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Strategy-proof social choice functions on Condorcet domains.

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STRATEGY-PROOF SOCIAL CHOICE FUNCTIONS ON CONDORCET DOMAINS

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DEDICATION

For Stefan. Creating this research has brought me happiness. I wish you were here to see it.

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ABSTRACT

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A social choice function is said to be strategy-proof if no voter has any motivation to lie about their true preference. Strategy-proofness is a desirable property of social choice functions so we consider here functions that always satisfy this property. We add to this property the additional desirable conditions of anonymity and neutrality and present domains on which we can get a characterization of majority rule as the only social choice function that satisfies these three properties. Furthermore, we consider what functions look like when we drop the condition of anonymity.

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CHAPTER 1
INTRODUCTION

Whenever a collection of people want to vote on a decision there is the obvious question of how should the winner of such a vote be chosen. Whether that decision is something as trivial as choosing food to have at an event or as significant as choosing a leader of a nation, all voters have opinions that should be brought together and collectively assessed to designate a fair winner. A common candidate considered to be a fair winner of a vote is the majority candidate. Another name for the majority candidate, is the Condorcet alternative. This name comes from Marie Jean Antoine Nicolas Caritat, Marquis de Condorcet - more commonly known as Nicolas de Condorcet. In his *Essai sur l'Application de l'Analyse la Probabilité des Décisions Rendues la Pluralité des Voix* (*Essay on the Application of Analysis to the Probability of Decisions Rendered by a Plurality of Votes*), Condorcet presented what is now referred to as Condorcet's paradox which shows that when there are three or more options a majority candidate is not guaranteed to exist [7]. Below we see an example of such a paradox.

Example 1.1 *Suppose we have three people submitting a ranking of their choice of pizza topping for a lunch. They can choose from pepperoni, sausage, and cheese. Notice, if the three voters give the following lists, there is no clear majority winner. See Table 1.1.*

This dissertation will look at collective decision making on a collection of voter preferences in which there is always a "pairwise comparison majority winner".

	Voter 1	Voter 2	Voter 3
First choice	pepperoni	cheese	sausage
Second choice	cheese	sausage	pepperoni
Third choice	sausage	pepperoni	cheese

Table 1.1: Condorcet's paradox

Since a Condorcet alternative will always exist, we refer to this collection as the Condorcet domain. The language of Condorcet domains has its origin in the work of Monjardet [16], [15] [17]. Monjardet's notion of a Condorcet domains is slightly different than the one in this dissertation. Monjardet defines Condorcet domains as sets of linear orders where Condorcet's paradox does not occur, whereas we define our notion of the Condorcet domain more generally since we just require that a Condorcet alternative must exist. Below is an example to help illustrate the difference in these notions.

Example 1.2 *Suppose the three people voting on their pizza order in Example 1.1 found out there was a special pizza being made that day. They are now voting again on these four types of pizza and everybody agrees that the special is their first choice but keeps everything else the same. Now the vote looks like the following table. Notice that there is a pizza that would be called the Condorcet alternative - the special - since it clearly is preferred to all other options by a majority of people. But the vote still exhibits the cyclical behavior of Condorcet's paradox for the three other pizza types.*

The vote in Example 1.2 would not be a part of Monjardet's notion of a Condorcet domain but we will consider this type of vote in our work.

	Voter 1	Voter 2	Voter 3
First choice	special	special	special
Second choice	pepperoni	cheese	sausage
Third choice	cheese	sausage	pepperoni
Fourth choice	sausage	pepperoni	cheese

Table 1.2: Vote that has a Condorcet alternative and exhibits Condorcet's paradox

1.1 Preliminary Definitions and Results

We mentioned the notion of a domain above but now give an explicit definition of a domain for our work. Additionally we define the notion of the rule by which we select the winner of a vote, namely a social choice function. We will also sometimes use the phrase voting rule when referring to a social choice function[1]. We let A stand for the set of alternatives, i.e. the objects being voted on, and $L(A)^N$ is the set of all profiles. A profile $P = (P_1, P_2, \ldots, P_n)$ is an n-tuple such that $P_i \in L(A)$ for $i = \{1, 2, \ldots, n\}$ where $L(A)$ is the set of all linear orders on A.

