability of a functional equation. We emphasize this notion of stability as we will see a novel way of viewing stability using partial orders once we have a construction of Δ^+. As such, we are ready to narrow our focus on Δ^+ and the stability of Cauchy's equation. We will do this in three parts. First, we will define and develop an understanding of Δ^+. Second, we will discuss solutions to Cauchy's equation on Δ^+, and third, we will produce new results centering on stability of Cauchy's equation in both the Hyers-Ulam sense and in an order theoretic sense.

CHAPTER 2
PRELIMINARIES FOR Δ^+

We wish to situate the question of the stability of Cauchy's equation in the context of the space Δ^+ which was introduced by Karl Menger in his foundational paper [29] on probabilistic metric spaces (titled by him statistical metrics). The goal of the paper was to generalize the notion of metric spaces to include some uncertainty in precision of measurement. In particular, he emphasized the utility of considering uncertainty in cases where it is not a guarantee that distinct objects are distinguishable as is the case with microscopic measurements and absolute thresholds of sensation [29]. The notion of probabilistic metric spaces also underpins notions of Fuzzy normed spaces and probabilistic normed spaces which has become a contemporary topic of interest [22] [33] [41] [51].

In [29], Menger introduced a set S and a collection of "probability functions" $F_{pq}(x)$, which measured the certainty that two objects in S, p and q, were at most distance x apart. We will informally refer to the collection of functions as Δ^+, reserving a more contemporary and precise definition for later. Together S and Δ^+ form the underpinnings of a probabilistic metric space. Under this interpretation, a few properties naturally follow. The first is that our certainty that p and q are distance x apart should only grow (or at least not decrease) with x. A second, is that since any two objects must be within some finite distance of one another. A third, is that with the exception of a "few" thresholds where certainty may increase rapidly (which can often occur in sense perception), the certainty of a measurement should increase gradually. These notions are encapsulated by the following properties:

1. If $x \leq y$ then $F_{pq}(x) \leq F_{pq}(y)$

2. $F_{pq}(\infty) = 1$

3. F_{pq} is left continuous

Menger imposed some additional regularity conditions on Δ^+, some of which have been changed over the years to mathematical utility without sacrificing the underlying intuition. One such condition is that $F_{qq} = 1$. While it is certainly desirable to say with certainty that everything is certainly distance zero away from itself, we can state that fact separately and remove the functions of the form $F_{qq}(x)$ from Δ^+. In doing so, we can instead impose the condition that $F_{pq}(0) = 0$ which essentially makes our informal Δ^+ a collection of cumulative distribution functions. Since it is a mathematician's habit to investigate a space with the broadest interpretation possible, the current definition of Δ^+ encapsulates all such functions with the properties described and not just the collection which apply to a specific set S. Therefore, we have the following definition:

Definition 3 *The space Δ^+ is the set of functions $F : [0, \infty] \to [0, 1]$ that satisfy*

1. $F(0) = 0$

2. F *is left continuous on* $(0, \infty)$

3. F *is nondecreasing*

4. $F(\infty) = 1$

Here, we observe that codified in the original definition of a probabilistic metric space were rules for comparing and combining members of Δ^+. In particular, the contexts which originally motivated probabilistic metric spaces it would be necessary to make (an analog of) the following statements,

- There are pairs of objects, p, q and r, s such that the objects p and q are certainly closer to one another than r and s are. That is,

$$\forall x \; F_{pq}(x) \geq F_{rs}(x)$$

- If there is a 100% that p and q are within 3 microns of one another, and a 70% chance that q and r are within 5 microns of each other, then there is at least a 70% that p and r are within 8 microns of one another.

This suggests that Δ^+ should be equipped with some sort of order structure and some sort of algebraic structure which would indeed justify the notion Δ^+ as a space. We will first investigate the order structure of Δ^+, and after reviewing several results useful for understanding that structure, we will turn our attention to the algebraic structure of Δ^+.

Order Structure

For functions defined over the same domain (as is the case in Δ^+), a common ordering is the pointwise partial order. That is, we will order functions so that $F \leq G$ is equivalent to the statement $F(x) \leq G(x)$ for every x in their domain. This sort of ordering also aligns with our intuition in probabilistic metric spaces that there are objects p, q, r, s such that $F_{pq}(x) \geq F_{rs}(x)$ for all x. Since there are functions which each exceed one another somewhere over their domain, there will be functions which are incomparable to one another (which would also be expected in probabilistic metric spaces). In [42], Powers investigates pointwise partial order in the setting of $L(a, b; c, d)$ which is defined below

Definition 4 *Let $[a, b]$ and $[c, d]$ be subintervals of the extended real line, $[-\infty, \infty]$. The set $L(a, b; c, d)$ is the set of functions $F : [a, b] \to [c, d]$ which satisfy*

1. $F(a) = c$

2. F *is left continuous on* (a, b)

3. F *is nondecreasing*

4. $F(b) = d$

This setting is will be useful to us as $\Delta^+ = L(0,1;0,\infty)$. In Definition 4, we referred to L as a lattice, which implies that under its partial order it contains infima(meets) and suprema(joins) of any finite subset. In fact, as is justified in [42], L is a complete lattice in the sense that it contains the infimum and supremum of any subset. The completeness of L will be important in our investigation of stability, so we will introduce several results from [42] supplying additional justification where necessary to build intuition about Δ^+.

We wish to emphasize a point of caution. For nonempty $\mathscr{F} \subseteq L$, the sets $\mathscr{F}_x = \{F(x)|F \in \mathscr{F}\}$ are sets of real numbers, and therefore have always infima and suprema. It follows then that $S(x) := \sup \mathscr{F}_x$ and $I(x) := \inf \mathscr{F}_x$ are well defined functions, and for the remainder of the section on order structure, we will preserve these definitions. Given that the partial ordering of L involves pointwise comparison, it should seem likely that S has some relationship to $\sup \mathscr{F}$, the least upper bound of $\mathscr{F}$ with respect to the partial order of L, and that a similar relationship exists between I and $\inf \mathscr{F}$, the greatest lower bound of $\mathscr{F}$ under the partial order of L. While there is certainly a relationship, we will see that there is a technical distinction when it comes to infima. As such, we will emphasize the supremum under pointwise partial order and infimum under pointwise partial order of $\mathscr{F}$ as the order theoretic supremum and order theoretic infimum of $\mathscr{F}$.

With the point of caution noted, we start with proving that L contains order theoretic suprema of its nonempty sets, and that they are equivalent to the pointwise supremum of those sets as given in [42].

Theorem 10 *Let $\mathscr{F} \subseteq L(a, b; c, d)$ be nonempty. The pointwise supremum of $\mathscr{F}$ is a member of L, and a function F is the order theoretic supremum of $\mathscr{F}$ if and only if it is the pointwise supremum.*

Proof:

Let $\mathscr{F}$ be a nonempty subset of $L(a, b; c, d)$ and S be the pointwise supremum function. It is immediate that $S(a) = c$ and $S(b) = d$. We will demonstrate that S is nondecreasing and that it is left continuous. This gives $S \in L$, and we will demonstrate this is sufficient for both directions of the theorem.

Let $a \leq x < y \leq b$. For an arbitrary $F \in \mathscr{F}$, we have $S(y) \geq F(y) \geq F(x)$. Since F is arbitrary, we have that $S(y)$ is an upper bound of the set $\mathscr{F}_x$ and thus $S(y) \geq S(x)$

Let $x \in [a, b]$, x_n be an increasing sequence which converges to x, and $\varepsilon > 0$. Since S is the pointwise supremum, we may choose $F \in \mathscr{F}$ such that $|S(x) - F(x)| < \frac{\varepsilon}{4}$. Since F is left continuous, we choose $N \in \mathbb{N}$ such that $n > N$ implies $|F(x) - F(x_n)| < \frac{\varepsilon}{4}$. Because S is nondecreasing, we know $S(x) > S(x_n)$ for all n, and by definition of S we know $S(y) - F(y) \geq 0$ for all $y \in [a, b]$. Therefore

$$S(x) - F(x_n) \geq S(x_n) - F(x_n) = |S(x_n) - F(x_n)|.$$

Taking $n > N$ we have

$$\begin{aligned} |S(x) - S(x_n)| &\leq |S(x) - F(x_n)| + |S(x_n) - F(x_n)| \\ &\leq |S(x) - F(x_n)| + |S(x) - F(x_n)| \\ &\leq 2(|S(x) - F(x)| + |S(x) - F(x_n)|) \\ &< \varepsilon \end{aligned} \tag{2.1}$$

Thus S is continuous on (a, b)

By the pointwise nature of the partial order of L, the fact that S is an upper bound of $\mathscr{F}$ is automatic. Let G be distinct from S with $G \leq S$. It follows that there exists $x \in [a,b]$ such that $S(x) - G(x) > 0$. By construction of S, there must be $F \in \mathscr{F}$ such that $S(x) - F(x) < S(x) - G(x)$, thus S is the least upper bound of $\mathscr{F}$ under pointwise partial order. ■

The above theorem implies that L is a join semi-lattice, which is to say L contains suprema of all nonempty finite sets. Before turning discussion to the containment and construction of infima, we first examine the collection of all join irreducible elements which will be useful in understanding infima and the order structure of L more broadly.

Definition 5 *A member, a, of a join semi-lattice is join irreducible if whenever $a = \sup\{b, c\}$ either $a = b$ or $a = c$.*

Definition 6 *The set of δ functions denoted $L_\delta(a, b; c, d)$ (or Δ^+_δ in the case of Δ^+) is the set of all functions of the form*

$$\delta_{u,v}(x) = \begin{cases} c & a \leq x \leq u \\ v & u < x < b \\ d & x = b \end{cases}$$

when $(u, v) \in [a, b) \times [c, d]$.

We observe for $u \in [a, b)$

$$\delta_{a,c} = \delta_{u,c} = \begin{cases} c & a \leq x < b \\ d & x = b \end{cases}$$

and will therefore use $\delta_{a,c}$ when referring to such a function. Powers [42] showed that $L_\delta(a, b; c, d)$ constitutes the entire set of join irreducible elements. Given the equivalence of pointwise and lattice theoretic suprema, we also have insight into the following statement which shows that any member of L can be expressed as the supremum of join irreducible elements.

Corollary 3 *If $F \in L(a, b; c, d)$, then $F = \sup\{\delta_{t,F(t)} | t < b\}$.*

This follows immediately from the fact that any F is nondecreasing and left continuous.

The set $L_\delta(a, b; c, d)$ is also useful for understanding infima. Let $u \in (a, b)$ and $S = \{\delta_{t,1} | t < c\}$. Observe that the pointwise infimum of S is the function

$$d(x) = \begin{cases} c & x < \frac{a+b}{2} \\ d & x \geq \frac{a+b}{2} \end{cases}$$

which is not a member of $L(a, b; c, d)$ since it is not left continuous. If the order theoretic infimum of a set is contained in L, we cannot rely on the pointwise infimum to create it directly. However, the above example suggests that pointwise infima may only differ from a member of L at points of discontinuity. Additionally, a nonempty set $\mathscr{F}$ has a nonempty set of order theoretic lower bounds (observe $\delta_{a,c}$ is in all lower bound sets), $l(\mathscr{F})$, and since we know L contains its order theoretic suprema, $\sup l(\mathscr{F})$ is also a likely candidate for $\inf \mathscr{F}$. Indeed, both intuitions hold, and are encapsulated in the following theorem from [42].

Theorem 11 *Let $\mathscr{F} \subseteq L(a, b; c, d)$ be nonempty, $l(\mathscr{F})$ be the set of lower bounds of $\mathscr{F}$, and I be the pointwise infimum of $\mathscr{F}$. The following are equivalent*

1. *F is the pointwise supremum of $l(\mathscr{F})$.*

2. *$F(a) = c$, for all $x \in (a, b)$ $F(x) = \lim_{t \to x^-} I(t)$, and $F(b) = d$.*

3. *F is the order theoretic infimum of $\mathscr{F}$.*

Proof:

Let M be the pointwise supremum of the lower bound set and I be the pointwise infimum of $\mathscr{F}$. Before justifying the equivalence outlined in the theorem, we first prove a few useful claims.

The first is that $M \in l(\mathscr{F})$. Suppose to the contrary that M is not a lower bound of $\mathscr{F}$. Then there exists $x \in (a, b)$ and $G \in \mathscr{F}$ such that $M(x) - G(x) > 0$. Because M is the pointwise supremum of $l(F)$, there exists $H \in l(\mathscr{F})$ such that $M(x) - H(x) < M(x) - G(x)$, but this would imply $H(x) > G(x)$ which is a contradiction.

The second claim is that when $a \leq x < y \leq b$, we have $M(x) \leq I(x) \leq M(y)$. Let $G \in \mathscr{F}$ and observe

$$\delta_{x,I(x)}(x) \leq I(x) \leq G(x).$$

So $\delta_{x,I(x)} \in l(\mathscr{F})$ which in turn gives

$$I(x) = \delta_{x,I(x)}(y) \leq M(y).$$

On the other hand, since $M \in l(\mathscr{F})$, it follows that $M(x) \leq G(x)$, and since I is the pointwise infimum of $\mathscr{F}$, $M(x) \leq I(x)$.

The third claim is that I is nondecreasing. Suppose to the contrary that there is $y < x$ such that $0 < I(x) - I(y)$. Then there is $G \in \mathscr{F}$ such that $G(y) - I(y) < I(x) - I(y) < G(x) - I(y)$, which means $G(y) < G(x)$ a contradiction.

The fourth claim is that $M(x) = I(x)$ at all continuity points of M. Suppose now that M is continuous at x, and let $\varepsilon > 0$. Since M is increasing and continuous at x we choose η such that

$$0 \leq M(x + \eta) - M(x) < \varepsilon.$$

Appealing to our second claim we have

$$0 \leq I(x) - M(x) < M(x + \eta) - M(x) < \varepsilon.$$

$(1 \Rightarrow 2)$ Let $\varepsilon > 0$. If $F = M$, then $F(a) = M(a) = c$ and $F(b) = M(b) = d$. If x is a continuity point of F,

$$F(x) = I(x) = \lim_{t \to x^-} I(t).$$

Otherwise, by monotonicity of F there is a $w \in (a, x)$ such that F is continuous on (w, x). Using left continuity of F, we choose η_1 such that $0 \leq F(x) - F(x - \eta_1) < \varepsilon$. Taking $\eta = \min\left\{\eta_1, \frac{x-w}{2}\right\}$ we have

$$0 \leq F(x) - I(x - \eta) = F(x) - F(x - \eta) < \varepsilon.$$

($2 \Rightarrow 3$) Since I is continuous on the same set as M, it follows immediately that the function $F(x) = \lim_{t \to x^-} I(t)$ is left continuous and monotone on (a, b). By hypothesis satisfies $F(a) = c$ and $F(b) = d$, so $F \in L(a, b; c, d)$. Let $G \in \mathscr{F}$. If F is continuous at x then

$$F(x) = I(x) \leq G(x).$$

If on the other hand F is not continuous at x, then by monotonicity of I

$$F(x) = \lim_{t \to x^-} I(x) < I(x) \leq G(x)$$

therefore, $F \in l(\mathscr{F})$.

Taking $H \in l(\mathscr{F})$ we have that $H(x_0) \leq I(x_0)$ for any value x_0 for which I is continuous. If I is not continuous at x_0 suppose by way of contradiction that $F(x_0) \leq H(x_0) \leq I(x_0)$. By left continuity of F and H, there exists a η_1 for which $F(x) < H(x)$ when $x \in (x_0 - \eta_1, x_0)$, and by monotonicity, there exits η_2 for which both functions are continuous on the interval $(x_0 - \eta_2, x_0)$. Thus taking $\eta = \min\{\eta_1, \eta_2\}$ we have that both H and F are continuous on $(x_0 - \eta, x_0)$ while $H(x) > F(x)$ which contradicts the fact that $F(x) \geq H(x)$ at all points of continuity of F. We therefore conclude F is the order theoretic maximum of $l(\mathscr{F})$.

($3 \Rightarrow 1$) Let $F = \inf \mathscr{F}$ and $\varepsilon > 0$. Let $x \in [a, b]$ and $l(\mathscr{F}_x) = \{H(x) | H \in l(\mathscr{F})\}$. Since F is the order theoretic maximum of $l(\mathscr{F})$, for any $H \in l(\mathscr{F})$, the inequality $H \leq F$ holds which in turn implies $H(x) \leq F(x)$. On the other hand, by definition F is a member of $l(\mathscr{F})$, so there exists an $H \in l(\mathscr{F})$ such that $|F(x) - H(x)| < \varepsilon$, namely $H = F$. ■

By the prior two theorems, $L(a, b; c, d)$ equipped with pointwise partial order contains suprema and infima of all non empty sets. In particular, this means that $\sup L$ and $\inf L$ exists. In fact, $\sup L = \delta_{a,d}$ and $\inf L = \delta_{a,c}$. Recalling the convention in partially ordered sets that $\inf \emptyset = \sup L$ and $\sup \emptyset = \inf L$, every subset of L has both an infimum and supremum, so L is a complete lattice.

