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Effects of a protection zone in a reaction-diffusion model with strong Allee effect.

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EFFECTS OF A PROTECTION ZONE IN A REACTION-DIFFUSION MODEL WITH STRONG ALLEE EFFECT

By

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B.S., West Virginia Wesleyan College, 2017
M.A., University of Louisville, 2019

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DEDICATION

This dissertation is dedicated to my parents

Mr. Jeff Johnson

and

Mrs. Jody Johnson

for instilling my work ethic and supporting my education.

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ABSTRACT

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A protection zone model represents a patchy environment with positive growth over the protection zone and strong Allee effect growth outside the protection zone. Generally, these models are considered through the corresponding eigenvalue problem, but that has certain limitations. In this thesis, a general protection zone model is considered. This model makes no assumption on the direction of the traveling wave solution over the Strong Allee effect patch. We use phase portrait analysis of this protection zone model to draw conclusions about the existence of equilibrium solutions. We establish the existence of three types of equilibrium solutions and the necessary conditions for each to exist. Then, through numerical techniques, we further explore the existence and behavior of these equilibrium solutions.

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CHAPTER 1

INTRODUCTION

Mathematical ecology introduces mathematical tools and techniques to understand ecology by pursuing methods for studying the mechanisms underpinning observed patterns involving organisms and their surroundings [36]. One way this is done is by modelling the population dynamics of a species through its habitat over time. In particular, we will be focusing on the theoretical conditions and mechanisms to preserve a stable species population over a given environment. In practice, wildlife preserves and other protected areas have long been used to support animal populations against hunting and the encroachment of urban environments [33] [5] [14] [1]. The total areas protected have increased recently, and significant portions of the planet are now protected [1]. However, this increase in protected area, has not slowed the decrease in biodiversity and species extinction [1] [35]. This indicates that the preservation of a species within protection zones is more complicated than just protecting the land.

It has been shown theoretically and observed that protected areas can be used to support the persistence of a species, but only when the conditions are suitable [39] [28] [8] [17]. The conditions that allow for persistence are not always obvious and the causes for protected areas failing to support the intended species are not precisely understood. There are many possible reasons that explain a protection zone not supporting the persistence of a species; some are caused by humans and others are not. For instance, a protection zone may 'fail' because the laws are not enforced; perhaps logging, hunting, or mining are allowed to continue illegally. For example, a size limit was implemented on the harvest of smallmouth bass (*Micropterus dolomieu*) on Lake Sharpe in South Dakota [12]. The size restriction was intended to increase the population of smallmouth bass in the lake. However, after several years of the size limit, there did not appear to be an increase in the local population. This is largely attributed to noncompliance from the anglers in the area [12]. Protection zones can fail without illicit activities, and this is primarily where we will focus. In particular, we are concentrating on the size of protection zones and the relationship between the size and efficacy of the protection zone. The effectiveness of protected areas to support the persistence of a species has been observed to be dependent on the habitability of the surrounding areas [35] [28] [8] [17]. Many species move from inside the protection zone to outside the protection zone as they mature or arbitrarily throughout their life. One such example is the smallmouth bass, a common sport fish. Gunderson Vanarum, et. al. studied the movement patterns of the smallmouth bass with the intention of determining the size of a protection zone necessary to support a local population of the fish [16]. They determined a protection zone of length two linear kilometers would cover the migratory range of 55% of smallmouth bass, seven kilometers would cover 75% of migratory range, and eighteen kilometers would cover 95% of migratory range of the local smallmouth bass population [16]. In the case of the smallmouth bass, if the protection zone is too small, the fish may migrate into areas that are not protected, allowing them to be harvested through fishing.

The efficacy of protection zones is commonly and incorrectly assumed to be independent of the surrounding area. Thus, indicating if the area near a protection zone is not sufficiently hospitable, the protection zone may not be as effective. An example of the exterior of a protected area affecting its success is Huron-Manistee National Forest in Michigan, which is home to the Kirtland's Warbler (*Dendrocia kirtlandii*), a federally listed endangered species [31]. The Kirtland's Warbler population in Huron-Manistee is negatively effected by the increase in human housing near the park. The increase in housing around the park has also increased the nest parasitism from brown-headed cowbirds (*Molothrus ater*) which depresses warbler reproductive success and therefore limit the efficacy of the protected area [31]. Thus, future protection zones for the Kirtland's warbler might consider exterior influences when determining the size of the protected space, in order to limit the effect human encroachment has on the warbler population inside the protected area. Habitats bordering protected areas can be made more hostile by increased human population density and a lack of natural land cover, which may result in the protected area becoming isolated [7] [29] [9]. Isolation refers to the protected area becoming disconnected from other similar habitat. This increase in isolation has been shown to increase the number of extinctions, particularly in smaller reserves [28] [27]. As protected areas tend to be geographically dissimilar to the surrounding environment, limiting isolation is increasingly important [34] [18] [17]. To limit the effects of isolation, the relationship between the exterior of the protected area and the size of the protected area must be taken into account when deciding on how large a protection zone should be.

The need for models that study species population dynamics has really increased in the last century. Early models only considered the population with respect to time, but it is clear that the movements of different species greatly influences how their population behave. In particular, the need for models to take into consideration both time and space becomes apparent. A popular category of spatial-time model is the reaction-diffusion equation. Reaction-diffusion equations are semi-linear parabolic partial differential equations. Specifically, reaction-diffusion equations are time-continuous spatial models that allow for many interesting dynamics and results [15]. Some of the interesting dynamics produced by reaction-diffusion models were first noticed in 1937, when

R.A. Fisher proposed using reaction-diffusion equations of the form:

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = ru(1-u)$$

to model the spread of an advantageous allele where $u(x,t)$ is the fraction of the population with an advantageous allele at location x and time t[13]. The model shows how this gene can disperse through a population. The Fisher model has the following constant values: D the diffusion coefficient, r the intrinsic growth rate, and 1 is the scaled carrying capacity. In particular, it was determined, given specific initial conditions, there would exist a traveling wave solution $u(x-ct)$ moving at speed $c \geq 2\sqrt{Dr}$ where in front of the wave $u(x,t) = 0$ and behind the wave $u(x,t) = 1$ [13]. This notion is biologically and mathematically significant as it can be used to represent the invasion of a species into a new environment. In the same year, Kolmogorov, Petrovsky, Piskunov introduced a more general version of the Fisher model:

$$\frac{\partial u}{\partial t} - \frac{\partial^2 u}{\partial x^2} = F(u)$$

with $F(0) = F(1) = 0, F'(0) = r > 0$ and $F(u) > 0, F'(u) < r$ for all $0 < u < 1$, where minimal wave speed is given by $2\sqrt{\frac{dF}{du}|_{u=0}}$[21]. Since then the reaction-diffusion model has been used regularly in mathematical biology. traveling waves have become a significant field of study, and are often used to model the advance or retreat of a species through an environment.

More recently the focus has been on models with non-homogeneous environments or growth rates. This is biologically significant as it allows for the study of more realistic population dynamics. These are reaction-diffusion models of the form:

$$\frac{\partial u}{\partial t} - D\frac{\partial^2 u}{\partial x^2} = F(u,x)$$

where $F(u,x)$ is a piecewise function. Each 'piece' of this function $F(u,x)$ is a different growth function over a different portion of the x-domain of $u(x,t)$. These 'pieces' in the x-domain are typically referred to as patches. Interesting dynamics emerge from the interaction between the different growth functions. A commonly cited model of this form supposes

$$F(u,x) = \begin{cases} ru(1-u), & x \in (-l,l), \\ -\beta u, & x \notin (-l,l) \end{cases}$$

where the patch $(-l,l) \in \mathbb{R}$ has positive growth. In this example, the species is shown to persist only if $|l-(-l)| > \frac{\pi}{\sqrt{\frac{r}{D}}}$ [10] [6]. This model has two distinct growth functions $ru(1-u)$ and $-\beta u$ respectively. In particular, the first is Logisitic growth which represents positive growth of the species over the patch with a carrying capacity of 1. Positive growth is regularly modelled using a logistic growth equation. It has been established Logistic growth is well suited for representing positive growth for many species[38]. Outside the patch, the growth function $-\beta u$ governs growth of the species which correspond to exponential decay with intrinsic growth rate $-\beta$ [6].

Recently, another reaction diffusion model has been considered by Du. et. al., again this is a reaction-diffusion equation similar in form to the previous model [11]. The fundamental difference, this model assumes strong Allee effect growth outside the patch instead of exponential decay. This model is defined far more generally:

$$
\begin{cases}
u_t = u_{xx} + f(u), & t > 0, -L < x < L, \\
u_t = u_{xx} + g(u), & t > 0, x \in \mathbb{R} - [-L, L], \\
u(t, -L - 0) = u(t, -L + 0), & t > 0, \\
u(t, L - 0) = u(t, L + 0), & t > 0, \\
u_x(t, -L - 0) = u_x(t, -L + 0), & t > 0, \\
u_x(t, -L - 0) = u_x(t, -L + 0), & t > 0, \\
u(0, x) = u_0(x) \geq 0, & x \in \mathbb{R}
\end{cases}
$$

$$
\begin{aligned}
& f(0) = f(1) = 0 < f'(0), \\
& f'(1) < 0, \\
& (1 - u)f(u) > 0, \;\; \forall u > 0, u \neq 1, \\
& g(0) = g(\theta) = g(1), \;\; \theta \in (0, 1), \\
& g'(0) < 0, \\
& g'(1) < 0, \\
& g(u) = \begin{cases} < 0, & (0, \theta) \\ > 0, & (\theta, 1) \\ < 0, & (1, \infty) \end{cases}, \\
& \int_0^1 g(s)ds > 0.
\end{aligned}
$$

The function $g(u)$ produces strong Allee effect growth and $f(u)$ positive growth with behavior similar to Logistic growth [11]. Allee effect was first observed in 1931, when W.C. Allee noticed gold fish populations exhibited lower individual fitness at lower total populations [2]. We will be focused on strong Allee effect, where a threshold exists such that the species population declines when the population density is below this threshold and increases when above the threshold, referred to as θ by Du, et. al. [11]. This is referred to as the Allee threshold, and plays a critical role in the population dynamics of a species with strong Allee growth. Mechanisms have been established to cause the existence of Allee effect, including: predator swamping, antipredator aggression, modifications of environment, and social facilitation of reproduction [37]. Notice, these mechanisms all effect individual fitness at lower population densities. Allee effect has recently been shown to possibly contribute to the declines in populations of the crested ibis [26]. Du. et. al. focused on asymptotic behavior of the solutions to their model. In particular, they established protection zone sizes that controlled the resulting behavior of the solutions. The first critical patch size $L_* = \frac{1}{\sqrt{f'(0)}} \arctan(\sqrt{-\frac{g'(0)}{f'(0)}})$ and $L^* \leq \int_0^{\theta^*} \frac{1}{\sqrt{2\int_r^{\theta^*} f(s)ds}} dr < \infty$ is the other. Where $\theta^* \in (0,1)$ is a constant such that $\int_0^{\theta^*} g(s)ds = 0$. These values L_* and L^* resulted in the following conditions:

1. If $0 < L < L_*$ then there are three possible solution types depending on initial conditions: vanishing, transition, or spreading.

2. If $L_* < L < L^*$ only two solution types exist: transition or spreading.

3. If $L^* < L$ then only spreading exists.

Biologically, a vanishing solution corresponds to global extinction, a transition solution is a stable solution supported by the protection zone, and a spreading solution corresponds to a species that spreads far beyond the protection zone and approaches the carrying capacity everywhere. Du, et.al. suppose the initial conditions are symmetric about the origin, therefore it is only necessary to consider the solutions over $[0, \infty)$. Where mathematically a vanishing solution is: $\lim_{t\to\infty} u(x,t) = 0$ uniformly over $[0, \infty)$, spreading solution is: $\lim_{t\to\infty} u(x,t) = 1$ locally uniformly over $[0, \infty)$, and transition solution is : $\lim_{t\to\infty} |u(x,t) - U(x)| = 0$ locally uniformly over $[0, \infty)$ where $U(x)$ is a ground state of the model [11].

Another model similar to Du. et.al. [11] was consdiered by Jin, et.al. [20]. They also considered protection zones with a single species reaction-diffusion model however their model was bounded and made no assumptions about the value $\int_0^1 g(s)ds$.

$$\begin{cases} u_t = u_{xx} + F(u), & x \in [0, L], \\ u(x,0) = u_0(x) > 0, & x \in [0, L], \end{cases}$$

$$F(u) = \begin{cases} f(u), & x \in (\alpha, \alpha + l) \\ g(u), & x \in (0, \alpha) \cup (\alpha + l, L) \end{cases},$$

Where $f \in C^1([0, +\infty)), f'(0) > 0 = f(0) = f(1)$, $f(u) > 0$ for all $u \in (0, 1)$, and $f(u) < 0$ for all $u \notin [0, 1]$. $g \in C^1([0, +\infty)], g'(0) < 0 = g(0) = g(a) = g(1)$, $g(u) > 0$ for $u \in (a, 1)$ and $g(u) < 0$ otherwise. The habitat is bounded at the values $x = 0$ and $x = L$ by general Robin type bounds:

$$\alpha_1 u_x(t,0) - \alpha_2 u(t,0) = 0,\ \alpha_1 \geq 0,\ \alpha_2 \geq 0,\ \alpha_1^2 + \alpha_2^2 \neq 0,\ t > 0$$

$$\beta_1 u_x(t,L) - \beta_2 u(t,L) = 0,\ \beta_1 \geq 0,\ \beta_2 \geq 0,\ \beta_1^2 + \beta_2^2 \neq 0,\ t > 0$$

Since the habitat is bounded, Jin, et.al. considered the location of the protection zone and its relative size. Where $(\alpha, \alpha + l)$ is the protection zone of length l, $f(u)$ is positive growth, and $g(u)$ strong Allee effect growth. They proved the principle eigenvalue of the eigenvalue problem related to the linearization of the system about the trivial solution can be used to determine persistence or extinction. In particular, they showed the eigenvalue is a decreasing function in terms of the protection zone length which shows the larger the protection zone, the more likely persistence is across the whole domain [20].

Continuing the same notation, $f(u)$ is Logistic growth and $g(u)$ is strong Allee effect growth. It has been well established that given nontrivial initial conditions $u_t = u_{xx} + f(u)$ will approach the steady state solution $u = 1$ as $t \to \infty$ [3][4]. It is also well established, $u_t = u_{xx} + g(u)$ has conditional solutions: if the initial population density is $u_0(x) < a$ for all $x \in \mathbb{R}$ the solution uniformly approaches $u(x) = 0$ as $t \to \infty$ and if $u_0(x) > a$ for all $x \in \mathbb{R}$ the solution uniformly approaches $u(x) = 1$ for all $x \in \mathbb{R}$. However, for example, if $\int_0^1 g(s)ds < 0$ the model has a solution $u(x - ct)$ where $c < 0$, $u(x - ct) = 1$ as $x \to (\infty)$, and $u(x - ct) = 0$ as $x \to (-\infty)$ i.e. a retreating wave solution with wave speed c. If $\int_0^1 g(s)ds = 0$ the wave does not advance or retreat. If $\int_0^1 g(s)ds > 0$ the model has solution $u(x - ct)$ where $c > 0$, so $u(x - ct) = 1$ as $x \to \infty$, $u(x - ct) = 0$ as $x \to -\infty$, and the wave front moves toward $-\infty$ as $t \to \infty$ i.e. an advancing wave solution [22] [32].

We propose a reaction-diffusion model with an unbounded habitat and positive growth over a protection zone and strong Allee effect growth outside the protection zone. Similar to, [20] [11], the other protection zone papers. The biological literature indicates the surrounding environment is critical for the efficacy of a protection zone [35] [28] [8] [17]. Thus it is critical for our model to consider the effect the area outside the protection zone will have on the persistence of our species. Our model will be of the form:

$$u_t - Du_{\chi\chi} = \begin{cases} \bar{f}(u), & \chi \in \Omega' \\ \bar{g}(u), & \chi \in \mathbb{R}\backslash\Omega' \end{cases}$$

with

$$u(-l^-) = u(-l^+),\ u(l^-) = u(l^+),\ u'(-l^-) = u'(-l^+),\ and\, u'(l^-) = u'(l^+),$$

where $\Omega' = (-l, l)$ is an interval in $\mathbb{R}$ representing the protection zone. The function $\bar{f}(u)$ is positive growth over the protection zone such that $\bar{f} \in C^1([0, +\infty)), \bar{f}(0) = \bar{f}(k_f) = 0 < \bar{f}'(0), \bar{f}'(k_f) < 0, (k_f - u)\bar{f}(u) > 0$. The function $\bar{g}(u)$ is strong Allee effect growth such that $\bar{g} \in C^1([0, +\infty))\bar{g}(0) = \bar{g}(A) = \bar{g}(k_g) = 0 > \bar{g}'(0), \bar{g}'(k_g) > 0, \bar{g}(u) = \begin{cases} < 0, & \in (0, A), \\ > 0, & \in (A, k_{\bar{g}}), \\ < 0, & \in (k_{\bar{g}}, \infty). \end{cases}$ K_g and K_f are the respective carrying capacity and A is the Allee threshold. Notice this habitat is unbounded, and no assumptions are made about the magnitude of the integral $\int_0^1 g(s)ds$. Intuitively this will allow for advancing, retreating, and stationary wave behavior outside the protection zone. Notice, when $\int_0^1 g(s)ds > 0$, we are considering the same model as [11]. We will explore the effects of varying the Allee threshold in the surrounding environments have on the protection zone. Intuitively, the more hostile (larger Allee threshold) the surrounding environments, the larger the a protection zone must be to allow for continued persistence.

CHAPTER 2

MODEL

We will now be defining our reaction diffusion model over the domain $\mathbb{R}$. More precisely, the interval $\Omega' \in \mathbb{R}$ corresponds to the protection zone and denotes the portion of the graph where positive growth occurs. For computations later in the paper, we will be using $[-l, l]$ in place of Ω or Ω'. We will need to define the constants on our environment $k_{\bar{f}}$ and $k_{\bar{g}}$ are carry capacity over Ω (inside the protection zone) and $\mathbb{R}\backslash\Omega$ (outside the protection zone) respectively and A the Allee threshold.

$$u_t - Du_{\chi\chi} = \begin{cases} \bar{f}(u), & \chi \in \Omega' \\ \bar{g}(u), & \chi \in \mathbb{R}\backslash\Omega' \end{cases}$$

with

$$u(-l^-) = u(-l^+),\ u(l^-) = u(l^+),\ u'(-l^-) = u'(-l^+),\ u'(l^-) = u'(l^+).$$

The following conditions ensure positive growth over the protection zone with carrying capacity k_f and strong Allee effect growth with carrying capacity k_g. over the unbounded area outside the protection zone respectively. A similar set of conditions can be found in Du, et. al. [11].

$$\bar{f} \in C^1([0,+\infty))$$
$$\bar{f}(0) = \bar{f}(k_f) = 0 < \bar{f}'(0),$$
$$\bar{f}'(k_f) < 0,$$
$$(k_f - u)\bar{f}(u) > 0,$$

$$\bar{g} \in C^1([0,+\infty))$$
$$\bar{g}(0) = \bar{g}(A) = \bar{g}(k_g) = 0 > \bar{g}'(0),$$
$$\bar{g}'(k_g) > 0,$$
$$\bar{g}(u) = \begin{cases} < 0, & \in (0, A) \\ > 0, & \in (A, k_{\bar{g}}) \\ < 0, & \in (k_{\bar{g}}, \infty) \end{cases} .$$

Our model can be simplified considerably by scaling our variables. We will suppose $u = wk_{\bar{g}}$, $A = ak_{\bar{g}}$, $K = \frac{k_{\bar{g}}}{k_{\bar{f}}}$, and $\chi = \sqrt{D}x$. By scaling our variables accordingly, we move from three parameters to two. We now have new growth functions $f(w)$ and $g(w)$ with new but similar properties which we can consider accordingly.

$$w_t - w_{xx} = \begin{cases} f(w), & x \in (-l, l) \\ g(w), & x \in \mathbb{R}\backslash(-l, l) \end{cases}$$

with

$$w(-l^-) = w(-l^+),\ w(l^-) = w(l^+),\ w'(-l^-) = w'(-l^+),\ w'(l^-) = w'(l^+).$$

These new conditions on f and g still ensure the type of growth we were looking for, but will be far simpler to analyze later in this paper. Specifically, we have scaled the diffusion coefficient to 1 and the carrying capacities into a single unique carrying capacity, K.

$$
\begin{aligned}
&f \in C^1([0,+\infty)),\\
&f(0)=f(K)=0<f'(0),\ 0>f'(K),\ (K-w)f(w)>0,\\
&g \in C^1([0,+\infty)),\\
&g(0)=g(a)=g(1)=0>g'(0),\ g'(1)>0\\
&g(w)=\begin{cases} <0, & w\in(0,a)\\ >0, & w\in(a,1),\\ <0, & w\in(1,\infty).\end{cases}
\end{aligned}
$$

These conditions have been simplified considerably, but the unique carrying capacity K still allows for some computational and theoretical challenges. In particular, there exist three cases to consider for K: when $K=1$, when $K<1$, and when $K>1$. Du, et.al. assumed $K=1$ in their model [11]. Changes in carrying capacity are observed to occur during changes in location and changes in time, in particular in response to variation in habitat and predation [19]. It seems likely then the carrying capacity over a protection zone would be different than the carrying capacity outside, as the amount of predation and type of environment may be quite different.

CHAPTER 3

ANALYTIC RESULTS

We will be using the phase plane to draw conclusions regarding the existence and conditions for various types of equilibrium solutions. In order to do this however, we will need to define some specific equations and functions in the phase plane. Fortunately for us, these equations do not change much through the various cases and as a result will only need to be defined in the following section.