Definition 1.1 *A **strict linear ordering**, or a ranking, on a set A is a complete, transitive, and antisymmetric binary relation, R, on A.*

By complete, we mean that for all $a_i, a_j \in A$, a_iRa_j and/or a_jRa_i. By transitive, we mean that for all $a_i, a_j, a_k \in A$ if a_iRa_j and a_jRa_k, then a_iRa_k. Finally, by antisymmetric we mean if a_iRa_j and a_jRa_i then $a_i = a_j$.

Additionally we want to define the notion of rank in a linear order P_i. The kth ranked alternative in a linear order, denoted $r_k(P_i)$, is the alternative $x \in A$ such that $k - 1$ alternatives are listed above x in P_i.

[1]Some authors will use these two terms to refer to distinct concepts (see [23] for example) but this dissertation will use them interchangeably.

Definition 1.2 *A nonempty subset D of $L(A)^N$ is called a* **domain** *and a* ***social choice function*** *is a map $f : D \to A$ that assigns to each profile P belonging to D a unique alternative $f(P) \in A$.*

The definition below is an example of a type of social choice function.

Definition 1.3 *A social choice function $f : D \to A$ is a* ***dictatorship*** *if there exists $j \in N$ such that, for any profile $P = (P_1, P_2, \ldots, P_n)$, $f(P) = r_1(P_j)$. In this case, individual j is the* ***dictator****. We will say that a social choice function is* **non-dictatorial** *if it is not a dictatorship.*

One property that is desirable for social choice functions is that they are strategy-proof. By this we mean that no individual has any motivation to lie about their true preferences when voting because doing so does not produce a better result for that individual. A dictatorship is always strategy-proof. Another property that a dictatorship satisfies is stated below.

Definition 1.4 *A social choice function f satisfies* ***unanimity*** *if, for any profile $P = (P_1, P_2, \ldots, P_n)$, $r_1(P_i) = x$ for all $i \in N$ implies $f(P) = x$.*

If we want our social choice functions to satisfy both the properties of strategy-proofness and unanimity then we are likely to have a dictatorship. In 1973 and 1975, Gibbard and Satterthwaite independently established the following seminal result showing this fact[2]. In Theorem 1.1 below, the domain, D, is often referred to as the *Universal Domain*, that is the domain of all possible linear orders.

Theorem 1.1 (Gibbard, 1973; Satterthwaite 1975) *Suppose there are at least three alternatives and that for each individual any strict linear ordering is permissible. Then the only unanimous, strategy-proof social choice function is a dictatorship.*

[2]Alternative phrasings of the Gibbard-Satterthwaite Theorem exist. One such equivalent phrasing assumes that the social choice function is onto rather than assuming it is unanimous.

Clearly a dictatorship does not fully consider the prefernences of all individuals when determining the social output as it is based on one specific voter's choice. Thus we would say a dictatorship is not anonymous, that is, the order of the voters matters in the result. We wish to consider social choice functions that do treat all individual's preferences more fairly, i.e. functions that are anonymous. Another desirable property for social choice functions is that they are neutral, meaning that the social output does not depend on the name of the alternatives. To look at functions that satisfy these properties, as well as are strategy-proof, we must consider a different domain than what is stated in Theorem 1.1. One such function is the rule that selects the majority candidate, or Condorcet alternative, often called majority rule. Hence the overarching theme of this dissertation is to consider on what domains we can get a characterization of majority rule as the only social choice function that satisfies the three properties of strategy-proofness, anonymity, and neutrality. Additionally we consider what happens when we swap the anonymity condition for a function that must be non-dictatorial as well as what happens when we completely drop the anonymity condition.

We begin with domain of profiles of all linear orders where the Condorcet alternative exists. In Chapter Three we extend our work to consider a domain that can model realistic representations of individual's beliefs, that is the single-peaked domain. In Chapter Four we begin to consider what happens when we do not require that our domain only consist of strict linear orderings. Finally, we conclude with some remaining questions that we hope to pursue in the future.