The last aspect of $L(a, b; c, d)$ we wish to discuss is the set of order isomorphisms over L. These functions are useful not just in a comparative sense, but also for examining a lattice itself. As an example, a well known result in lattice theory is that the set of join irreducible elements is always fixed by order automorphisms [42]. Therefore we introduce the following definitions:

Definition 7 *An order isomorphism is an order preserving bijection from one lattice to another, whose inverse is also order preserving. If the image set is the same as the domain, the mapping is said to be an order automorphism.*

Definition 8 *A dual isomorphism is an order reversing bijection from one lattice to another whose inverse has the same properties. If the image set is identical to the domain the mapping is a dual automorphism.*

Definition 9 *For $F \in L(a, b; c, d)$, we define the quasi inverse of F, $F^{\vee}$, to be*

$$F^{\vee}(y) = \begin{cases} a & y = c \\ \inf\{x : F(x) > y\} & c < y < d \\ b & y = d \end{cases}$$

Definition 10 *Let A be a partially ordered set. A set $P \subset A$ is a principal down-set if there exists $a \in A$ such that $P = \{x \in A | x \leq a\}$.*

Definition 11 *A mapping F between two partially ordered sets A and B is called residuated if the preimage of every principal down-set is itself a principal down* *set. In particular, if A and B are real intervals, F is residuated if it maps the left endpoint of A to the left endpoint of B and is left continuous.*

Definition 12 *If F is a residual mapping between two partially ordered sets A and B, then G is the residual of F if G is monotone mapping from B to A which for all $a \in A$ and $b \in B$, $F \circ G(b) \leq b$ and $G \circ F(a) \geq a$. In particular if A and B are real intervals $G = F^{\vee}$.*

The following theorems are central results of [42] and as such, thorough proofs can be found there.

Theorem 12 *The mapping φ is an order automorphism of $L(a, b; c, d)$ if and only if for all $F \in L(a, b; c, d)$ one of the following holds:*

1. *$\varphi(F) = \theta \circ F \circ \gamma$ where θ is an order automorphism of $[c, d]$ and γ is an order automorphism of $[a, b]$*

2. *$\varphi(F) = \alpha \circ F^{\vee} \circ \beta$ where α and β are dual isomorphisms from $[a, b]$ into $[c, d]$*

We will refer to order automorphisms as type one and type two order automorphisms if they are of the first or second form above. The next theorem concerns weak convergence, which is to say pointwise convergence of sequences of functions at all points of continuity. As we have observed, in our investigation of the lattice structure of $L(a, b; c, d)$, pointwise suprema and infima behave well at points of continuity. As a result, monotone sequences which are weakly convergent will agree with the pointwise supremum or infimum of the sequence. This suggests a relationship between weakly continuous maps on $L(a, b; c, d)$ and order preserving maps. The following result from [42] confirms that insight.

Theorem 13 *Let F_n be a sequence in $L(a, b; c, d)$, $F \in L(a, b; c, d)$, and φ be an order automorphism of L. If F_n converges weakly to F then $\varphi(F_n)$ converges weakly to $\varphi(F)$.*

As we conclude discussion on the order structure of $L(a,b;c,d)$ we summarize some key points to be framed in the space of our primary interest, Δ^+.

1. Δ^+ is a complete lattice where at all points of continuity, the suprema and infima of sets are the same as the pointwise suprema and infima. Since members of Δ^+ are left continuous and monotone, it follows that save for a countable, nowhere dense set, identifying order theoretic suprema and infima is the same as finding pointwise suprema and infima.

2. The least element of Δ^+ is $\delta_{0,0}$ and the greatest is $\delta_{0,1}$.

3. Type one order automorphisms are of the form $\varphi(F) = \theta \circ F \circ \gamma$ where θ is an order automorphism of the unit interval and γ is an order automorphism of $[0,\infty]$. In particular, they fix Δ^+_δ in the following way

$$\begin{aligned}\varphi(\delta_{a,b})(x) = \theta \circ \delta_{a,b} \circ \gamma(x) &= \begin{cases} \theta(0) & \gamma(x) \le a \\ \theta(b) & a < \gamma(x) < \infty \\ \theta(1) & \gamma(x) = \infty \end{cases} \\ &= \begin{cases} 0 & x \le \gamma^{-1}(a) \\ \theta(b) & \gamma^{-1}(a) < x < \infty \\ 1 & x = \infty \end{cases} \\ &= \delta_{\gamma^{-1}(a),\theta(b)}\end{aligned} \tag{2.2}$$

4. Type two order automorphisms are of the form $\varphi = \alpha \circ F^\vee \circ \beta$ where α and β are dual isomorphisms from $[0,\infty]$ into the unit interval. In particular they

fix Δ^+_δ in the following way

$$\begin{aligned}
\varphi(\delta_{a,b})(x) &= \alpha \circ \delta^{\vee}_{a,b} \circ \beta(x) \\
&= \begin{cases} \alpha(\infty) & b \leq \beta(x) \\ \alpha(a) & \beta(\infty) < \beta(x) < b \\ \alpha(0) & \beta(\infty) = \beta(x) \end{cases} \\
&= \begin{cases} 0 & 0 < x \leq \beta^{-1}(b) \\ \alpha(a) & \beta^{-1}(b) < x < \infty \\ 1 & x = \infty \end{cases} \\
&= \delta_{\beta^{-1}(b),\alpha(a)}
\end{aligned} \tag{2.3}$$

Topology

Before turning our attention to the algebraic structure of Δ^+, the end of the section on its order structure and its compatibility with weak convergence suggests a natural topology to apply to Δ^+. From [53], we have that in the modified Levy metric weak convergence is equivalent to metric convergence. Furthermore, Δ^+ is essentially a collection of cumulative distribution functions, so modified Levy convergence on Δ^+ is akin to convergence in distribution of some sequence of random variables reinforcing its desirability in application to Δ^+. We define the Levy metric and present evidence supporting our clam in sequel.

Definition 13 *Let $F, G \in \Delta^+$. The modified Levy Distance between F and G, $d_L(F,G)$, is*

$$\inf\{h | \forall x \in \left(0, \frac{1}{h}\right) F(x+h) + h \geq G(x) \text{ and } G(x+h) + h \geq F(x)\}$$

The following is a theorem from [53]

Theorem 14 *The function d_L is a metric on Δ^+, for which convergence in d_L is equivalent to weak convergence. Furthermore, (Δ^+, d_L) is compact.*

Since order automorphisms are weakly continuous, we have the following corollary:

Corollary 4 *Order automorphisms are uniformly continuous on (Δ^+, d_L).*

The construction of the modified Levy metric may seem strange especially in comparison to the original Levy metric presented below.

Definition 14 *Let $F, G \in \Delta^+$. The the Levy Distance between F and G is*

$$\inf\{h|F(x-h) - h \leq G(x) \leq F(x+h) + h \textit{ for all } x\}$$

However, we do have need for this modification as Δ^+ is not complete in the Levy metric. As an example, take the sequence $\delta_{n,1}$. It converges weakly and in the modified Levy metric to $\delta_{0,0}$ while in the Levy metric it does not converge in Δ^+.

As we shall see, we will also have applications for the modified Levy metric and its induced topology in our discussion of algebraic operations on Δ^+.

Algebraic Structure

Focusing now on algebraic structure, we recall the prototype statement which motivated imposing an algebraic structure on Δ^+. "If there is a 100% chance that p and q are within 3 microns of one another, and a 70% chance that q and r are within 5 microns of each other, then there is at least a 70% that p and r are within 8 microns of one another." Intentionally, this was not rephrased in precise notation, but we will make a first pass here to initiate commentary:

$$F_{pq}(3) \cdot F_{qr}(5) \leq F_{pr}(8)$$

If we take d_1 to be the distance between p and q, d_2 to be the distance between q and r, and d_3 to be the distance between p and r, we observe that F_{pq}, F_{qr}, and F_{pr}

are analogous to the cumulative distribution functions of random variables which estimate d_1, d_2, and d_3. Therefore, our first pass can be cast in the following way

$$P(d_1 \leq 3) \cdot P(d_2 \leq 5) \leq P(d_3 \leq 8)$$

On the other hand, applying conventional triangle inequality to the three values we have

$$d_3 \leq d_1 + d_2$$

so we observe

$$P(d_1 \leq 3 \text{ and } d_2 \leq 5) \leq P(d_1 + d_2 \leq 8) \leq P(d_3 \leq 8)$$

Since $P(d_1 \leq 3) = 1$ in our example, we can see that this last inequality is equivalent to what we started with. Further, if d_1 and d_2 were independent random variables, the equivalence would not depend on the bound of d_1. In this case we would have for $x, y \in [0, \infty]$

$$F_{pq}(x) \cdot F_{qr}(y) \leq F_{pr}(x + y)$$

which is a generalization of triangle inequality.

More generally, if d_1 and d_2 were not independent, we would like a way to cast the joint probability $P(d_1 \leq x \text{ and } d_2 \leq y)$ in terms of the distributions functions $F_{pq}(x)$ and $F_{qr}(y)$. For this, we introduce the t-norm

Definition 15 *A t-norm , T, is a binary operation on $I = [0, 1]$ which is commutative, associative, nondecreasing in each place, and has identity 1.*

Here we observe that t-norms have many valuable features that capture intuitions around the relationship between a joint distribution of random variables and their marginal distributions.

1. Let $x_1 \leq x_2$ and $y_1 \leq y_2$. Since any t-norm T is nondecreasing in each place, we have $T(F(x_1), G(y_1)) \leq T(F(x_2), G(y_2))$ which captures the notion that a joint distribution is increasing in each variable.

2. Since any t-norm T is nondecreasing and has identity 1, if $F(x) = 0$, we may conclude $T(F(x), G(y)) \leq T(F(x), 1) = 0$. This captures the notion that a joint probability is zero if either of the marginal probabilities are zero.

3. Since logical conjunction and intersection are associative and commutative, it is appropriate that T is associative commutative.

4. Product is a t-norm, so t-norms have the capacity to model the joint distribution of independent random variables.

Here a tempting and valid operation to apply to Δ^+ is a pointwise application of t-norms, $\Pi_T(F, G)(x) = T(F(x), G(x))$. However, recalling the desire of our operation to model the lower bound of the generalized triangle inequality

$$P(d_1 \leq x \text{ and } d_2 \leq y) \leq P(d_1 + d_2 \leq x + y)$$

we propose an additional operation for consideration as well. Let $u, v \in [0, \infty]$ such that $u + v = x + y$ and observe that

$$\sup_{u+v=x+y} P(d_1 \leq u \text{ and } d_2 \leq v) \leq P(d_1 + d_2 \leq x + y)$$

would follow from the prior inequality, and would give the tightest lower bound. Therefore, taking $\tau_T(F, G)(x) = \sup_{u+v=x} T(F(u), G(v))$ we model the tightest lower bound that can be expressed in terms of marginal distributions.

With several natural operations to impose on Δ^+, we introduce a general class of operations, triangle functions (in reference to the generalized triangle inequality).

Definition 16 *A function $\tau : \Delta^+ \times \Delta^+ \to \Delta^+$ is a triangle function if and only if*

1. $\tau(F, G) = \tau(G, F)$ *i.e.* τ *is commutative*

2. $\tau(F, \tau(G, H)) = \tau(\tau(F, G), H)$ *i.e.* τ *is associative*

3. $\tau(\delta_{0,1}, F) = F$ *i.e.* τ *has identity* $\delta_{0,1}$

4. If $F \leq G$, then $\tau(F,H) \leq \tau(G,H)$ i.e. τ is nondecreasing in each place

Commutativity of τ justifies the claim of $\delta_{0,1}$ as identity monotonicity in each place. Using 3 and 4 we may verify that $\delta_{0,0}$ is a zero. Therefore (Δ^+, τ) is a commutative monoid with 0. It is also readily verified for a t-norm T, both Π_T and τ_T are triangle functions. Here we also introduce the notation, F^n as a stand in for iterated operation of F with itself under τ.

Because of the rich order structure of Δ^+, it is often desirable that our monoid operation be sup continuous, that is for $\mathscr{F} \subseteq \Delta^+$, $\sup_{F\in\mathscr{F}} \tau(F,G) = \tau(\sup_{F\in\mathscr{F}} F, G)$. Fortunately, if T is a left continuous t-norm then τ_T and Π_T are sup continuous as justified below in a result from [55].

Theorem 15 *If T is a left continuous t-norm then both Π_T and τ_T are sup continuous. Furthermore, $(\Delta^+_\delta, \tau_T)$ and (Δ^+_δ, Π_T) are submonoids of (Δ^+, τ_T) and (Δ^+, Π_T).*

Proof:

Let T be a left continuous t-norm, $\mathscr{F} \subseteq \Delta^+$, and $S = \sup \mathscr{F}$. For $n \in \mathbb{N}$, we define

$$F_n(x) = \begin{cases} 0 & x = 0 \\ \frac{n-1}{n} S(x) & x \in (0,\infty) \\ 1 & x = \infty \end{cases}$$

We have that $\lim_{n\to\infty} F_n = \sup_{F\in\mathscr{F}} F$ and for $x \in [0,\infty]$ $(F_n(x))$ is a nondecreasing sequence which converges to $S(x)$. Since T is left continuous for $G \in \Delta^+$ we have

$$T(\sup_{F\in\mathscr{F}} F(u), G(v)) = T(\lim_{n\to\infty} F_n(u), G(v)) = \lim_{n\to\infty} T(F_n(u), G(v))$$

Let $n \in \mathbb{N}$, then there exists $H \in \mathscr{F}$ such that $F_n(u) = S(u) - \frac{1}{n}S(u) \leq H(u)$. Since T is nondecreasing

$$\lim_{n\to\infty} T(F_n(u), G(v)) \leq T(H(u), G(v)) \leq \sup_{F\in\mathscr{F}} T(H(u), G(v))$$

Thus

$$T(\sup_{F\in\mathscr{F}} F(u), G(v)) \leq \sup_{F\in\mathscr{F}} T(F(u), G(v))$$

On the other hand, for $H \in \mathscr{F}$, $T(\sup_{F\in\mathscr{F}} F(u), G(v)) \geq T(H(u), G(v))$ which makes it an upper bound. So

$$\sup_{F\in\mathscr{F}} T(F(u), G(v)) \leq T(\sup_{F\in\mathscr{F}} F(u), G(v))$$

Lastly, since $S(x)$ is the pointwise supremum of $\mathscr{F}$,

$$\sup_{u+v=x} \sup_{F\in\mathscr{F}} T(F(u), G(v)) = \sup_{F\in\mathscr{F}} \sup_{u+v=x} T(F(u), G(v))$$

Thus

$$\sup_{F\in\mathscr{F}} \tau_T(F, G) = \sup_{F\in\mathscr{F}} \sup_{u+v=x} T(F(u), G(v)) = \sup_{u+v=x} T(\sup_{F\in\mathscr{F}} F(u), G(v)) = \tau_T(\sup_{F\in\mathscr{F}} F, G)$$

which is sufficient for the sup continuity claim.

To show that $(\Delta_\delta^+, \tau_T)$ is a monoid, we observe that it is sufficient to show closure as all other properties follow from the fact that τ_T is a monoid operation on Δ^+. To that end, let $\delta_{a,b}, \delta_{c,d} \in \Delta_\delta^+$ and $a, c < \infty$. Let $x \leq a + c$ and $y > a + c$. Since τ_T is nondecreasing, we have that $\tau_T(\delta_{a,b}, \delta_{c,d})(x) \leq \tau_T(\delta_{a,b}, \delta_{c,d})(a + c)$. So

$$\tau_T(\delta_{a,b}, \delta_{c,d})(x) \leq \sup_{u+v=a+c} T(\delta_{a,b}(u), \delta_{c,d}(v)) \leq T(\delta_{a,b}(x), \delta_{c,d}(c+1)) = 0$$

while taking $\varepsilon = \frac{y-(a+c)}{2}$ we have

$$\tau_T(\delta_{a,b}, \delta_{c,d})(y) = T(\delta_{a,b}(a+\varepsilon), \delta_{c,d}(c+\varepsilon)) = T(b, d)$$

So $\tau_T(\delta_{a,b}, \delta_{c,d}) = \delta_{a+c,T(b,d)}$. A near identical proof holds for Π_T ■

Not only do left continuous t-norms induce sup continuous triangle norms, a near converse holds if we relax the definition of τ_T as seen in a result from [44]:

Theorem 16 *If τ is a sup continuous triangle function which satisfies*

1. *$\tau(\delta_{a,b}, F)$ is an injective map for all nonzero $\delta_{a,b}$*

2. (Δ^+_δ, τ) *is a monoid*

3. *If* $\mathscr{F} \subseteq \Delta^+$ *with* $\inf \mathscr{F} = \delta_{0,0}$, *then for all* $\delta_{a,b} \in \Delta^+_\delta$, $\inf_{F\in\mathscr{F}} \tau(\delta_{a,b}, F) = \delta_{0,0}$

Then there is a commutative, associative, strictly increasing in each variable, and right continuous function $L : [0,\infty) \times [0,\infty) \to [0.\infty)$ *and left continuous strictly increasing in each variable t-norm,* T *such that*

$$\tau(F,G)(x) = \sup_{L(t,s)=x} T(F(t), G(s))$$

Here we note that it is a nontrivial concern that a triangle function be sup continuous. In particular, if we define $F * G(x) = \int_{(0,\infty)} F(x-t)\, dG(t)$ for $x \in (0,\infty)$ we have that $*$ is a triangle norm that is not sup continuous [49].