3.1 Phase Plane Trajectories

In the last chapter, we defined our model and the conditions on the growth functions. We will now use the above conditions to derive the phase portrait of our model. In order to achieve this, we first consider the phase portraits that correspond to $f(w)$ and $g(w)$ respectively. We begin by considering equilibrium solutions, which results in the following second order ODEs:

$$w'' + f(w) = 0, x \in (-l, l),$$
$$w'' + g(w) = 0, x \notin (-l, l),$$

with

$$w(-l^-) = w(-l^+),\ w(l^-) = w(l^+),\ w'(-l^-) = w'(-l^+),\ w'(l^-) = w'(l^+).$$

These ODEs correspond to steady-state solutions of the original system, i.e. when the curve $w(x,t)$ is not changing with respect to time. We will refer to these solutions as $w(x)$ since they are not dependent on time by definition. This allows us to consider the global solutions of the system without needing to consider the effects of time. We can now rewrite these equations as the following two variable systems of first order ODEs respectively:

$$\begin{aligned} w' &= v \\ v' &= -f(w), \end{aligned} \tag{3.1}$$

$$\begin{aligned} w' &= v \\ v' &= -g(w), \end{aligned} \tag{3.2}$$

with

$$w(-l^-) = w(-l^+),\ w(l^-) = w(l^+),\ w'(-l^-) = w'(-l^+),\ w'(l^-) = w'(l^+).$$

Recall from the definitions of f and g, there exist multiple critical points to both of these systems. We will need to thoroughly consider all of the critical points to track down the possible steady-state solutions to the overall system. First, notice the point $(0,0)$ is a critical point to both 3.1 and 3.2. 3.1 also has critical point $(K,0)$ and 3.2 has critical points $(a,0)$ and $(1,0)$. We begin our phase plane analysis of these systems by considering the Jacobians of each system separately.

$$\nabla f(w,v) = \begin{bmatrix} 0 & 1 \\ -f'(w) & 0 \end{bmatrix},$$
$$\nabla g(w,v) = \begin{bmatrix} 0 & 1 \\ -g'(w) & 0 \end{bmatrix}.$$

Then the eigenvalues of these Jacobian matrices at each critical value based on the assumptions on $f(w)$ and $g(w)$ are as follow:

$$\begin{aligned}
\lambda_{f.1} &= \pm\sqrt{-f'(0)} \in \mathbb{C}, \\
\lambda_{f.2} &= \pm\sqrt{-f'(K)} \in \mathbb{R}, \\
\lambda_{g.1} &= \pm\sqrt{-g'(0)} \in \mathbb{R}, \\
\lambda_{g.2} &= \pm\sqrt{-g'(a)} \in \mathbb{C}, \\
\lambda_{g.3} &= \pm\sqrt{-g'(1)} \in \mathbb{R}.
\end{aligned} \tag{3.3}$$

Therefore the critical value $(0,0)$ is a center and $(K,0)$ is a saddle in the phase plane of 3.1, and the critical value $(0,0)$ is a saddle, $(a,0)$ is a center, and $(1,0)$ is a saddle in the phase plane of 3.2. This is ideal, as our goal is to 'glue' these phase portraits together and interpret the resulting phase portrait. This analysis relies heavily on considering when, where, and how the two component phase portraits intersect. Naively it appears the trajectories will intersect as we need. In order to prove this intersection can exist, or consider the conditions necessary for its existence; we begin to describe the specific equations of the trajectories in the phase plane. In order to find the specific equations in the phase plane, we will use separation of variables to solve the second order ordinary differential equations from above.

$$\begin{aligned}
\frac{dw}{dv} &= \frac{v}{(-f(w))} \\
\int v dv &= \int -f(w) dw \\
\frac{v^2}{2} &= \int -f(w) dw
\end{aligned}$$

$$\begin{aligned}
\frac{dw}{dv} &= \frac{v}{(-g(w))} \\
\int v dv &= \int -g(w) dw \\
\frac{v^2}{2} &= \int -g(w) dw
\end{aligned}$$

For simplicity of notation, we are going to define two new functions in order to discuss these trajectories in a more efficient way.

$$\tau_f(w) = -2\int_0^w f(s)ds \tag{3.4}$$

$$\tau_g(w) = -2\int_0^w g(s)ds \tag{3.5}$$

We now focus on gluing these separate phase portraits together in such a way that is representative of global steady-state solutions to the one patch model discussed above. A similar approach is used commonly [6][25] [23]. The first important consideration is "Do these phase trajectories intersect, and if so where?". In order to answer this question, we will define the trajectories in the phase plane as the sets T_f and T_g:

$$T_{f,c_1} = \{(w,v) \in (0,\infty) \times \mathbb{R} : v^2 = \tau_f(w) + c_1\}, \tag{3.6}$$

$$T_{g,c_2} = \{(w,v) \in (0,\infty) \times \mathbb{R} : v^2 = \tau_g(w) + c_2\}. \tag{3.7}$$

We have defined the trajectories in the phase plane and considered some basic properties. It remains to consider the intersections of these trajectories and what they may mean in regards to global equilibrium solutions to the system. In particular, we will be looking for the conditions necessary for non-trivial solutions i.e. solutions other than $w(x,t) = 0$ for all $x \in \mathbb{R}$. We will need three specific trajectories for later in the paper T_{f,c_1} the trajectory corresponding the protection zone with arbitrary c_1 values, $T_{g,0}$ the trajectory outside the protection zone which passes through $(0,0)$ in the phase plane, and $T_{g,1}$ the trajectory outside the protection zone which passes through $(1,0)$ in the phase plane:

$$T_{f,c_1} = \{(w,v) \in (0,\infty) \times \mathbb{R} : v^2 = \tau_f(w) + c_1\},$$
$$T_{g,0} = \{(w,v) \in (0,\infty) \times \mathbb{R} : v^2 = \tau_g(w) + 0\},$$
$$T_{g,1} = \{(w,v) \in (0,\infty) \times \mathbb{R} : v^2 = \tau_g(w) - \tau_g(1)\}.$$

Many of the equilibrium we will be considering later will be symmetric about the w-axis in phase plane. This makes considering just the positive portion of T_{f,c_1}, $T_{g,0}$, and $T_{g,1}$ necessary. It is important to note, the positive portion of these sets are functions, but for simplicity of notation we will continue a similar style:

$$T^+_{f,c_1}(w) = \{(w,v) \in (0,\infty) \times \mathbb{R} : v = \sqrt{\tau_f(w) + c_1}\},$$
$$T^+_{g,0}(w) = \{(w,v) \in (0,\infty) \times \mathbb{R} : v = \sqrt{\tau_g(w) + 0}\},$$
$$T^+_{g,1}(w) = \{(w,v) \in (0,\infty) \times \mathbb{R} : v = \sqrt{\tau_g(w) - \tau_g(1)}\}.$$

Lemma 3.1.1. *$T_{g,0}$ is defined over an interval in $[0,1]$.*

Proof. We need to consider three specific cases, $\int_0^1 g(s)ds < 0$, $\int_0^1 g(s)ds = 0$, and $\int_0^1 g(s)ds > 0$. Recall our conditions on the function $g(w)$. In particular, $g(0) = g(a) = g(1) = 0$, $g(w) < 0$ for $w \in (0,a)$, and $g(w) > 0$ for $w \in (a,1)$. Then by the fundamental theorem of calculus, $0, 1$ are local minimums and a is a local maximum over the interval $[0,1]$ for the function $\tau_g(w)$.

case i: Suppose $\int_0^1 g(s)ds < 0$. We are going to begin by considering the function $\tau_g(w) + c_2$ at $w = 0, a, 1$.

$$\tau_g(0) = -2\int_0^0 g(s)ds = 0,$$
$$\tau_g(a) > 0,$$
$$\tau_g(1) = -2\int_0^1 g(s)ds > 0.$$

Then $\tau_g(w) \geq 0$ for all $w \in [0, 1]$. Therefore $T_{g,0}$ is defined over the interval $[0, 1]$.

case ii: Suppose $\int_0^1 g(s)ds = 0$. We will consider the values $w = 0, a, 1$:

$$\tau_g(0) = -2\int_0^0 g(s)ds = 0,$$
$$\tau_g(a) > 0,$$
$$\tau_g(1) = -2\int_0^1 g(s)ds = 0.$$

Then $\tau_g(w) \geq 0$ for all $w \in [0, 1]$. Therefore $T_{g,0}$ is defined over the interval $[0, 1]$.

case iii: Suppose $\int_0^1 g(s)ds > 0$. We will again consider the values $w = 0, a, 1$:

$$\tau_g(0) = -2\int_0^0 g(s)ds = 0,$$
$$\tau_g(a) = \tau_g(a) > 0,$$
$$\tau_g(1) = -2\int_0^1 g(s)ds < 0.$$

Then there exists a value $w \in (a, 1)$ such that $\tau_g(w) = 0$. Going forward, we will refer to this value as a^*. Then notice $\tau_g(w) \geq 0$ for all $w \in [0, a^*]$. Therefore $T_{g,0}$ is defined over the interval $[0, a^*]$. Thus $T_{g,0}$ is defined over an interval in $[0, 1]$ regardless of Allee condition. □

Lemma 3.1.2. *$T_{g,1}$ is defined over an interval in $[0, 1]$.*

Proof. We need to consider three specific cases, $\int_0^1 g(s)ds < 0$, $\int_0^1 g(s)ds = 0$, and $\int_0^1 g(s)ds > 0$. First, recall from the definition of $\tau_g(w)$, $-\tau_g(1) = -(-2\int_0^1 g(s)ds) = 2\int_0^1 g(s)ds$. Recall our conditions on the function $g(w)$. In particular, $g(0) = g(a) = g(1) = 0$, $g(w) < 0$ for $w \in (0, a)$, and $g(w) > 0$ for $w \in (a, 1)$. Then by the fundamental theorem of calculus, $0, 1$ are local minimums and a is a local maximum over the interval $[0, 1]$ for the function $\tau_g(w)$.

case i: Suppose $\int_0^1 g(s)ds < 0$. We are going to begin by considering the function $\tau_g(w) + c_2$ at $w = 0, a, 1$.

$$
\begin{aligned}
&\tau_g(0) - \tau_g(1) = \int_0^0 g(s)ds + c_2 = c_2 < 0,\\
&\tau_g(a) - \tau_g(1) = \tau_g(a) - \tau_g(1) > 0,\\
&\tau_g(1) - \tau_g(1) = 0.
\end{aligned}
$$

Therefore, there exists a value $w \in (0, a)$ such that $\tau_g(w) + c_2 = 0$. We will be referring to this value as a_* going forward. Since $\tau_g(w)$ is strictly increasing over $(0, a)$ and strictly decreasing over $(a, 1)$, $\tau_g(w) + c_2 \geq 0$ for $w \in [a_*, 1]$. Thus if $\int_0^1 g(s)ds < 0$, $T_{g,1}$ is defined over $[a_*, 1]$ a sub-interval of $[0, 1]$.

case ii: Suppose $\int_0^1 g(s)ds = 0$. In this case, notice $\tau_g(1) = -2\int_0^1 g(s)ds = 0$. Then consider the values $w = 0, a, 1$:

$$\tau_g(0) + 0 = -2\int_0^0 +0 = 0,$$
$$\tau_g(a) + 0 = \tau_g(a) + 0 = \tau_g(a) > 0,$$
$$\tau_g(1) + 0 = -2\int_0^1 +0 = 0.$$

Therefore, $\tau_g(w) \geq 0$ for all $w \in [0, 1]$. Thus $T_{g,1}$ is defined over the interval $[0, 1]$.

case iii: Suppose $\int_0^1 g(s)ds > 0$. We will again consider the values $w = 0, a, 1$:

$$\tau_g(0) - \tau_g(1) = \int_0^0 g(s)ds + c_2 = c_2 > 0,$$
$$\tau_g(a) - \tau_g(1) = \tau_g(a) - \tau_g(1) > 0,$$
$$\tau_g(1) - \tau_g(1) = 0.$$

Therefore, $\tau_g(w) \geq 0$ for all $w \in [0, 1]$. Thus $T_{g,1}$ is defined over an interval in $[0, 1]$.

$\square$

Much of the work later in the paper will involve considering the intersections between T_{f,c_1}, $T_{g,0}$, and $T_{g,1}$. Notice from the definitions, these intersections occur when $\sqrt{\tau_f(w) + c_1} = \sqrt{\tau_g(w)}$ or $\sqrt{\tau_f(w) + c_1} = \sqrt{\tau_g(w) - \tau_g(1)}$ for instance. We need to construct some functions to discuss these intersections later in the paper. We will use the following functions to discuss the intersections between T_{f,c_1} and $T_{g,0}$ and the intersections between T_{f,c_1} and $T_{g,0}$:

$$\Delta_{\tau,0}(w, c_1) = \tau_f(w) + c_1 - \tau_g(w),$$

$$\Delta_{\tau,1}(w, c_1) = \tau_f(w) + c_1 - (\tau_g(w) - \tau_g(1)).$$

3.2 Equilibrium Types

We will be considering the possible equilibrium solutions for our equation. First, we must define the types of equilibrium that may exist.

Trivial Equilibrium

The trivial equilibrium is guaranteed to exist. This equilibrium type occurs when $w(x) = 0$ for all $x \in (-\infty, \infty)$. This is not overall a very interesting equilibrium, but it will always exist. In nature, this equilibrium corresponds to global extinction.

3.2.1 Type I Equilibrium

The Type I equilibrium is the first nontrivial equilibrium type we must consider. The type I equilibrium occurs when $w(x) > 0$ for all x in the protection zone and $w(x) \to 0$ as $x \to \pm\infty$. In this the phase plane, this equilibrium type will form a circuit from $(0, 0)$ to an intersection between $T_{g,0}$ and T_{f,c_1} to T_{f,c_1} w-intercept over $(0, K)$ in the first quadrant and return to $(0, 0)$ through similar points in quadrant four. In nature, this equilibrium corresponds to the protection zone forming an 'island' where the species persists over the protection zone and the area around the protection zone but does not thrive outside the protection zone. More importantly, in this case the persistence of the species is entirely dependent on the existence of the protection zone.

3.2.2 Type II Equilibrium

The Type II equilibrium refers to the circuit in the phase plane that starts and ends at $(1, 0)$. This refers to an equilibrium solution where $w(x) \to 1$ as $x \to \pm\infty$. In our phase plane analysis, this will be broken into three case: $K < 1$, $K > 1$, and $K = 1$. The magnitude of K will have an effect on the behavior and existence of this equilibrium type. In nature, this equilibrium type refers to a population that thrives across the entire environment.

3.2.3 Type III Equilibrium

In a type III equilibrium, $w(x) > 0$ over the protection zone and outside the protection zone $w(x) \to 0$ as $x \to -\infty$ and $w(x) \to 1$ as $x \to \infty$ or $w(x) \to 0$ as $x \to \infty$ and $w(x) \to 1$ as $x \to -\infty$. In the case of a retreating wave in particular, the protection zone 'stops' the extinction of a species by stopping the wave from retreating toward $\pm\infty$. In the phase plane, this equilibrium refers path that connects the critical points $(0, 0)$ and $(1, 0)$ in either the first or fourth quadrant of the phase plane.

3.3 Equilibrium Solutions with Retreating Wave Condition $\int_0^1 g(s)ds < 0$

The first case we will need to consider is the integral condition $\int_0^1 g(s)ds < 0$. This corresponds to the retreating wave condition for a strong Allee effect model, results in the most hostile environment we will consider.

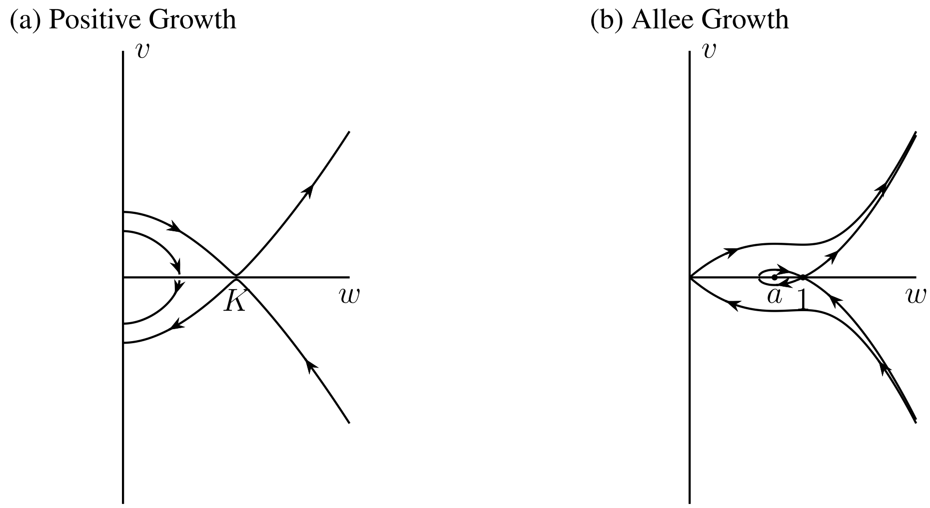


Figure 3.1: Figure (a) shows two trajectories in the phase plane over the protection zone governed by a positive growth function. Figure (b) shows the two trajectories in the phase plane outside the protection zone governed Allee effect growth with condition $\int_0^1 g(s)ds < 0$ that pass through $(0, 0)$ and $(1, 0)$.

3.3.1 Type I Equilibrium

The analysis of these phase planes will begin by considering the specific $T_{g,0}$ trajectory that passes through the origin, i.e. $c_2 = 0$. This trajectory corresponds to a solution where the population density of the protection zone and 'local' area are nonzero, but $w(x) \to 0$ as $x \to \pm\infty$. We have defined the trajectories in the phase plane as sets, since they exist in both the first and fourth quadrant in the w,v-plane. Fortunately, they are symmetric about the w-axis, so largely we will be able to treat them similarly to functions. We will define a function $0 < w_1(c_1) < K$ where c_1 is the constant term of $\tau_f(w) + c_1$. This function corresponds to the w-intersection point in the phase plane, which is the maximal value of the solution $w(x)$ in a type I equilibrium.

$$\tau_f(w_1(c_1)) + c_1 = 0$$

This $w_1(c_1)$ is an important function for us, but first we will need to prove that it exists and the conditions necessary for its existence. Notice, we have defined $w_1(c_1)$ as dependent on $\tau_f(w) + c_1$, so we will need to show $\tau_f(w) + c_1$ is invertible over at least a portion of its domain. We also would like to show there is no value $w \in (0, w_1(c_1))$ such that $\tau_f(w) + c_1 = 0$. Thus we will need to show $\tau_f(w) + c_1$ is strictly decreasing, which shows invertibility and uniqueness of $w_1(c_1)$.

Lemma 3.3.1. $\tau_f(w)$ *is strictly decreasing for* $w \in (0, K)$.

Proof. Recall from definition, $\tau_f(w) = -2\int_0^w f(s)ds + c_1$. By the fundamental theorem of calculus, $\frac{d}{dw}[\tau_f(w)] = -2f(w)$. We assumed $(K - w)f(w) > 0$, so $\frac{d}{dw}[\tau_f(w)] < 0$ for $w \in (0, K)$. Therefore $\tau_f(w)$ is strictly decreasing for $w \in (0, K)$. $\square$

$\tau_f(w) + c_1$ being strictly decreasing is a very strong condition. In particular, this shows $\tau_f(w) + c_1$ is injective over this domain so we have a notion of inverse, and this allows for the existence of a formula for $w_1(c_1)$, i.e. $w_1(c_1) = \tau_f^{-1}(-c_1)$. We have defined $w_1(c_1)$, such that $w_1(c_1)$ corresponds to the maximum value of our solution function $w(x)$ over the phase plane, so $w_1(c_1) \leq K$. This results in a restriction on c_1, as we have defined $w_1(c_1)$ to depend on c_1. Then define $\tilde{c_1} = -\tau_f(K)$, this new constant $\tilde{c_1}$ forms an upper bound on c_1. Notice $w_1(\tilde{c_1}) = \tau_f^{-1}(-(-\tau_f(K))) = K$.

Lemma 3.3.2. $w_1(c_1) : (0, \tilde{c_1}) \to (0, K)$ *is strictly increasing*

Proof. Recall $w_1(c_1) = \tau_f^{-1}(-c_1)$ for $c_1 \in (0, \tilde{c_1})$ and from Lemma 3.3.1 $\tau_f(w) : (0, w_1(\tilde{c_1}) \to (0, \tilde{c_1})$ is strictly decreasing. Then $\tau_f^{-1}(w) : (0, \tilde{c_1}) \to (0, w_1(\tilde{c_1})$ must also be strictly decreasing. Therefore $w_1(c_1) = \tau_f^{-1}(-c_1)$ must be strictly increasing for $c_1 \in (0, \tilde{c_1})$. $\square$

Corollary 3.3.2.1. *If* $0 < c_1 < \tilde{c_1}$ *then* $0 < w_1(c_1) < K$ *and* $\tau_f(w_1(c_1)) + c_1 = 0$ *over the interval* $(0, K)$.

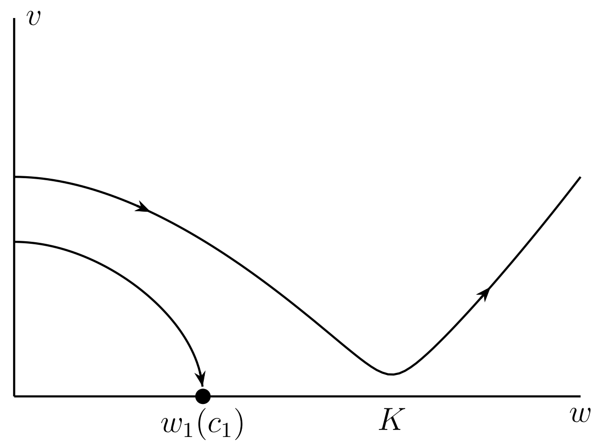


Figure 3.2: This figure shows quadrant I of the phase portrait of T_{f,c_1} with two trajectories with various c_1 values. One of the trajectories shows the intersection $w_1(c_1)$.

Proof. First, recall $\tau_f(w)$ is strictly decreasing for $w \in (0, K)$ by Lemma 3.3.1, so $\tau_f(w) + c_1$ is strictly decreasing as well for $w \in (0, K)$. Now recall our definition of $w_1(c_1) = \tau_f^{-1}(-c_1)$, so $w_1(c_1)$ is strictly increasing. Thus we only need to consider $\tau_f(w) + c_1$ to draw conclusions about $w_1(c_1)$. We can now infer $\sup\{\tau_f(w) : w \in (0, K)\} = \tau_f(0) = 0$ and $\inf\{\tau_f(w) : w \in (0, K)\} = \tau_f(K) = -\tilde{c_1}$. Notice then $\tau_f(0) + c_1 > 0$ and $\tau_f(K) + c_1 < 0$ by assumption $0 < c_1 < \tilde{c_1}$. Finally, since $\tau_f(w)$ is continuous and strictly decreasing over $(0, K)$, by the intermediate value theorem there exists precisely one $w_1(c_1) \in (0, K)$ such that $\tau_f(w_1(c_1)) + c_1 = 0$ over the interval $(0, K)$. Notice, by the definition of w_1, we can see $\tau_f(w)+0 = 0$ and $\tau_f(w)-\tau_f(K) = 0$ correspond to the minimal and maximal w_1 values. Thus if $0 < c_1 < \tilde{c_1}$ then $0 < w_1(c_1) < K$. $\square$

Notice $w_1(c_1) \in (0, K)$ is dependant on c_1, so we are going to consider the w-intercept as a function in terms of c_1. Recall the definition of (3.6), in particular, in order to consider v in terms of w from this equation we will need to establish the conditions on c_1 and w such that $\sqrt{\tau_f(w) + c_1}$ is a real number. For us to show these phase portraits intersect in a meaningful way, we will need to show (3.6) is defined for $w \in [0, w_1(c_1)]$ in both the first and fourth quadrant of the w, v-plane. In order to do this, we really only need to show $\tau_f(w) + c_1 \geq 0$ for $w \in [0, w_1(c_1)]$

Lemma 3.3.3. *$\tau_f(w) + c_1 \geq 0$ for $w \in [0, w_1(c_1)]$ when $0 < c_1 < \tilde{c_1}$.*

Proof. Suppose $0 < c_1 < \tilde{c_1}$. By Corollary 3.3.2.1, $\tau_f(w_1(c_1)) + c_1 = 0$ and $w_1(c_1) \in (0, K)$. Recall $\tau_f(0) + c_1 > 0$ and $\tau_f(w)$ is continuous and strictly decreasing, so $\tau_f(w) + c_1 \geq 0$ for $w \in [0, w_1(c_1)]$ □

Theorem 3.3.4. *T_{f,c_1} is defined over $[0, w_1(c_1)]$ when $0 < c_1 < \tilde{c_1}$.*

Proof. Recall definition of T_{f,c_1}, in particular $v = \pm\sqrt{\tau_f(w) + c_1}$. Thus for T_f to be defined over $[0, w_1(c_1)]$, $\tau_f(w) + c_1 \geq 0$ for $w \in [0, w_1(c_1)]$. Recall $\tau_f(0) + c_1 = c > 0$, $\tau_f(w_1) + c_1 = 0$, and τ_f is strictly decreasing, therefore the absolute maximum of $\tau_f(w) + c_1$ occurs at $w = 0$ and the absolute minimum occurs at $w = w_1(c_1)$.
Thus T_{f,c_1} is defined over $[0, w_1(c_1)]$ when $0 < c_1 < \tilde{c_1}$. □

We have established the necessary and sufficient conditions for type I equilibrium on equation (3.6), i.e. for a type I equilibrium: $w_1(c_1) \in (0, K)$ and T_{f,c_1} is defined over $[0, w_1(c_1)]$. We will now need to consider $T_{g,0}$ through the properties of $\tau_g(w)$. We have shown the necessary and sufficient conditions for (3.7) to be defined over the interval $[0, 1)$. Similarly to before, that relied heavily on showing $\tau_g(w) > 0$ for all $w \in [0, 1)$.

Notice, Lemma 3.1.1 has the conditions to ensure (3.7) is defined over $[0, 1)$. Then we can begin considering combining the two phase planes. Recall the function $\Delta_{\tau,0}$.