1.2 One Final Comment

The results of this dissertation are on domains where the Condorcet alternative always exists thus they are called Condorcet domains. There is a notion of a social choice function being what is referred to as Condorcet consistent which is slightly different than what we are considering here. We say a social choice function is Condorcet consistent if when a majority element exists, the function always selects that element as the winner. We could look at extending our work to Condorcet consistent rules. But for this work, we will always consider that the majority element exists for all profiles in our domain.

CHAPTER 2
THE CONDORCET DOMAIN

We begin our work by looking at the largest domain where the majority element always exists. This domain is called the Condorcet Domain, named after Nicolas de Condorcet. By restricting our focus to this domain, we guarantee that the majority element is always defined which allows us to focus on the social choice function which always outputs this element, namely majority rule or the Condorcet rule.

2.1 Notation and Terminology

Let A denote a finite set of alternatives with $|A| = m$. Let $N = \{1, 2, \dots, n\}$ be a set of individuals. Each individual has a strict ranking of the alternatives. The notation $L(A)$ is for the set of all linear orders on A. A profile is an n-tuple $P = (P_1, P_2, \dots, P_n)$ such that $P_i \in L(A)$ for $i = 1, 2, \dots, n$. The set of all profiles is denoted by $L(A)^N$. For any profile $P = (P_1, P_2, \dots, P_n)$, individual i has a strict ranking, P_i, of the alternatives with the notation xP_iy representing the fact that i prefers alternative x to alternative y. The notation $r_k(P_i)$ represents the kth ranked alternative of individual i.

To define the Condorcet Domain we need some additional notation. For any $P = (P_1, P_2, \dots, P_n)$ belonging to $L(A)^N$ and for any subset $\{x, y\}$ of A,

$$N_{xy}(P) = \{i \in N : xP_iy\}$$

is the set of individuals that rank x above y in the profile P.

Definition 2.1 *Given a profile P of linear orders, an alternative x is the* ***strict Condorcet alternative*** *if, for any other alternative y, $|N_{xy}(P)| > |N_{yx}(P)|$. The set of all profiles of linear orders where the Condorcet alternative exists is called the* ***Condorcet domain*** *and is denoted by $\mathscr{L}_C$.*

Notice that x is a strict Condorcet alternative with respect to a profile P if more individuals rank x above y than y above x for any $y \neq x$. Throughout this chapter, we will assume that the domain D of a social choice function is the Condorcet domain $\mathscr{L}_C$. Therefore, we will drop the qualifier "strict" when referring to the strict Condorcet alternative[1].

The Condorcet domain always admits a majority winner - the Condorcet alternative. This alternative seems an obvious potential alternative to be the output of a social choice function. Of course this statement relies on the belief that a majority winner is the best one (see Felsenthal and Machover [10] and Risse [21] for arguments in favor of the Condorcet alternative being the winning alternative). With the Condorcet alternative seeming an obvious social output, it is interesting to consider if voting rules admit different alternatives as the social output, or if the Condorcet alternative always wins and thus we can get a characterization of majority rule. By focusing on the domain $\mathscr{L}_C$, are we limited in the possibility of surprise outputs? That is, if the Condorcet alternative exists within a domain, can it be guaranteed that that alternative is the social output?

Below we give a formal definition of the rule that always outputs the Condorcet alternative and follow with some inherent properties of this function.

Definition 2.2 *The* **Condorcet rule***, or* **majority rule***, is the social choice func-*

[1]There is a notion of a weak Condorcet alternative, x, where for any other alternative y, $|N_{xy}(P)| \geq |N_{yx}(P)|$. There can be more than one weak Condorcet alternative for a profile of linear orders. See [9] for an interesting comparison of strong Condorcet winners and weak Condorcet winners.

tion $f_C : \mathscr{L}_C \to A$ defined as follows: for any $P \in \mathscr{L}_C$, $f_C(P) = x$ where x is the Condorcet alternative with respect to P.