In addition to sup continuity, it is desirable for τ to be continuous in the (product) topology of Δ^+ equipped with the modified Levy metric. Fortunately, the above theorem and its converse from [44] holds with only minor modification in that context.

Theorem 17 *The operation* τ *is a sup continuous and continuous triangle function which satisfies*

1. $\tau(\delta_{a,b}, F)$ *is an injective map for all nonzero* $\delta_{a,b}$

2. (Δ^+_δ, τ) *is a monoid*

3. *If* $\mathscr{F} \subseteq \Delta^+$ *with* $\inf \mathscr{F} = \delta_{0,0}$, *then for all* $\delta_{a,b} \in \Delta^+_\delta$, $\inf_{F\in\mathscr{F}} \tau(\delta_{a,b}, F) = \delta_{0,0}$

If and only if there is a commutative, associative, strictly increasing in each variable, and continuous function $L : [0,\infty) \times [0,\infty) \to [0.\infty)$ *and continuous strictly increasing in each variable t-norm,* T *such that*

$$\tau(F,G)(x) = \sup_{L(t,s)=x} T(F(t), G(s))$$

The above theorem along with the role t-norms have in the development of triangle functions makes examining the structure of (Δ^+, τ_T) where T is a continuous t-norm a point of interest. Since τ_T is largely dependent on T, we will now develop the theory of continuous t-norms. Two important types of continuous T norms are Archimedean t-norms and strict t-norms, which are defined below.

Definition 17 *We say T is an Archimedean t-norm if*

1. $T(x, x) < x$ *for* $x \in (0, 1)$

We further classify T as a strict t-norm if

1. *T is Archimedean*
2. *T is continuous*
3. *T is strictly increasing on* $(0, 1) \times (0, 1)$

These notions are distinct, and as an example of each type, we have the Lukasiewicz t-norm and the product t-norm T_p which are defined below

$$T_L(a, b) = \max\{0, a + b - 1\} \text{ and } T_p(x, y) = xy$$

As a way to emphasize that a continuous Archimedean t-norm is not strict, we observe that for T_L, all members of $[0, 1)$ are nilpotent. That is to say, for all $x \in [0, 1)$ there exists an $n \in \mathbb{N}$ such that $T_L^n(x)$, x operated with itself n times under T_L, will be equal to 0. This gives rise to the following definition

Definition 18 *A t-norm T is nilpotent if for all* $x \in [0, 1)$ *there exists an* $n \in \mathbb{N}$ *such that* $T^n(x) = 0$

Archimedean t-norms admit a useful representation that also provides another way of distingushing the special set of strict t-norms from the broader set of

Archimedean t-norms. For $a > 0$ there is a dual isomorphism, $g : [0,1] \to [0,a]$ and

$$k_g(x) = \begin{cases} g^{-1}(x) & x \leq a \\ 0 & a < x \leq 2a \end{cases}$$

such that

$$T(x,y) = k_g(g(x) + g(y))$$

if and only if T is Archimedean, and we call g an inner additive generator of T. Inner additive generators are not unique since for $c > 0$, $h(x) = cg(x)$ will also be a generator. If g maps onto $[0,\infty]$, then we have that T is strict and $k_g = g^{-1}$ [23].

Furthermore, when T is strict, we have that for any $x_0 \in (0,1)$ if

$$g(x) = \inf\left\{ \frac{m-n}{k} \middle| m,n,k \in \mathbb{N} \text{ and } T^m(x_0) < T\Big(T^n(x_0), T^k(x)\Big) \right\}$$

for all $x \in (0,1]$ (and $g(0) = 0$) then g is an inner additive generator of T [23]. This gives rise to the following example which is a constructive way of identifying the inner additive generator of the product t-norm.

Example 1 *Let $T(x,y) = xy$ and choose $x_0 = \frac{1}{2}$. Observe $\left(\frac{1}{2}\right)^m < \left(\frac{1}{2}\right)^n x^k$ is equivalent to $2^{\frac{m-n}{k}} > \frac{1}{x}$, and*

$$\left\{ \frac{m-n}{k} \middle| m,n,k \in \mathbb{N} \right\} = \mathbb{Q}$$

Since $T^m(x_0) < T^n(x_0)$ is implied by $T^m(x_0) < T\Big(T^n(x_0), T^k(x)\Big)$, we have that $m > n$, so we have that

$$g(x) = \inf\left\{ q \in \mathbb{Q}^+ | q > -\log_2(x) \right\} = -\log_2(x)$$

so $-\log_2(x)$ is a generator of T.

The inner additive generator construction also reveals the following fact:

Theorem 18 *An Archimedean t-norm is either nilpotenet or strict.*

Proof:

Let T be an Archimedean t-norm and g be an additive generator of T. If T is strict, then for $x \in (0,1)$ $g(x) \in (0,\infty)$, so $n(g(x)) < \infty$ for all $n \in \mathbb{N}$. Therefore

$$T^n(x,\ldots,x) = g^{-1}(ng(x)) > 0$$

On the other hand if T is not stirct, take $[0,a]$ to be the codomain of g. For $x \in (0,1)$, $g(x) > 0$, so there exists $n \in \mathbb{N}$ such that $ng(x) > a$ giving

$$T^n(x,\ldots,x) = k_g(ng(x)) = 0$$

■

The following definition will be useful in classifying all continuous t-norms and their relationship to Archimedean t-norms.

Definition 19 *We say that two t-norms, T and T', are isomorphic if there is an order automorphism f of $[0,1]$ such that*

$$f^{-1}(T(f(x),f(y))) = T'(x,y)$$

With that definition, we have the following result due to [23]:

Theorem 19 *Let T be an Archimedean t-norm and g an inner additive genrator then the following are equivalent*

1. *T and T' are isomorphic*

2. *There is an order automorphism f on $[0,1]$ such that $g \circ f$ is an inner additive generator of T'*

3. *T' is Archimedean. Furthermore, either T and T' are both strict, or both are nilpotent.*

Proof:

$1 \Rightarrow 2$ If g is an inner additive generator of T and let f be the automorphism that makes T and T' isomorphic. Then

$$\begin{aligned} T'(x,y) &= f^{-1}(T(f(x),f(y))) \\ &= f^{-1}(k(g(f(x))+g(f(y)))) \\ &= (f^{-1}\circ k)(g\circ f(x)+g\circ f(y)) \end{aligned}$$

Since $(f^{-1}\circ k_g)(x) = (g\circ f)^{-1}(x)$ whenever $x \leq a$ and is 0 otherwise we have

$$T'(x,y) = k_{g\circ f}(g\circ f(x)+g\circ f(y))$$

$2 \Rightarrow 3$ Since f is an order automorphism, $g\circ f$ is a dual isomorphism from $[0,1]$ into $[0,a]$ where $[0,a]$ is the range of g. This is sufficent to conclude that T' is Archimedean. If $a=\infty$ then clearly both T and T' are stirct. Otherwise, for all $x<1$ we have $g(x)>0$ and therefore the existence of $n\in\mathbb{N}$ such that $a< ng(x)\leq 2a$ which gives us that all $x<1$ are nilpotent under T. Since f is an order automorphism we have for all $x<1$ that $f(x)<0$ and that $g(f(x))>0$ allowing us to conclude that all $x<1$ are nilpotent under T'.

$3 \Rightarrow 1$ Let h be the inner additive generator of T', $a=g(0)$ and $b=h(0)$. Since T and T' share nilpotency, $\frac{b}{a}$ is well defined if we adopt the convention in this context that $\frac{\infty}{\infty}=1$. Let $d(x)=\frac{a}{b}x$, and $f=g^{-1}\circ d\circ h$.

$$f^{-1}(T(f(x),f(y))) = f^{-1}(k_g(g(f(x))+g(f(y)))) = f^{-1}\left(k_g\left(\frac{a}{b}h(x)+\frac{a}{b}h(y)\right)\right)$$

Observe that $f^{-1}(0)=0$ and for $x\leq a$, we have $f^{-1}(k_g(x)) = h^{-1}(\frac{b}{a}(x))$. Thus

$$f^{-1}(T(f(x),f(y))) = f^{-1}\left(k_g\left(\frac{a}{b}(h(x)+h(y))\right)\right) = h^{-1}(h(x)+h(y)) = T'(x,y)$$

Since f is an order automorphism of $[0,1]$ the two t-norms are isomorphic. ■

With thorough understanding of the structure of Δ^+, we are ready to work on functional equations in the space.

CHAPTER 3
CAUCHY'S EQUATION ON Δ^+

With a motivation for examining Δ^+ and an understanding of the space from several perspectives, we may now turn our attention to solutions of Cauchy's equation on Δ^+. Once we have established solutions of Cauchy's equation, we will be able to meaningfully discuss notions of stability. Here we will principally follow the results of [45]. Save for Lemma 2 which comes from [44] , all results in this chapter either come from [45] or are immediate corollaries of those results.

As we established in the prior section, for a triangle function τ, the pair (Δ^+, τ) is a monoid with zero, therefore, we may naturally define $\varphi : \Delta^+ \to \Delta^+$ to be a solution to Cauchy's equation if

$$\varphi(\tau(F, G)) = \tau(\varphi(F), \varphi(G)).$$

In keeping with convention, we will disregard the trivial solutions $\varphi(F) \equiv \delta_{0,1}$ and $\varphi(F) \equiv \delta_{0,0}$ unless specified otherwise.

By definition, we have the following universal properties of solutions of Cauchy's Equation:

Lemma 1 *Let φ satisfy Cauchy's equation for τ then*

1. *$\varphi(\delta_{0,1})$ is the identity in $Ran(\varphi)$*

2. *$\varphi(\delta_{0,0})$ is the zero in $Ran(\varphi)$*

3. *φ maps idempotents to idempotents*

4. φ *preserves* $n-th$ *powers, i.e.* $\varphi(H^n) = \varphi(H)^n$ *for all* $H \in \Delta^+$

5. φ *maps nilpotents to nilpotents*

6. φ *maps any element with* $n-th$ *root to an element with an* $n-th$ *root i.e. if there exists* $F, G \in \Delta^+$ *such that* $F = G^n$ *then there exists* $H \in \Delta^+$ *such that* $\varphi(F) = H^n$.

We also have the following solutions:

Theorem 20 *Let* H *be an idempotent of* τ*. The following are solutions for Cauchy's equation for* τ

- $\varphi_H(F) = \tau(F, H)$*, in particular, the identity map is a solution where* $H = \delta_{0,1}$
- $\varphi(F) = F^n$

As emphasized in prior sections, an important set of functions on Δ^+ are the order automorphisms. We will first present a sequence of lemmas useful to central results concerning order automorphisms, and what properties of an order automorphism are necessary for it to be a solution to Cauchy's equation.

Lemma 2 *Let* T *be a continuous Archimedean t-norm,* g *an inner additive generator of* T*, and* θ *a mapping from* $[0, 1]$ *into* $[0, 1]$*, then* θ *is a solution to Cauchy's equation for* T *if and only if there is a* $c > 0$ *such that* $g^{-1}(c \cdot g(x))$.

Lemma 3 *Let* T *be a nilpotent Archimedean t-norm and* g *be an inner additive generator of* T*. Then* θ *is a solution of Cauchy's equation for* T *if and only if there is* $c \geq 1$ *such that* $\theta(x) = k_g(c \cdot g(x))$

Corollary 5 *Let* T *be a nilpotent Archimedean t-norm. If* θ *is a solution of Cauchy's equation for* T *then* θ *is an order automorphism if and only if* θ *is the identity map*

Proof:

The identity map is clearly an order automorphism, so we need only prove coverse. We prove via contrapositive. Suppose θ is not the identity map. From Lemma 3, we have that there is an inner additive generator of T, g, such that $\theta(x) = k_g(c \cdot g(x))$ by virtue of being a solution of Cauchy's equation. Since θ is not the identity map we further have $c > 1$ Since T is nilpotent, we have that there is an $a \in (0, \infty)$ such that $g(0) = a$. Since g is injective and decreasing, there exists $x \in (0, 1)$ such that $cg(x) = a$. Therefore, there exists $x \in (0, 1)$ such that

$$\theta(x) = k_g(c \cdot g(x)) = k_g(a) = 0$$

Therefore, θ is zero on the interval $(0, x)$ which means it is not an order automorphism

■

Theorem 21 *Let T be a continuous t-norm and let φ be a type one order automorphism with θ and γ as in Theorem 12. The map φ satisfies Cauchy's equation for τ_T if and only if*

$$\gamma(a + b) = \gamma(a) + \gamma(b) \textit{ for all } a, b \in \mathbb{R}^+$$

and

$$\theta(T(c, d)) = T(\theta(c), \theta(d))$$

Theorem 22 *Let T be a continuous t-norm and let φ be a type two order automorphism with α and β as in Theorem 12. The map φ satisfies Cauchy's equation for τ_T if and only if for all $x, y \in [0, 1]$*

$$\alpha(\alpha^{-1}(x) + \alpha^{-1}(y)) = T(x, y)$$

and

$$\beta(\beta^{-1}(x) + \beta^{-1}(y)) = T(x, y)$$

Furthermore, a solution exists if and only if T is a strict t-norm.

It is also possible to find solutions to Cauchy's equation which are not order automorphisms. Again, using results from [45], we give a characterization of sup continuous (but not necessarily bijective, nor order preserving under inverse) solutions to Cauchy's equation for τ_T. In order to do so, we introduce some notation and define the set of T log concave functions. For a function $F \in \Delta^+$, the values a_F and b_F are as follows:

$$a_F = \sup\{x \in \mathbb{R}^+ | F(x) = 0\}$$

$$b_F = \lim_{x\to\infty} F(x)$$

Definition 20 *Let T be a strict t-norm and g be any inner additive generator of T. Then the set of T log concave elements of Δ^+ is*

$$\Delta_T^+ = \{F \in \Delta^+ | g \circ F \text{ is convex on } (a_F, \infty)\}$$

Here we emphasize as in [45] that convexity is not affected by multiplication with a positive scalar, so choice of g is irrelevant. In order to characterize the sup continuous solutions to Cauchy's equation, we need that exponentiation is well defined over Δ_T^+. For that, we have the following two theorems

Theorem 23 *Let T be a strict t-norm with inner additive generator g and suppose $F \in \Delta^+ \setminus \{\delta_{0,0}\}$. For any $\mu \geq 0$, let F^μ be defined by*

$$F^\mu(x) = g^{-1}\left(\mu \cdot g\left(F\left(\frac{x}{\mu}\right)\right)\right) \qquad \text{for } 0 < \mu < \infty,$$

$$F^0 = \lim_{\mu\to 0} F^\mu = \delta_{0,1},$$

$$F^\infty = \lim_{\mu\to\infty} F^\mu = \begin{cases} \delta_{0,0} & F \neq \delta_{0,1} \\ \delta_{0,1} & F = \delta_{0,1}. \end{cases}$$

Then F^μ is in Δ^+ for any $\mu, \nu \geq 0$, we have

$$\tau_T(F^\mu, F^\nu) = F^{\mu+\nu},$$

$$(F^{\mu})^{\nu} = (F^{\nu})^{\mu} = F^{\mu\nu},$$

and in particular for $n \in \mathbb{N}$,

$$\tau_T^n(F, \ldots, F) = F^n$$

that is, nth powers in the algebraic sense agree with the μ power defined above where $\mu = n$.