Theorem 3.3.5. *If $0 < c_1 < \tilde{c_1}$, then T_{f,c_1} intersects $T_{g,0}$ over the interval $(0, K)$.*

Proof. Suppose $0 < c_1 < \tilde{c_1}$. Recall the definitions of $\Delta_{\tau,0}(w, c_1)$, T_{f,c_1}, intersects $T_{g,0}$. In particular, $\Delta_{\tau,0}(w, c_1) = 0$ implies intersection between T_{f,c_1} intersects $T_{g,0}$ for a specific c_1 value. Notice $\Delta_{\tau,0}(0, c_1) = c_1 > 0$, by our condition on c_1. Next $\Delta_{\tau,0}(w_1(c_1), c_1) = 0 - \tau_g(w_1(c_1)) < 0$ by our definition of $w_1(c_1)$ and conditions on $g(w)$. Therefore T_{f,c_1} intersects $T_{g,0}$ over the interval $(0, w_1(c_1))$ by the intermediate

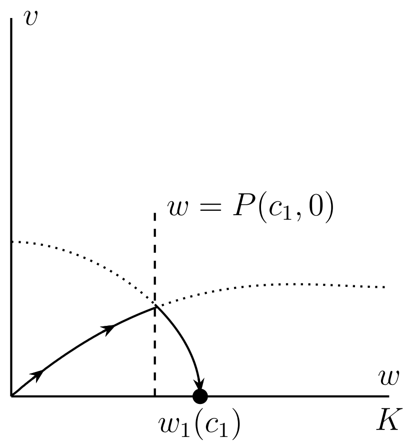


Figure 3.3: Quadrant I of the combined phase portraits of a type I equilibrium with $w_1(c_1)$ and $P(c_1, 0)$ labelled.

value theorem. Finally, by Corollary 3.3.2.1 and $0 < c_1 < \tilde{c_1}$, T_{f,c_1} intersects $T_{g,0}$ over $(0, K)$. □

Above, we have shown that the trajectories (3.6) and (3.7) have at least one intersection point over the interval $(0, w_1(c_1))$. There exists the possibility of multiple intersection points between the two trajectories over the interval $(0, K)$. However, for simplicity, we will be assuming there is only one intersection over the interval $(0, K)$. We will be referring to this intersection point as $P(c_1, 0)$, as it is dependent on the constant c_1 and the constant $c_2 = 0$. We have defined $P(c_1, 0)$ as the w-value of the intersection of T_{f,c_1} and $T_{g,0}$ in the interval $(0, K)$. It remains to establish a formula for $P(c_1, 0)$ that represents the correct intersection. Then it is critical to consider our definition of T_{f,c_1} and $T_{g,0}$, recall they are only defined over the w-interval $(0, \infty)$. Our condition $0 < c_1 < \tilde{c}$ ensures there is at least one positive intersection between T_{f,c_1} and $T_{g,0}$ over the interval $(0, w_1(c_1))$. Thus any other intersection occurs over $[w_1(c_1), \infty)$. Then we can say our intersection $P(c_1, 0)$ is the minimal intersection between T_{f,c_1} and $T_{g,0}$ over $(0, \infty)$. We can define a formula for an intersection between T_{f,c_1} and T_{g,c_1}:

$$P(c_1, c_2) = \min\{w \in (w, \pm v) : (w, \pm v) = T_{f,c_1} \cap T_{g,c_2}\}$$

Therefore the w-value of first intersection between T_{f,c_1} and $T_{g,0}$ is the following formula:

$$P(c_1, 0) = \min\{w \in (w, \pm v) : (w, \pm v) = T_{f,c_1} \cap T_{g,0}\}$$

We are using this notation for the smallest intersection between T_{f,c_1} and $T_{g,0}$ because later it will be useful to study the smallest intersection between T_{f,c_1} and $T_{g,1}$ which we can denote similarly as $P(c_1, -\tau_g(1))$.

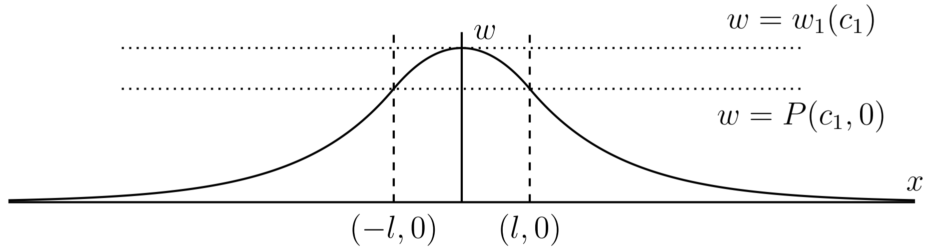


Figure 3.4: An example of a type I steady-state solution $w(x)$ which corresponds 3.3 in the phase plane.

We have established the trajectories in the phase plane (3.6) and (3.7) intersect over the interval $(0, w_1(c_1))$ and by definition $0 < P(c_1, 0) < w_1(c_1)$. This intersection $P(c_1, 0)$ and the value $w_1(c_1)$ will form the bounds of an integral that we will use to calculate the minimum patch size. Recall, we referred to the protection zone as $\Omega \in \mathbb{R}$. We are attempting to establish the length of Ω that ensures persistence of our general species, a similar problem is considered in [10]. We are going to define the length of Ω as $|\Omega| = l(c_1) - (-l(c_1)) = 2l(c_1) = L(c_1)$. Where $l(c_1)$ is a function for half the length of the protection zone depending on the value c_1. We can calculate the minimum patch size by considering the following equations:

$$
\begin{aligned}
v &= \sqrt{\tau_f(w) + c_1} \\
\int_0^L dx &= \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
l(c_1) &= \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
L(c_1) &= 2\int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}.
\end{aligned}
$$

We can now consider the function $L(c_1)$ in order to determine when the minimal patch size occurs. $L(c_1)$ can be shown to be continuous for $c_1 \in (0, \tilde{c_1})$ (see Lemma 2.5 in [24]). This also established $L(c_1)$ is well-defined for $c_1 \in (0, \tilde{c_1})$ however it would not be well defined for $c_1 = \tilde{c_1}$. Then it remains to establish $L(c_1) \to +\infty$ as $c_1 \to \tilde{c_1}$. Notice $T_{f,\tilde{c_1}}$ corresponds to a stable manifold in the phase plane since this trajectory passes through the point $(K, 0)$ in the phase plane, thus $L(c_1) \to \infty$ as $c_1 \to \tilde{c_1}$. This implies a nonzero equilibrium solution for $c_1 \in (0, \tilde{c_1})$. A similar argument for considering minimal patch length is considered in Pouchol, et.al [30]. Then we can see the minimal patch size corresponds to the following value:

$$
L_1^* = 2 \inf_{0<c_1<\tilde{c_1}} \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}.
$$

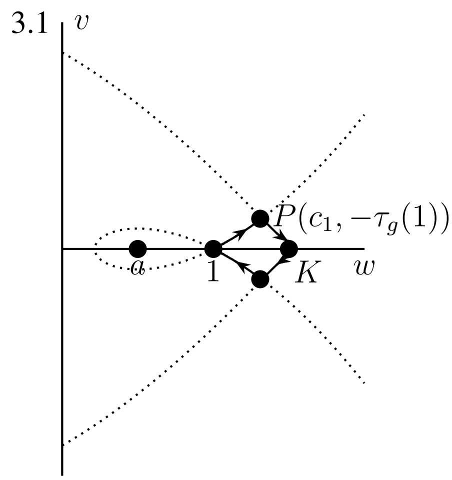


Figure 3.5: This figure is an example of a phase plane of type II equilibrium with $K > 1$ and $\int_0^1 g(s)ds < 0$. The portion of the trajectories that correspond to type II equilibrium are shown with solid lines.

3.3.2 Type II Equilibrium

We will now consider the conditions necessary for a type II equilibrium solution. This is the most complicated of the three solutions and requires the consideration of three cases, $K > 1$ and $K < 1$ and $K = 1$. In type I and III solutions the trajectories do not occur 'near' the carrying capacity in the phase plane, so the conditions do not need to depend as much on the carrying capacity. In this case, the relative size of the carrying capacities determine the behavior of the solutions in phase space.

$K > 1$

In this case, we will have to again consider a specific trajectory in the $T_{g,1}$ phase plane. This trajectory begins and ends at $w = 1$. In order for this trajectory to exist, $T_{g,1}$ must be defined for $w \in (1, \infty)$. We will need to define the conditions necessary for this trajectory to exist. Recall, Lemma 3.1.2 shows $T_{g,1}$ is defined over the interval $[a_*, 1]$, but it remains to show this trajectory is defined over $(1, \infty)$.

Lemma 3.3.6. *$T_{g,1}$ is defined over the interval $[a_*, \infty)$.*

Proof. Recall Lemma 3.1.2 shows $T_{g,1}$ is defined over $[a_*, 1]$. It remains to show $\tau_g(w) - \tau_g(1) \geq 0$ for $w > 1$. Again recall $g(w) < 0$ for $w \in (1, \infty)$ from our definition of $g(w)$. Then by the fundamental theorem of calculus, $\tau_g'(w) = -2g(w)$, thus $\tau_g(w) - \tau_g(1) > \tau_g(1) - \tau_g(1) = 0$ for all $w \in (1, \infty)$. Then $\tau_g(w) - \tau_g(1) \geq 0$ for all $w \in [a_*, \infty)$. Therefore $T_{g,1}$ is defined over the interval $[a_*, \infty)$. $\square$

Next, we need to consider the conditions on c_1 that ensure T_{f,c_1} and $T_{g,1}$ intersect in a meaningful way in the phase plane. In particular, we will need to ensure $w_1(c_1) \in (1, K)$.

$$\tau_f(1) + c_1 = 0$$
$$c_1 = -\tau_f(1)$$

Then notice, if $-\tau_f(1) < c_1 < \tilde{c}_1$ then $w_1(c_1) \in (1, K)$, this follows from Corollary 3.3.2.1. We have now arrived at the necessary conditions for a Type II solution to exist.

Theorem 3.3.7. *If $-\tau_f(1) < c_1 < \tilde{c}_1$ then T_{f,c_1} and $T_{g,1}$ intersect over the interval $(1, K)$.*

Proof. Suppose $-\tau_f(1) < c_1 < \tilde{c}_1$. Recall the function $\Delta_{\tau,1}(w) = \tau_f(w) + c_1 - (\tau_g(w) - \tau_g(1))$. Notice the intersection will occur over the interval $(1, w_1(c_1)) \subset (1, K)$ specifically. We will consider the values $w = 1$ and $w = w_1(c_1)$. Recall $\Delta_{\tau,1}(w) = \tau_f(w) + c_1 - (\tau_g(w) - \tau_g(1))$, $\tau_f(w) + c_1 > 0$ for all $w \in (0, K)$, and $\tau_g(w) - \tau_g(1) > 0$ for all $w \in (1, \infty)$.

$$\Delta_{\tau,1}(1) = \tau_f(1) + c_1 - (\tau_g(1) - \tau_g(1)) = \tau_f(1) + c_1 > 0,$$

$$\Delta_{\tau,1}(w_1(c_1)) = \tau_f(w_1(c_1)) + c_1 - (\tau_g(w_1(c_1)) - \tau_g(1)) = 0 - (\tau_g(w_1(c_1)) - \tau_g(1)) < 0.$$

Thus by the intermediate value theorem there exists $w \in (1, w_1(c_1)) \subset (1, K)$ such that $\Delta_{\tau,1}(w) = 0$ Therefore, T_{f,c_1} and $T_{g,1}$ intersect over the interval $(1, K)$. □

Then by Theorem 3.3.7, we have established the following path in the phase plane:

$$(1, 0) \to P(c_1, -\tau_g(1)) \to (K, 0) \to P(c_1, -\tau_g(1)) \to (1, 0).$$

Notice, this path corresponds to the existence of a type II equilibrium. Then it remains to establish the conditions on the patch that allow for this equilibrium type to exist.

$$\begin{aligned}
v &= \sqrt{\tau_f(w) + c_1} \\
\int_0^L dx &= \int_{P(c_1,-\tau_g(1))}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
l_2(c_1) &= \int_{P(c_1,-\tau_g(1))}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
L_2(c_1) &= 2\int_{P(c_1,-\tau_g(1))}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}.
\end{aligned}$$

Notice $L_2(c_1)$ can be shown to be well-defined for $c_1 \in (-\tau_f(1), \tilde{c_1})$ through a similar argument to the type I case. Also similar, as c_1 approaches $\tilde{c_1}$, T_{f,c_1} approaches the stable manifold $T_{f,\tilde{c_1}}$ and thus $L(c_1) \to +\infty$ as $c_1 \to \tilde{c_1}$. Therefore the following minimal

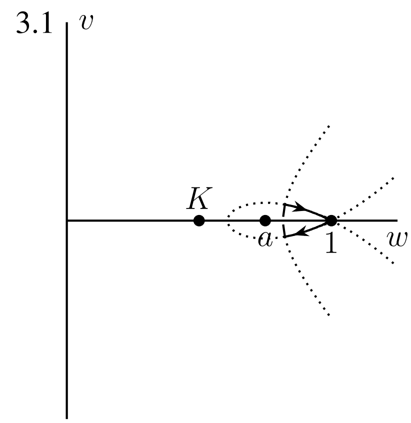


Figure 3.6: This figure shows a type II equilibrium with $K < 1$ and $\int_0^1 g(s)ds < 0$. The type II equilibrium is shown as the solid lines that begins and ends at $(1, 0)$.

patch size allows for the existence of a type II equilibrium:

$$L_2^* = 2 \inf_{-\tau_f(1)<c_1<\tilde{c_1}} \int_{P(c_1,-\tau_g(1))}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w)+c_1}}.$$

$K < 1$

In this section we will be considering another version of the Type II equilibrium solution. We have already established that $T_{g,1}$ is defined over $[a_*, 1]$. So it remains to consider the conditions on c_1 to ensure intersection over the interval $(K, 1)$. In this case, we will be considering a different part of the T_{f,c_1} trajectory. We will typically only consider the curve T_{f,c_1} over the interval $[0, w_1(c_1)]$. But we will now need to consider the second w-intercept. By our assumptions, $\tau_f(w)$ is strictly increasing for $w \in (K, \infty)$.

Lemma 3.3.8. *$\tau_f(w)$ is strictly increasing for $w \in (K, \infty)$.*

Proof. Recall $\tau_f'(w) = -2f(w)$ and $(K - w)f(w) > 0$. Notice if $w \in (K, \infty)$ then $K - W < 0$, so $f(w) < 0$ for $w \in (K, \infty)$. Therefore $\tau_f'(w) > 0$ for all $w \in$

(K, ∞). $\square$

$\tau_f(w)$ is injective if we restrict its domain to (K, ∞), so there exists $w_2(c_1) = \tau_f^{-1}(-c_1)$. We can now use this function $w_2(c_1)$ to determine the conditions on c_1 similar to how w_1 was used in the type I section. Notice by definition, $w_2(\tilde{c_1}) = K = w_1(\tilde{c_1})$, so $\tilde{c_1}$ forms the upper bound of acceptable c_1 values for this solution type as well. It remains to find the lower bound, i.e. the c_1 value such that $w_2(c_1) = 1$.

$$\begin{aligned} w_2(c_1) &= 1 \\ \tau_f^{-1}(-c_1) &= 1 \\ -c_1 &= \tau_f(1) \\ c_1 &= -\tau_f(1) \end{aligned}$$

Lemma 3.3.9. *If* $-\tau_f(1) < c_1 < \tilde{c_1}$, $w_2(c_1) \in (K, 1)$.

Proof. Recall $w_2(c_1) = \tau_f^{-1}(-c_1)$ for c_1 such that $w_2 \in (K, \infty)$ and $w_2(\tilde{c_1}) = K = w_1(\tilde{c_1})$. Since $\tau_f(w)$ is strictly increasing over (K, ∞), we can say $\tau_f^{-1}(-c_1)$ is strictly decreasing therefore the upper bound is $\tilde{c_1} > 0$ and the lower bound $-\tau_f(1)$.
Thus $w_2(c_1) \in (K, 1)$ since $w_2(c_1)$ is strictly decreasing and the upper/lower bounds are known. $\square$

It is important to note, $w_2(c_1) \in (K, 1)$ does not ensure the existence of a type II equilibrium. We will need to further restrict c_1 to ensure the existence of this solution type when $K < 1$. This leads to an interesting result, i.e. in certain cases the existence of a type II equilibrium is not guaranteed. Biologically this case may not be very likely, but we are referring to a species where the carrying capacity over the protection zone is small relative to the carrying capacity outside the protection.

Lemma 3.3.10. *If* $-\tau_f(1) < c_1 < -\tau_f(a_*)$, $w_2(c_1) \in (a_*, 1)$

Proof. Suppose $-\tau_f(1) < c_1 < -\tau_f(a_*)$. Recall from the proof of Lemma 3.3.9, $w_2(c_1)$ is strictly decreasing so $-\tau_f(1)$ forms the lower bound of c_1. Now, notice $w_2(-\tau_f(a_*)) = a_*$ by definition, however $w_2(c_1) \in (K, \infty)$ and a_* is not necessarily greater than K, so we must consider a couple cases.

case i: Suppose $K \leq a_*$. Then $w_2(c_1) \in (a_*, 1)$.

case ii: Suppose $K > a_*$. Then by definition, $w_2(c_1) \in (K, 1)$, so $w_2(c_1) \in (a_*, 1)$ by set containment. $\square$

Theorem 3.3.11. *If* $-\tau_f(1) < c_1 < \min\{\tilde{c_1}, -\tau_f(a_*)\}$, *a type II equilibrium exists.*

Proof. Suppose $-\tau_f(1) < c_1 < \min\{-\tau_f(a_*), \tilde{c_1}\}$, then

$$w_2(c_1) \in (a_*, 1)$$

and $w_2(c_1) > K$ by Lemma 3.3.10 and Lemma 3.3.9 respectively. We will consider $\Delta_{\tau,1}(w, c_1)$:

$$\Delta_{\tau,1}(w_2(c_1), c_1) = \tau_f(w_2(c_1)) + c_1 - (\tau_g(w_2(c_1), c_1) - \tau_g(1) = 0 - (\tau_g(w_2(c_1)) - \tau_g(1)) < 0,$$
$$\Delta_{\tau,1}(1, c_1) = \tau_f(1) + c_1 - (\tau_g(1) - \tau_g(1)) = \tau_f(1) + c_1 > 0.$$

Therefore by the intermediate value theorem, there exists an intersection between T_{f,c_1} and $T_{g,1}$. Notice by our assumptions on f and g, $\tau_f(w)$ is strictly increasing and g is concave down over the interval $(\max\{K, a_*\}, 1)$ respectively. Thus there exists precisely one intersection between T_{f,c_1} and $T_{g,1}$ over $(\max\{K, a_*\}, 1)$ referred to as $P_2(c_1, -\tau_g(1))$. Therefore we have the following circuit in the phase plane:

$$(1,0) \to P_2(c_1, -\tau_g(1)) \to (w_2(c_1), 0) \to P_2(c_1, -\tau_g(1)) \to (1,0)$$

This circuit starts and ends at $(1, 0)$, thus a type II equilibrium exists. $\square$

We have established the existence of $P_2(c_1, -\tau_g(1))$, but we must introduce a formula that differentiates this value from $P(c_1, -\tau_g(1))$. The key difference is this intersection occurs over the interval (K, ∞):

$$P_2(c_1, -\tau_g(1)) = \min\{w \in (w, v) : (w, v) = T_{f,c_1} \cap T_{g,-\tau_g(1)} \in ((K, \infty) \times (-\infty, \infty))\}.$$

As mentioned earlier, these conditions indicate that a type two equilibrium does not always exist when $K < 1$. In particular, a type II equilibrium may not exist when $K < a_*$.

$$\begin{aligned}
v &= \sqrt{\tau_f(w) + c_1} \\
\int_0^l dx &= \int_{w_2(c_1)}^{P_2(c_1,-\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
l_2(c_1) &= \int_{w_2(c_1)}^{P_2(c_1,-\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
L_2(c_1) &= 2\int_{w_2(c_1)}^{P_2(c_1,-\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}}.
\end{aligned}$$

There are two cases to consider here, when $K \geq a_*$ and when $K < a_*$. First, when

$K \geq a_*$, $\min\{-\tau_f(a_*), \tilde{c_1}\} = \tilde{c_1}$. Thus $-\tau_f(1) < c_1 < \tilde{c_1}$. Then through a similar argument to the $K > 1$ case, $L_2(c_1)$ is well-defined for $c_1 \in (-\tau_f(1), \tilde{c_1})$ and as $c_1 \to \tilde{c_1}$ then T_{f,c_1} approaches the stable manifold $T_{f,\tilde{c_1}}$, so $L_2(c_1) \to +\infty$. Thus there exists a nonzero equilibrium solution for $c_1 \in (-\tau_f(1), \tilde{c_1})$. Then the minimal patch length to allow for the existence of the type II equilibrium when $a_* \leq K < 1$ is the following value:

$$L_2^*(c_1) = \inf_{-\tau_f(1) < c_1 < \tilde{c_1}} 2 \int_{w_2(c_1)}^{P_2(c_1, -\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$

Finally, we can consider $K < a_*$. Then $\min\{-\tau_f(a_*), \tilde{c_1}\} = -\tau_f(a_*)$, so $-\tau_f(1) < c_1 < -\tau_f(a_*)$. However, the stable manifold $T_{f,\tilde{c_1}}$ does not pass through the $T_{g,1}$ trajectory, so we can only say the type II equilibrium exists when the protection length L is in the range of $L_2(c_1)$ for $-\tau_f(1) < c_1 < -\tau_f(a_*)$. Therefore the critical values of $L(c_1)$ can not be determined through the same logic as the previous cases. In this case, there is the potential for maximal and minimal protection zone to allow for the existence of a type II equilibrium. This case is explored more thoroughly in the case study and simulations chapter.

$K = 1$

In this case, the existence of the type II equilibrium is guaranteed in the phase plane.

Theorem 3.3.12. *If* $K = 1$ *a type II equilibrium exists, and specifically* $w(x, t) = 1$.

Proof. Suppose $K = 1$ and $w(x, t) = 1$. Recall an equilibrium solution occurs when $w_t = 0$

$$
\begin{aligned}
w_t &= w_{xx} + \begin{cases} f(w), & x \in [-l, l] \\ g(w), & x \notin [-l, l] \end{cases}, \\
w_t &= 0 + \begin{cases} f(1), & x \in [-l, l] \\ g(1), & x \notin [-l, l] \end{cases}, \\
w_t &= 0.
\end{aligned}
$$

By our assumptions on f and g respectively we can see $w_{xx} + \begin{cases} f(w), & x \in [-l, l] \\ g(w), & x \notin [-l, l] \end{cases} = 0$ when $w(x,t) = 1$ for all $x \in \mathbb{R}$.

Therefore a type II equilibrium exists. $\square$

Notice, the argument above is not dependent on the integral condition of $T_{g,1}$, and therefore hold for all three cases. It is also not dependent on the size of the protection zone, i.e. any protection zone size will allow for a type II equilibrium if $K = 1$.

3.3.3 Type III Equilibrium

Recall from above, our conditions ensure a retreating wave outside of the protection zone i.e. outside of the protection zone has the potential to tend toward extinction. In this section, we are going to explore the conditions necessary for a positive solution where $w(x,t) \to 0$ as $x \to -\infty$, $w(x,t) \to 1$ as $x \to \infty$ or $w(x,t) \to 0$ as $x \to \infty$, $w(x,t) \to 1$ as $x \to -\infty$. We are looking for the minimal protection zone that stops the traveling wave from advancing toward extinction and allows for a thriving population on one side of the protection zone.

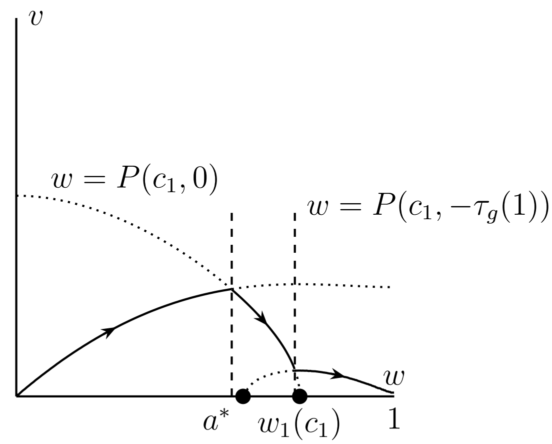


Figure 3.7: This figure shows quadrant I of the combined phase portraits of a type III equilibrium where $\int_0^1 g(s)ds < 0$. Demonstrating the intersection between T_{f,c_1} and $T_{g,0}$ and the intersection between T_{f,c_1} and $T_{g,1}$.