Definition 2.3 *A social choice function $f : \mathscr{L}_C \to A$ has **full range**, or is said to be **surjective**, if for every alternative $x \in A$ there exists a profile P such that $f(P) = x$.*

Definition 2.4 *A social choice function f satisfies* **anonymity**, *or is said to be **anonymous**, if for every permutation $\sigma : N \to N$ and for every profile $P = (P_1, \ldots, P_n)$, $f(P_\sigma) = f(P)$ where*

$$P_\sigma = (P_{\sigma(1)}, \ldots, P_{\sigma(n)}).$$

Definition 2.5 *A social choice function f satisfies **neutrality**, or f is said to be* **neutral**, *if for any profile $P = (P_1, \ldots, P_n)$ and for any permutation $\phi : A \to A$, $f(\phi(P)) = \phi(f(P))$ where*

$$\phi(P) = (\phi(P_1), \ldots, \phi(P_n)).$$

The notation $\phi(P_i)$ is a linear order formed by applying the permutation ϕ to the alternatives listed in P_i.

These two aforementioned properties can be best delineated by noting that anonymity requires that the social output does not depend on the names of the voters whereas neutrality requires that the social output not depend on the names of the alternatives.

Below we present two examples to help illustrate these properties.

Example 2.1 *Let $A = \{a, b, c, d\}$ and $n = 4$. Consider the two social choice functions defined below.*

$$f(P) = \begin{cases} x & \textit{if } |\{i \in N : r_1(P_i) = x\}| \geq 3 \\ a & \textit{otherwise} \end{cases}$$

$$g(P) = \begin{cases} x & \textit{if } |\{i \in N : r_1(P_i) = x\}| \geq 3 \\ r_1(P_1) & \textit{otherwise} \end{cases}$$

Notice how the definition of f relies on a specific alternative but not a specific voter's preference, hence f is is not neutral but it is anonymous. On the other hand, g relies on a specific voter's preference but not on a specific alternative, hence g is not anonymous but is neutral.

To make clear some of the notation we have introduced, consider Example 2.2. In this example, we use the idea of a transposition. Let $x, y \in A$. A transposition, $\phi = (xy)$ is a permutation that exchanges the two alternatives x and y while all other alternatives remain fixed.

Example 2.2 *Let $P = (P_1, \ldots, P_n)$ be a profile as shown below.*

$$P = \begin{pmatrix} a & a & b & c \\ c & b & a & a \\ b & c & c & b \end{pmatrix}$$

To make clear the notation $r_k(P_i)$ consider here $k = 2$ and $i = 1$ so $r_2(P_1) = c$. That is, the second ranked alternative of the first voter is c. We also want to note the following observation. For any permutation $\phi : A \to A$, for any linear order $\ell \in L(A)$, and for any $j \in \{1, 2, \ldots, m\}$,

$$r_j(\phi(\ell)) = \phi(r_j(\ell))$$

We can make this more clear by letting $\phi = (ac)$ then the permuted profile is shown below.

$$\phi(P) = \begin{pmatrix} c & c & b & a \\ a & b & c & c \\ b & a & a & b \end{pmatrix}$$

Now we can see that $r_2(\phi(P_1)) = a = \phi(r_2(P_1)) = \phi(c)$.

To give a formal definition of what it means for a social choice function to be strategy-proof, we must first introduce the following notation. Given a profile $P = (P_1, P_2, \ldots, P_n)$ and an individual $i \in N$, if P_i is replaced by P_i' in the profile P, then the resulting profile is denoted by (P_i', P_{-i}). In addition, (P_i, P_{-i}) is another way of writing the profile P.

Definition 2.6 *A social choice function $f : \mathscr{L}_C \to A$ is said to be* ***manipulable*** *at a profile $P = (P_1, P_2, \ldots, P_n)$ by individual i if there exists $P_i' \in L(A)$ such that $(P_i', P_{-i}) \in \mathscr{L}_C$ and $f(P_i', P_{-i})\ P_i\ f(P_i, P_{-i})$.*

Definition 2.7 *A social choice function f is said to be* ***strategy-proof*** *if it is not manipulable at any profile P belonging to the domain $\mathscr{L}_C$.*

An example of a strategy-proof social choice function is given in the next section.