Corollary 6 *Let T be a strict t-norm and $G \in \Delta_T^+ \setminus \{\delta_{0,0}\}$. Then for any $\mu > 0$, there exists a unique $H \in \Delta_T^+$ such that*

$$G = H^{\mu},$$

namely $H = G^{1/\mu}$

We may now turn our attention to sup continuous solutions to Cauchy's equation and their properties

Corollary 7 *Let T be a strict t-norm and φ a sup continuous solution of Cauchy's equation for τ_T and $F \in \Delta_T^+ \setminus \{\delta_{0,0}\}$. If for all positive integers n, $\varphi(F^{1/n}) \in \Delta^+ \setminus \{\delta_{0,0}\}$ then for all $\mu \geq 0$,*

$$\varphi(F^{\mu}) = [\varphi(F)]^{\mu}$$

Corollary 8 *Let $\delta_{a,b} \neq \delta_{0,0}$ and let T be a strict t-norm with inner additive generator g. Then for any $c \in (0,1)$, $\delta_{a,b}$ admits the decomposition*

$$\delta_{a,b} = \tau_T(\delta_{1,1}^{a}, \delta_{0,c}^{g(b)/g(c)})$$

Corollary 9 *Let T be a strict t-norm with inner additive generator g and let φ be a sup continuous solution of Cauchy's equation for τ_T. If for some $c \in (0,1)$ and all positive integers n, $\varphi(\delta_{0,c}^{1/n})$ and $\varphi(\delta_{1,1}^{1/n})$ are in $\Delta_T^+ \setminus \{\delta_{0,0}, \delta_{0,1}\}$ then for all $F \in \Delta^+$,*

$$\varphi(F) = \sup_{t \in \mathbb{R}^+} \tau_T([\varphi(\delta_{1,1})]^t, [\varphi(\delta_{0,c})]^{kg(F(t))}),$$

where $k = \frac{1}{g(c)}$.

Theorem 24 *Let T be a strict t-norm with generator g. Let G and H in $\Delta_T^+ \setminus \{\delta_{0,0}, \delta_{0,1}\}$ and $c \in (0,1)$. If $\varphi : \Delta^+ \to \Delta^+$ is defined by*

$$\varphi(F) = \sup_{t \in \mathbb{R}^+} \tau_T(G^t, H^{kg(F(t))}) \quad \textit{for all} F \in \Delta^+$$

where $k = \frac{1}{g(c)}$. Then φ is a sup continuous solution of Cauchy's equation for τ_T. Moreover, $G = \varphi(\delta_{1,1})$, $H = \varphi(\delta_{0,c})$, and for all positive integers n, $\varphi(\delta_{1,1}^{1/n})$ and $\varphi(\delta_{0,c}^{1/n})$ are in $\Delta_T^+ \setminus \{\delta_{0,0}, \delta_{0,1}\}$

Corollary 10 *Let T be a strict t-norm with inner additive generator g and let φ be an order automorphism solution of Cauchy's equation for τ_T. If for some $c \in (0,1)$ and all positive integers n, $\varphi(\delta_{0,c}^{1/n})$ and $\varphi(\delta_{1,1}^{1/n})$ are in $\Delta_T^+ \setminus \{\delta_{0,0}, \delta_{0,1}\}$ then*

$$\varphi(\delta_{1,1}) = \delta_{\gamma^{-1}(1),1} \textit{ and } \varphi(\delta_{0,c}) = \delta_{0,\theta(c)} \textit{ if } \varphi \textit{ is type one}$$

and

$$\varphi(\delta_{1,1}) = \delta_{0,\alpha(1)} \textit{ and } \varphi(\delta_{0,c}) = \delta_{\beta^{-1}(c),1} \textit{ if } \varphi \textit{ is type two}$$

Corollary 11 *Let T be a strict t-norm with generator g. Let $G = \delta_{a,1}$ and $H = \delta_{0,b}$ where $a \in (0,\infty)$, $b \in (0,1)$. Let $c \in (0,1)$. If $\varphi : \Delta^+ \to \Delta^+$ is defined by*

$$\varphi(F) = \sup_{t \in \mathbb{R}^+} \tau_T(G^t, H^{kg(F(t))}) \quad \textit{for all} F \in \Delta^+$$

where $k = \frac{1}{g(c)}$. Then φ is a type one order automorphism solution of Cauchy's equation for τ_T. Moreover, $G = \varphi(\delta_{1,1})$, $H = \varphi(\delta_{0,c})$, and for all positive integers n, $\varphi(\delta_{1,1}^{1/n})$ and $\varphi(\delta_{0,c}^{1/n})$ are in $\Delta_T^+ \setminus \{\delta_{0,0}, \delta_{0,1}\}$

If instead $G = \delta_{0,b}$ and $H = \delta_{a,1}$, φ is a type two order automorphism solution.

CHAPTER 4
NEW RESULTS IN STABILITY

Framing Stability

With sufficient knowledge of Cauchy's equation on Δ^+, we are now ready to begin discussion of the principle focus of this dissertation, stability in the context of Δ^+. Since Δ^+ is equipped with the modified Levy metric, it is certainly natural to investigate the Hyers Ulam Stability of Cauchy's equation. So, we pose the question, when is there a relationship between metric quasi solutions of Cauchy's equation and approximate solutions of Cauchy's equation in metric in the context of Δ^+?

One may have hope of finding solutions using general stability theorems like Theorem 8. Alas, Theorem 8 and those like it require a degree of compatibility between the group operation and the metric (see also [36]). Here we recall theorem 8 has the hypothesis that the range of our mapping, $(X, +, d)$, be a complete metric abelian semigroup uniquely divisible by 2 for which $d(2x, 2y) = cd(x, y)$ for some $c > 1$. In the context of (Δ^+, τ), we may not have unique divisibility when τ is not sup continuous. Even if we take T to be the product triangle norm and $\tau = \tau_T$, for which a great many regularity conditions are satisfied, the compatibility condition, $d(2x, 2y) = cd(x, y)$, often fails as we see below:

$$\frac{d_L(\tau_T(\delta_{0,0.2}, \delta_{0,0.2}), \tau_T(\delta_{0,0.4}, \delta_{0,0.4}))}{d_L(\delta_{0,0.2}, \delta_{0,0.4})} = \frac{d_L(\delta_{0,0.04}, \delta_{0,0.16})}{d_L(\delta_{0,0.2}, \delta_{0,0.4})} = \frac{0.12}{0.2} < 1$$

We therefore must look elsewhere to understand Hyers Ulam stability. Here we observe that part of the reason for our trouble is that our group operation is in many ways analogous to multiplication of numbers on $[0, 1]$ which decreases over iteration rather than increases. This observation also motivates a new perspective on stability. In particular, we recall that in the phrasing of Hyers original investigation of the question of stability, the one dimensional case would be

$$|f(x+y) - (f(x) + f(y))| < \varepsilon$$

which is equivalent to

$$f(x) + f(y) - 2\varepsilon < f(x+y) - \varepsilon < f(x) + f(y)$$

Since Δ^+ is partially ordered and for all $F, G \in \Delta^+$ $\tau(F, G) \leq F$, we have an analog to the above inequality in

$$\tau(\tau(\varphi(F), \varphi(G)), \tau(H, H)) \leq \tau(\varphi(\tau(F, G)), H) \leq \tau(\varphi(F), \varphi(G))$$

Unlike in the case of the inequality for real valued functions which defines the standard metric over $\mathbb{R}$, this inequality doesn't define the Levy metric. Therefore, this way of viewing quasi solutions of a functional equation is conceptually distinct from the Hyers Ulam view. Since Cauchy's Equation was originally phrased concerning the preservation of addition, we will refer to functions which satisfy this inequality as order quasi additive with error H, and if for all $F \in \Delta^+$, if φ satisfies the inequality

$$\tau(\varphi(F), \tau(H, H)), \leq \tau(\varphi'(F), H) \leq \varphi(F)$$

for some solution of Cauchy's equation for τ, then we will say φ is an order additive approximator with error H. The relationship between order quasi additive functions and order additive approximators is the order stability of Cauchy's equation.

Order stability not only gives us a different perspective on stability, as we shall see, it also gives insight into stability in the sense of Hyer's and Ulam.

Therefore, we begin with results involving order stability. Our focus is finding results in order stability involving those triangle functions generated by t-norms. In order to do so, we will make use of the following Lemmas which will allow us to simplify cases:

Triangle functions generated over Strict T

Lemma 4 *Let $F, G, H \in \Delta^+$, $H \neq \varepsilon_\infty$, and T be a strict t-norm. It follows that*

$$\tau_T(F, H) \leq G \text{ if and only if } \tau_T(F, H^2) \leq \tau_T(G, H)$$

Proof:

Since τ is non decreasing in each place, one direction of the statement is trivial. For the converse, we suppose to the contrary that there is $x \in (0, \infty)$ such that $\tau_T(F, H)(x) > G(x)$. By left continuity of G we have that there is $\delta > 0$ such that

$$\tau_T(F, H)(x) > G(x + \delta)$$

Then by strictness of T we have for all finite $y > a_H$

$$T(\tau_T(F, H)(x), H(y)) > T(G(x + \delta), H(y))$$

Let $w = x + a_H + \delta$ then

$$\tau_T(F, H^2)(w) \geq T(\sup_{u+v=x} T(F(u), H(v)), H(\delta + a_H)) > T(G(x + \delta), H(\delta + a_H))$$

Since $H(\delta - e + a_H) = 0$ if $e \geq \delta$

$$\begin{aligned} T(G(x + \delta), H(\delta + a_H)) &\geq \sup_{e \in [0, \delta)} T(G(x + e), H(\delta - e + a_H)) \\ &= \sup_{u+v=w} T(G(u), H(v)) \\ &= \tau_T(G, H)(w) \end{aligned}$$

Therefore $\tau_T(G, H)(w) < \tau_T(F, H^2)(w)$ which is a contradiction ■

Corollary 12 *Let $F, G, H \in \Delta^+$, $H \neq \varepsilon_\infty$, and T be a strict t-norm. It follows that*

$$\tau_T(F, H) = \tau_T(G, H) \iff F = G$$

For the following lemma, we recall the notation from Chapter 3

$$a_F = \sup\{x \in \mathbb{R}^+ | F(x) = 0\}$$

$$b_F = \lim_{x \to \infty} F(x) F(x)$$

and introduce the notation

$$F_\delta := \delta_{a_F, b_F}$$

Lemma 5 *Let $F, G \in \Delta^+$ and T be strict. We define $\mathscr{H}$ to be the set of function $H \in \Delta^+$ such that*

$$\tau_T(F, H^2) \leq \tau_T(G, H) \leq F$$

If $S = \sup \mathscr{H}$, then

1. *$S \in \mathscr{H}$*

2. *$H \leq S$ if and only if $H \in \mathscr{H}$*

3. *If $F, G \in \Delta^+_\delta$ then $H \in \mathscr{H}$ if and only if $H_\delta \in \mathscr{H}$. In particular $S_\delta = S \in \mathscr{H}$*

Proof:

Observe that $\mathscr{H}$ is nonempty as ε_∞ is always a member. Since T is strict, we may cancel in an H in the first part of the compound inequality, so

$$\tau_T(F, H) \leq G \text{ and } \tau_T(G, H) \leq F$$

for an $H \in \mathscr{H}$. Since τ_T is sup continuous and G is an upper bound of $\tau_T(F, H)$ for all $H \in \mathscr{H}$ we have

$$\tau_T(F, S) \leq G$$

similar argumentation gives

$$\tau_T(G,S) \leq F$$

which is conclusion 1.

Clearly, $H \in \mathscr{H}$ implies $H \leq S$, so observing that when $H \leq S$,

$$\tau_T(F,H) \leq \tau_T(F,S) \leq G \text{ And } \tau_T(G,H) \leq \tau_T(G,S) \leq F$$

yielding conclusion 2.

If $F, G \in \Delta_\delta^+$ then $H \in \mathscr{H}$ gives us $a_h + a_f \geq a_g$ and $a_h + a_g \geq a_f$. For all $x \in (0,\infty)$ it must be the case that $T(H(x), b_f) \leq b_g$ and vice versa. Since T is continuous this gives $H_\delta \in \mathscr{H}$. Since $S \leq S_\delta$, and $S \in \mathscr{H}$ we must have $S = S_\delta$. ■

The above lemma means that for strict T whenever $\varphi(F)$ and $\varphi(G)$ are delta functions, we can reduce the order quasi solution inequality to one only involving delta functions. As a result, we have the following theorems

Theorem 25 *Let $H \in \Delta^+ \setminus \{\varepsilon_\infty\}$ and T be a strict t-norm. If for all $F, G \in \Delta_\delta^+$ a type one order automorphism $\varphi : \Delta^+ \to \Delta^+$ satisfies*

$$\tau_T(\tau_T(\varphi(F), \varphi(G)), H^2) \leq \tau_T(\varphi(\tau_T(F,G)), H) \leq \tau_T(\varphi(F), \varphi(G))$$

Then there exists a unique type one order automorphism φ_1 satisfying

1. *$\forall F, G \in \Delta^+$ $\tau_T(\varphi_1(F), \varphi_1(G)) = \varphi_1(\tau_T(F,G))$ (φ_1 is additive)*

2. *$\forall F \in \Delta^+[\tau_T(\varphi(F), H) \leq \varphi_1(F)$ and $\tau_T(\varphi_1(F), H) \leq \varphi(F)]$ (φ and φ_1 approximate one another)*

Proof:

By Lemma 5, it is sufficient to consider $H \in \Delta_\delta^+$. Let φ satisfy our hypothesis, then there exists $\gamma \in Aut(\mathbb{R}^+)$ and $\theta \in Aut([0,1])$ such that $\varphi(F) =$

$\theta \circ F \circ \gamma$. Let $F = \delta_{a_f,b_f}, G = \delta_{a_g,b_g}$, and $H = \delta_{a_h,b_h}$. By hypothesis, we then have the following

$$\begin{aligned}\delta_{\gamma^{-1}(a_f)+\gamma^{-1}(a_g)+2a_h,T\left(T(\theta(b_f),\theta(b_g)),T(b_h,b_h)\right)} &\leq \delta_{\gamma^{-1}\left(a_f+a_g\right)+a_h,T(\theta(T(b_f,b_g)),b_h)} \\ &\leq \delta_{\gamma^{-1}(a_f)+\gamma^{-1}(a_g),T(\theta(b_f),\theta(b_g))}\end{aligned}$$

Which yields the following two inequalities

$$\gamma^{-1}(a_f) + \gamma^{-1}(a_g) \leq \gamma^{-1}(a_f + a_g) + a_h \leq \gamma^{-1}(a_f) + \gamma^{-1}(a_g) + 2a_h \tag{4.1}$$

$$T\left(T\Big(\theta(b_f), \theta(b_g)\Big), b_h^2\right) \leq T\left(\theta\Big(T(b_f, b_g)\Big), b_h\right) \leq T\Big(\theta(b_f), \theta(b_g)\Big) \tag{4.2}$$

Let z be the inner additive generator of T. Since T is strict and z is a decreasing function, application of z to (4.2) yields

$$\begin{aligned}z\Big(\theta(b_f)\Big) + z\Big(\theta(b_g)\Big) + z(b_h^2) &\geq z\left(\theta\left(z^{-1}\Big(z(b_f) + z(b_g)\Big)\right)\right) + z(b_h) \\ &\geq z\Big(\theta(b_f)\Big) + z\Big(\theta(b_g)\Big)\end{aligned}$$

Notice that $z(b_h^2) = z(z^{-1}(z(b_h) + z(b_h)))$ and let $\psi = z \circ \theta \circ z^{-1}$ and observe that the inequality becomes

$$\begin{aligned}\psi\Big(z(b_f)\Big) + \psi\Big(z(b_g)\Big) + 2z(b_h) &\geq \psi\Big(z(b_f) + z(b_g)\Big) + z(b_h) \\ &\geq \psi\Big(z(b_f)\Big) + \psi\Big(z(b_g)\Big)\end{aligned} \tag{4.3}$$

We now wish to find additive functions that approximate ψ and γ^{-1}. In order to do so, we first establish particular facts about ψ. By hypothesis, we have $b_h \neq 0$ meaning that $z(b_h) < \infty$. Knowing that the inner additive generator, z is a surjection onto $\mathbb{R}^+$ and that (4.3) holds for all choices of $b_f, b_g \in [0,1]$ means that for some fixed $c < \infty$ and for all $x, y \in [0, \infty)$

$$|\psi(x + y) - (\psi(x) + \psi(y))| \leq c \tag{4.4}$$

Observe that (4.1) implies a similar statement for γ^{-1} and what follows will be sufficient argumentation for both γ^{-1} and ψ . Let $x = 2^n u$ and $y = 2^n v$. Then (4.4) gives us that $|\psi(2^n(u+v)) - (\psi(2^n u) - \psi(2^n v))| = o(2^n)$ and convergence of the series

$$\sum_{i=1}^{\infty} 2^{-i}|\psi(2^{i+1}u) - 2\psi(2^i u)|.$$

To satisfy all hypotheses of Theorem 8 we now need only show that there are some $\tilde{x}, \tilde{y} \in \mathbb{R}^+$ such that $\liminf[2^{-n}|\psi(2^n\tilde{x}) - \psi(2^n\tilde{y})|] > 0$. In order to do so, we first establish by induction $\psi(2^n x) \geq 2^n(\psi(x) - c)$. As a base case let $n = 1$ then by (4.4) we have

$$\psi(2x) \geq \psi(x) + \psi(x) - c \geq 2\psi(x) - 2c$$

Suppose now for $n > 1$ we have $\psi(2^n x) \geq 2^n(\psi(x) - c)$ then

$$\begin{aligned}
\psi(2^{n+1}x) &\geq 2\psi(2^n x) - c \\
&\geq 2\Big(2^n\psi(x) - (2^n - 1)c\Big) - c \\
&\geq 2^{n+1}\psi(x) - 2^{n+1}c + 2c - c \\
&= 2^{n+1}\psi(x) - (2^{n+1} - 1)c \\
&= 2^{n+1}(\psi(x) - c)
\end{aligned}$$

Since $z^{-1}(0) = 1$, $\theta(1) = 1$ and $z(1) = 0$, we have that $\psi(0) = 0$. Since ψ is a surjection onto $\mathbb{R}^+$ we know there exists $\tilde{x}$ such that $\psi(\tilde{x}) > c$ so

$$\liminf[2^{-n}|\psi(2^n\tilde{x}) - \psi(2^n \cdot 0)| \geq \liminf[2^{-n} \cdot 2^n(\psi(\tilde{x}) - c)] > 0$$

Therefore, by Theorem 8 we may conclude that there are unique non constant additive functions ψ_1, γ_1 such that

$$|\psi_1(x) - \psi(x)| \leq \sum_{i=1}^{\infty} 2^{-i}z(b_h) = z(b_h) \text{ and } |\gamma_1^{-1}(x) - \gamma^{-1}(x)| \leq \sum_{i=1}^{\infty} 2^{-i}a_h = a_h$$

We now turn our attention towards constructing an additive order automorphism on Δ^+ with the desired properties. We define $\theta_1 = z^{-1} \circ \psi_1 \circ z$. Since z is an order reversing bijection from $[0,1]$ into $\mathbb{R}^+$ and ψ_1 is an additive function on $\mathbb{R}^+$ admitting the representation $\psi_1(x) = c_1 x$ which is an automphism of $\mathbb{R}^+$, θ_1 is an automorphism on $[0,1]$. Similarly, $\gamma_1^{-1}(x) = c_2 x$, and thus γ_1, so γ_1 is an order automorphism on $\mathbb{R}^+$. Therefore we can conclude the function $\varphi_1(F) := \theta_1 \circ F \circ \gamma_1$ is an order automorphism on Δ^+.