Notice $\tau_g(0) - \tau_g(1) < 0$ and $\tau_g(a) - \tau_g(1) > 0$, by our conditions on $g(w)$ and the definition of $\tau_g(w)$. Then there must exist a w-intercept for $\tau_g(w) - \tau_g(1)$ over the interval $(0, a)$. We will be referring to this w-intercept as a_*. Unfortunately, the equation $\tau_g(w) - \tau_g(1) = 0$ has multiple solutions so correctly writing an equation for a_* will be tough. We have shown this intercept exists and is unique, Lemma 3.1.1.

We considered the existence and behavior of two T_{g,c_2} trajectories in the plane. We must now construct the necessary conditions on c_1 to ensure the T_{f,c_1} intersects both $T_{g,0}$ and $T_{g,1}$. First, we need to define a new constant for this equilibrium type, $\bar{c_1} = \min\{\tilde{c_1}, -\tau_f(1)\}$. We have already established an intersection between T_{f,c_1} and $T_{g,0}$ over the interval $(0, K)$, when $0 < c_1 \leq \tilde{c_1}$. Then all we need consider is the conditions for an intersection between T_{f,c_1} and $T_{g,1}$. We will be able to discover these conditions largely by looking at the function $w_1(c_1)$. Naively, we will need to restrict c_1 such that $a_* < w_1(c_1) < 1$. Another consideration here, if $K < a_*$ a type III equilibrium can not exist. This is because $a_* < w_1(c_1) < 1$ is the necessary and sufficient condition for the existence of a type III equilibrium. Then we will assume $K \geq a_*$ for this section We must first prove $w_1(c_1)$ is an increasing function on c_1 over the interval $(0, \bar{c_1})$.

Lemma 3.3.13. *If $K < a_*$ a type III equilibrium can not exist.*

Proof. Suppose $K < a_*$. Recall a type III equilibrium occurs when T_{f,c_1} connects $T_{g,0}$ and $T_{g,1}$ in either the first or fourth quadrant form a path in the phase plane between the points $(0,0)$ and $(1,0)$. In forming this path, the flow of the trajectories is important in the phase plane. Notice all three trajectories we are considering are traveling from the v-axis toward $+\infty$ in quadrant I and from $+\infty$ to the v-axis in quadrant IV. The next critical thing to consider is the relationship between $T_{g,0}$ and $T_{g,1}$. For simplicity going forward, we will only be considering the quadrant I trajectories, by definition they are symmetric about the w-axis. Thus anything established in the first quadrant also holds in quadrant IV. Notice $T^+_{g,0}(w) > T^+_{g,1}(w)$ for all $w \in (0, \infty)$ by our definitions for these trajectories. Therefore $P(c_1, 0) < P(c_1, -\tau_g(1))$ and $T^+_{f,c_1}(P(c_1, 0)) > T^+_{f,c_1}(P(c_1, -\tau_g(1)))$. Thus the portion of $T^+_{f,c_1}(w)$ that connects $T^+_{g,0}$ and $T^+_{g,1}$ must be decreasing. If it were to be increasing, the flow of $T^+_{g,0}$ and T^+_{f,c_1} would both be traveling to the point $P(0, c_1)$ which would not allow for the existence of a type III equilibrium. T^+_{f,c_1} is only decreasing over $(0, w_1(c_1)]$. Thus the intersections $P(c_1, 0), P(c_1, -\tau_g(1))$ must occur over the interval $(0, w_1(c_1)] \subset (0, K)$. Thus by our assumption, $K < a_*$, T_{f,c_1} can not intersect $T_{g,1}$ as $T_{g,1}$ is only defined over the interval $[a_*, \infty)$.Therefore a type III equilibrium can not exist if $K < a_*$. $\square$

We have shown $w_1(c_1)$ is strictly increasing in Lemma 3.3.2, so the lower bound is c_1 such that $w_1(c_1) = a_*$ and the upper bound is c_1 such that $w_1(c_1) = 1$. Therefore our new condition is $c_1 \in (-\tau_f(a_*), -\tau_f(1))$.

Theorem 3.3.14. *If $c_1 \in (-\tau_f(a_*), \bar{c_1})$ then T_{f,c_1} intersects $T_{g,1}$ over the interval $(a_*, 1)$.*

Proof. Suppose $c_1 \in (-\tau_f(a_*), -\tau_f(1))$. First, recall from the proof of Lemma 3.1.2, $\tau_g(w) - \tau_g(1) > 0$ for $w \in (a_*, 1)$. Then $\Delta_{\tau,1}(a_*) = \tau_f(a_*) + c_1 - 0 > 0$ and

$\Delta_{\tau,1}(1) = \tau_f(1)+c_1-(\tau_g(1)-\tau_g(1)) = \tau_f(1)+c_1 < 0$. Thus there exists a $w \in (a_*, 1)$ such that $\Delta_{\tau,1}(w) = 0$ which implies intersection of $T_{g,-\tau_g(1)}$ and T_{f,c_1} over the same interval. $\square$

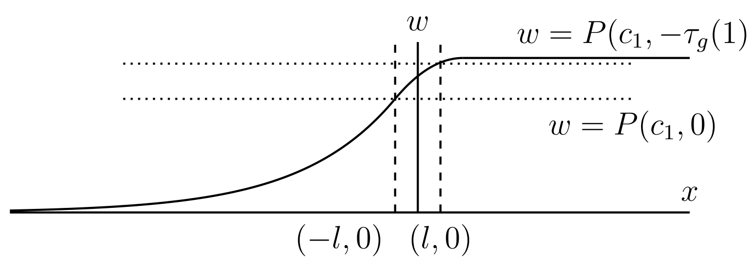


Figure 3.8: This figure shows a possible solution of the protection zone model that corresponds to a type III equilibrium. Notice $w(x) \to 0$ as $x \to -\infty$ and $w(x) \to 1$ as $x \to \infty$ which corresponds to our definition of a type III equilibrium.

We have established the conditions on c_1 to ensure the existence of a Type III solution, $K \geq a_*$ and $c_1 \in (-\tau_f(a_*), -\tau_f(1))$. Similar to the type I solution, this type III solution is dependent on the size of the protection zone. Consider the following equation which was derived similarly in the type I section:

$$L_3(c_1) = \int_{P(c_1,-\tau_g(1))}^{P(c_1,0)} \frac{dw}{\sqrt{\tau_f(w)+c_1}}$$

Again, we have two cases to consider, when $a_* \leq K \leq 1$ and when $K > 1$. First, if $a_* \leq K \leq 1$. Then $L_3(c_1)$ is well-defined for $c_1 \in (-\tau_f(a_*), -\tau_f(1))$, and as $c_1 \to \tilde{c_1}$, T_{f,c_1} approaches the stable manifold $T_{f,\tilde{c_1}}$ and $L_3(\tilde{c_1} \to +\infty$ as $c_1 \to \tilde{c_1}$. This implies a nonzero equilibrium solution for $c_1 \in (-\tau_f(a_*), -\tau_f(1))$. Then the following value is the minimal patch length required for a type III equilibrium to exist:

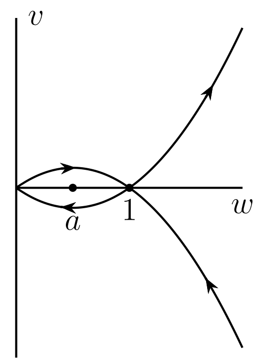


Figure 3.9: This figure shows the $T_{g,0}$ phase portrait with integral condition $\int_0^1 g(s)ds = 0$

$$L_3^* = \inf_{c_1 \in (-\tau_f(a_*), -\tau_f(1))} \int_{P(c_1,-\tau_g(1))}^{P(c_1,0)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$

Now, we can consider $K > 1$. Notice the stable manifold no longer forms the type III equilibrium in the phase plane. Thus all we can say is the type III equilibrium exists when L is in the range of $L_3(c_1)$ for $c_1 \in (-\tau_f(a_*), -\tau_f(1))$.

3.4 Equilibrium Solutions with Stationary Wave Condition $\int_0^1 g(s)ds = 0$

In this section, we will be consider the case where $\int_0^1 g(s)ds = 0$. This condition allows for a stationary wave solution outside the protection zone, i.e. a wave solution $w(x + tc)$ where $c = 0$.

3.4.1 Type I Equilibrium

We will again be considering the phase portraits. In particular, we will be paying special attention to the area around the origin. Fortunately for us, the general behavior does not change much in this case.

With the previous integral conditions the trajectory $T_{g,0}$ and the trajectory $T_{g,1}$ were different, but under our new assumption they are the same i.e. $\tau_g(0) = 0$ and $\tau_g(1) = -2\int_0^1 g(s)ds = 0$. Thus for this case, we will only need to consider the intersections of T_{f,c_1} and $T_{g,0}$. We will need to restrict c_1 such that $w_1(c_1) \in (0,1)$. Recall from Theorem 3.3.5, we had the conditions necessary for a type I solution if $c_1 \in (0, \tilde{c_1})$. We defined $\tilde{c_1}$ so that the condition will also hold when $\int_0^1 g(s)ds = 0$.

Lemma 3.4.1. *If* $0 < c_1 < \bar{c_1}$ *then* $w_1(c_1) \in (0,1)$.

Proof. Recall, $w_1(c_1)$ is strictly increasing, and therefore injective. Consider, $w_1(f\tau_f(1) = 1$ and $w_1(\tilde{c_1}) = K$. Thus, if $0 < c_1 < \bar{c_1}$, then $w_1(c_1) \in (0, K)$ if $K < 1$ or $w_1(c_1) \in (0,1)$ if $1 < K$.
Therefore, $w_1(c_1) \in (0,1)$. □

Theorem 3.4.2. *If* $0 < c_1 < \bar{c_1}$*, then* $T_{f,0}$ *intersects* $T_{g,0}$ *over the interval* $(0, w_1(c_1))$.

Proof. Suppose $0 < c_1 < \bar{c_1}$. We will consider $\Delta_{\tau,0}(w) = \tau_f(w) + c_1 - (\tau_g(w) + 0)$, and in particular look for a value $P(c_1, 0)$ such that $\Delta_{\tau,0}(P(c_1,0)) = 0$. First, notice $\Delta_\tau(0, c_1) = 0 + c_1 - 0 = c_1 > 0$ and now we will consider $\Delta_{\tau,0}(w_1(c_1), c_1) = 0 - \tau_g(w_1(c_1))$. By Lemma 3.4.1, $w_1(c_1) \in (0,1)$ so $\tau_g(w_1(c_1)) > 0$ and $\Delta_{\tau,0}(w_1(c_1), c_1) < 0$. Therefore by the intermediate value theorem, there exist an intersection $P(c_1, 0)$ between $T_{f,0}$ and $T_{g,0}$ over the interval $(0, w_1(c_1))$ □

Then we again have shown the existence of a type I equilibrium. We assume $P(c_1, 0) \in (0, w_1(c_1))$ is unique over the interval $(0,1)$. There are two cases to consider when analyzing the minimal length of the protection zone in this case, when $K \leq 1$ and $K > 1$. When $K \leq 1$, the stable manifold $T_{f,\tilde{c_1}}$ forms the type I equilibrium in the phase plane. In the retreating wave type I equilibrium, we argued $L_1(c_1)$ is well-defined, and through a similar argument $L_1(c_1)$ is well defined for $c_1 \in (0, \tilde{c_1})$. Then it remains to consider

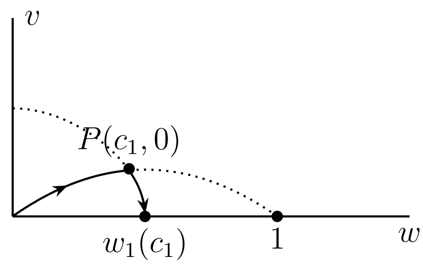


Figure 3.10: This figure shows quadrant I of a type I equilibrium phase portrait with integral condition $\int_0^1 g(s)ds = 0$. In particular, this figure shows the intersection between $T_{g,0}$ and T_{f,c_1}.

$L_1(\tilde{c_1})$, as $c_1 \to \tilde{c_1}$, $T_{f,c_1} \to T_{f,\tilde{c_1}}$ then $L_1(c_1) \to +\infty$. Thus when $K \leq 1$ the calculations of the minimal patch length are very similar to the retreating wave case, so the resulting formula for L_1^* is the same.

$$L_1(c_1) = 2\int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$
$$L_1^* = 2\inf_{0<c_1<\tilde{c_1}} \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$

It remains to consider when $K > 1$. Notice, in this case the stable manifold $T_{f,\tilde{c_1}}$ does not intersect $T_{g,0}$ over the interval $(0, 1)$. Then we can say a type I equilibrium exists when L is in the range of $L_1(c_1)$ for $c_1 \in (0, -\tau_f(1))$.

3.4.2 Type II Equilibrium

Again we must consider three cases: $K > 1$, $K < 1$, and $K = 1$. These cases end up being quite similar to the cases from the retreating wave section. This is to be expected as the general behavior around the critical value $(1, 0)$ is relatively similar in both cases.

$K > 1$

From the previous section, we have shown the necessary conditions for intersection and ultimately the existence of the type II equilibrium when $K > 1$. Recall Theorem 3.3.7, we established the conditions for an intersection between T_{f,c_1} and $T_{g,1}$ over the interval $(1, K)$. The only significant change is $-\tau_g(1) = 0$ in this case, but the proof from the previous section still holds. Thus an intersection between T_{f,c_1} and $T_{g,0}$ exists if $-\tau_f(1) < c_1 < \tilde{c_1}$. The general behavior of $T_{g,1}$ is the same over the interval $(1, \infty)$ regardless of integral conditions.

Similar to the retreating wave type II equilibrium when $K > 1$, as $c_1 \to \tilde{c_1}$, T_{f,c_1} approaches the stable manifold $T_{f,\tilde{c_1}}$ and thus $L(c_1) \to +\infty$. Therefore the infimum corresponds to the minimal patch size. Thus the following value allows for the existence of a type II equilibrium:

$$L_2^* = 2 \inf_{-\tau_f(1)<c_1<\tilde{c_1}} \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}.$$

$K < 1$

This assumption $K < 1$ is considerably different from the previous section. With our new integral condition, the speed of a traveling wave solution is 0, and this effects our type II equilibrium when $K < 1$. In particular, we only need to establish the intersection of T_{f,c_1} and $T_{g,0}$ over the interval $(K, 1)$ for the type II equilibrium to exist. We have already established $w_2(c_1) \in (K, 1)$ if $-\tau_f(1) < c_1 < \tilde{c_1}$ 3.3.9.

Theorem 3.4.3. *If $-\tau_f(1) < c_1 < \tilde{c_1}$ then a type II equilibrium exists.*

Proof. Suppose $-\tau_f(1) < c_1 < \tilde{c_1}$. We will consider the function $\Delta_{\tau,1}(w)$.

$$\begin{aligned}
\Delta_{\tau,1}(w_2(c_1)) &= \tau_f(w_2(c_1)) + c_1 - (\tau_g(w_2(c_1)) - \tau_g(1)) \\
&= 0 - \tau_g(w_2(c_1)) + \tau_g(1) < 0 \\
\Delta_{\tau,1}(1) &= \tau_f(1) + c_1 - (\tau_g(1) - \tau_g(1)) \\
&= \tau_f(1) + c_1 > 0
\end{aligned}$$

Then by the intermediate value theorem, there exists $P_2(c_1, 0) \in (K, 1)$ such that $\Delta_{\tau,1}(P_2(c_1, 0)) = 0$. Therefore an intersection between T_{f,c_1} and $T_{g,0}$ occurs at the w-value $P_2(c_1, 0)$. This forms the following circuit:

$$(1, 0) \to P_2(c_1, 0) \to (w_2(c_1), 0) \to P_2(c_1, 0) \to (1, 0)$$

$\square$

Through a similar argument to the $K > 1$ case, as $c_1 \to \tilde{c_1}$ then T_{f,c_1} approaches the stable manifold $T_{f,\tilde{c_1}}$, so $L_2(\tilde{c_1}) \to +\infty$. Then the minimal patch length to allow for the existence of the type II equilibrium when $a_* \le K < 1$ is the following value:

$$L_2^*(c_1) = \inf_{0 < c_1 < -\tau_f(1)} 2 \int_{w_2(c_1)}^{P_2(c_1, -\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$

.

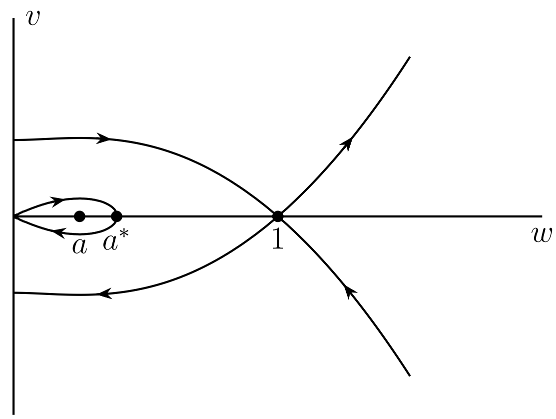


Figure 3.11: This figure shows the $T_{g,0}$ and $T_{g,1}$ phase portrait with integral condition $\int_0^1 g(s)ds > 0$

3.5 Equilibrium Solutions with Advancing Wave Condition $\int_0^1 g(s)ds > 0$

This section will focus on the model with the condition $\int_0^1 g(s)ds > 0$. This integral condition ensure the traveling wave will be advancing when it exists. This case is the basis of the paper [11]. This is a very complex case similar to $\int_0^1 g(s)ds < 0$ and will require very careful phase plane analysis. It is also quite different from the first two cases. So first, let us define a new constant $a < a^* < 1$ such that:

$$\int_0^{a^*} g(s)ds = 0$$

Lemma 3.5.1. $a < a^* < 1$ *exists such that* $\int_0^{a^*} g(s)ds = 0$.

Proof. Recall from the proof of Lemma 3.1.1, we showed there exists $a^* \in (a, 1)$ such that $\tau_g(a^*) = 0$. By definition, $\tau_g(a^*) = -2\int_0^{a^*} g(s)ds = 0$, therefore $\int_0^{a^*} g(s)ds = 0$. □

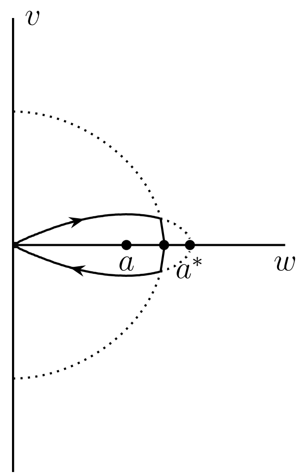


Figure 3.12: This shows the intersection of $T_{g,0}$ and T_{f,c_1} forming a type I equilibrium phase portrait with integral condition $\int_0^1 g(s)ds > 0$.

3.5.1 Type I Equilibrium

We now need to consider the conditions necessary for the existence of a type I equilibrium. Similar to the previous two cases, we will need to consider the necessary conditions for the existence of an intersection between $T_{g,0}$ and T_{f,c_1}. We showed $T_{g,0}$ is defined over the interval $(0, a^*)$ in Lemma 3.1.1.

We now need to restrict c_1 to ensure the existence of an intersection between T_{f,c_1} and $T_{g,0}$. Specifically an intersection that will ensure a type I solution. So We will need to restrict c_1 such that $w_1(c_1)$ is in the interval $(0, a^*)$. Then $0 < c_1 < -\tau_f(a^*)$ implies $w_1(c_1) \in (0, a^*)$. Much like we did in the previous case, we will add this new restriction to $\bar{c_1} = \min\{\tilde{c_1}, -\tau_f(a^*)\}$. Notice the definition of $\bar{c_1}$ is slightly different here as a^* replaces 1 as the largest defined value for $T_{g,0}$. This also implies for a type I equilibrium to exist, $w_1(c_1) \in (0, a^*)$.

Theorem 3.5.2. *If* $c_1 \in (0, \bar{c_1})$ *then* $T_{g,0}$ *intersect* T_{f,c_1} *over the interval* $(0, w_1(c_1))$.

Proof. Suppose $c_1 \in (0, \bar{c_1})$. Then $w_1(c_1) \in (0, a^*)$ by the definition of w_1. Then we can consider $\Delta_{\tau,0}(w, c_1)$.

$$\Delta_{\tau,0}(0, c_1) = \tau_f(0) + c_1 - \tau_g(0) = c_1 > 0$$

and

$$\Delta_{\tau,0}(w_1(c_1), c_1) = \tau_f(w_1(c_1)) + c_1 - (\tau_g(w_1(c_1))) = 0 - \tau_g(w_1(c)) < 0$$

Therefore by the intermediate value theorem, there exists $w \in (0, w_1(c_1))$ such that $\Delta_{\tau,0}(w) = 0$. This implies there exists an intersection between T_{f,c_1} and $T_{g,0}$. Therefore $T_{g,0}$ and T_{f,c_1} over the interval $(0, w_1(c_1))$. □

Then a type I equilibrium can exist when $c_1 \in (0, \bar{c_1})$. It remains to consider the length of the protection zone that ensures type I equilibrium. There are two cases to consider when considering the minimal length of the protection zone in this case, when $K \leq a^*$ and $K > a^*$. When $K \leq a^*$, the stable manifold $T_{f,\tilde{c_1}}$ forms a type I equilibrium in the phase plane. Thus when $K \leq a^*$ the calculations of the minimal patch length are very similar to the retreating wave case, so the resulting formula for L^* is the same.

$$L(c_1) = 2\int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$
$$L_1^* = 2\inf_{0<c_1<\tilde{c_1}} \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$

It remains to consider when $K > a^*$. Notice, in this case the stable manifold $T_{f,\tilde{c_1}}$ does not intersect $T_{g,0}$. Then the type I equilibrium exists when L is in the range of $L(c_1)$ for $c_1 \in (0, -\tau_f(a^*))$. In this case, there is also likely a minimal and maximal

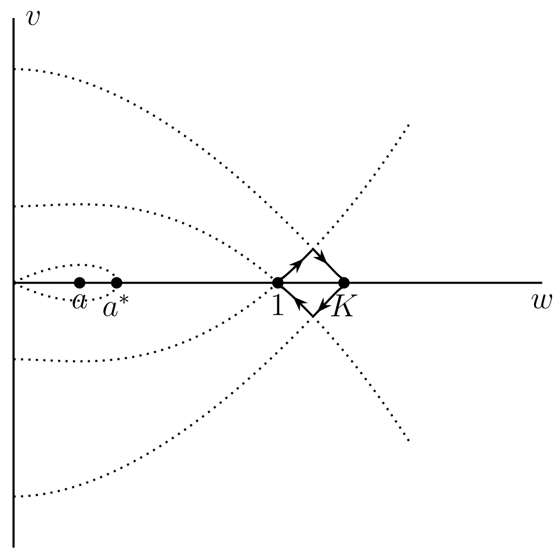


Figure 3.13: This figure shows the intersection of T_{f,c_1} and $T_{g,1}$ forming a type II equilibrium with $K > 1$ and $\int_0^1 g(s)ds > 0$.

patch length for a type I equilibrium to exist. This is the case considered by Du. et. al., and they established depending on the patch size and initial conditions there could be a trivial equilibrium, type I equilibrium, or a spreading solution [11].

3.5.2 Type II Equilibrium

We will consider the type II equilibrium with advancing wave conditions. This case will end up have very similar results to the stationary wave condition. Again, we will be considering three specific cases $K > 1$, $K < 1$, and $K = 1$.

$K > 1$

We begin this case by establishing $T_{g,1}$ exists over the interval $(1, \infty)$.

Lemma 3.5.3. *$T_{g,1}$ is defined over the interval* $(0, \infty)$

Proof. In order to show $T_{g,1}$ is defined, it is equivalent to show $\tau_g(w) - \tau_g(1) \geq 0$ for all $w \in (0, \infty)$. We first consider the critical values of $\tau_g(w)$: $0, a, 1$. In particular, $0, 1$ are local minimums of $\tau_g(w)$. Notice $\tau_g(0) = 0$ and $\tau_g(1) = -2\int_0^1 g(s)ds < 0$, thus $\tau_g(w) - \tau_g(1) \geq 0$ for all $w \in (0, 1)$. Finally, recall $\tau_g(w)$ is increasing over the interval $(1, \infty)$. thus $\tau_g(w) > 0$ for all $w \in (1, \infty)$. Therefore $T_{g,1}$ is defined over the interval $(0, \infty)$. $\square$

It is an immediate consequence of Lemma 3.5.3, that $T_{g,1}$ is also defined over the interval $(1, \infty)$ as we need.