With these preliminaries in mind, we now turn our focus to characterizing majority rule on the Condorcet Domain.

2.2 Characterizations of Majority Rule on $\mathscr{L}_C$

Our work to characterize majority rule stems from results due to Campbell and Kelly [4], [6]. They established the following:

Theorem 2.1 (Campbell and Kelly, 2003) *Assume $m \geq 3$ is odd and $n \geq 3$. If $f : \mathscr{L}_C \to A$ is strategy-proof, non-dictatorial, and surjective, then f is majority rule, f_C.*

The assumption that n is odd necessary as exhibited by the following example which satisfies the same three properties stated in Campbell and Kelly's theorem - i.e. it is strategy-proof, non-dictatorial, and surjective - yet is distinct from majority rule. This example is due to Merrill [14].

Example 2.3 *Let $n = 4$ and let $A = \{x, y, z\}$. We define the social choice function $g : \mathscr{L}_C \to A$ as follows:*

$$g(P) = \begin{cases} r_1(P_1) & \textit{if } f_C(P) = x \\ r_1(P_2) & \textit{if } f_C(P) = y \\ r_1(P_3) & \textit{if } f_C(P) = z \end{cases}$$

It is not hard to see that this example is non-dictatorial and satisfies unanimity. Since n is even, no single voter can change their preference and change the Condorcet alternative. So no single voter can manipulate g. Hence g is strategy-proof. Moreover, g is not neutral. Consider the profile $P = (P_1, P_2, P_3, P_4)$ below:

$$P = \begin{pmatrix} y & x & x & z \\ x & y & z & x \\ z & z & y & y \end{pmatrix}$$

The Condorcet alternative is x, thus $g(P) = r_1(P_1) = y$. Let $\phi = (x\ z)$ and consider the resulting profile $\phi(P)$.

$$\phi(P) = \begin{pmatrix} y & z & z & x \\ z & y & x & z \\ x & x & y & y \end{pmatrix}$$

The Condorcet alternative of $\phi(P)$ is z, thus $g(\phi(P)) = r_1(P_3) = z$ but neutrality would give that $g(\phi(P)) = \phi(g(P)) = \phi(y) = y$.

In addition to this example, Merrill presented another function that is strategy-proof and non-dictatorial yet is distinct from majority rule [14]. This second example fails anonymity.

As Campbell and Kelly sought to extend their characterization to include the case where the number of voters is even, they combined their previous work to that of Merrill and arrived at the following characterization of majority rule [5].

Theorem 2.2 (Campbell and Kelly, 2015) *(a) For $m \geq 3$ and $n = 4$ or $n = 4k+2$ where $k \geq 0$, if f is an anonymous, neutral, and strategy-proof social choice function on $\mathscr{L}_C$, then $f = f_C$.*

(b) For $m \geq 4$ and $n = 4k$ where $k \geq 1$, if f is an anonymous, neutral, and strategy-proof social choice function on $\mathscr{L}_C$, then $f = f_C$.

The case where $m = 3$ and n is a multiple of four is missing from this work. This missing case was addressed by Powers and Wells in their recent paper. The work to get this complete characterization of majority rule comes from first establishing the following results [20].

Theorem 2.3 (Powers and Wells, 2023) *Assume that $m \geq 3$ and $n \geq 3$. A social choice function $f : \mathscr{L}_C \to A$ is strategy-proof, non-dictatorial, neutral, and satisfies unanimity if and only if f is majority rule, i.e., $f = f_C$.*

Lemma 2.1 (Powers and Wells, 2023) *Assume $f : \mathscr{L}_C \to A$ is strategy-proof and neutral. If $P = (P_1, \dots, P_n) \in \mathscr{L}_C$ satisfies $x\ P_i\ z$ for each $i = 1, \dots, n$ and x is not the Condorcet alternative, then $f(P) \neq z$.*

Theorem 2.3 combined with the Lemma 2.1 allows Powers and Wells to arrive at a proof of the missing case from Theorem 2.2 and thus fully characterize majority rule as the only anonymous, neutral, and strategy-proof function on $\mathscr{L}_C$. Formally we have the following characterization [20].