Next we establish the additivity of φ_1. Let $u, v \in [0,1]$. Then

$$\begin{aligned}
\theta_1(T(u,v)) &= z^{-1} \circ \psi_1 \circ z \circ z^{-1}(z(u) + z(v)) \\
&= z^{-1}(\psi_1(z(u)) + \psi_1(z(v))) \\
&= z^{-1}(z \circ \theta_1(u) + z \circ \theta_1(v)) \\
&= T(\theta_1(u), \theta_1(v))
\end{aligned}$$

Let $F, G \in \Delta^+$. It follows that

$$\begin{aligned}
\varphi_1(\tau_T(\delta_{x,F(x)}, \delta_{y,G(y)}) &= \delta_{\gamma_1^{-1}(x+y), \theta_1(T(F(x),G(y))} \\
&= \delta_{\gamma_1^{-1}(x)+\gamma_1^{-1}(y), T(\theta_1(F(x)), \theta_1(G(y)))} \\
&= \tau_T(\varphi_1(\delta_{x,F(x)}), \varphi_1(\delta_{y,G(y)}))
\end{aligned}$$

Using sup continuity of order automorphisms we have the following

$$\begin{aligned}
\sup_y[\sup_x[\varphi_1(\tau_T(\delta_{x,F(x)}, \delta_{y,G(y)})]] &= \sup_y[\varphi_1(\tau_T(\sup_x[\delta_{x,F(x)}], \delta_{y,G(y)})) \\
&= \sup_y[\varphi_1(\tau_T(F, \delta_{y,G(y)})] = \varphi_1(\tau_T(F,G))
\end{aligned}$$

While on the other hand

$$\begin{aligned}
\sup_y[\sup_x[\varphi_1(\tau_T(\delta_{x,F(x)}, \delta_{y,G(y)})]] &= \sup_y[\sup_x[\tau_T(\varphi_1(\delta_{x,F(x)}), \varphi_1(\delta_{y,G(y)})]] \\
&= \tau_T(\varphi_1(F), \varphi_1(G))
\end{aligned}$$

Therefore allowing us to conclude that φ_1 is additive.

Now we demonstrate that φ_1 approximates φ on all of Δ^+. Let $x \in \mathbb{R}^+$ and observe without loss of generality that

$$0 \leq z \circ \theta \circ z^{-1}(x) - z \circ \theta_1 \circ z^{-1}(x) \leq z(b_h).$$

Adding $z \circ \theta_1 \circ z^{-1}(x)$ and applying z^{-1} we obtain

$$\theta_1(z^{-1}(x)) \geq \theta(z^{-1}(x)) \geq T\Big(\theta_1\Big(z^{-1}(x)\Big), b_h\Big).$$

Let $u = z^{-1}(x)$ since x was chosen arbitrarily we have for all $u \in [0,1]$ that

$$T(\theta_1(u), b_h) \leq \theta(u) \text{ and } T(\theta(u), b_h) \leq \theta_1(u). \tag{4.5}$$

We may further conclude

$$\gamma_1^{-1}(x) + a_h \geq \gamma^{-1}(x) \text{ and } \gamma^{-1}(x) + a_h \geq \gamma_1(x). \tag{4.6}$$

Let $H \in \Delta_\delta^+$ as before and $F \in \Delta^+$. It follows from (4.5) and (4.6) that

$$\tau_T(\varphi_1(\delta_{x,F(x)}), H) = \delta_{\gamma_1^{-1}(x)+a_h, T(\theta_1(F(x)), b_h)} \leq \delta_{\gamma^{-1}(x), \theta(F(x))} = \varphi(\delta_{x,F(x)}).$$

Similar argumentation justifies $\tau_T(\varphi(\delta_{x,F(x)}), H) \leq \varphi_1(\delta_{x,F(x)})$. Taking the supremum over $x \in \mathbb{R}^+$ in both inequalities gives us approximation on all of Δ^+.

Finally, to justify uniqueness, suppose φ_2 is an additive type one order automorphism that satisfies $\tau_T(\varphi(F), H) \leq \varphi_1(F)$ and $\tau_T(\varphi_1(F), H) \leq \varphi(F)$ for all $F \in \Delta^+$. Since φ_2 is a type one order automorphism, it follows that for $\gamma_2 \in Aut(\mathbb{R}^+)$ and $\theta_2 \in Aut([0,1])$, $\varphi_2(F) = \theta_2 \circ F \circ \gamma_2^{-1}$. Therefore, for $F \in \Delta^+$ and $x \in \mathbb{R}^+$

$$\delta_{\gamma^{-1}(x)+a_h, T(\theta(F(x)), b_h)} = \tau_T(\varphi(\delta_{x,F(x)}), H) \leq \varphi_2(\delta_{x,F(x)}) = \delta_{\gamma_2^{-1}(x), \theta_2(F(x))} \tag{4.7}$$

and

$$\delta_{\gamma_2^{-1}(x)+a_h, T(\theta_2(F(x)), b_h)} \leq \delta_{\gamma^{-1}(x), \theta(F(x))}. \tag{4.8}$$

Allowing us to conclude that for all x: $|\gamma_2^{-1}(x) - \gamma^{-1}(x)| \leq a_h$ but the uniqueness condition from Theorem 8 gives us that $\gamma_2 = \gamma_1$. We now define $\psi_2 = z \circ \theta_2 \circ z^{-1}$. Let $u = F(x)$ then from (4.7) and (4.8) we have

$$z^{-1}(\psi(z(u)) + z(b_h)) = T(\theta(u), b_h) \leq \theta_2(u) = z^{-1}(\psi_2(z(u))$$

and

$$z^{-1}(\psi_2(z(u)) + z(b_h)) \leq z^{-1}(\psi(z(u)).$$

Applying z to each of the above inequalities gives us

$$\psi(z(u)) - z(b_h) \leq \psi_2(z(u)) \leq \psi(z(u)) + z(b_h)$$

Since our choice of F and x are free and z is a surjection, we can conclude $|\psi_2(y) - \psi(y)| \leq z(b_h)$ for all $y \in \mathbb{R}^+$. Therefore Lemma 1 guarantees that $\psi_2 = \psi_1$. Furthermore, the bijectvity of z and z^{-1} gives us that $\theta_2 = \theta_1$. Since φ_2 is completely determined by θ_2 and γ_2 we conclude that $\varphi_1 = \varphi_2$. ■

We have the same result for type two order automorphisms:

Theorem 26 *Let $H \in \Delta^+ \setminus \{\varepsilon_\infty\}$ and T be a strict t-norm. If for all $F, G \in \Delta^+_\delta$ a type two order automorphism $\varphi : \Delta^+ \to \Delta^+$ satisfies*

$$\tau_T(\tau_T(\varphi(F), \varphi(G)), H^2) \leq \tau_T(\varphi(\tau_T(F,G)), H) \leq \tau_T(\varphi(F), \varphi(G))$$

Then there exists a unique type two order automorphism φ_1 satisfying

1. *$\forall F, G \in \Delta^+$ $\tau_T(\varphi_1(F), \varphi_1(G)) = \varphi_1(\tau_T(F,G))$ (φ_1 is additive)*
2. *$\forall F \in \Delta^+[\tau_T(\varphi(F), H) \leq \varphi_1(F)$ and $\tau_T(\varphi_1(F), H) \leq \varphi(F)]$ (φ and φ_1 approximate one another)*

Proof:

By Lemma 5, it is sufficient to consider $H \in \Delta_\delta^+$. Let φ satisfy our hypothesis, then there exists strictly decreasing functions $\alpha, \beta : \mathbb{R}^+ \to [0,1]$ such that $\varphi(F) = \alpha \circ F^\vee \circ \beta$. Let $F = \delta_{a_f,b_f}$, $G = \delta_{a_g,b_g}$, and $H = \delta_{a_h,b_h}$. By hypothesis, we then have the following

$$\begin{aligned}\delta_{\beta^{-1}(b_f)+\beta^{-1}(b_g)+2a_h, T\left(T(\alpha(a_f),\alpha(a_g)),T(b_h,b_h)\right)} &\le \delta_{\beta^{-1}\left(T(b_f,b_g)\right)+a_h, T(\alpha(a_f+a_g),b_h)} \\ &\le \delta_{\beta^{-1}(b_f)+\beta^{-1}(b_g), T(\alpha(a_f),\alpha(a_g))}.\end{aligned}$$

Which yields the following two inequalities:

$$\beta^{-1}(b_f) + \beta^{-1}(b_g) \le \beta^{-1}\big(T(b_f, b_g)\big) + a_h \le \beta^{-1}(b_f) + \beta^{-1}(b_g) + 2a_h \tag{4.9}$$

$$T\left(T\Big(\alpha(a_f), \alpha(a_g)\Big), b_h^2\right) \le T\left(\alpha(a_f + a_g), b_h\right) \le T\Big(\alpha(a_f), \alpha(a_g)\Big) \tag{4.10}$$

Let z be the inner additive generator of T. Since T is strict and z is a decreasing function, application of z to the second inequality yields:

$$z\Big(\alpha(a_f)\Big) + z\Big(\alpha(a_g)\Big) + z(b_h^2) \ge z\Big(\alpha(a_f + a_g)\Big) + z(b_h) \ge z\Big(\alpha(a_f)\Big) + z\Big(\alpha(a_g)\Big)$$

Notice that $z(b_h^2) = z(z^{-1}(z(b_h) + z(b_h)))$ and let $\psi = z \circ \alpha$ and observe that the inequality becomes

$$\psi(a_f) + \psi(a_g) + 2z(b_h) \ge \psi(a_f + a_g) + z(b_h) \ge \psi(a_f) + \psi(a_g). \tag{4.11}$$

Define $\varpi = \beta^{-1} \circ z^{-1}$ and observe that (4.9) becomes

$$\begin{aligned}\varpi\Big(z(b_f)\Big) + \varpi\Big(z(b_g)\Big) &\le \varpi\Big(z(b_f) + z(b_g)\Big) + a_h \\ &\le \varpi\Big(z(b_f)\Big) + \varpi\Big(z(b_g)\Big) + 2a_h.\end{aligned} \tag{4.12}$$

We now wish to find additive functions that approximate ψ and ϖ. In order to do so, we first establish particular facts about ψ. By hypothesis, we have $b_h \ne 0$

meaning that $z(b_h) < \infty$. Knowing that (4.11) holds for all choices of $a_f, a_g \in [0, \infty)$ means that for some fixed $c < \infty$ and for all $x, y \in [0, \infty)$

$$|\psi(x+y) - (\psi(x) + \psi(y))| \leq c. \tag{4.13}$$

Observe that the inner additive generator, z, is a surjection onto $\mathbb{R}^+$ and $a_h < \infty$ which means a similar statement for ϖ. Therefore what follows will be sufficient argumentation for both ϖ and ψ. Let $x = 2^n u$ and $y = 2^n v$. Equation (4.13) gives us that

$$|\psi(2^n(u+v)) - (\psi(2^n u) - \psi(2^n v))| = o(2^n)$$

and convergence of the series

$$\sum_{i=1}^{\infty} 2^{-i} |\psi(2^{i+1}u) - 2\psi(2^i u)|.$$

To satisfy all hypotheses of Theorem 8 we now need only show that there are some $\tilde{x}, \tilde{y} \in \mathbb{R}^+$ such that $\liminf[2^{-n}|\psi(2^n\tilde{x}) - \psi(2^n\tilde{y})|] > 0$. Following the previous proof, we have for all $n \in \mathbb{N}$ that $\psi(2^n x) \geq 2^n(\psi(x) - c)$. Since $\alpha(0) = 1$ and $z(1) = 0$, we have that $\psi(0) = 0$. Since ψ is a surjection onto $\mathbb{R}^+$ we know there exists $\tilde{x}$ such that $\psi(\tilde{x}) > c$ so

$$\liminf[2^{-n}|\psi(2^n\tilde{x}) - \psi(2^n \cdot 0)| \geq \liminf[2^{-n} \cdot 2^n(\psi(\tilde{x}) - c)] > 0.$$

By Theorem 8 we may conclude that there are unique non constant additive functions ψ_1, ϖ_1 such that

$$|\psi_1(x) - \psi(x)| \leq \sum_{i=1}^{\infty} 2^{-i} z(b_h) = z(b_h)$$

and

$$|\varpi_1(x) - \varpi(x)| \leq \sum_{i=1}^{\infty} 2^{-i} a_h = a_h.$$

We now turn our attention towards constructing an additive order automorphism on Δ^+ with the desired properties. We define $\alpha_1 = z^{-1} \circ \psi_1$ and $\beta_1 = z^{-1} \circ \varpi_1^{-1}$.By Corollary 2, we have that there exists $c_1, c_2 > 0$ such that $\psi_1(x) = c_1 x$ and

$\varpi(x) = c2x$. Therefore both ψ_1 and ϖ_1^{-1} are strictly increasing bijections of $\mathbb{R}^+$. Since z^{-1} is an order reversing bijection from $\mathbb{R}^+$ into $[0,1]$, α_1 is a strictly decreasing function from $\mathbb{R}^+$ into $[0,1]$. Similarly, β_1 is a strictly decreasing function from $\mathbb{R}^+$ into $[0,1]$. Therefore we can conclude the function $\varphi_1(F) := \alpha_1 \circ F^\vee \circ \beta_1$ is an order automorphism on Δ^+.