Theorem 3.5.4. *A type II equilibrium exists when* $\int_0^1 g(s)ds > 0$ *and* $K > 1$.

Proof. Recall the proof Theorem 3.3.7, this proof establishes the intersections necessary for a type II equilibrium to exist when $K > 1$. $\square$

Similar to the retreating wave type II equilibrium when $K > 1$, as $c_1 \to \tilde{c_1}$, T_{f,c_1} approaches the stable manifold $T_{f,\tilde{c_1}}$ and thus $L(c_1) \to +\infty$. Therefore the infimum corresponds to the minimal patch size. Thus the following value allows for the existence of a type II equilibrium:

$$L_2^* = 2 \inf_{0<c_1<\tilde{c_1}} \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}.$$

$K < 1$

When $K < 1$, the type II equilibrium will exist very similarly to the stationary wave case when $K < 1$. Notice the key feature of both is the $T_{g,1}$ trajectory being defined over the entire interval $(0, 1]$.

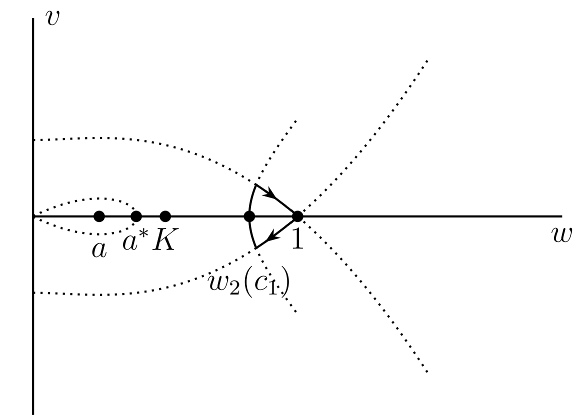


Figure 3.14: This figure shows the intersection of T_{f,c_1} and $T_{g,1}$ forming a type II equilibrium with $K < 1$ and $\int_0^1 g(s)ds > 0$.

Theorem 3.5.5. *A type II equilibrium exists when* $\int_0^1 g(s)ds > 0$ *and* $K < 1$.

Proof. Since we have established $T_{g,1}$ is defined over $(0, 1)$. The existence of a type II equilibrium in this case follows from Theorem 3.4.3. □

Through a similar argument to the $K > 1$ case, as $c_1 \to \tilde{c_1}$ then T_{f,c_1} approaches the stable manifold $T_{f,\tilde{c_1}}$, so $L_2(\tilde{c_1}) = \infty$. Then the minimal patch length to allow for the existence of the type II equilibrium when $a^* \leq K < 1$ is the following value:

$$L_2^*(c_1) = \inf_{-\tau_f(1)<c_1<-\tau_f(1)} 2\int_{w_2(c_1)}^{P_2(c_1,-\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w)+c_1}}$$

.

3.5.3 Type III Equilibrium

For this type III equilibrium, we need to carefully consider the necessary conditions for it to exist, and the conditions that ensure it can not exist. Intuitively, this equilibrium appears unlikely to exist, but we will investigate many possibilities in that regard. We will also need to consider two distinct cases $K > a^*$ and $K \leq a^*$.

$K > a^*$

We must first consider the necessary conditions for the existence of an intersection between T_{f,c_1} and $T_{g,1}$. We have established the existence of a Type I equilibrium when $\int_0^1 g(s)ds > 0$ and therefore established the existence of an intersection between T_{f,c_1} and $T_{g,0}$. Notice, since $\tau_f(w)$ is strictly decreasing over $(0, K)$, both of these intersections must occur between 0 and $w_{1(c_1)}$.

Lemma 3.5.6. *If* $c_1 \in (\tau_g(0) - \tau_g(1), \bar{c_1})$ *there exists an intersection between* T_{f,c_1} *and* $T_{g,1}$.

Proof. Suppose $c_1 \in (-\tau_g(1), \bar{c_1})$. Then $w_1(c_1) \in (0, K)$ by Corollary 3.3.2.1. Then we must consider $\Delta_{\tau,1}(w)$:

$$
\begin{aligned}
\Delta_{\tau,1}(0) &= \tau_f(0) + c_1 - (\tau_g(0) - \tau_g(1)) \\
&= c_1 + \tau_g(1) > 0, \\
\Delta_{\tau,1}(w_1(c_1)) &= \tau_f(w_1(c_1)) + c_1 - (\tau_g(w_1(c_1) - \tau_g(1)), \\
&= 0 - (\tau_g(w_1(c_1) - \tau_g(1)) < 0.
\end{aligned}
$$

Then by the intermediate value theorem, there exists an intersection between T_{f,c_1} and $T_{g,1}$. $\square$

Corollary 3.5.6.1. *There exists precisely one intersection between* T_{f,c_1} *and* $T_{g,1}$.

Proof. Recall our proof of the uniqueness of the intersection between T_{f,c_1} and $T_{g,0}$. The argument is dependent on the relationship between the slope of both curves. In particular, the proof is dependent on considering where either curve is increasing/decreasing, and $T_{g,0}$ and $T_{g,1}$ are increasing and decreasing at the same places. Therefore T_{f,c_1} and $T_{g,1}$ intersect at precisely one place. □

Theorem 3.5.7. *A type III equilibrium does not exist if* $\int_0^1 g(s)ds > 0$ *and* $K > a^*$.

Proof. We have established that if $c_1 \in (-\tau_g(1), \bar{c_1})$ then T_{f,c_1} intersects both $T_{g,0}$ and $T_{g,1}$ over the interval $(0, w_1(c_1))$. This creates the possibility of a path from $(0,0)$ to $(1,0)$ in the phase plane by connecting the trajectories that pass through those points respectively. To this point, it has been assumed the trajectories in the phase plane are moving in the direction necessary for the existence of the given solution type. In particular, all of the curves in quadrant I are moving left to right, i.e. from 0 toward positive infinity. Thus for a type III equilibrium to exist we must construct a continuous curve from $(0,0)$ to $(1,0)$ where the direction of this curve is always moving left to right. Now, we will consider the curves $T^+_{g,0}$ and $T^+_{g,1}$. Notice $T^+_{g,0} < T^+_{g,1}$ for all $w \in (0,1)$ and recall T^+_{f,c_1} is strictly decreasing over the interval $(0, w_1(c_1)$. Then the relationship between the intersection points $P(c_1, -\tau_g(1))$ and $P(c_1, 0)$ is $P(c_1, -\tau_g(1)) < P(c_1, 0)$. Then the trajectory T_{f,c_1} is moving from $P(c_1, -\tau_g(1)$ to $P(c_1, 0)$ and the trajectory $T_{g,0}$ is moving from $(0,0)$ to $P(c_1, 0)$. Therefore a type III equilibrium does not exist if $\int_0^1 g(s)ds > 0$ and $K > a^*$. □

$K \leq a^*$

In this case, we will need to establish the conditions necessary for an intersection between T_{f,c_1} and $T_{g,0}$ over the interval (K, a^*) and for an intersection between T_{f,c_1} and $T_{g,1}$ over the interval $(K, 1)$.

Lemma 3.5.8. *If* $c_1 \in (-\tau_f(a^*), \tilde{c_1})$*, there exists an intersection between* T_{f,c_1} *and* $T_{g,0}$ *over the interval* (K, a^*).

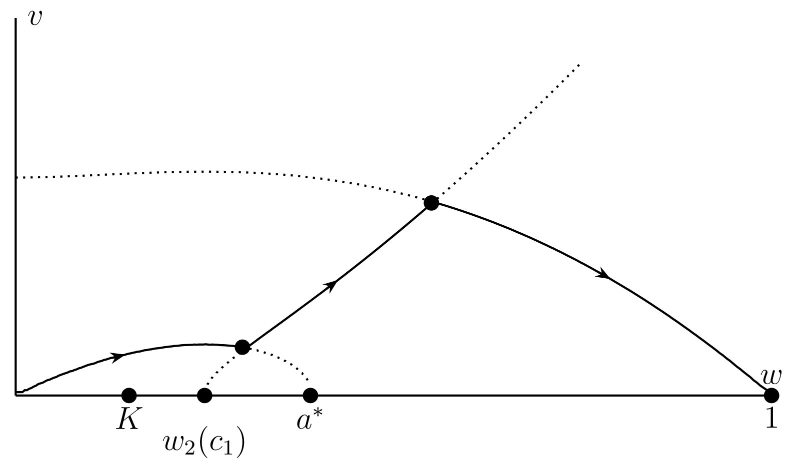


Figure 3.15: This figure shows the intersection of T_{f,c_1}, $T_{g,0}$, and $T_{g,1}$ forming a type III equilibrium with $K < a^*$ and $\int_0^1 g(s)ds > 0$.

Proof. Suppose $c_1 \in (-\tau_f(a^*), \tilde{c_1})$. Notice $w_2(c_1) \in (K, a^*)$ by definition of $w_2(c_1)$. Recall $\tau_f(w) + c_1$ is increasing over the interval (K, ∞). Then we can consider $\Delta_{\tau,0}(w)$.

$$\begin{aligned}
\Delta_{\tau,0}(w_2(c_1) &= \tau_f(w_2(c_1)) + c_1 - (\tau_g(w_2(c_1))) \\
&= 0 - (\tau_g(w_2(c_1))) < 0, \\
\Delta_{\tau,0}(a^*) &= \tau_f(a^*) + c_1 - (\tau_g(a^*)) \\
&= \tau_f(a^*) + c_1 > 0.
\end{aligned}$$

Therefore by the intermediate value theorem, there exists an intersection between T_{f,c_1} and $T_{g,0}$ over the interval (K, a^*). $\square$

Lemma 3.5.9. *If $c_1 \in (-\tau_f(a^*), \tilde{c_1})$, there exists an intersection between T_{f,c_1} and $T_{g,1}$ over the interval $(K, 1)$.*

Proof. Suppose $c_1 \in (-\tau_f(a^*), \tilde{c_1})$. We will consider $\Delta_{\tau,1}(w)$:

$$\begin{aligned}
\Delta_{\tau,1}(w_2(c_1)) &= \tau_f(w_2(c_1)) + c_1 - (\tau_g(w_2(c_1)) - \tau_g(1)) \\
&= 0 - (\tau_g(w_2(c_1)) - \tau_g(1)) < 0, \\
\Delta_{\tau,1}(1) &= \tau_f(1) + c_1 - (\tau_g(1) - \tau_g(1)) \\
&= \tau_f(1) + c_1 > 0.
\end{aligned}$$

Therefore by the intermediate value theorem, there exists an intersection between T_{f,c_1} and $T_{g,1}$ over the interval $(K, 1)$. $\square$

Theorem 3.5.10. *If* $c_1 \in (-\tau_f(a^*), \tilde{c_1})$*, there exists a type III equilibrium when* $\int_0^1 g(s)ds > 0$ *and* $K \leq a^*$*.*

Proof. Suppose $c_1 \in (-\tau_f(a^*), \tilde{c_1})$. By Lemma 3.5.8 and Lemma 3.5.9, we have established a continuous path from $(0,0)$ to $(1,0)$ in the phase plane. Therefore a type III equilibrium exists when $\int_0^1 g(s)ds > 0$ and $K \leq a^*$. $\square$

Then it remains to consider the minimal patch size necessary for this type III equilibrium to exist. First, notice the the stable manifold $T_{f,\tilde{c_1}}$ will intersect both $T_{g,0}$ and $T_{g,1}$ when $K \leq a^*$. Then we can see as $c_1 \to \tilde{c_1}$, the trajectory T_{f,c_1} approaches the trajectory $T_{f,\tilde{c_1}}$ and $L(c_1)$ is well defined for $c_1 \in (-\tau_f(a^*), \tilde{c_1})$. Thus through a similar argument as before, $L_3(c_1) \to +\infty$, and therefore the minimal patch size corresponds to the infimum of the range of $L_3(c_1)$.

$$L_3(c_1) = 2\int_{P_2(c_1,0)}^{P_2(c_1,-\tau_g(1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$

$$L_3^* = 2 \inf_{-\tau_f(a^*)<c_1<\tilde{c_1}} \int_{P_2(c_1,0)}^{P_2(c_1,-\tau_g(1)} \frac{dw}{\sqrt{\tau_f(w) + c_1}}$$

CHAPTER 4

CASE STUDY AND SIMULATIONS

This chapter will be focused on affirming the conclusions drawn in the previous chapter. We will be considering a protection zone model with specific growth functions. This allows for making calculations and using computers to simulate the model.

4.1 Specific Growth Functions

In order to build intuition about the general growth functions, we will also be considering some common specific growth functions that match the behavior of the general growth functions. Similar growth equations were considered in [11] and [20]. We will be applying the information gathered above to these specific models where calculations and estimations can be made. We will be making some assumptions that ensure somewhat easier calculations. In particular, we will be assuming the carrying capacity is the same inside and outside the protection zone, a similar assumption is made in [11]. We will also be using polynomial growth functions that ensure somewhat more intuitive behavior. The specific model is defined below:

$$
\begin{aligned}
u_t &= Du_{\chi\chi} + \begin{cases} \bar{f}(u) & \chi \in (-l, l) \\ \\ \bar{g}(u) & \chi \notin (-l, l) \end{cases}, \\
\bar{f}(u) &= ru(1 - \frac{u}{K}), \\
\bar{g}(u) &= ru(\frac{u}{A} - 1)(1 - \frac{u}{K}),
\end{aligned}
$$

with

$$u(-l^-) = u(-l^+),\ u(l^-) = u(l^+),\ u'(-l^-) = u'(-l^+),\ u'(l^-) = u'(l^+).$$

Similar to the general case, we will be scaling our variables. Suppose $w = \frac{u}{K}$, $\tau = rt$, $\sqrt{\frac{r}{D}}x = \chi$, and $A = aK$.

$$
\begin{aligned}
\frac{\frac{\partial w}{\partial t}}{r} &= D\frac{\frac{\partial^2 w}{\partial x \partial x}}{r} + \frac{rw(1-w)}{r} \\
\frac{\partial w}{\partial rt} &= D\frac{\partial^2 w}{\partial \sqrt{\frac{r}{D}}x \partial \sqrt{\frac{r}{D}}x} + w(1-w) \\
\frac{\partial w}{\partial \tau} &= \frac{\partial^2 w}{\partial \chi \partial \chi} + w(1-w),
\end{aligned}
$$

$$\begin{aligned}
\frac{\frac{\partial w}{\partial t}}{r} &= D\frac{\frac{\partial^2 w}{\partial x \partial x}}{r} + \frac{rw(\frac{wK}{A}-1)(1-w)}{r}\\
\frac{\partial w}{\partial rt} &= D\frac{\partial^2 w}{\partial\sqrt{\frac{r}{D}}x\partial\sqrt{\frac{r}{D}}x} + w(\frac{wK}{A}-1)(1-w)\\
\frac{\partial w}{\partial \tau} &= \frac{\partial^2 w}{\partial \chi^2} + w(\frac{w}{a}-1)(1-w).
\end{aligned}$$

After scaling our variables, we have simplified our system from three parameters to one.

$$w_t = w_{xx} + \begin{cases} f(w) & x \in (-l,l) \\ g(w) & x \notin (-l,l) \end{cases},$$

$$\begin{aligned}
f(w) &= w(1-w),\\
g(w) &= w(w-a)(1-w),
\end{aligned}$$

with

$$w(-l^-) = w(-l^+),\ w(l^-) = w(l^+),\ w'(-l^-) = w'(-l^+),\ w'(l^-) = w'(l^+).$$

Similar to the general case, this integral condition controls the behavior of traveling wave solutions. Much like the general growth functions case, we will be considering the three cases $\int_0^1 g(w)dw < 0$, $\int_0^1 g(w)dw = 0$, $\int_0^1 g(w)dw > 0$. However, these integral conditions are entirely dependent on the constant a, so the retreating wave case corresponds to $0.5 < a < 1$, the stationary wave corresponds to $a = 0.5$, and the advancing wave corresponds to $0 < a < 0.5$.

4.1.1 Trajectories

We will again be using the phase plane trajectories as the basis of our analysis of the system. In the previous chapter, we derived the formulas $\tau_f(w)$, $\tau_g(w)$, T_{f,c_1}, and T_{g,c_2}. These trajectory formulas involve the general growth functions from the previous chapter, $f(w)$ and $g(w)$, but we can replace them with our new growth functions $f(w) = w(1-w)$ and $g(w) = w(w-a)(1-w)$. Thus, in the case of the specific growth functions, we are given the following formulas:

$$T_{g,c_2} = \{(w,v) : v^2 = 2(\frac{w^4}{4} - \frac{(1+a)w^3}{3} + \frac{aw^2}{2}) + c_2\}, \tag{4.1}$$

$$T_{f,c_1} = \{(w,v) : v^2 = 2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1\}. \tag{4.2}$$

With the corresponding $\tau_f(w)$ and $\tau_g(w)$ as follows:

$$\tau_g(w) = 2(\frac{w^4}{4} - \frac{(1+a)w^3}{3} + \frac{aw^2}{2}), \tag{4.3}$$

$$\tau_g'(w) = 2(w^3 - (1+a)w^2 + aw),$$

$$\tau_f(w) = 2(-\frac{w^2}{2} + \frac{w^3}{3}), \tag{4.4}$$

$$\tau_f'(w) = 2(-w + w^2).$$

Again, we will be considering the T_{g,c_2} trajectories that pass through the points $(0,0)$ and $(1,0)$. In particular, the trajectories where $c_2 = 0$ and $c_2 = -\tau_g(1)$ respectively.

$$T_{g,0} = \{(w,v) : v^2 = 2(\frac{w^4}{4} - \frac{(1+a)w^3}{3} + \frac{aw^2}{2})\},$$
$$T_{g,1} = \{(w,v) : v^2 = 2(\frac{w^4}{4} - \frac{(1+a)w^3}{3} + \frac{aw^2}{2}) - 2(\frac{1}{4} - \frac{(1+a)}{3} + \frac{a}{2})\}.$$

We have now defined the trajectories in the phase plane. These will be used in a similar way to their general counterparts in the previous chapter.

4.2 Equilibrium Solutions with Retreating Wave Condition $0.5 < a < 1$

As before, we will begin by considering the retreating wave case i.e. $\int_0^1 g(s)ds < 0$. We now have a specific $g(s)$ function, so we can see this condition corresponds to $0.5 < a < 1$. We have established the conditions necessary for the three equilibrium types to exist in the previous chapter. In this chapter, we will calculate the specific constant values discussed previously.

4.2.1 Type I Equilibrium

Recall from the previous chapter, the existence of a type I equilibrium is dependent on the existence of an intersection between T_{f,c_1} and $T_{g,0}$ over the interval $(0, K)$ where in this case we assume $K = 1$. Previously we began this by find the w-intercept of T_{f,c_1} depending on c_1. We have established $\tau_f(w_1(c_1)) + c_1 = 0$ over the interval $(0, 1)$ when $c_1 \in (0, -\tau_f(1))$ Corollary 3.3.2.1. For concise notation, we will say $\bar{c_1} = -\tau_f(1)$. Notice, with the growth functions we chose, $-\tau_f(1) = -\tau_f(K)$ and $\tau_g(1) - \tau_f(1) \geq \tau_f(1)$

therefore $-\tau_f(1) = \min\{-\tau_f(K), -\tau_f(1), \tau_g(1) - \tau_f(a)\}$. It remains to find the formula for $w_1(c_1)$:

$$
\begin{aligned}
\tau_f(w) &= 0 \\
2(\frac{1}{3}w^3 - \frac{1}{2}w^2) + c_1 &= 0 \\
\frac{2}{3}w^3 - w^2 + c_1 &= 0 \\
2w^3 - 3w^2 + 3c_1 &= 0, \\
w &= x + \frac{1}{2}, \\
2x^3 + 3x^2 + \frac{3}{2}x + \frac{1}{4} - 3x^2 - 3x - \frac{3}{4} + 3c_1 &= 0 \\
x^3 - \frac{3}{4}x - \frac{1}{4} + \frac{3c_1}{2} &= 0,
\end{aligned}
$$

$$
\begin{aligned}
x_n &= \cos\left[\frac{1}{3}\arccos(1 - 6c_1) - \frac{2\pi n}{3}\right] \; t = 0, 1, 2 \\
w_n &= x_n + \frac{1}{2}.
\end{aligned}
$$

We now need to establish which w_n value corresponds to the w-intersect that occurs over $(0, 1)$. In order to ensure this intersect exists, we will need to consider the conditions on c_1. Notice, our w-intersect of $\sqrt{\tau_f(w) + c_1}$ occurs over $(0, 1)$ if and only if $\sqrt{\tau_f(0) + c_1} > 0$ and $\sqrt{\tau_f(1) + c_1} < 0$. If we analyze this function $\tau_f(w) + c_1$, notice in order for there to be a w-intersection over $(0, 1)$ then there are two other intersections.

In Corollary 3.3.2.1, we have proven that if $0 < c_1 < \bar{c_1}$ then the w-intersect of

$\tau_f(w) + c_1$ occurs over $(0, 1)$. This implies our w-intersect is the second largest w_t value. Therefore

$$w_1(c_1) = \cos\left[\frac{1}{3}\arccos(1 - 6c_1) - \frac{2\pi}{3}\right] + \frac{1}{2}$$

We have now established a formula for $w_1(c_1) \in (0, 1)$. We have already shown T_{f,c_1} and $T_{g,0}$ are defined over the interval $(0, w_1(c_1))$ and $(0, 1)$ by Theorem 3.3.4 and Lemma 3.1.1 respectively. Then we have shown the sufficient conditions for the required intersection to exist by Theorem 3.3.5. Similarly to before, we will consider the following function:

$$\begin{aligned}\Delta_{\tau,c_2}(w) &= \tau_f(w) + c_1 - (\tau_g(w) + c_2) \\ &= \frac{-w^4}{2} + \frac{4 + 2a}{3}w^3 - (1 + a)w^2 + c_1 - c_2. \qquad (4.5)\end{aligned}$$

Recall from the proof of Theorem 3.3.5, the intersection of T_{f,c_1} and $T_{g,0}$ occurs when $\Delta_{\tau,w}(w) = 0$. In the general case, we had to be careful to avoid multiple $w \in (0, 1)$ such that $\Delta_{\tau,0}(w) = 0$.

Corollary 4.2.0.1. *If $0 < c_1 < \bar{c}_1$, then there exists precisely one intersection between T_{f,c_1} and $T_{g,0}$ over $(0, 1)$.*

Proof. Suppose $0 < c_1 < \bar{c}_1$. Consider the derivative of $\Delta_{\tau,0}(w)$. $\Delta'_{\tau,0}(w) = -2w^3 + (4 + 2a)w^2 - 2(a + 1)w$. Notice $\Delta'_{\tau,0}(w)$ has critical values $0, 1, 1 + a$, thus $\Delta_{\tau,0}(w)$ is strictly decreasing between $(0, 1)$. Therefore, there exists precisely one w-value in $(0, 1)$ such that $\Delta_{\tau,0}(w) = 0$. Then there must exist precisely one intersection point between T_{f,c_1} and $T_{g,0}$. $\square$

We now have a combined phase plane. It remains to find some of the specific values on this phase plane. We have already determined the function $w_1(c_1)$. We will need to determine the formula for the intersection point between T_{f,c_1} and $T_{g,0}$ in the first and fourth quadrant of the phase plane. Recall, this intersection point will be dependant on the constant value c_1, we will refer to this intersection as the function value $P(c_1, 0)$. For future reference, we will be defining $P(c_1, c_2)$ as the general function to determine intersections between T_{f,c_1} and T_{g,c_2} over the interval $(0, K)$. Recall we defined this function originally in the previous chapter for the general growth functions.

$$P(c_1, c_2) = \min\{w : (w, \pm v) = T_{f,c_1} \cap T_{g,c_2}\}$$

Recall an equivalent way to investigate the intersection if the trajectories $T_{g,0}$ and T_{f,c_1} is by considering the w-values such that $\Delta_{\tau,0}(w) = 0$. In particular, the smallest w-value such that $\Delta_{\tau,0}(w) = 0$ corresponds to our $P(c_1, 0)$. Then the specific definition for a type I solution, $P(c_1, 0)$ is:

$$\begin{aligned} P(c_1, 0) &= \min\{w : (w, \pm v) = T_{f,c_1} \cap T_{g,0}\}, \\ &= \min\{w \in (0, \infty) : \Delta_{\tau,0}(w) = 0\}. \end{aligned}$$

Calculating $P(c_1, 0)$ is possible, but requires using the quartic formula, the degree four extension of the quadratic formula. However, this formula is very unwieldy and as such it does not impart much useful information. We can calculate $P(c_1, 0)$ with a computer fairly easily if needed. Now we can use the values calculated above to find the minimal protection zone that guarantees the existence of a type I equilibrium:

$$v = \sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3K}) + c_1}$$
$$\frac{dw}{dx} = \sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}$$
$$dx = \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}}$$
$$L_1(c_1) = 2\int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}}$$
$$L_1^* = 2 \inf_{c_1 \in (0,\frac{1}{3})} \{\int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}}\}.$$

This integral is not easily solved analytically if it can be solved analytically at all. We will instead focus on numerical approximation. Notice Figure 4.1, This figure shows the $L_1(c_1)$ for various $c_1 \in (0, \frac{1}{3})$ and results in $L^* \approx 1.2$ when $a = 0.75$. We use a left Riemann sum to approximate this value L^*, so our approximated L^* is likely somewhat less than the actual value. The approximation becomes increasingly less accurate as c_1 becomes close to 0. This is an issue as the the actual value L^* likely occurs very close to $c_1 = 0$, but this is to be expected. If we consider L_1^* as $\lim_{c_1 \to 0} l_1(c_1)$, it is in an indeterminate form. The limits of integration both approach 0 as the integrand approaches ∞.