Theorem 2.4 (Powers and Wells 2023) *For any $m \geq 3$ and for any $n \geq 3$, a voting rule $f : \mathscr{L}_C \to A$ is strategy-proof, neutral, and anonymous if and only if $f = f_C$.*

Restricting $m \geq 3$ in Theorem 2.4 is necessary. If $m = 2$ and n is even then the rule that selects the Condorcet loser (i.e. the alternative that is not the

Condorcet alternative) is also strategy-proof, neutral, and anonymous as noted in [5]. We will later address the two alternative case more thoroughly.

2.3 Strategy-Proof and Neutral Social Choice Functions

In the previous section we presented results that showed that majority rule is the only strategy-proof, neutral, and anonymous social choice function on the Condorcet domain, $\mathscr{L}_C$. We now consider what social choice functions on the Condorcet domain will look like if we drop the condition of anonymity and by doing so arrive at a full characterization of strategy-proof and neutral rules on the Condorcet domain (see Theorem 2.7). This problem was pointed out by a referee for the Powers and Wells paper [20].

The characterizations in this section are delineated by the parity of n. We know from the work of Campbell and Kelly that when n is odd, if a social choice function is strategy-proof and surjective that it must satisfy unanimity [4]. By Theorem 2.3, if we wish to satisfy this property as well as strategy-proofness and neutrality, we would necessarily have a social choice function that is either a dictatorship or is majority rule. Before moving into the definitions and theorems of this section, we will show that if our function violates unanimity, the output will never be the Condorcet alternative.

For the proof of this lemma we need some notation. Let $R \in L(A \setminus \{x\})^N$. Then we define the profile $R^x \in \mathscr{L}_C$ as the profile where we take each preference $R_i \in R$ and add x to the top, keeping the rest of R_i the same. So this profile looks like this

$$R^x = \begin{pmatrix} x, & x, & \dots, & x \\ R_1, & R_2, & \dots, & R_n \end{pmatrix}$$

Additionally we define below the notion of a standard sequence. For any two profiles $P = (P_1, \dots, P_n)$ and $Q = (Q_1, \dots, Q_n)$ in $\mathscr{L}_C$ we define the standard sequence

$\{P^t : t = 0, 1, 2, \ldots, n\}$ of profiles from P to Q as $P^0 = P$ and for $t > 0$, P^t is the profile $P_i^t = P_i^{t-1}$ for all $i \neq t$ and $P_t^t = Q_t$. This is illustrated more clearly below.

$$\begin{aligned}
P^0 &= (P_1, P_2, P_3, \ldots, P_{n-1}, P_n) \\
P^1 &= (Q_1, P_2, P_3, \ldots, P_{n-1}, P_n) \\
P^2 &= (Q_1, Q_2, P_3, \ldots, P_{n-1}, P_n) \\
&\vdots \\
P^{n-1} &= (Q_1, Q_2, Q_3, \ldots, Q_{n-1}, P_n) \\
P^n &= (Q_1, Q_2, Q_3, \ldots, Q_{n-1}, Q_n)
\end{aligned}$$

Lemma 2.2 *For any $m \geq 3$ and for n an even integer, if $f : \mathscr{L}_C \to A$ is strategy-proof, neutral, and violates unanimity, then $f(P) \neq f_C(P)$ for all $P \in \mathscr{L}_C$.*

Proof:

Assume that there exists a profile $P = (P_1, \ldots, P_n)$ belonging to $\mathscr{L}_C$ such that $f(P) = f_c(P) = x$. Let $R^x = (R_1^x, \ldots, R_n^x)$ be the profile derived from P such that $r_1(R_i^x) = x$ and $R_i^x|_{A\setminus\{x\}} = P_i|_{A\setminus\{x\}}$ for $i = 1, \ldots, n$. Let $\{P^t : t = 0, \ldots, n\}$ be the standard sequence of profiles starting with P and ending with R^x. So $P^0 = P$ and $P^n = R^x$. Note that $f_c(P^t) = x$ for $t = 0, \ldots, n$ and so all these profiles belong to the Condorcet domain $\mathscr{L}_C$. Moreover, by strategy-proofness, $f(P^t) = x$ implies $f(P^{t+1}) = x$ for $t = 0, \ldots, n-1$. Therefore, $f(R^x) = x$.