Next we establish the additivity of φ_1. Let $x, y \in \mathbb{R}^+$ and $u, v \in [0,1]$. Then

$$
\begin{aligned}
\alpha_1(x+y) &= z^{-1} \circ \psi_1(x+y) \\
&= z^{-1}(\psi_1(x) + \psi_1(y)) \\
&= z^{-1}(z \circ \alpha_1(x) + z \circ \alpha_1(y)) \\
&= T(\alpha_1(x), \alpha_1(y))
\end{aligned}
$$

and

$$
\begin{aligned}
\beta_1^{-1}(T(u,v)) &= \varpi_1 \circ z(T(u,v)) \\
&= \varpi_1(z(u) + z(v)) \\
&= \varpi_1(z(u)) + \varpi_1(z(v)) \\
&= \beta_1^{-1}(u) + \beta_1^{-1}(v).
\end{aligned}
$$

Let $F, G \in \Delta^+$. It follows that

$$
\begin{aligned}
\varphi_1(\tau_T(\delta_{x,F(x)}, \delta_{y,G(y)}) &= \delta_{\beta_1^{-1}(T(F(x),F(y))),\alpha_1(x+y)} \\
&= \delta_{\beta_1^{-1}(F(x))+\beta_1^{-1}(y),T(\alpha_1(x),\alpha_1(y))} \\
&= \tau_T(\varphi_1(\delta_{x,F(x)}), \varphi_1(\delta_{y,G(y)}))
\end{aligned}
$$

Using sup continuity of order automorphisms we have the following

$$
\begin{aligned}
\sup_y[\sup_x[\varphi_1(\tau_T(\delta_{x,F(x)}, \delta_{y,G(y)})]] &= \sup_y[\varphi_1(\tau_T(\sup_x[\delta_{x,F(x)}], \delta_{y,G(y)}))\\
&= \sup_y[\varphi_1(\tau_T(F, \delta_{y,G(y)})] = \varphi_1(\tau_T(F,G))
\end{aligned}
$$

While on the other hand

$$
\begin{aligned}
\sup_y[\sup_x[\varphi_1(\tau_T(\delta_{x,F(x)}, \delta_{y,G(y)})]] &= \sup_y[\sup_x[\tau_T(\varphi_1(\delta_{x,F(x)}), \varphi_1(\delta_{y,G(y)})]] \\
&= \tau_T(\varphi_1(F), \varphi_1(G))
\end{aligned}
$$

Therefore allowing us to conclude that φ_1 is additive.

Now we demonstrate that φ_1 approximates φ on all of Δ^+. Let $x \in \mathbb{R}^+$ and observe without loss of generality that

$$0 \leq z \circ \alpha(x) - z \circ \alpha_1(x) \leq z(b_h).$$

Adding $z \circ \alpha_1(x)$ and applying z^{-1} we obtain

$$\alpha_1(x) \geq \alpha(x) \geq T\Big(\alpha_1(x), b_h\Big).$$

Therefore, we conclude for all $x \in \mathbb{R}^+$

$$T(\alpha_1(x), b_h) \leq \alpha(x) \text{ and } T(\alpha(x), b_h) \leq \alpha_1(x). \tag{4.14}$$

We may further conclude

$$\varpi_1(x) + a_h \geq \varpi(x) \text{ and } \varpi(x) + a_h \geq \varpi_1(x).$$

Since the above is true for all $x \in \mathbb{R}^+$, we have for all $u \in [0,1]$.

$$\beta^{-1}(u) \leq \beta_1^{-1}(u) + a_h \text{ and } \beta_1^{-1}(u) \leq \beta^{-1}(u) + a_h \tag{4.15}$$

Let $H \in \Delta_\delta^+$ as before and $F \in \Delta^+$. It follows from (4.14) and (4.15) that

$$\tau_T(\varphi_1(\delta_{x,F(x)}), H) = \delta_{\beta_1^{-1}(F(x))+a_h, T(\alpha_1(x), b_h)} \leq \delta_{\beta^{-1}(F(x)), \alpha(x)} = \varphi(\delta_{x,F(x)})$$

and

$$\tau_T(\varphi(\delta_{x,F(x)}), H) = \delta_{\beta^{-1}(F(x))+a_h, T(\alpha(x), b_h)} \leq \delta_{\beta_1^{-1}(F(x)), \alpha_1(x)} = \varphi_1(\delta_{x,F(x)}).$$

Application of a supremum to both inequalities gives us approximation on all of Δ^+.

Finally, to justify uniqueness, suppose φ_2 is an additive type two order automorphism that satisfies $\tau_T(\varphi(F), H) \leq \varphi_1(F)$ and $\tau_T(\varphi_1(F), H) \leq \varphi(F)$ for all $F \in \Delta^+$. Since φ_2 is a type two order automorphism, it follows that there are

$\alpha_2, \beta_2 : \mathbb{R}^+ \to [0,1]$ such that $\varphi_2(F) = \alpha_2 \circ F^{\vee} \circ \beta_2$. Therefore, for $F \in \Delta^+$ and $x \in \mathbb{R}^+$

$$\delta_{\beta^{-1}(F(x))+a_h, T(\alpha(x), b_h)} = \tau_T(\varphi(\delta_{x,F(x)}), H) \leq \varphi_2(\delta_{x,F(x)}) = \delta_{\beta_2^{-1}(F(x)), \alpha_2(x)} \tag{4.16}$$

and

$$\delta_{\beta_2^{-1}(F(x))+a_h, T(\alpha_2(x), b_h)} \leq \delta_{\beta^{-1}(F(x)), \alpha(x)}. \tag{4.17}$$

Allowing us to conclude that for all x

$$|\beta_2^{-1}(F(x)) - \beta^{-1}(F(x))| \leq a_h.$$

Since $F(x) \in [0,1]$ there exists $y \in \mathbb{R}^+$ such that $z^{-1}(y) = F(x)$,so

$$|\beta_2^{-1}(z^{-1}(y)) - \beta^{-1}(z^{-1}(y))| \leq a_h.$$

Since our choice of F and x are arbitrary we have the previous inequality for all choices of y. Therefore for all $y \in \mathbb{R}^+$ we have $|\varpi_2^{-1}(y) - \varpi^{-1}(y)| \leq a_h$ where $\varpi_2 = \beta_2^{-1} \circ z^{-1}$ but the uniqueness condition from Theorem 8 gives us that $\varpi_2 = \varpi_1$ which in turn guarantees $\beta_1 = \beta_2$. Similar reasoning gives us that $\alpha_2 = \alpha_1$. Since φ_2 is completely determined by α_2 and β_2 we conclude that $\varphi_1 = \varphi_2$. ■

Here we observe that we have yet to justify that there are order quasi solutions which are in anyway distinct from actual solutions of Cauchy's equation. That is to say, that it could be the case that the only way to approximate solutions of Cauchy's equation is to actually be a solution. In the case of both type one and type two order automorphisms, we have that there are indeed order quasi additive functions which are not solutions of Cauchy's equation. Below are examples for each case:

Example 2 *Let* $\theta(x) = \sqrt{x}e^{-0.1\sin(\pi x)}$ *and* $\gamma(x) = 2x + 0.05\sin(x)$. *We have that both* θ *and* γ *are continuous strictly increasing functions over the interior of their respective domains and both are surjections, hence each are order automorphisms.*

Since γ is an order automorphism, so is γ^{-1}, thus $\varphi(F) = \theta \circ F \circ \gamma^{-1}$ is an order automorphism of Δ^+.

Further, the map θ is bounded by $\sqrt{x}$ and $0.9\sqrt{x}$, while γ^{-1} is bounded bounded by the functions $\frac{1}{2}x - 0.05$ and $\frac{1}{2}x + 0.05$. Thus if we take T to be product, $H = \delta_{0.15,0.81}$, and $\varphi(F) = \theta \circ F \circ \gamma^{-1}$ we have

$$\tau_T(\tau_T(\varphi(F), \varphi(G)), H^2) \leq \tau_T(\varphi(\tau_T(F,G)), H) \leq \tau_T(\varphi(F), \varphi(G)).$$

However, we also have

$$\varphi(\tau_T(\delta_{\frac{\pi}{2},0.5}, (\delta_{\frac{\pi}{2},0.5})) = \delta_{2\pi, 0.5e^{0.1\frac{\sqrt{2}}{2}}}$$

while

$$\tau_T(\varphi(\delta_{\frac{\pi}{2},0.5}), \varphi(\delta_{\frac{\pi}{2},0.5})) = \delta_{2\pi+0.1,0.5e^{-0.2}}$$

Therefore φ is strictly order quasi additive.

By Theorem 25 we should also be able to produce a unique additive order automorphism which φ approximates with error $\delta_{0.15,0.81}$, and it is readily verified that $\varphi_1(F)(x) := \sqrt{F(\frac{1}{2}x)}$ is the approximated solution.

Example 3 *Let $\alpha(x) = e^{-2x+0.1|\sin(x)|}$, $\beta = e^{-2x}$ and $\varphi(F) = \alpha \circ F^{\vee} \circ \beta$. Since both α and β are continuous, strictly decreasing, send 0 to 1 and ∞ to 0, we have that φ is an order automorphism. If we take $H = \delta_{0,0.81}$ we can see that φ satisfies*

$$\tau_T(\tau_T(\varphi(F), \varphi(G)), H^2) \leq \tau_T(\varphi(\tau_T(F,G)), H) \leq \tau_T(\varphi(F), \varphi(G)).$$

However, φ is not a solution of Cauchy's equation. Therefore φ is strictly order quasi additive, and by Theorem 26 $\varphi_1(F) := e^{2F^{\vee}(e^{-2x})}$ is the unique order automorphism solution that φ approximates.

The following is a theorem concerning the similarity of stability between triangle functions which are generated by isomorphic continuous t-norms. First we note that occasionally, it is useful to treat order automorphisms of $[0,1]$ as order automorphisms of Δ^+. We may do so because whenever f is an order automorphism of $[0,1]$ we have that $f(F(x)), f^{-1}(F(x)) \in \Delta^+$, and $F(x) \leq G(x)$ happens if and only if $f(F(x)) \leq f(G(x))$. Therefore, when we write $f \circ F$ for $F \in \Delta^+$ we are viewing f as an order automorphism of Δ^+ and when we write $f(F(x))$ we are emphasizing f as an order automorphism of $[0,1]$.

Theorem 27 *Let T and T' be continuous t-norms that are isomorphic under f*

1. *The function φ is order quasi additive on (Δ^+, τ_T) for error function H if and only if $\psi(F) = f \circ \varphi(f \circ F))$ is order quasi additive on $(\Delta^+, \tau_{T'})$ for error function $f \circ H$.*

2. *The function φ is an order additive approximator on (Δ^+, τ_T) for error function H if and only if $\psi(F) = f \circ \varphi(f^{-1} \circ F))$ is an order additive approximator on $(\Delta^+, \tau_{T'})$ for error function $f \circ H$.*

Proof:

We first show part 1. Let $F, G \in \Delta^+$ and $x \in \mathbb{R}^+$. By isomorphism under f we have

$$\tau_T(F,G)(x) = \sup_{u+v=x} T(F(u), G(v))(x) = \sup_{u+v=x} f^{-1}(T'(f(F(u)), f(G(v))))(x).$$

Since f^{-1} is an order autmorphism, we may pass the supremum through it yeliding

$$\tau_T(F,G)(x) = f^{-1} \circ \tau_{T'}(f \circ F, f \circ G)(x)$$

Application of f to both sides and substituting $f^{-1} \circ F$ and $g^{-1} \circ G$ for F and G respectively yeilds

$$\tau_{T'}(F,G) = f \circ \tau_T(f^{-1} \circ F, f^{-1} \circ G).$$

Similar rearrangement also gives the following two equations:

$$\tau_T(f^{-1} \circ F, f^{-1} \circ G) = f^{-1} \circ \tau_{T'}(F,G)$$

$$\tau_{T'}(f \circ F, f \circ G) = f \circ \tau_T(F, G)$$

So for all $F, G \in \Delta^+$ we observe the following where $U = f \circ F$ and $V = f \circ G$

$$\tau_T(\varphi(F), \varphi(G)) = f^{-1} \circ \tau_{T'}(f \circ \varphi(F), f \circ \varphi(G)) = f^{-1} \circ \tau_{T'}(\psi(U), \psi(V))$$

$$\varphi(\tau_T(F, G)) = f^{-1} \circ \psi \circ f \circ f^{-1}\Big(\tau_{T'}(f \circ F, f \circ G)\Big) = f^{-1} \circ \psi(\tau_{T'}(U, V))$$

$$\begin{aligned}\tau_T(\varphi(\tau_T(F, G)), H) &= \tau_T(f^{-1} \circ \psi(\tau_{T'}(U, V)), H) \\ &= f^{-1} \circ \tau_{T'}(\psi(\tau_{T'}(U, V)), f \circ H)\end{aligned}$$

$$\tau_T(\tau_T(\varphi(F), \varphi(G)), \tau_T(H, H)) = f^{-1} \circ \tau_{T'}(\tau_{T'}(\psi(U), \psi(V)), \tau_{T'}(f \circ H, f \circ H))$$

Since φ is an order quasi additive function with error H, we have the following inequalities:

$$\begin{aligned}f^{-1} \circ \tau_{T'}(\tau_{T'}(\psi(U), \psi(V)), \tau_{T'}(f \circ H, f \circ H)) &= \tau_T(\tau_T(\varphi(F), \varphi(G)), \tau_T(H, H)) \\ &\leq \tau_T(\varphi(\tau_T(F, G)), H) \\ &= f^{-1} \circ \tau_{T'}(\psi(\tau_{T'}(U, V)), f \circ H)\end{aligned}$$

$$\begin{aligned}f^{-1} \circ \tau_{T'}(\psi(\tau_{T'}(U, V)), f \circ H) &= \tau_T(\varphi(\tau_T(F, G)), H) \\ &\leq \tau_T(\varphi(F), \varphi(G)) \\ &= f^{-1} \circ \tau_{T'}(\psi(U), \psi(V))\end{aligned}$$

Since f is order preserving, we may apply f to all parts of the above inequalities and obtain

$$\tau_{T'}(\tau_{T'}(\psi(U), \psi(V)), \tau_{T'}(f \circ H, f \circ H)) \leq \tau_{T'}(\psi(\tau_{T'}(U, V)), f \circ H) \leq \tau_{T'}(\psi(U), \psi(V))$$

for all $U, V \in f^{-1}(\Delta^+)$. Since composition with f^{-1} is a bijection, the above holds for all $U, V \in \Delta^+$. For the converse, suppose ψ is an order quasi additive function with error $f \circ H$. We have

$$\begin{aligned} f \circ \tau_T(\tau_T(\varphi(F), \varphi(G)), \tau_T(H, H)) &= \tau_{T'}(\tau_{T'}(\psi(U), \psi(V)), \tau_{T'}(f \circ H, f \circ H)) \\ &\leq \tau_{T'}(\psi(\tau_{T'}(U, V)), f \circ H) \\ &= f \circ \tau_T(\varphi(\tau_T(F, G)), H) \end{aligned} \tag{4.18}$$

and

$$\begin{aligned} f \circ \tau_T(\varphi(\tau_T(F, G)), H) &= \tau_{T'}(\psi(\tau_{T'}(U, V)), f \circ H) \\ &\leq \tau_{T'}(\psi(U), \psi(V)) \\ &= f \circ \tau_T(\varphi(F), \varphi(G)). \end{aligned} \tag{4.19}$$

Applying f^{-1} to both (4.18) and (4.19) completes part 1.

For part 2, let φ be an order additive approximator over τ_T with error H, and φ' to be a solution which φ approximates. Then, taking $\psi' = f \circ \varphi' \circ f^{-1}$

$$\begin{aligned} \tau_{T'}(\psi'(F), \psi'(G)) &= \tau_{T'}(f \circ \varphi'(f^{-1} \circ F), f \circ \varphi'(f^{-1} \circ G)) \\ &= f \circ \tau_T(\varphi'(f^{-1} \circ F), \varphi'(f^{-1} \circ G)) \\ &= f \circ \varphi'(\tau_T(f^{-1} \circ F, f^{-1} \circ G)) \\ &= f \circ \varphi'(f^{-1} \circ \tau_{T'}(F, G)) \\ &= \psi'(\tau_{T'}(F, G)). \end{aligned}$$

So ψ' is a $\tau_{T'}$ additive function and

$$f^{-1} \circ \tau_{T'}(f \circ \psi(F), f \circ H) = \tau_T(\varphi(f^{-1} \circ F), H) \leq \varphi'(f^{-1} \circ F) = f^{-1} \circ \psi(F)$$

$$f^{-1} \circ \tau_{T'}(f \circ \psi'(F), f \circ H) = \tau_T(\varphi'(f^{-1} \circ F), H) \leq \varphi(f^{-1} \circ F) = f^{-1} \circ \psi(F)$$

Applying f to both inequalities completes the forward direction.

For the converse, suppose ψ is an order additive approximator with error $f \circ H$, and take ψ' to be the approximated solution. Therefore,

$$\tau_T(\varphi(f^{-1} \circ F), H) = f^{-1} \circ \tau_{T'}(f \circ \psi(F), f \circ H) \leq f^{-1} \circ \psi(F) = \varphi'(f^{-1} \circ F)$$

$$\tau_T(\varphi'(f^{-1} \circ F), H) = f^{-1} \circ \tau_{T'}(f \circ \psi'(F), f \circ H) \leq f^{-1} \circ \psi(F) = \varphi(f^{-1} \circ F)$$

This completes the proof. ∎

Since we have a notion of the existence of an order stability relationship in (Δ^+, τ_T) when T is strict, this theorem gives us the ability to conclude a similar result for Hyers Ulam stability when T is strict.