Consider Figure 4.3 and Figure 4.2. These figures are representative of a protection zone models with protection zones of length 1 and 2 respectively. Both figures have the same initial conditions $w_0(x) = \begin{cases} 1, & x \in [40, 60] \\ 0, & x \notin [40, 60] \end{cases}$. The model with protection zone length $L = 1$, Figure 4.3, tends toward extinction while the model with protection zone length $L = 1.5$, Figure 4.2b, persists. They show the minimal protection zone

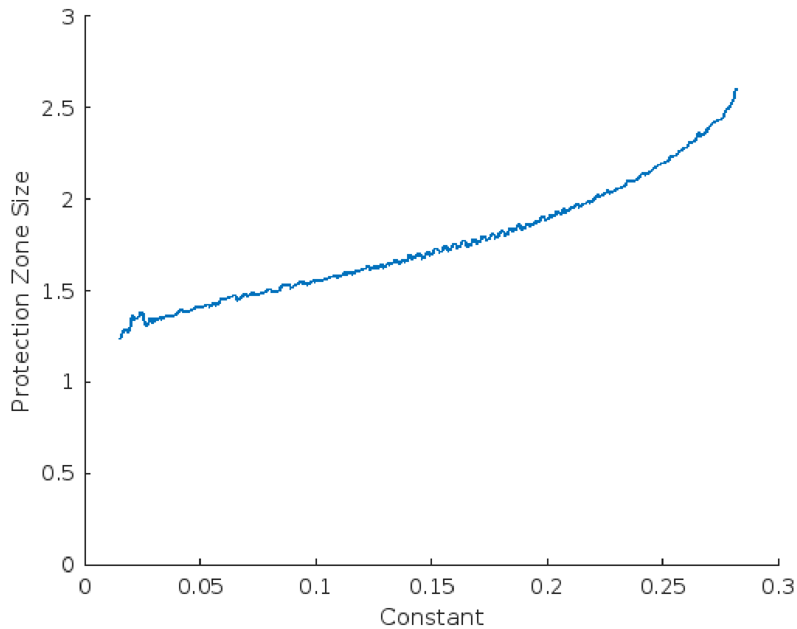


Figure 4.1: This figure is a graph of the function $L_1(c_1)$ when $a = 0.75$.

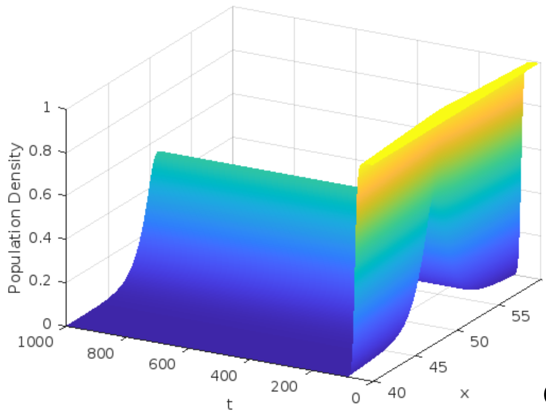
required to guarantee a type I equilibrium is between 1 and 1.5 which corresponds to our approximation, $L^* \approx 1.2$.

4.2.2 Type II Equilibrium

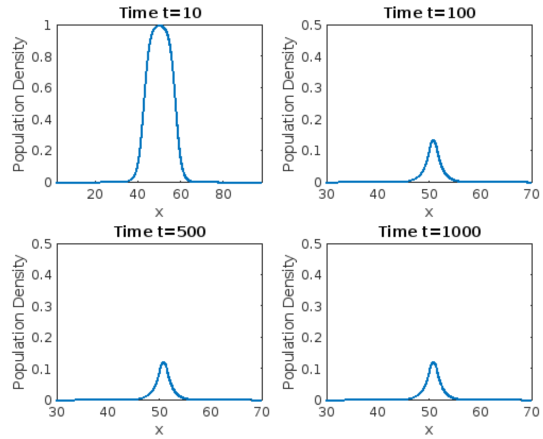
In the type II equilibrium, we established the existence of this solution is dependent on the relationship between K and 1. In particular, we showed in the general case $K \geq 1$ implies the existence of a type II equilibrium likely regardless of protection zone size. So the focus of this section will be the case $K < 1$. In this case, there is not a guarantee of a type II equilibrium. In order to consider this case, we need to redefine the growth function $f(w)$. Let us redefine $f(w)$ as follows:

$$f(w) = w(K - w)$$

where $K < 1$. We established the location of the second w-intersection of $\tau_f(w)$

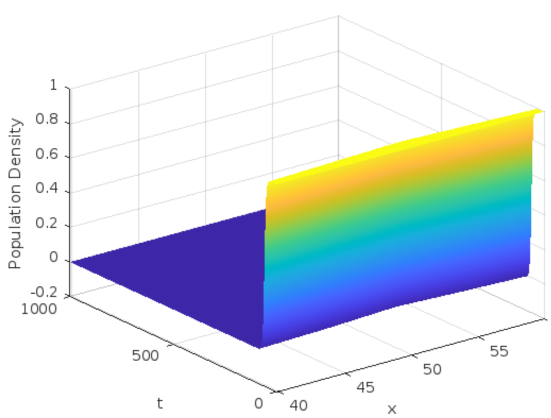


(a) Protection zone of length 2.

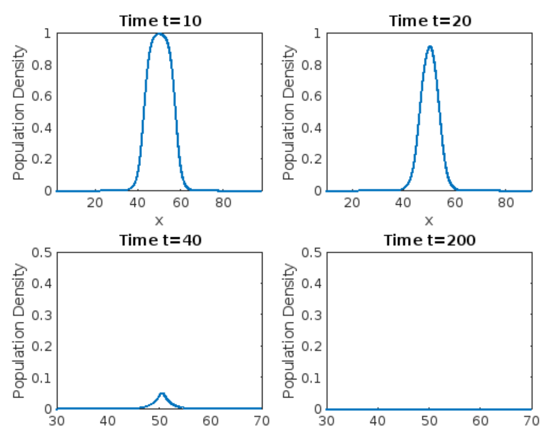


(b) This figure has time slices of a protection zone with type I equilibrium and protection zone of length 1.5.

Figure 4.2: Both figures are models of a protection zone large enough to support persistence with $a = 0.75$. Demonstrating the existence of a type I equilibrium when $L > L^*$.



(a) Protection zone of length 1.



(b) This figure has time slices of a protection zone model without type I equilibrium.

Figure 4.3: Both figures are models of a protection zone $(50, 51)$ with $a = 0.75$. Demonstrating the failure to persist when $L < L^*$.

by Lemma 3.3.10. Then we can say, if $-\tau_f(1) < c_1 < -\tau_f(a_*)$ then $w_2(c_1) \in (a_*, 1)$. This gives us sufficient conditions for the existence of a type two equilibrium. We have already proved the existence of a type II equilibrium Theorem 3.3.11. It remains to calculate two constants for the for this case. The first is $w_2(c_1)$. We will apply a similar methodology to finding $w_1(c_1)$:

$$\frac{2w^3}{3} - Kw^2 + c_1 = 0$$
$$2w^3 - 3Kw^2 + 3c_1 = 0$$
$$w = x + \frac{K}{2}$$
$$x^3 - \frac{3K^2}{4}x + \frac{K^3}{4} - \frac{3K^3}{8} + \frac{3c_1}{2} = 0$$
$$w_n = K\cos(\frac{1}{3}\arccos(1 - \frac{6c_1}{K^3}) - \frac{2\pi n}{3}) + \frac{K}{2}\ n = 0, 1, 2$$

It remains to determine which of the three options is the intersection over the interval (K, ∞). Recall $\tau_f(w)$ has at most three real roots, and we are looking for the largest of the three. Thus, we can say:

$$w_2(c_1) = K\cos(\frac{1}{3}\arccos(1 - \frac{6c_1}{K^3})) + \frac{K}{2}$$

Then we can calculate the final constant $P_2(c_1, -\tau_f(1))$. Again, an explicit formula depends on solving a quartic equation and will result in a not so useful equation. We can consider a formula for P_2 in terms of set intersection between T_{f,c_1} and $T_{g,1}$:

$$P_2(c_1, -\tau_f(1)) = \min\{w \in (w,v) : (w,v) = (T_{f,c_1} \cap T_{g,1}) \cap (K,\infty) \times (0,\infty)\}.$$

We can now use these values to calculate the maximal protection zone for a type II equilibrium to exist. This maximal protection zone occurs in the range of $L_2(c_1)$, but we can not say for sure which value specifically as established in the general case. We will refer to this value as L_2*.

$$\begin{aligned}
v &= \sqrt{\tau_f(w) + c_1} \\
\int_0^l dx &= \int_{w_2(c_1)}^{P_2(c_1,-\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
l_2(c_1) &= \int_{w_2(c_1)}^{P_2(c_1,-\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}} \\
L_2(c_1) &= 2\int_{w_2(c_1)}^{P_2(c_1,-\tau_g(1))} \frac{dw}{\sqrt{\tau_f(w) + c_1}}.
\end{aligned}$$

Again this integral is not easily calculated and will require computer approximation. In Figure 4.7, this shows time slices of a type II equilibrium with $a = 0.75$, initial conditions $w_0(x) = 0.8$, and the protection zone is length $L = 1$. Figure 4.4, shows time slices of a model with $a = 0.75$, protection zone $L = 1$, and initial conditions $w_0(w) = 0.8$. This demonstrates a failed type II equilibrium, where the carrying capacity over the protection zone, $K = 0.25$, is too small and creates a spreading solution. In this case the spreading solution refers to the waves retreating in both directions from the protection zone toward $\pm\infty$. Figure 4.5, shows a type II equilibrium where $K = 0.25$ with $w_0(w) = 0.8$. This happens because $L = 0.1$ which is sufficiently small to allow

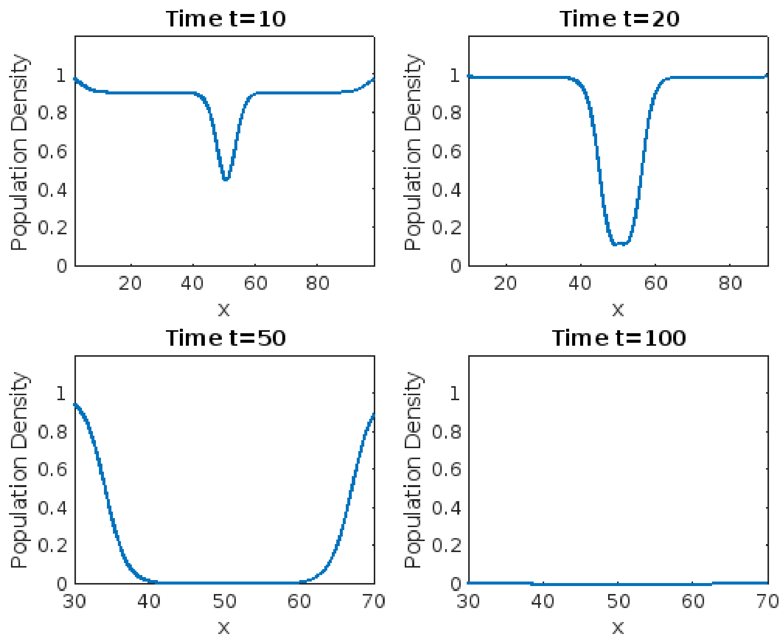


Figure 4.4: This figure shows a protection zone with carrying capacity $K = 0.25$ and length 1 which tends toward extinction.

the existence of the type II equilibrium. These figures show the existence of a type II equilibrium is dependent on the size of the protection zone in relation to the carrying capacity over the protection zone. Another interesting thing to consider, in the case Figure 4.4, the species does not persist. This occurs because the protection zone is smaller than the minimal type I protection zone L_1^*. In Figure 4.6, the protection zone is length $L = 5$ which is large enough to support the existence of a type I equilibrium. This is an interesting outcome, as this shows transition from type II and type I equilibrium depending on the size of the protection zone.

4.2.3 Type III Equilibrium

Recall our definition of a type III equilibrium from a previous chapter. The existence of a type III equilibrium was shown in Theorem 3.3.14. We have already considered the constant $P(c_1, 0)$, so it remains to calculate the constant value $P(c_1, -\tau_g(1))$.

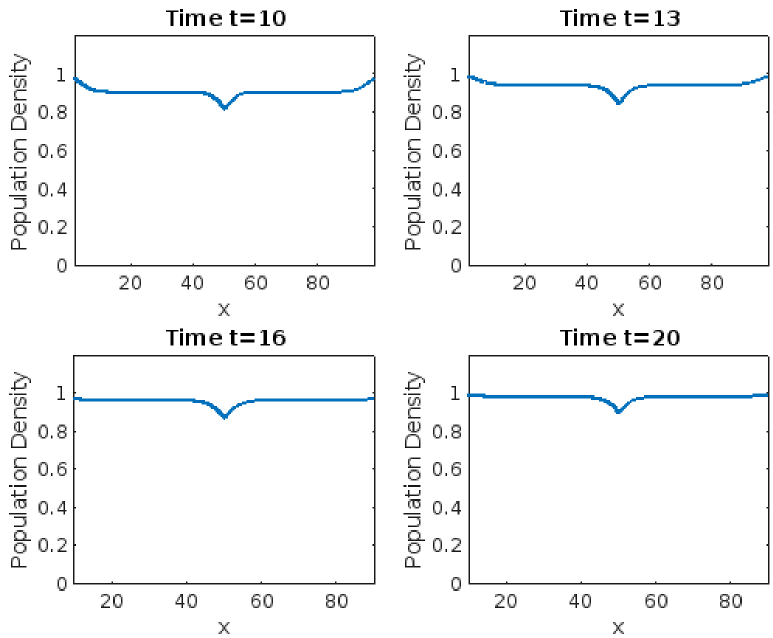


Figure 4.5: This figure shows protection zone with carrying capacity $K = 0.25$ length 0.1 which persists.

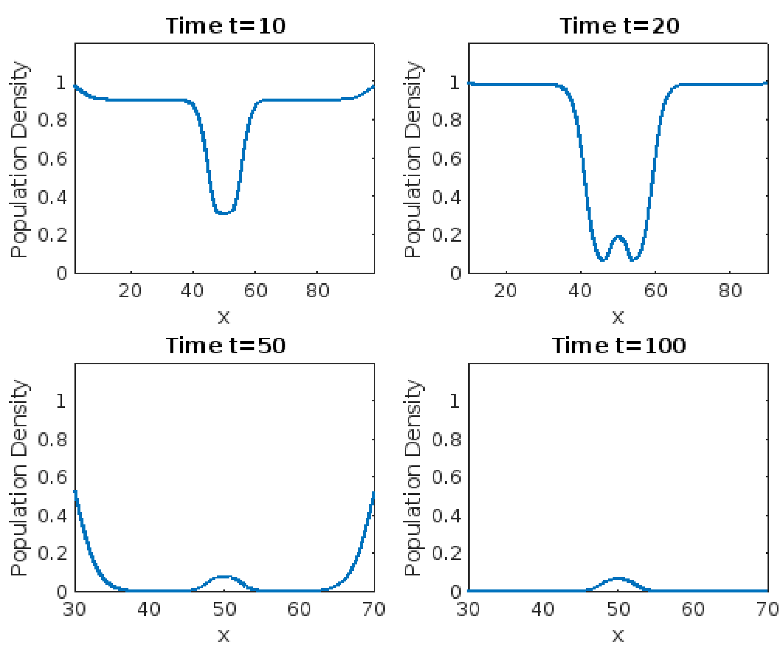


Figure 4.6: This figure shows a transition from type II equilibrium to type I. The carrying capacity is $K = 0.25$ with protection zone too large to support a type II equilibrium, but also large enough to support a type I.

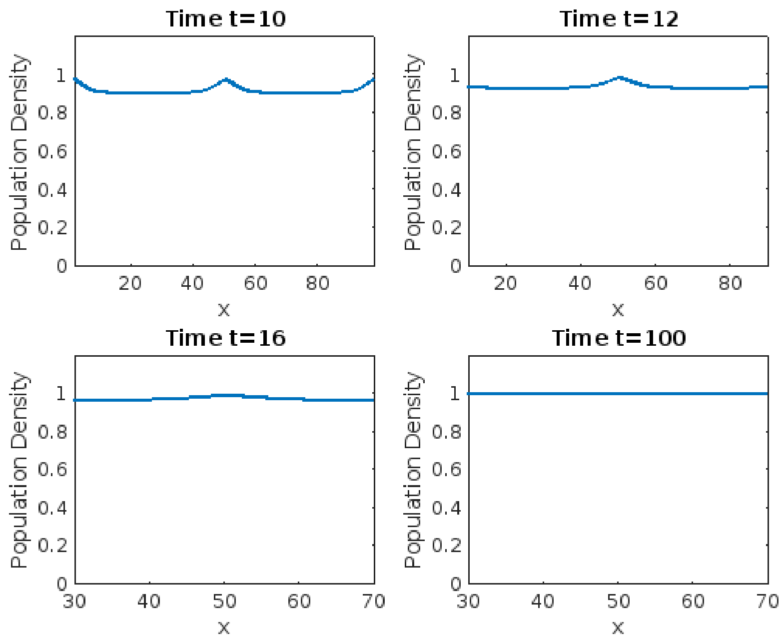


Figure 4.7: This figure shows a protection zone model with a type II equilibrium. The carrying capacity over the protection zone is $K = 1$.

$$P(c_1, -\tau_g(1)) = \min\{w \in (w, v) | (w, v) = (T_{f,c_1} \cap T_g, -\tau_g(1)) \cap ((0, K) \times (0, \infty))\}$$

Then from our previous chapter, we have already calculated the necessary values, so we can consider the minimal protection zone to allow for the existence of a type III equilibrium.

$$L_3(c_1) = \int_{P(c_1,0)}^{P(c_1,-\tau_g(1))} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}}$$

$$L_3^* = \inf_{c_1 \in (0,\frac{1}{3})} \{ \int_{P(c_1,-\tau_g(1))}^{P(c_1,0)} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}} \}$$

Figure 4.8, shows an approximation of L_3^* where $L_3^* \approx 1$ when $a = 0.75$. Figure 4.10 and Figure 4.9 both assume $a = 0.75$ and $w_0(x) = \begin{cases} 1, & x > 50 \\ 0, & x \leq 0 \end{cases}$. Figure 4.9 shows a type III equilibrium with protection zone $L = 2$. In this figure the wave is stopped from retreating by a sufficiently large protection zone. Thus allowing for persistence of the species over half of its domain. Figure 4.10 shows time slices of a model with a protection zone length $L = 0.8$. Notice this is smaller than L_*, and therefore does not stop the wave from retreating. Then the species tends toward extinction.

4.3 Equilibrium Solutions with Stationary Wave Condition $a = 0.5$

We can now consider the stationary wave case. This refers to the $\int_0^1 g(s)ds = 0$, which with our specific growth functions implies $a = 0.5$. We will start by inspecting the value $-\tau_g(1)$ to build some intuition about $T_{g,1}$.

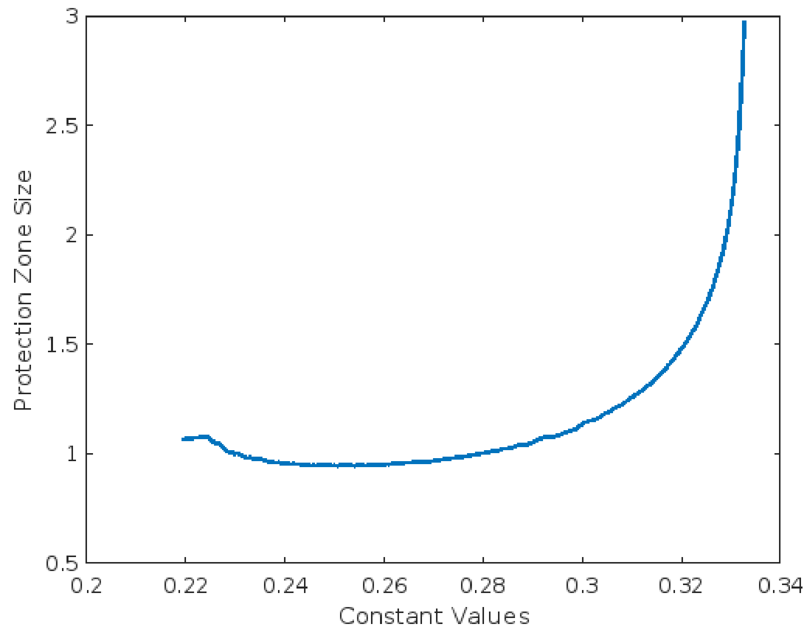


Figure 4.8: This figure shows a graph of approximations of $L_3(c_1)$

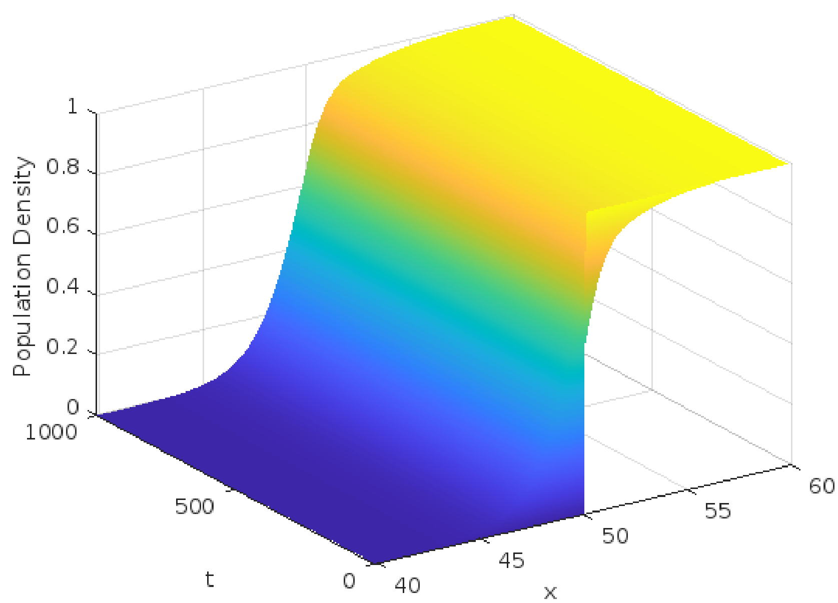


Figure 4.9: This figure shows a protection zone model with Type III equilibrium.

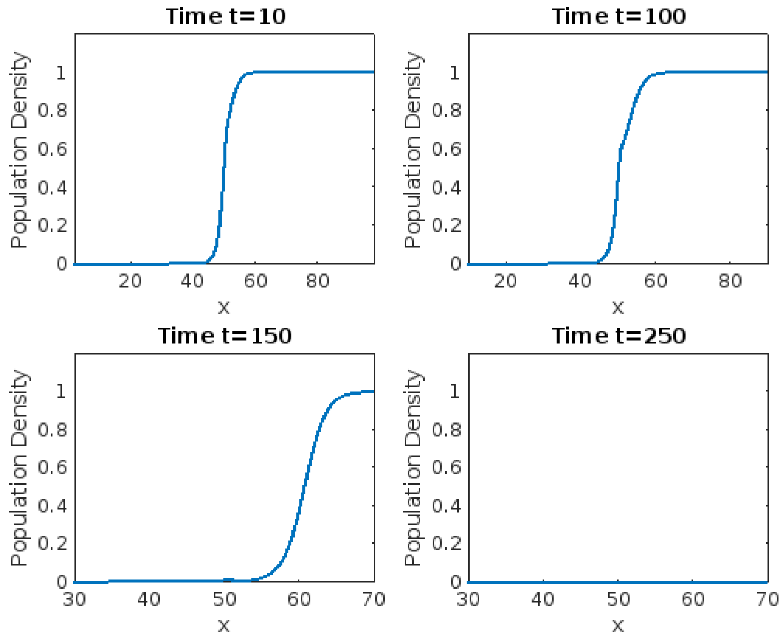


Figure 4.10: This figure shows a protection zone model that can not support a type III equilibrium.

$$\begin{aligned}\tau_g(1) &= 2(\frac{1^4}{4} - \frac{1.5}{3}1^3 + \frac{0.5}{2}1^2) \\ &= 2(\frac{1}{4} - \frac{1}{2} + \frac{1}{4}) \\ &= 0\end{aligned}$$

Thus the trajectory $T_{g,0}$ is equivalent to the trajectory $T_{g,1}$. Therefore in the stationary wave case, there are only two distinct trajectories to consider T_{f,c_1} and $T_{g,0}$.

4.3.1 Type I Equilibrium

The Type I equilibrium is very similar to the Type I equilibrium in the retreating wave case, as established in the general growth function section. We have established the existence of the intersection between T_{f,c_1} and $T_{g,0}$ in the retreating wave section.

We will explore the conditions on and existence of $P(c_1, 0)$.

In the retreating wave case, we spent some time establishing the existence of $w_1(c_1)$. Fortunately, this value is not dependent on a, so our proof of its existence previously will suffice for this section. We also established in the general case the conditions on c_1 for the intersection $P(c_1, 0)$ to exist. Similar to the retreating wave case, the intersection $P(c_1, 0)$ is dependent on c_1 being restricted to the interval $(0, \frac{1}{3})$ where $-\tau_f(1) = \frac{1}{3}$.

Then we must calculate the minimum patch size.

$$
\begin{aligned}
v &= \sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3K}) + c_1} \\
\frac{dw}{dx} &= \sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1} \\
dx &= \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}} \\
l_1(c_1) &= \int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}} \\
L_1(c_1) &= 2\int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}} \\
L_1^* &= 2 \inf_{c_1 \in (0, \frac{1}{3})} \{\int_{P(c_1,0)}^{w_1(c_1)} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}}\}
\end{aligned}
$$

We must again calculate this L_1^* numerically. From the graph of $L_1(c_1)$ Figure 4.11, we can approximate $L_1^* \approx 1$. As with L_1^* in the retreating wave case, the approximation becomes poor as $c_1 \to 0$. Now Figure 4.13 and Figure 4.12 both show models with

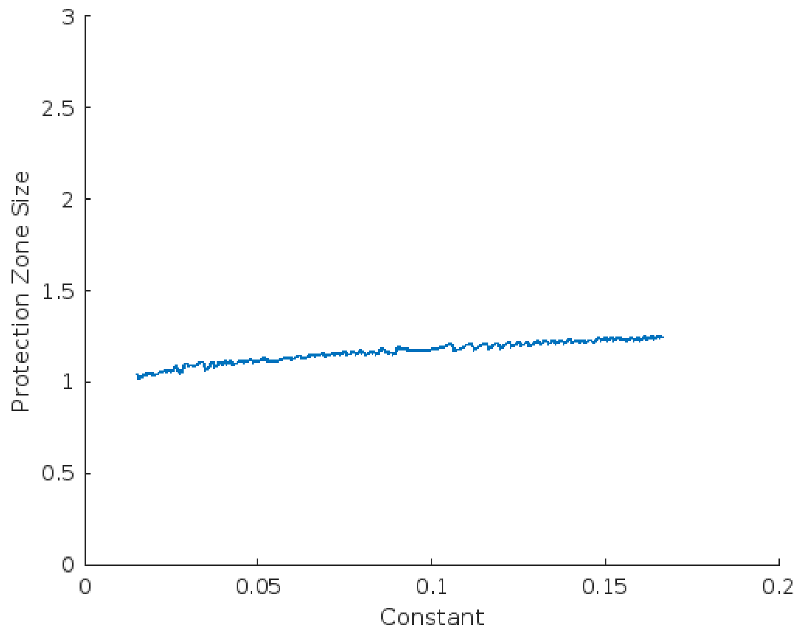


Figure 4.11: This figure shows the graph of $L_1(c_1)$ with $c_1 \in (0.01, 0.17)$ when $a = 0.5$.

$a = 0.5$ and $w_0(x) = \begin{cases} 1, & x \in [40, 60] \\ 0, & x \notin [40, 60] \end{cases}$. Figure 4.13 is a model with $L = 0.8$, so the species tends toward extinction which matches our intuition as $L < L^*$. Figure 4.12 has $L = 1.2$, and the model reaches a type I equilibrium. Our approximation of L_1^* may not be precise, but Figure 4.13 and Figure 4.12 indicate $0.8 < L_1^* < 1.2$.

4.3.2 Type II Equilibrium

In the general case, we established the existence of a type II equilibrium is guaranteed when $\int_0^1 g(s)ds = 0$. Considering the specific case is not necessary as the existence is not dependent on the the size of the protection zone. Notice both Figure 4.15 and Figure 4.14 demonstrate a type two equilibrium. Figure 4.15 shows time slices of a type II equilibrium with $a = 0.5$, protection zone $L = 1$, and initial conditions $w_0(x) = \begin{cases} 0.55, & x \in [0, 100] \\ 1, & x \notin [0, 100] \end{cases}$. In the general case, we established the existence of

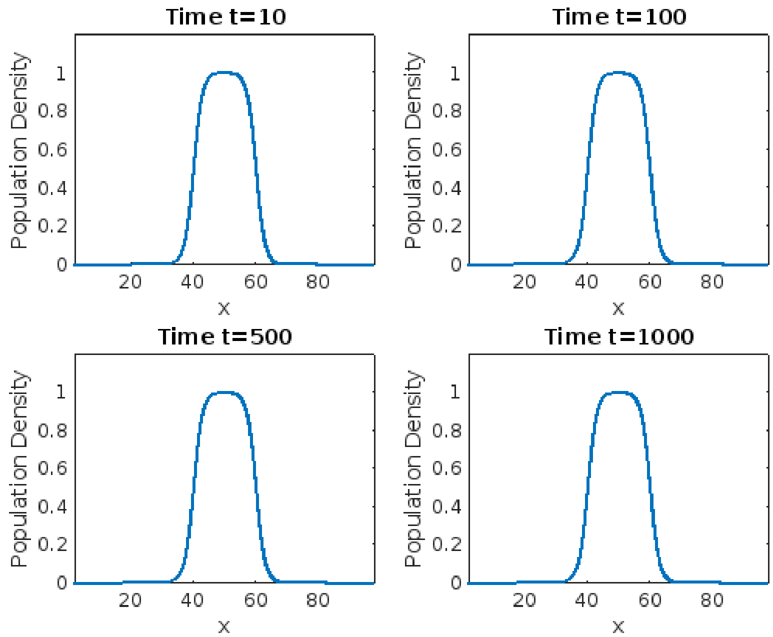


Figure 4.12: This figure shows time slices of a model with protection zone of length 1.5.

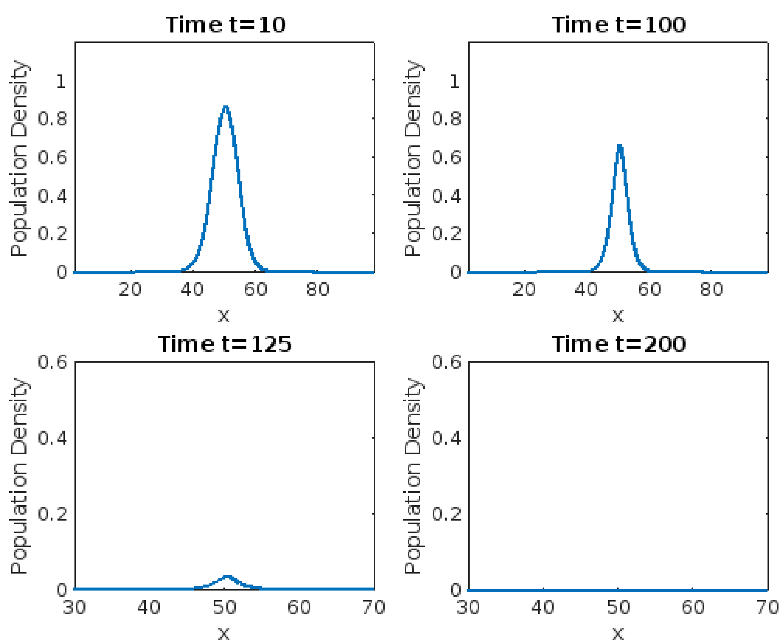


Figure 4.13: This figure shows time slices of a model with protection zone of length 0.8.

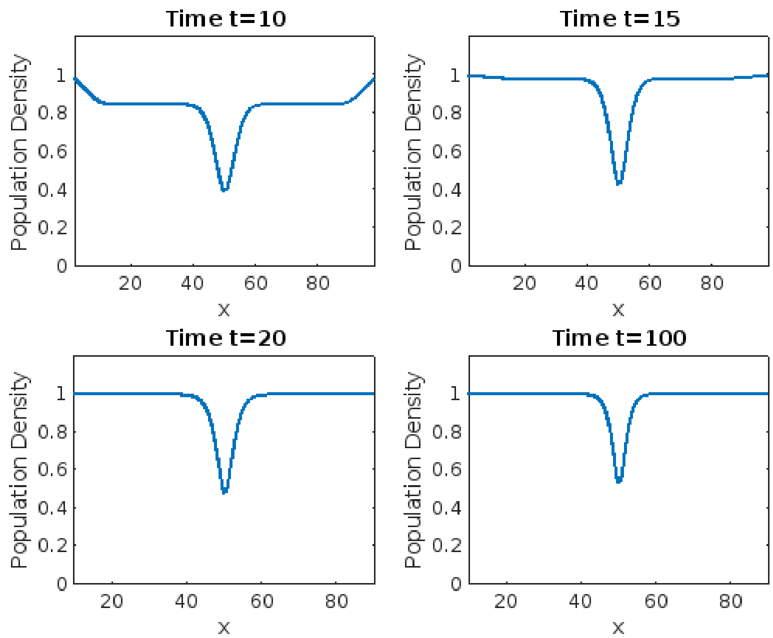


Figure 4.14: This figure shows time slices of a type II equilibrium with protection zone length 2 and carrying capacity $K = 0.25$.

type II equilibrium do not depend on the carrying capacity over the patch in the stationary wave case. Figure 4.14 has the same assumptions as Figure 4.15, but the carrying capacity over the protection zone is $K = 0.25$. Both Figure 4.15 and Figure 4.14 reach equilibrium which matches our results from phase plane analysis of a general protection zone model.

4.4 Equilibrium Solutions with Advancing Wave Condition $0 < a < 0.5$

We will consider the integral condition $\int_0^1 g(s)ds > 0$. This condition correspond to the existence of an 'advancing' wave solution. This is the most 'hospitable' exterior environment we will consider. Recall from the definition of the problem, the advancing wave occurs when $0 < a < 0.5$. In the general case, we established the need for a constant value $a^* \in (a, 1)$ such that $\int_0^{a^*} g(s)ds = 0$.

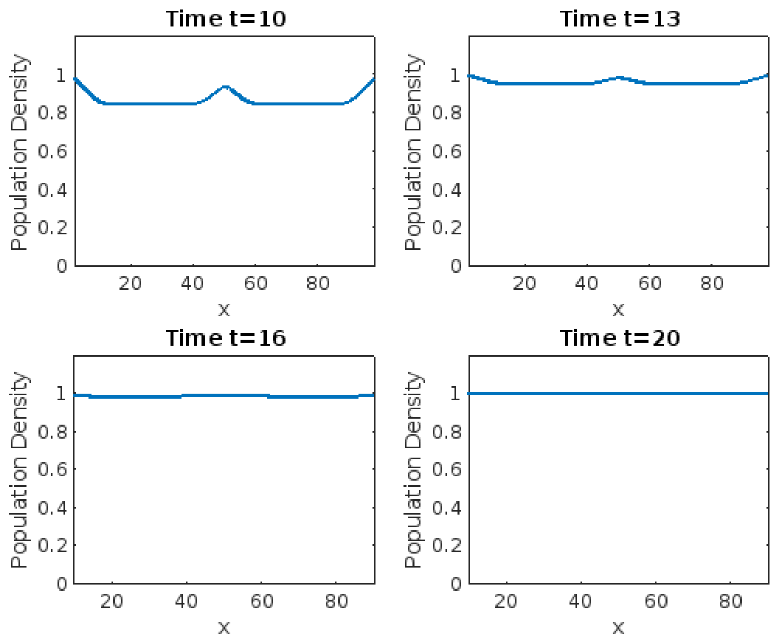


Figure 4.15: This figure shows time slices of a type II equilibrium with protection zone length 1.5

$$\int_0^{a^*} g(s)ds = 0$$

$$-\frac{(a^*)^4}{4} + \frac{(1+a)(a^*)^3}{3} - \frac{a(a^*)^2}{2} = 0$$

$$(a^*)^2(-\frac{(a^*)^2}{4} + \frac{(1+a)(a^*)}{3} - \frac{a}{2}) = 0$$

Then we can see $a^* = 0$ or $a^* = \frac{2}{3}((1+a) \pm \sqrt{\frac{2a^2-5a+2}{2}})$. However, we are looking for a value in the interval $(a,1)$, so $a^* \neq 0$. Then we need to determine which $\frac{2}{3}((1+a) - \sqrt{\frac{2a^2-5a+2}{2}})$ or $\frac{2}{3}((1+a) + \sqrt{\frac{2a^2-5a+2}{2}})$ occurs over $(a,1)$.

$$
\begin{aligned}
a &< 0.5 \\
a &< \frac{6}{12} \\
\frac{-3}{2}a &> \frac{-3}{4} \\
a^2 - \frac{5}{2}a + 1 &> a^2 - a + \frac{1}{4} \\
\sqrt{a^2 - \frac{5}{2}a + 1} &> \frac{1}{2} - a \\
1 + a + \sqrt{a^2 - \frac{5}{2}a + 1} &> \frac{3}{2} \\
\frac{2}{3}(1 + a + \sqrt{a^2 - \frac{5}{2}a + 1}) &> 1
\end{aligned}
$$

Then we can say for certain $a^* \neq \frac{2}{3}(1 + a + \sqrt{a^2 - \frac{5}{2}a + 1})$. It remains to show $\frac{2}{3}((1 + a) - \sqrt{\frac{2a^2-5a+2}{2}}) \in (a, 1)$. In the mean time, we will refer to $\frac{2}{3}((1 + a) - \sqrt{\frac{2a^2-5a+2}{2}})$ as a^*. Notice, $a^* < \frac{2(1+a)}{3} < 1$ so we can use $\frac{2(1+a)}{3}$ as the upper bound of a^* instead of 1.

$$
\begin{aligned}
a &< 2 \\
1 &> \frac{1}{2}a \\
1 + a &> \frac{3}{2}a \\
\frac{2(1 + a)}{3} &> a
\end{aligned}
$$

Then it remains to show $a^* > a$.

$$
\begin{aligned}
0 &< a < 2 \\
\frac{3}{4}a^2 - \frac{3}{2}a &< 0 \\
a^2 - \frac{5}{2}a + 1 &< 1 - a + \frac{1}{4}a^2 \\
\sqrt{a^2 - \frac{5}{2}a + 1} &< (1 - \frac{1}{2}a) \\
-\sqrt{a^2 - \frac{5}{2}a + 1} &> \frac{1}{2}a - 1 \\
1 + a - \sqrt{a^2 - \frac{5}{2}a + 1} &> \frac{3}{2}a \\
\frac{3}{2}(1 + a - \sqrt{a^2 - \frac{5}{2}a + 1}) &> a
\end{aligned}
$$

Thus, we can say $a < a^* < 1$. In the general case, we established that $T_g, 0$ is defined over $(0, a^*)$ by 3.1.1.

It now remains to establish the existence of $T_{g,1}$ and the interval over which it exists. Remember, this trajectory in the phase plane that contains the critical point $(1, 0)$. So we need to calculate the c_2 value that ensure the inclusion of $(1, 0)$ in $T_{g,1}$. In the general case, we established $c_2 = -\tau_g(1)$ ensure the inclusion of $(1, 0)$.

$$
\begin{aligned}
-\tau_g(1) &= -2(\frac{1}{4} - \frac{(1+a)}{3} + \frac{a}{2}) \\
&= \frac{1}{6} - \frac{a}{3}
\end{aligned}
$$

In the general case, we determined $T_{g,1}$ is defined over the interval $(0, \infty)$ by 3.5.3. It remains to consider the specific requirements for the various equilibrium types.

4.4.1 Type I Equilibrium

In the previous section, we have established all of the necessary conditions on the trajectories for a type I equilibrium to exist. It remains to consider the appropriate intersections and the necessary conditions for their existence.

First, we need to restrict $w_1(c_1)$ to the interval $(0, a^*)$. Recall w_1 is strictly increasing and continuous over $(0, 1)$, so we just need to find the c_1 value such that $w_1(c_1) = a^*$.

$$\begin{aligned} w_1(c_1) &= a^* \\ \tau_f^{-1}(-c_1) &= a^* \\ -c_1 &= \tau_f(a^*) \\ c_1 &= -\tau_f(a^*) \end{aligned}$$

In the general case, we established if $c_1 \in (0, -\tau_f(a^*))$ then a type I equilibrium exists Theorem 3.5.2. We can then calculate the minimal protection zone that guarantees the existence of a type I equilibrium:

$$L_1^* = 2 \inf_{c_1 \in (0, -\tau_f(a^*))} \{ \int_{P(c_1,0)}^{w_{f,1}(c_1)} \frac{dw}{\sqrt{2(-\frac{w^2}{2} + \frac{w^3}{3}) + c_1}} \}.$$

We can approximate L_1^* in a similar manner to before, which results in $L_1^* \approx 1$. Notice in Figure 4.17, the population reaches equilibrium with a protection zone of length $L = 1.2$. In this advancing wave case, there are two ways a model can fail to have a type I equilibrium. The first results in extinction, this is a result of the protection zone being too small to support a population. This occurs in Figure 4.19, the protection zone is only $L = 0.8 < L_1^*$. The second results in persistence, the protection zone is large enough to support life, but the population becomes large enough that it begins to spread through the environment. We will refer to this as a spreading solution. This occurs in Figure 4.18, where the protection zone is $L = 1.2$. This is the same protection zone size as the Type I equilibrium, but the initial conditions are different. In the type I equilibrium, the initial conditions are $w_0 = \begin{cases} 0.3, & x \in (45,55) \\ 0, & x \notin (45,55) \end{cases}$. In the spreading solution, the initial conditions are $w_0 = \begin{cases} 0.5, & x \in (45,55) \\ 0, & x \notin (45,55) \end{cases}$. This result shows the positive solution types are sensitive to the initial conditions. It is an important distinction to show the model can reach equilibrium without becoming a spreading solution.

4.4.2 Type II Equilibrium

The phase plane analysis of a general protection zone model with $\int_0^1 g(s)ds > 0$, established the existence of a type II equilibrium is guaranteed. In particular, it is not dependent on the size of the protection zone. The persistence of this type equilibrium is dependent on the initial conditions. Figure 4.20 shows the occurrence of a type II equilibrium when $a = 0.45$ and $K = 1$ with initial conditions $w_0(x) = \begin{cases} 0.5, & x \in [0,100] \\ 1, & x \notin [0,100] \end{cases}$. The protection zone in this model is length $L = 1.25$, but as we can see the protection does not have an effect on the long term behavior of the model. In the phase plane analysis, we also considered what would happen if $K < 1$, and established the model

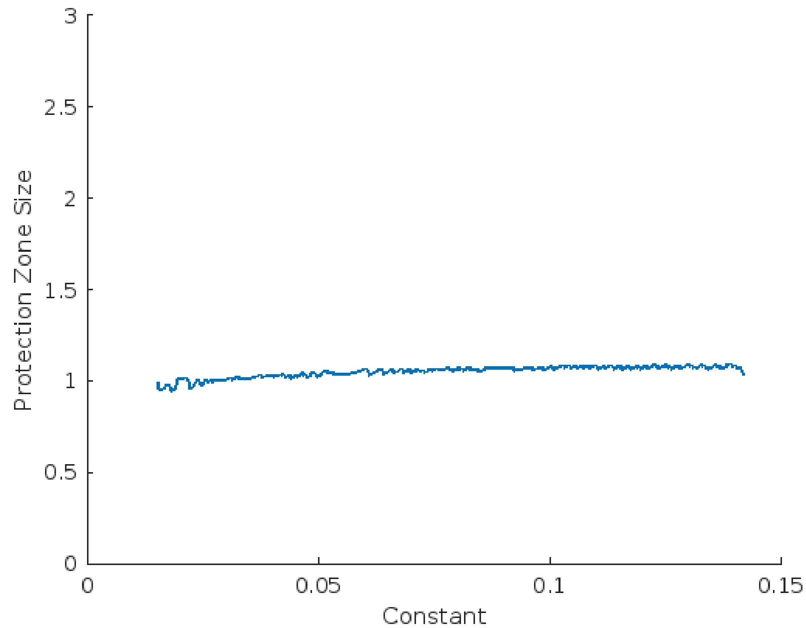


Figure 4.16: This figure shows the graph of $L_1(c_1)$ with $c_1 \in (0, 0.15)$ when $a = 0.45$.

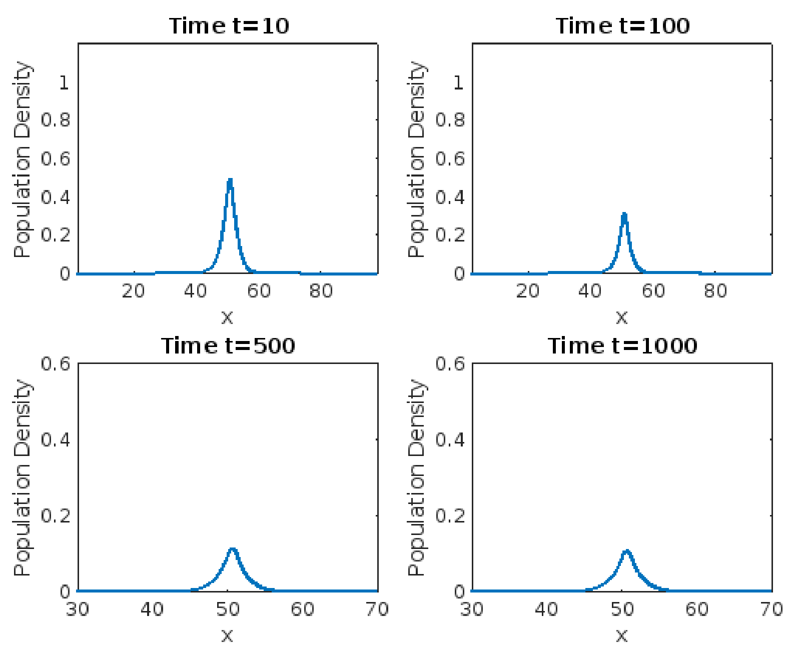


Figure 4.17: This figure shows a type I equilibrium with $a = 0.45$ and protection zone length 1.2.