Theorem 28 *Let $F, G \in \Delta^+$ and T be a strict t-norm whose isomorphism to the product norm is denoted f.*

1. *If φ is an order quasi additive function on (Δ^+, τ_T) with error $\delta_{a,b}$, then $\psi = f \circ \varphi \circ f^{-1}$ is metric quasi additive in (Δ^+, τ_p, d_L) with error bounded by* $\max\{a, 1 - f(b)\}$.

2. *If φ is an order additive approximator in (Δ^+, τ_T) with error $\delta_{a,b}$ then $\psi = f \circ \varphi \circ f^{-1}$ is a solution in metric approximation of Cauchy's equation in (Δ^+, τ_p, d_L) with uncertainty bounded by* $\max\{a, 1 - f(b)\}$.

Proof:

For the first claim, we have that

$$\begin{aligned} f(\tau_T(\tau_T(\varphi(F), \varphi(G)), \delta_{a,b})) &= \tau_p(f(\tau_T(\varphi(F), \varphi(G))), f(\delta_{a,b})) \\ &= \tau_p(\tau_p(f(\varphi(F)), f(\varphi(G))), \delta_{a,f(b)}) \end{aligned}$$

and

$$f(\tau_T(\varphi(\tau_T(F, G)), \delta_{a,b})) = \tau_p(f(\varphi(\tau_T(F, G))), \delta_{a,f(b)})$$

By order quasi additivity of φ and strictness of T we have that

$$\tau_T(\tau_T(\varphi(F), \varphi(G)), \delta_{a,b}) \leq \varphi(\tau_T(F, G)) \text{ and } \tau_T(\varphi(\tau_T(F, G)), \delta_{a,b}) \leq \tau_T(\varphi(F), \varphi(G))$$

Therefore, by applying f to both sides of each inequality we obtain

$$\tau_p(\tau_p(f \circ \varphi(F), f \circ \varphi(G)), \delta_{a,f(b)}) \leq f \circ \varphi(\tau_T(F, G))$$

and

$$\tau_p(f \circ \varphi(\tau_T(F, G)), \delta_{a,f(b)}) \leq \tau_p(f \circ \varphi(F), f \circ \varphi(G))$$

In what follows we may adopt the harmless convention that all members of Δ^+ are 0 on $(-\infty, 0)$ in order to avoid cases. For $W \in \Delta^+$ we have by left continuity that $\tau_p(W, \delta_{a,b})(x - a) = W(x) \cdot b$. Additionally, for any positive number y we have that $y - (1 - b) \leq y \cdot b$ when $b \in [0, 1]$. These observation grants us the following inequalities:

$$\tau_p(f \circ \varphi(F), f \circ \varphi(G))(x) \leq f \circ \varphi(\tau_T(F, G))(x + a) + (1 - f(b))$$

$$f \circ \varphi(\tau_T(F, G))(x) \leq \tau_p(f \circ \varphi(F), f \circ \varphi(G))(x + a) + (1 - f(b))$$

Since this observation holds for all $x \in \mathbb{R}^+$ and not just $x = \frac{1}{h}$ (where $h = \max\{a, 1-f(b)\}$) we may conclude

$$d_L(\tau_p(f \circ \varphi(F), f \circ \varphi(G)), f \circ \varphi(\tau_T(F, G))) \leq \max\{a, 1 - f(b)\}$$

Furthermore, we observe that since the above inequality is true for all $F, G \in \Delta^+$ and that left composition of f^{-1} is a bijection on Δ^+ we may replace F and G everywhere with $f^{-1} \circ F$ and $f^{-1} \circ G$. With the added observation that

$$f \circ \varphi(\tau_T(f^{-1} \circ F, f^{-1} \circ G)) = f \circ \varphi(f^{-1} \circ f(\tau_T(f^{-1} \circ F, f^{-1} \circ G))) = \psi(\tau_p(F, G))$$

we may conclude

$$d_L(\tau_p(\psi(F), \psi(G)), \psi(\tau_p(F, G))) \leq \max\{a, 1 - f(b)\}$$

For the second claim, we have the assumption that φ is an order additive approximator, so there is φ' such that for all $F \in \Delta^+$

$$\tau_T(\varphi(F), \delta_{a,b}) \leq \varphi'(F) \text{ and } \tau_T(\varphi'(F), \delta_{a,b}) \leq \varphi(F)$$

Further, application of f to these inequalities along with replacement of F by $f^{-1} \circ F$ gives

$$d_L(\psi(F), \psi'(F)) \leq \max\{a, 1 - f(b)\}$$

where $\psi' = f \circ \varphi' \circ f^{-1}$. Now we show ψ' is additive over the product triangle norm. To that end, the following suffices

$$\begin{aligned}
\tau_p(\psi'(F), \psi'(G)) &= f(\tau_T(\varphi'(f^{-1}(F)), \varphi'(f^{-1}(G)))) \\
&= f \circ \varphi'(\tau_T(f^{-1}(F), f^{-1}(G))) \\
&= \psi'(\tau_p(F, G))
\end{aligned}$$

■

Corollary 13 follows from Theorem 25 and Theroem 26. While the subsequent corollary follows from Corollary 13 and Lemma 5

Corollary 13 *Let $F, G \in \Delta^+$ and T be a strict t-norm whose isomorphism to the product norm is denoted f. If φ is an order automorphism which is order quasi additive with error $\delta_{a,b}$ then $\psi = f \circ \varphi \circ f^{-1}$ is a solution in metric approximation of Cauchy's equation in (Δ^+, τ_p, d_L) with uncertainty bounded by* $\max\{a, 1 - f(b)\}$.

Corollary 14 *Let $F, G \in \Delta^+$ and T be a strict t-norm whose isomorphism to the product norm is denoted f. If φ is an order automorphism which is order quasi additive with error $H \neq \delta_{0,0}$ then there exists $a \in \mathbb{R}^+$ and $b \in (0, 1]$ such that $\delta_{a,b} \geq H$ and $\psi = f \circ \varphi \circ f^{-1}$ is a solution in metric approximation of Cauchy's equation in (Δ^+, τ_p, d_L) with uncertainty bounded by* $\max\{a, 1 - f(b)\}$.

Triangle functions generated by nilpotent T

Now that we have results for strict t-norms, we turn our attention to gaining results for continuous nilpotent Archimedean t-norms. Here, the state of affairs is less straightforward. For one, type two order isomorphism solutions do not exist (recall Theorem 22). Furthermore, the relationship between order quasi additive type one order automorphisms and an order additive approximator order automorphisms isn't as secure as we shall see in the following sequence of results.

Theorem 29 *Let T be a continuous, nilpotent Archimedean t-norm isomorphic under f to the Lukasiewicz t-norm, $a \in \mathbb{R}^+$, and $b \in (0,1)$. If γ is a metric quasi additive order automorphism of $\mathbb{R}^+$ with error a, and θ is a convex order automorphism of $[0,1]$ that satisfies $\theta(z-1) \leq 2\theta(\frac{z}{2}) - b$ when $z > 1$ then for all $F, G \in \Delta^+$ $\psi(F) = f \circ \theta \circ f^{-1} \circ F \circ \gamma$ is an order quasi additive function on (Δ^+, τ_T) for error function $\delta_{a,f(b)}$.*

Proof:

By Theorem 27, ψ satisfies τ_T order quasi additivity for error $\delta_{a,f(b)}$ if and only if $\varphi = \theta \circ F \circ \gamma$ satisfies $\tau_{T'}$ order quasi additivity for error $\delta_{a,b}$ where T' is the Lukasiewicz t-norm. Therefore, we will show that φ is a quasi additive function on $(\Delta^+, \tau_{T'})$ for error function $\delta_{a,b}$. Since φ is sup continuous, it suffices to show that φ is order quasi additive on Δ^+_δ. To that end, suppose $x, y \in [0,1]$ and $w, z \in \mathbb{R}^+$. Letting b^2 denote $T'(b,b)$ it follows that

$$\tau_{T'}(\tau_{T'}(\varphi(\delta_{x,w}), \varphi(\delta_{y,z})), \delta_{2a,b^2}) = \delta_{\gamma^{-1}(w)+\gamma^{-1}(z)+2a, T'(T'(\theta(x),\theta(y)),b^2)}$$

$$\tau_{T'}(\varphi(\tau_{T'}(\delta_{x,w}, \delta_{y,z})), \delta_{a,b}) = \delta_{\gamma^{-1}(w+z)+a, T'(\theta(T'(x,y)),b)}$$

$$\tau_{T'}(\varphi(\delta_{x,w}), \delta_{y,z}) = \delta_{\gamma^{-1}(w)+\gamma^{-1}(z), T'(\theta(x),\theta(y))}$$

Since γ^{-1} is metric quasi additive with error a we have

$$\gamma^{-1}(w) + \gamma^{-1}(z) \leq \gamma^{-1}(w+z) + a \leq \gamma^{-1}(w) + \gamma^{-1}(z) + 2a$$

As such, we need only check θ satisfies the following inequality to verify quasi additivity.

$$T'(T'(\theta(x), \theta(y)), b^2) \leq T'(\theta(T'(x, y)), b) \leq T'(\theta(x), \theta(y))$$

Since θ is convex and $\theta(0) = 0$, we have that it is super additive on its domain (see [7]). In particular, taking $x + y \leq 1$ we have $\theta(x) + \theta(y) \leq \theta(x + y) \leq 1$. Thus when $x + y \leq 1$, we have that both $T'(\theta(x), \theta(y))$ and $\theta(T'(x, y))$ are 0.

Instead taking $1 < x + y < 2$ (while maintaining $x, y \in [0, 1]$) we have $0 < y - (1 - x)$. So by convexity of θ

$$\frac{\theta(y) - \theta(y - (1 - x))}{1 - x} \leq \frac{\theta(1) - \theta(y - (1 - x))}{1 - (y - 1 + x)} \leq \frac{\theta(1) - \theta(x)}{1 - x}$$

Therefore, examining the secant line of θ between $y - (1 - x)$ and y

$$\theta(y - (1 - x)) = \theta(y) - \frac{\theta(y) - \theta(y - (1 - x))}{1 - x}(1 - x) \geq \theta(y) - \frac{\theta(1) - \theta(x)}{1 - x}(1 - x)$$

So $\theta(x + y - 1) \geq \theta(x) + \theta(y) - 1$ which implies $\theta(x) + \theta(y) - 3 + 2b \leq \theta(x + y - 1) - 1 + b$. We claim this fact is sufficient to conclude that θ satisfies

$$T'(T'(\theta(x), \theta(y)), b^2) \leq T'(\theta(T'(x, y)), b)$$

by observing that the inequality may be rephrased as

$$\max\{\max\{\theta(x) + \theta(y) - 1, 0\} + 2b - 2, 0\} \leq \max\{\theta(\max\{x + y - 1, 0\}) + b - 1, 0\}$$

The left side of the rephrased inequality is nonzero only when the expression $\theta(x) + \theta(y) - 3 + 2b$ is greater than zero, and is equivalent to the expression in that case. Clearly then, the right side is necessarily greater in value.

Now we only need verify $T'(\theta(T'(x, y)), b) \leq T'(\theta(x), \theta(y))$. Assuming without loss of generality that $T'(\theta(T'(x, y)), b) > 0$ and applying the hypothesis that $\theta(x + y - 1) \leq 2\theta(\frac{x+y}{2}) - b$ we have

$$T'(\theta(T'(x, y)), b) = \theta(x + y - 1) - 1 + b \leq 2\theta(\frac{x + y}{2}) - 1$$

By convexity we have

$$T'(\theta(T'(x,y)),b) \leq 2\theta(\frac{x+y}{2}) - 1 \leq \theta(x) + \theta(y) - 1 = T'(\theta(x),\theta(y)$$

$\blacksquare$

Theorem 30 *Let T be a continuous, nilpotent Archimedean t-norm isomorphic to Lukasiewicz t-norm under f, $a \in \mathbb{R}^+$, and $b \in (0,1)$. If γ is a metric quasi additive order automorphism on $\mathbb{R}^+$ with error a, and θ is a concave order automorphism that satisfies $\theta(z-1) \geq 2\theta(\frac{z}{2}) + b - 2$ when $z > 1$ then for all $F, G \in \Delta^+$ $\psi(F) = f \circ \theta \circ f^{-1} \circ F \circ \gamma$ is an order quasi additive function on (Δ^+, τ_T) for error function $\delta_{a,f(b)}$.*

Proof:

Let, $\varphi = \theta \circ F \circ \gamma$, and T' be the Lukasiewicz t- norm. Following the proof of the prior theorem using the same properties of γ, we observe that it will be sufficient to show that θ satisfies

$$T'(T'(\theta(x),\theta(y)),b^2) \leq T'(\theta(T'(x,y)),b) \leq T'(\theta(x),\theta(y))$$

Here, we will show that concavity gives $T'(\theta(T'(x,y)),b) \leq T'(\theta(x),\theta(y))$. Since the inequality is clearly true when the left side is 0, we can assume that it is instead positive which in turn allows the assumption that $x + y > 1$. In this case, it is sufficient to show concavity gives the following

$$\theta(x+y-1) + b - 1 \leq \theta(x) + \theta(y) - 1$$

So by concavity of θ

$$\frac{\theta(y) - \theta(y-(1-x))}{1-x} \geq \frac{\theta(1) - \theta(y-(1-x))}{1-(y-1+x)} \geq \frac{\theta(1) - \theta(x)}{1-x}$$

Therefore, examining the secant line of θ between $y-(1-x)$ and y

$$\theta(y-(1-x)) = \theta(y) - \frac{\theta(y) - \theta(y-(1-x))}{1-x}(1-x) \leq \theta(y) - \frac{\theta(1) - \theta(x)}{1-x}(1-x)$$

So

$$\theta(x+y-1)+b-1 \leq \theta(x+y-1) \leq \theta(x)+\theta(y)-1$$

To complete the proof, we show that θ satisfies $T'(T'(\theta(x),\theta(y)),b^2) \leq T'(\theta(T'(x,y)),b)$. To that end, we again observe that we need only examine when $T'(T'(\theta(x),\theta(y)),b^2)$ is positive. In that case, it must also follow that $\theta(x)+\theta(y)+2b-3$ is positive. The following inequality demonstrates that $x+y \leq 1$ would therefore force $T'(T'(\theta(x),\theta(y)),b^2) \leq 0$

$$\theta(x)+\theta(y)+2b-3 \leq \theta(x+y)+2b-3 \leq 1+2b-3 \leq 0$$

So taking $T'(T'(\theta(x),\theta(y)),b^2) > 0$, we have $x+y>1$ which yields $\theta(x+y-1) \geq \theta(x)+\theta(y)-b$ under our hypothesis. Therefore,

$$\begin{aligned} T'(T'(\theta(x),\theta(y)),b^2) &\leq \theta(x)+\theta(y)+2b-3 \\ &\leq \theta(\frac{x+y}{2})+\theta(\frac{x+y}{2})+2b-3 \\ &\leq \theta(x+y-1)+b-1 \\ &\leq T'(\theta(T'(x,y)),b) \end{aligned} \tag{4.20}$$

■

Now, with sufficient theoretical basis, we can show that unlike the case of strict t norms, being an order quasi additive function with error function H is an insufficient condition for being an order additive approximator with error H. Taking T to be a t-norm isomorphic to the Lukasewicz t-norm, T' we have the following of counter example:

For $b \in (0,1)$, let

$$\theta(x) = \begin{cases} \frac{b-b^2}{2-b}x & 0 \leq x \leq 1-\frac{b}{2} \\ \frac{2-b+b^2}{b}x - \frac{2-2b+fb^2}{b} & 1-\frac{b}{2} < x. \end{cases}$$

Recalling Theorem 29, we have that θ is sufficient for $\varphi(F) = \theta \circ F$ to be an order quasi additive function on $(\Delta^+, \tau_{T'})$ for error function $\delta_{0,b}$. Furthermore, from

Theorem 27 we have that there is an order automorphism that is an order quasi additive function on (Δ^+, τ_T) with error $\delta_{0,f(b)}$, namely $\psi = f \circ \theta \circ f^{-1} \circ \varphi$.

However, we also recall from Corollary 5 that $\varphi'(F)$ is an order automorphism solution of Cauhcy's equation if and only if $\varphi'(F) = F \circ \gamma$ where γ is an order automorphism of $\mathbb{R}^+$. Therefore, for all $x \in [0,1]$ θ must satisfy both

$$T'(\theta(x), b) \leq x$$

and

$$T'(x, b) \leq \theta(x)$$

the latter fails in general since, for $x = 1 - \frac{b}{2}$:

$$\theta(1 - \frac{b}{2}) - T'(x, b) = \frac{b - b^2}{2} - (1 - \frac{b}{2} + b - 1) = \frac{b - b^2}{2} - \frac{b}{2} < 0$$

This means that ψ does not approximate an order automorphism solution of Cauchy's equation with error $\delta_{0,b}$. Furthermore by way of Theorem 27, this means that for all $\delta_{a,b}$ and all continuous nilpotent Archimedean t-norm there exists an order automorphism φ such that φ is an order quasi additive function on (Δ^+, τ_T) with error $\delta_{a,b}$ but is not an order additive approximator of an order automorphism solution with the same error.