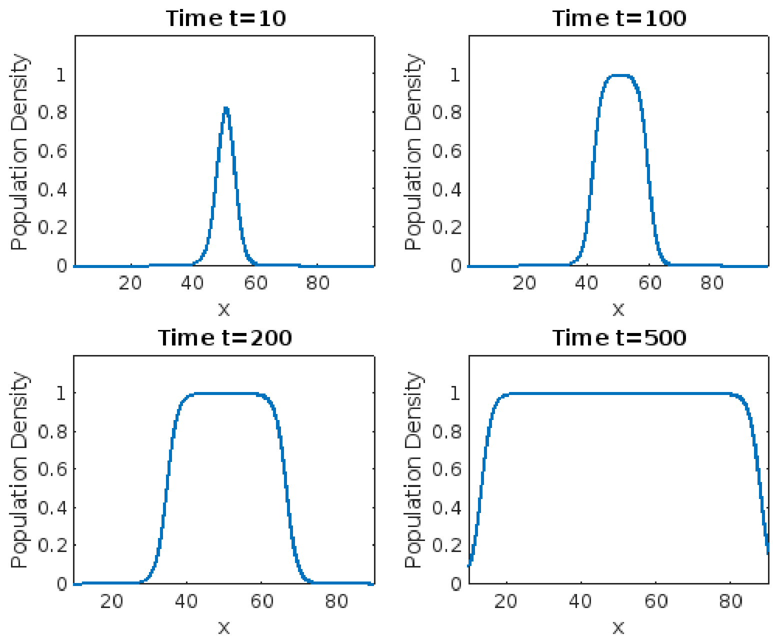


Figure 4.18: This figure shows a failed type I equilibrium with $a = 0.45$ and protection zone 1.2.

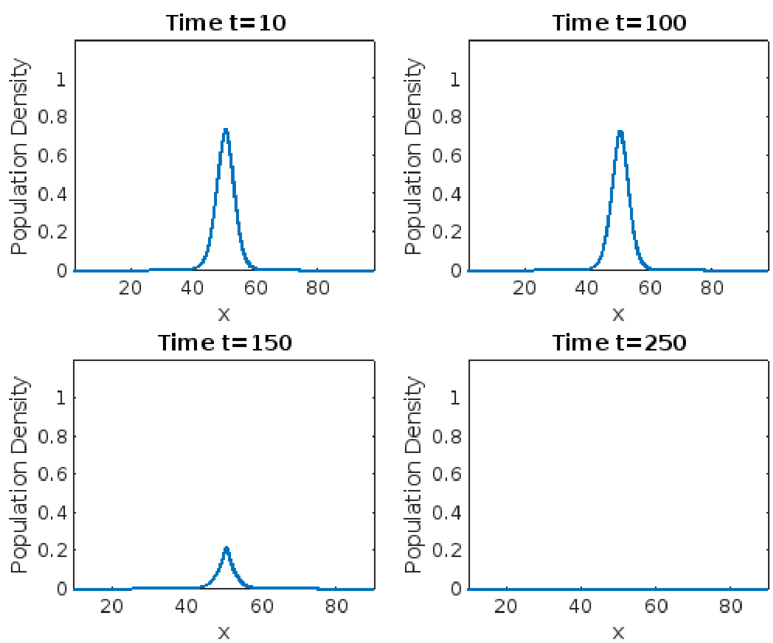


Figure 4.19: This figure shows a protection zone length 0.8 which is too small to support persistence.

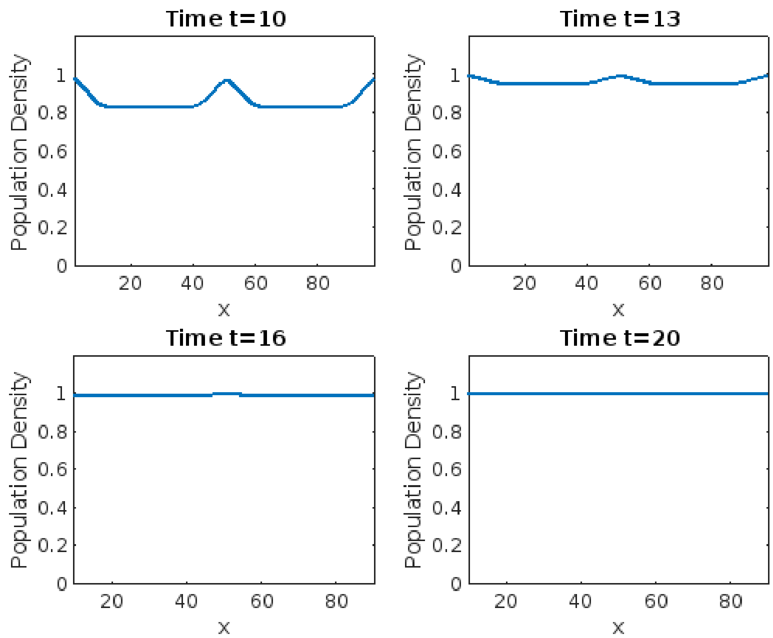


Figure 4.20: This figure shows time slices of a type II equilibrium with $a = 0.45$.

would persist. Figure 4.21 shows a protection zone model with $a = 0.45$, $L = 1.25$, and $K = 1.25$. The initial conditions are $w_0(x) = \begin{cases} 0.5, & x \in [0, 100] \\ 1, & x \notin [0, 100] \end{cases}$. Figure 4.21 reaching an equilibrium agrees with our results from the phase plane analysis.

4.4.3 Type III Equilibrium

We have considered the existence of the trajectories necessary for a type III equilibrium, but it remains to establish if this equilibrium type can exist and the necessary conditions. In the general case, we considered two cases, when $K \leq a^*$ and $K > a^*$.

$K > a^*$

Recall from the general case, if $K > a^*$, then there does not exist a type III equilibrium. Notice Figure 4.22, this figures shows the time slices of a model with $K = 1$,

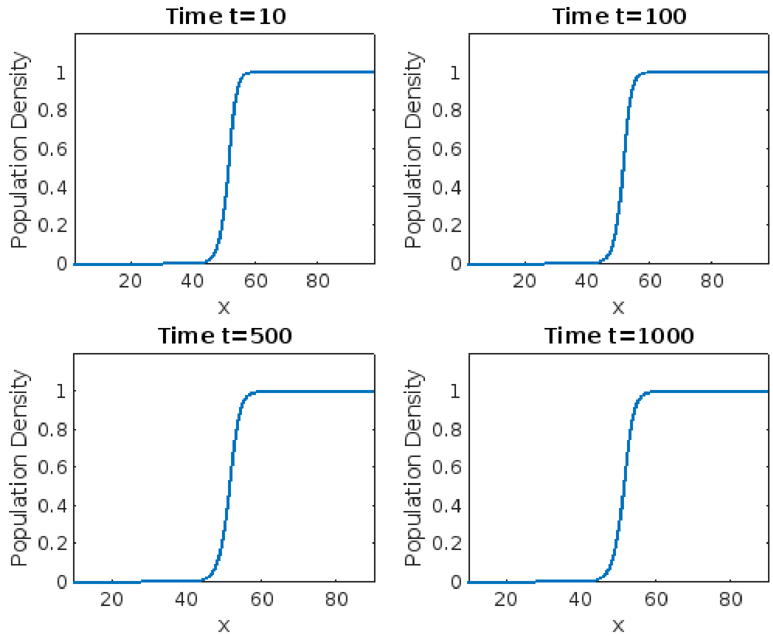


Figure 4.21: This figure shows time slices of a type II equilibrium with $a = 0.45$ where the carrying capacity over the protection zone is $K = 0.25$.

$a = 0.45$, and initial conditions $w_0(w) = \begin{cases} 1, & x > 50 \\ 0, & x \leq 0 \end{cases}$. The protection zone is the interval $(50, 51.25)$. Notice the wave continues to move across the time slices i.e. the protection zone does not stop the wave from advancing. Then we can say a type III does not exist with these conditions which agrees with our results from the general case.

$K \leq a^*$

We proved the existence of a type III equilibrium when $K \leq a^*$ in the general case. Specifically, a type III equilibrium exists when $c_1 \in (-\tau_f(a^*), \tilde{c_1})$ where $\tilde{c_1} = -\tau_f(K)$ and $K \leq a^*$. There is likely a minimal protection zone, such that the advancing wave is stopped by the protection zone. It remains to consider the formula for the size of this protection zone. Similar to previous cases, we need to find the limits of integration and apply them to the minimal protection zone formula.

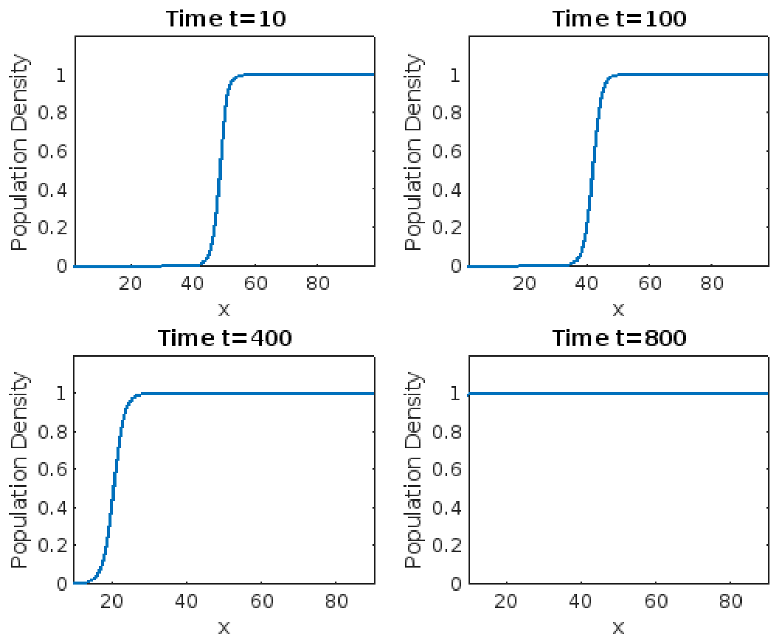


Figure 4.22: This figure shows a model where $K > a^*$ and thus a type III equilibrium can not exist.

$$\begin{aligned} P_2(c_1, c_2) &= \min\{w \in (w, v) : (w, v) = (T_{f,c_1} \cap T_{g,c_2}) \cap ([K, \infty] \times (0, \infty))\} \\ &= \min\{w \in (K, \infty) : \Delta_{\tau,0}(w) = 0\} \end{aligned}$$

We begin by finding the intersection between $T_{g,0}$ and T_{f,c_1}. First, notice this intersection occurs of the interval (K, a^*) and a separate intersection occurs over $(0, K)$, so we need to distinguish between these two intersections. The intersection over $(0, a^*)$ is $P(c_1, 0)$. To remain consistent in our naming practices, we will refer to the intersection over (K, a^*) as $P_2(c_1, 0)$.

$$P_2(c_1,0) = \min\{w \in (w,v) : (w,v) = (T_{f,c_1} \cap T_{g,0}) \cap ([K,\infty] \times (0,\infty))\}$$
$$= \min\{w \in (K,\infty) : \Delta_{\tau,0}(w) = 0\}$$

Then it remains to consider the intersection between the trajectories T_{f,c_1} and $T_{g,1}$:

$$P_2(c_1,-\tau_g(1)) = \min\{w \in (w,v) : (w,v) = (T_{f,c_1} \cap T_{g,-\tau_g(1)}) \cap ([K,\infty] \times (0,\infty))\}$$
$$= \min\{w \in (K,\infty) : \Delta_{\tau,1}(w) = 0\}.$$

We have established the limits of integration needed to calculate the minimal protection zone.

$$L_3(c_1) = \int_{P_2(c_1,-\tau_g(1))}^{P_2(c_1,0)} \frac{dw}{\sqrt{2(-\frac{w^2}{2}+\frac{w^3}{3})+c_1}}$$
$$L_3^* = \inf_{c_1 \in (-\tau_f(a^*),\tilde{c_1})} \{\int_{P_2(c_1,-\tau_g(1))}^{P_2(c_1,0)} \frac{dw}{\sqrt{2(-\frac{w^2}{2}+\frac{w^3}{3})+c_1}}\}$$

The following figures demonstrate the existence of the type III equilibrium with $0 < a < 0.5$. In Figure 4.23 and Figure 4.24, we show a type III equilibrium where the Allee threshold is $a = 0.45$ and the carrying capacity is $K = 0.25$. The protection zone in both figures is length $L = 1.25$ with the initial conditions $w_0(x) = \begin{cases} 1, & x \geq 50 \\ 0, & x < 50 \end{cases}$. In this case the protection zone halts the propagation of the species through the entire environment. Without the protection zone or with a protection zone with $l < L_3^*$, the

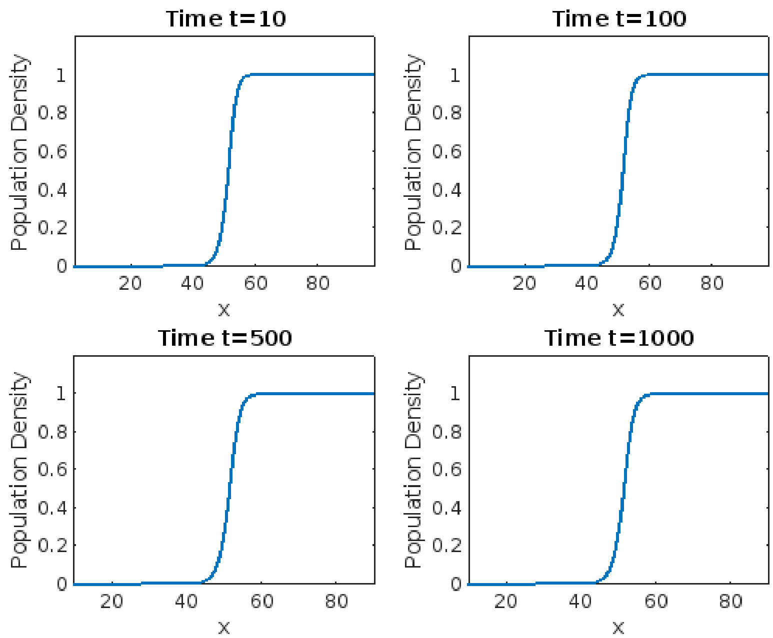


Figure 4.23: This figure shows time slices of a type III equilibrium with $a = 0.45$ and carrying capacity over the protection zone of $K = 0.25$.

solution wave would advance toward $-\infty$.

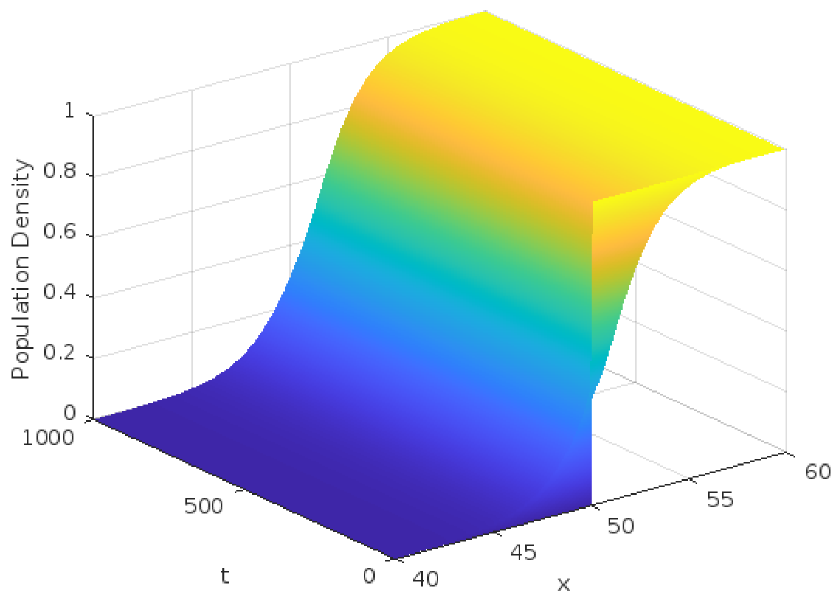


Figure 4.24: This figure shows a type III equilibrium with $a = 0.45$ and carrying capacity over the protection zone of $K = 0.25$.

CHAPTER 5

DISCUSSION

In the case study and simulations chapter, we considered a general protection zone model and the conditions on that model required for the existence of nontrivial steady state solutions. We found these solutions by considering paths in the combined phase plane that began and ended at the critical values of Allee effect growth portion of the phase plane. This resulted in three equilibrium solution types. The existence of each of these equilibrium types depend on varying conditions, so we will explore the results individually.

The existence of the type I equilibrium is not dependent on the traveling wave integral condition, i.e. it exists in much the same form for retreating, stationary, and advancing waves. The phase planes all look very similar, which indicates similar behavior Figure 3.3. Critically, we established if $L > L_1^*$ then there exists a type I equilibrium. This is most significant in the $\int_0^1 g(s)ds < 0$ case, where without the protection zone the species would not persist. This is demonstrated by Figure 4.3b, Figure 4.3a, two examples of a protection $L < L_1^*$. A population will persist with a sufficiently large protection zone, demonstrated in Figure 4.2a and Figure 4.2b. Something to notice, the relative size of the protection zone is dependent on the value $\int_0^1 g(s)ds$. In particular, the protection zone is smaller as this value decreases. The integral value determines how large the relative population density needs to be for the species to persist which indicates the relative hostility of the environment governed by strong Allee effect growth.

It is also important to consider the $\int_0^1 g(s)ds > 0$ case more closely. In particular, this integral condition creates an upper bound of the protection zone size for a type I equilibrium to exist. As shown, the protection zone length is a function on the constant value c_1. The type I equilibrium stops existing when $w_1(c_1) > a^*$. Thus the existence of a type I equilibrium is determined by the size of the protection zone. This result can be noticed in the simulation chapter Figure 4.18, Figure 4.17, where both models have a protection zone of length 1.25, but only one has initial conditions that support a type I equilibrium. This matches our intuition as the integral condition implies an advancing wave, so a large population over the protection zone would result in a spreading solution. This also matches the results obtained in [11], once the protection zone reaches a certain size only a spreading solution will exist. Though a spreading solution is not a steady state solution, the species will persist. Regarding the relationship between a type I equilibrium and persistence of the species. If $L > L^*$ then a type I equilibrium can exist and the species can persist.

The type II equilibrium must be considered in several cases. First, we will consider the retreating wave condition, $\int_0^1 g(s)ds < 0$. This can be broken down into three cases $K < 1$, $K > 1$, and $K = 1$. Unlike the type I equilibrium, the type II equilibrium is dependent on the carrying capacity's relative size. The phase plane behavior when $K > 1$ and $K = 1$ are different, but the resulting effects on the solution are largely the same. The existence of equilibrium solutions in both of these cases are not likely dependent on the size or even existence of a protection zone, but are instead dependent on initial conditions. Then the case $K < 1$, there are two sub-cases $K < a_*$ and $K \geq a_*$. When $K \geq a_*$, the type II equilibrium likely exists regardless of the protection zone. When $K < a_*$, then the existence of a type II equilibrium is dependent on the size of the protection zone. The the larger protection zone can restrict the existence of the type II equilibrium. If L is too large, then a type II equilibrium can not exist. It is important to consider the relationship between L^* and $L_2(c_1)$. If L^* is larger than the maximal value in the range of $L_2(c_1)$ then there exists a possibility of only the trivial equilibrium solution, otherwise a type I equilibrium will exist. This is explored in the simulations chapter, Figure 4.4 shows a model with $K = 0.25$ and protection zone length $L = 1$. In this figure, the species tends toward extinction as for a large L but $L < L^*$. Figure 4.5 also assumes $K = 0.25$, but the protection zone is only length $l = 0.1$. This is sufficiently small to allow the type II equilibrium to exist. Finally, Figure 4.6 demonstrates a protection zone of length $l = 5$ which is large enough to support a type I equilibrium when $K = 0.25$. Then, if we consider the other two integral conditions, the type II equilibrium will exist regardless of the protection zone lengths we assumed for our model.

The type III equilibrium results in stopping the traveling wave from advancing or retreating. In the phase plane, this is the most complicated solution type as it depends in multiple trajectory intersections. A type III equilibrium requires the connection between two distinct Allee effect trajectories in the same quadrant. This results in increased amount of conditions necessary for the existence of a type III equilibrium. First, suppose $\int_0^1 g(s)ds < 0$, the condition for a retreating wave. Intuitively, if there is a large enough protection zone, then it is possible to stop the wave from retreating toward $\pm\infty$ resulting in extinction. Our calculations agree with this intuition and result in $L > L_*$ being the sufficient condition for a type III equilibrium to exist when $\int_0^1 g(s)ds < 0$ and $K \geq a_*$. In the simulation section, we approximated the protection zone necessary for a type III equilibrium in the figure Figure 4.8 when $a = 0.75$. It appears the $L_3^* \approx 1$. In further simulations, Figure 4.10 and Figure 4.9, we demonstrate that a protection zone of length $L = 2$ is sufficiently large enough to stop a retreating wave and form a type III equilibrium but $L = 0.8$ is not. Notice, for a sufficiently small carrying capacity, $K < a_*$ over the protection zone, and the protection zone will not succeed in stopping the wave from retreating. Now, we can consider the final wave condition $\int_0^1 g(s)ds > 0$, an advancing wave condition. A type III equilibrium can only exist in this case, if the carrying capacity over the protection zone is sufficiently small, i.e. $K < a^*$. In the results chapter, we considered some models with carrying capacity $K = 0.25$ and Allee threshold $a = 0.45$ with protection zone length $L = 1.25$. With those assumptions and sufficient initial conditions Figure 4.24 and Figure 4.23 both demonstrate a type III equilibrium. Notice the protection zone in this equilibrium type is similar to a barrier zone considered by Li, et. al. [25]. In [25], Li, et. al. consider an integro-difference model with negative growth over the barrier zone which is surrounded by strong Allee effect growth on an unbounded domain. In particular, they assume the Allee effect growth has sufficient conditions for a traveling wave with positive speed. We showed that protection zone can form a barrier to expansion of a species, if the carrying capacity over the protection zone is sufficiently small when compared to the carrying capacity of the surrounding environment.

There are several ways this work could be advanced in the future. Consider first, a reaction-diffusion model with a moving protection zone. Berestycki, et. al. considered a similar model in [6]. Their reaction-diffusion model had an advancing patch with exponential decay growth surrounding it. A similar model with Allee effect growth outside of the patch could be imagined.

$$u_t - Du_{xx} = \begin{cases} \bar{f}(u, x - ct), & \chi \in \Omega \\ \bar{g}(u, x - ct), & \chi \in \mathbb{R} \backslash \Omega \end{cases}$$

This may be interesting to consider as the existence of traveling wave behavior outside the patch may allow for some interesting dynamics over the entire domain as the patch moves. In particular, a case where the patch is advancing slower than the wave generated by the Allee effect. It may be interesting to consider the effects on the population as the distance between the crest of the wave and the patch change. The relationship between speed of the shifting patch and the wave will likely be critical in studying the behavior of this system. Berestycki et. al. also considered the effects multiple patches would have on the persistence of the species. This may also be interesting to consider with a protection zone model, because protection zones are often fragmented.

Another possible direction to consider in the future, is a protection zone model with varying behavior outside the protection zone. We considered the model with fixed strong Allee effect growth. However, a model can be imagined with oscillating growth outside the protection zone for instance $\bar{g}(0) = \bar{g}(a(t)) = \bar{g}(1) = 0$ where $0 < a(t) < 1$. In this case, we may consider a species that persists year round in the protection zone, but only persists seasonally outside the protection zone. For instance, a species of fish that exists in a tropical location year round, but extends it's habitat during the summer when the water is warmer elsewhere. This may be interesting to consider as the habitability outside of protection zones is often not fixed and may worsen or improve over time [7] [29] [9].

There are several other ways protection zone research could be continued in the future outside of reaction-diffusion models. We could consider an integro-difference or integro-differential equation model with similar characteristics. These two options may give greater insight into how the dispersal of the species throughout its environment effects the overall population persistence. Another alternate model form to consider is a competition model with protection zone. Understanding how a predator and prey interact with a protection zone or how two species that compete for resources interact with a protection zone may be very informative biologically. It is not hard to imagine a predator species whose population is also dependent on the protection of it's prey over some portion of their shared environment.

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