While this example shows that the relationship between order quasi additive functions and order additive approximators is more tenuous in the context of triangle functions generated by nilpotent t-norms, it does not mean that nilpotent t-norms completely abbrogate the relationship between the two concepts either. If we instead construct $\psi_1 = \theta_1 \circ F$ and $\psi_2 = \theta_2 \circ F$ where

$$\theta_1(x) = x^{2-b} \text{ and } \theta(x) = x^b$$

we see via Theorems 29 and 30 that ψ_1 and ψ_2 are order quasi additive functions on $(\Delta^+, \tau_{T'})$ for error function $\delta_{0,b}$. Since θ_1 is convex and θ_2 is concave the below it is sufficient to verify that

$$T'(x,b) \leq \theta_1(x) \text{ and } T'(\theta_2(x),b) \leq x$$

hold in order to conclude ψ_1 and ψ_2 are order additive approximators. Let $i(x,b) = x^{2-b} - x - b + 1$. We have that

$$i(x,0) = x^2 - x + 1$$

which is positive for all $x \in [0,1]$ and that $i(x,1) = 0$ for all $x \in [0,1]$. Further

$$\frac{\partial i}{\partial b} = -ln(x)x^{2-b} - 1 \text{ and } \frac{\partial^2 i}{\partial b^2} = ln(x)x^{2-b}$$

The second partial derivative is clearly negative for all $x \in (0,1]$ and $b \in (0,1)$, and the first partial derivative is clearly negative for all $x \in (0,1]$ and $b = 0$ or $b = 1$ Therefore the first partial derivative is negative for all $x \in (0,1]$ and $b \in (0,1)$, so by extension i is positive for all $x, b \in [0,1]$. This gives the first inequality,and we can verify with similar argumentation that the second inequality is satisfied by θ_2.

Using Theorem 27 we further have for all continuous nilpotent Archimedean t-norms, T, there exists an order automorphism φ on (Δ^+, τ_T) which is an order quasi additive function for some error function H and is an order additive approximator of an order automorphism with error H. This motivates the following results:

Theorem 31 *Let $\varphi(F) = \theta \circ F \circ \gamma$ be a type 1 order automorphism and T be the Lukasiewicz t-norm. If γ is a metric quasi additive order automorphism with error a on $\mathbb{R}^+$ and $x^{2-b} \leq \theta(x) \leq x^b$, then φ is an order additive approximator on (Δ^+, τ_T) with error function $\delta_{a,b}$.*

Proof:

Since γ is metric quasi additive on $\mathbb{R}^+$ with error a, it is in particular a solution in metric approximation with error a. That is, there exists γ' which is a solution of Cauchy's equation on $\mathbb{R}^+$, such that $|\gamma(x) - \gamma'(x)| \leq a$. Further, we have that $\varphi'(F) = F \circ \gamma'$ is a solution of Cauchy's equation for (Δ^+, τ_T). Using the work in the example above we have

$$T(x, b) \leq x^{2-b} \leq \theta(x)$$

and

$$T(\theta(x), b) \leq T(x^b, b) \leq x$$

so for all $F \in \Delta^+_\delta$ we have that

$$\tau_T(\varphi(F), \delta_{a,b}) \leq \varphi'(F) \text{ and } \tau_T(\varphi'(F), \delta_{a,b}) \leq \varphi(F)$$

■

Corollary 15 *Let T b a continuous, nilpotent Archimedean t-norm isomorphic under f to the Lukasiewicz t-norm, $a \in \mathbb{R}^+$, and $b \in (0, 1)$. If γ is a metric quasi additive order automorphism on $\mathbb{R}^+$ with error a, and θ is a power function that satisfies $x^{2-b} \leq \theta(x) \leq x^b$ $\psi(F) = f \circ \theta \circ f^{-1} \circ F \circ \gamma$ is an order quasi additive function on (Δ^+, τ_T) for error function $\delta_{a,f(b)}$, and an order additive approximator for error function $\delta_{a,f(b)}$.*

CHAPTER 5
CONCLUSIONS

This dissertation has laid the groundwork for understanding the stability of Cauchy's equation in the setting of Δ^+ by contextualizing the problem, and it has furthered understanding by introducing a framework to enrich the notion of stability in a way that makes use of the unique properties of Δ^+ by introducing order stability. In keeping with tradition of Cauchy's equation, the results in Chapter Four emphasize the utility of regularity assumptions, and the close of Chapter Four demonstrates the necessity of certain assumptions.

The next steps to supplement this work would be to relax the assumptions of order automorphisms in many of the stability theorems to sup continuous maps of Δ^+ and to relax the hypothesis concerning triangle functions so that they are instead generated by continuous T norms which are not Archimedean.

Several other natural questions are raised by this dissertation as well. For one, in many other contexts, Cauchy's equation is key for understanding a broad array of other functional equations and their stability, so it is natural to ask whether the same is true in this context. It may also be desirable to understand if there is a meaningful generalization of the results of Rassias where approximation error is a function of the equation's variables.

In more general mathematics, the results of Chapter 4 hint at the possibility of investigating the interplay of topological, order, and algebraic structure. In particular, the use of order automorphisms which in this instance have topological and algebraic properties suggests the possibility of category theoretic generalizations.

This would have broad mathematical utility as well as allowing further development of probabilistic metric spaces.

REFERENCES

[1] Janos Aczel and J Dhombres, *Functional equations containing several variables*, Cambridge University Press, 1989.

[2] Janos Aczel and Paul Erdos, *The non-existence of a hamel-basis and the general solution of cauchy's functional equation for nonnegative numbers.*, Publ. Math. Debrecen (1966), 259–63.

[3] Muhammad ibn Musa al Khwarizmi, *The algebra of mohammed ben musa*, 0813.

[4] Claudi Alsina, *On the stability of a functional equation arising in probabalistic normed spaces*, International Series of Numerical Mathematics **80** (1987), 263–271.

[5] Guram Bezhanishvili, Mai Gehrke, John Harding, Carol Walker, and Elbert Walker, *Varieties of algebras in fuzzy set theory*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 321–344.

[6] Henry Briggs, *Arithmetica logarithmica*, 1624.

[7] Andrew Bruckner, *Minimal superadditive extensions of superadditive functions*, Pacific Journal of Mathematics **10** (1960), no. 4, 1155–1162.

[8] Jost Burgi, *Arithmetische und geometrishe progress tabulen*, University of Prague, 1620.

[9] Augustin-Louis Cauchy, *Cours d'analyse*, Debure frères, 1821.

[10] Bernard De Baets, Hans De Meyer, and Bart De Schuymer, *Transitive comparison of random variables*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 415–442.

[11] Glad Deschrijver and Etienne E. Kerre, *Triangular norms and related operators in l*-fuzzy set theory*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 231–259.

[12] W M Faucett, *Compact semigroups irreducibly connected between two idempotents*, Proceedings of the American Mathematical Society **6** (1955), no. 5, 741–747.

[13] Gian Luigi Forti, *An existence and stability theorem for a class of functional equations*, Stochastica **4** (1980), 23–30.

[14] Harald Fripertinger and Jens Schwaiger, *Some remarks on the stability of the cauchy equation and completeness*, Aequationes mathematicae **95** (2021), no. 6, 1243–1255.

[15] P. Gavruta, *A generalization of the hyers ulam rassias stability of approximately additive mappings.pdf*, Journal of Mathematical Analysis and Applications **184** (1992), 431–436.

[16] Ioan Goleţ, *Some remarks on functions with values in probabilistic normed spaces*, Mathematica Slovaca **57** (2007), no. 3, 259–270.

[17] Siegfried Gottwald and Petr Hájek, *Triangular norm-based mathematical fuzzy logics*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular

Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 275–299.

[18] Olga Hadžić and Endre Pap, *Triangular norms in probabilistic metric spaces and fixed point theory*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 443–472.

[19] Georg Hamel, *Eine basis aller zahlen und die unstetigen lösungen der funktionalgleichung f(x+y)=f(x)+f(y)*, Mathematische Annalen (1905), 459–462.

[20] D. H. Hyers, *On the stability of the linear functional equation*, Proceedings of the National Academy of Sciences **27** (1941), no. 4, 222–224.

[21] Ulrich Höhle, *Many-valued equalities and their representations*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 301–319.

[22] Jyotsana Jakhar, Renu Chugh, and Jagjeet Jakhar, *Stability of various iterative type functional equations in menger phi- normed spaces*, Bulletin of Mathematical Analysis and Applications **13** (2021), no. 1, 106–120.

[23] Erich Peter Klement, Radko Mesiar, and Endre Pap, *Triangular norms. position paper III: continuous t-norms*, Fuzzy Sets and Systems **145** (2004), no. 3, 439–454, 439.

[24] ______, *Semigroups and triangular norms*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 63–93.

[25] ———, *Triangular norms: Basic notions and properties*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 17–60.

[26] Donald Knuth, *Algorithms in modern mathematics and computer science*, Standford Department of Computer Science Reports (1980).

[27] R L Kruse and J J Deely, *Joint continuity of monotonic functions*, The American Mathematical Monthly **76** (1969), no. 1, 74–76.

[28] Gaspar Mayor and Joan Torrens, *Triangular norms on discrete settings*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 189–230.

[29] Karl Menger, *Statistical metrics*, Proceedings of the National Academy of Sciences of the United States of America **28** (1942), 535–537.

[30] Andrea Mesiarová, *Generators of triangular norms*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 95–111.

[31] Dorel Miheţ, *The stability of the additive cauchy functional equation in non-archimedean fuzzy normed spaces*, Fuzzy Sets and Systems **161** (2010), no. 16, 2206–2212.

[32] Dorel Miheţ and Viorel Radu, *On the stability of the additive cauchy functional equation in random normed spaces*, Journal of Mathematical Analysis and Applications **343** (2008), no. 1, 567–572.

[33] Dorel Miheţ and Reza Saadati, *On the stability of the additive cauchy functional equation in random normed spaces*, Applied Mathematics Letters **24** (2011), no. 12, 2005–2009.

[34] Alireza Kamel Mirmostafaee and Mohammad Sal Moslehian, *Fuzzy versions of hyers–ulam–rassias theorem*, Fuzzy Sets and Systems **159** (2008), no. 6, 720–729.

[35] Paul S. Mostert and Allen L. Shields, *On the structure of semigroups on a compact manifold with boundary*, The Annals of Mathematics **65** (1957), no. 1, 117.

[36] Zenon Moszner, *On the stability of functional equations*, Aequationes mathematicae **77** (2009), no. 1, 33–88.

[37] John Napier, *The description of the wonderful canon of logarithms*, Andro Hart, 1614.

[38] ———, *The construction of the wonderful canon of logarithms*, 1889.

[39] Mirko Navara, *Triangular norms and measures of fuzzy sets*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 345–390.

[40] Roger B. Nelsen, *Copulas and quasi-copulas: An introduction to their properties and applications*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 391–413.

[41] Alireza Pourmoslemi, Siavash Rajabi, Mehdi Salimi, and Ali Ahmadian, *Fuzzy routing protocol for d2d communications based on probabilistic normed spaces*, Wireless Personal Communications **122** (2022), no. 3, 2505–2520.

[42] Robert C. Powers, *Order automorphisms of spaces of nondecreasing functions*, Journal of Mathematical Analysis and Applications **136** (1980), 112–123.

[43] Themistocles Rassias, *On the stability of the linear mapping in banach spaces*, Proceedings of the American Mathematical Society **72** (1978), no. 2, 297–300.

[44] T. Riedel, *On sup-continuous triangle functions*, Journal of Mathematical Analysis and Applications **184** (1994), no. 2, 382–388.

[45] Thomas Riedel, *Cauchy's equation on delta +*, aequationes mathematicae **41** (1991), no. 1, 192–211, 192.

[46] Thomas Riedel and Kelly Wallace, *On a pexider type equation on delta+*, Annales Academiae Paedagogicae Cracoviensis (2000), 129–138.

[47] P. Saha, S. Guria, Samir Kumar Bhandari, and Binayak S. Choudhury, *A global optimality result in probabilistic spaces using control function*, Optimization **70** (2021), no. 11, 2387–2400.

[48] Wolfgang Sander, *Some aspects of functional equations*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 143–187.

[49] B. Schweizer and A. Sklar, *Probabilistic metric spaces*, Dover Publications, INC., 1983.

[50] Berthold Schweizer, *Triangular norms, looking back—triangle functions, looking ahead*, Logical, Algebraic, Analytic and Probabilistic Aspects of Triangular Norms (Erich Peter Klement and Radko Mesiar, eds.), Elsevier Science B.V., 2005, pp. 3–15.

[51] Mausumi Sen, Soumitra Nath, and Binod Chandra Tripathy, *Best approximation in quotient probabilistic normed space*, Journal of Applied Analysis **23** (2017), no. 1.

[52] Zhiqiang Shen and Dexue Zhang, *A note on the continuity of triangular norms*, Fuzzy Sets and Systems **252** (2014), 35–38.

[53] David A. Sibley, *A metric for weak convergence of distribution functions*, Rocky Mountain Journal of Mathematics **1** (1971), no. 3, 427–430.

[54] Frank Swetz, *Mathematical treasure: John napier's mirifici logarithmorum*, 2013.

[55] Robert M. Tardiff, *Topologies for probabilistic metric spaces.*, Pacific Journal of Mathematics **65** (1976), no. 1, 233–251, Publisher: Pacific Journal of Mathematics, A Non-profit Corporation.

CURRICULUM VITAE

Holden Wells

Education

University of Louisville *August 2014 - Present*
Doctor of Philosophy in Industrial and Applied Mathematics *Expected August 2023*
Master of Arts in Mathematics *Earned December 2019*
Bachelor of Science in Mathematics *Earned April 2018*

Research Experience, Publications, and Presentations

Research for Dissertation *January 2019 - Present*
Mathematics Department: University of Louisville
Description
Presentation Holden Wells. (2022, November) "Cauchy's Equation on Delta Plus". 42^{nd} Western Kentucky University Mathematics Symposium.
Presentation Holden Wells. (2022, April) "Super and Sub Additivity on the Space of Distribution Functions". KYMAA Spring Meeting

Teaching Experience

Graduate Teaching Assistant
Mathematics Department, University of Louisville *August 2018 - Present*

- Instructed business calculus, math for elementary education, pre calculus, quantitative reasoning, and college algebra. Across all subjects, I taught fundamental skills in mathematics and logical reasoning. Additionally, I instructed students on communicating mathematics in informal settings like group discussions and in formal settings like projects and exams.

- Led recitation sections in college algebra, quantitative reasoning, and business calculus.

Undergraduate Teaching Assistant
Mathematics Department, University of Louisville *August 2016 - April 2018*

- Led recitation sections in quantitative reasoning, college algebra, and business calculus.

Replacement Instructor
Saint Xavier High School *January 2022 - May 2022*

- I taught an introduction to statistics course where students where introduced to skills in statistical interpretation and representation.

Adjunct Professor
Mathematics Department, Bellarmine University *August 2021 - December 2021*

- I taught Foundations of Mathematics, where students learned to investigate the deep construction of topics typically taught in elementary classrooms. Students were taught to come up with effective models of these common topics and to rigorously evaluate the models of others,all while examining fundamental problem solving skills.

Math Tutor
Resources for Academic Achievement (REACH), University of Louisville *August 2015 - May 2018*

- Gen 103/104 Assistant: I worked as an in class assistant for a course in elementary algebra and math study strategies where I worked one on one with students to develop fundamental concepts important to success in future courses.
- Drop in tutor: I staffed a drop in tutoring center where students came to receive individual assistance on a broad spectrum of math topics including algebra, calculus, and math education. I also served as a resource for less experienced tutors later in my tenure.

Academic Achievements and Campus Involvement

- University of Louisville American Mathematical Society Chapter President *August 2020 - May 2021*
- Passed University of Louisville Probability Qualifying Exam *December 2019*
- Passed University of Louisville Analysis Qualifying Exam *August 2019*
- Passed University of Louisville Algebra Qualifying Exam *August 2019*
- University of Louisville Varsity Track and Field Member *August 2016 - May 2018*
- Robert J. Bickel Scholarship *2017*
  Awarded to a rising Junior and rising senior in the department of Mathematics based off of mathematics GPA, general transcript, and letters of support from Mathematics faculty.
- Lois Pedigo Scholarship *2017*
  Awarded to a rising Junior and rising senior in the department of Mathematics based off of mathematics GPA and letters of support from Mathematics faculty.

- Eagle Scout Award *2014*
  Since combined with trustees competitive scholarships. Awarded to some Eagle Scout applicants based off of GPA and ACT considerations