Since f violates unanimity, there exists a profile $Q = (Q_1, \ldots, Q_n)$ such that $r_1(Q_i) = y$ for $i = 1, \ldots, n$ and $f(Q) \neq y$. It is possible that $y = x$. If $y \neq x$, then apply the permutation $\phi = (xy)$ to Q to get the profile $\phi(Q)$ such that $r_1(\phi(Q_i)) = x$ for $i = 1, \ldots, n$. Since $f(Q) \neq y$ it follows from neutrality that $f(\phi(Q)) \neq x$. Now let $\{R^t : t = 0, \ldots, n\}$ be the standard sequence of profiles from R^x to $\phi(Q)$ if $y \neq x$ or from R^x to Q if $y = x$. Note that $\{R^t : t = 0, \ldots, n\}$ is a subset of $\mathscr{L}_C$ with the strict Condorcet alternative always equaling x. Now

$R^0 = R^x$ and so $f(R^0) = f(R^x) = x$. By strategy-proofness, $f(R^t) = x$ implies $f(R^{t+1}) = x$ for $t = 0, \ldots, n-1$. Thus, $f(R^n) = x$. If $y \neq x$, then $R^n = \phi(Q)$ and so $f(R^n) = f(\phi(Q)) \neq x$. If $y = x$, then $R^n = Q$ and $f(R^n) = f(Q) \neq x$. This contradiction completes the proof of the lemma.

$\square$

The following definition is a strange example of a voting rule, but it shows how unanimity can always be violated.

Definition 2.8 *We will say that $g : \mathscr{L}_C \to A$ is a **majority avoiding dictatorship** (MAD) if there exists $j \in N$ such that $g(P) = r_1(P_j|_{A-f_C(P)})$.*

By $P_j|_{A-f_C(P)}$ we mean the preference of voter j where the linear order is now of $m-1$ alternatives with the Condorcet alternative deleted from the ranking and, if necessary, every alternative ranked below $f_C(P)$ shifted up one place in the linear order.

To understand Definition 2.8 more clearly, consider the following example.

Example 2.4 *Let $m \geq 3$ and $n = 2k \geq 4$. Define $g : \mathscr{L}_C \to A$ by*

$$g(P) = r_1\big(P_1|_{A-f_C(P)}\big) = \begin{cases} r_2(P_1) & \text{if } f_C(P) = r_1(P_1) \\ r_1(P_1) & \text{if } f_C(P) \neq r_1(P_1) \end{cases}$$

This rule is neutral and strategy-proof. Below we prove that this is indeed true.

Let $P \in \mathscr{L}_C$ and let ϕ be a permutation on A. Since majority rule is neutral, $f_C(\phi(P)) = \phi(f_C(P))$. Then by the definition of g we have

$$g(\phi(P)) = \begin{cases} r_2(\phi(P_1)) & \text{if } f_C(\phi(P)) = r_1(\phi(P_1)) \\ r_1(\phi(P_1)) & \text{if } f_C(\phi(P)) \neq r_1(\phi(P_1)) \end{cases}$$

Then by our observation in Example 2.2 we see $r_2(\phi(P_1)) = \phi(r_2(P_1))$ and $r_1(\phi(P_1)) = \phi(r_1(P_1))$. Also

$$f_C(\phi(P)) = r_1(\phi(P_1)) \iff \phi(f_C(P)) = \phi(r_1(P_1)) \iff f_C(P) = r_1(P_1)$$

since ϕ is a one-to-one function. Thus we can re-write $g(\phi(P))$ as

$$g(\phi(P)) = \begin{cases} \phi(r_2(P_1)) & \textit{if } f_C(P) = r_1(P_1) \\ \phi(r_1(P_1)) & \textit{if } f_C(P) \neq r_1(P_1) \end{cases} = \phi(g(P))$$

Now to see that g is strategy-proof, suppose $P \in \mathscr{L}_C$ and $P_i' \in L(A)$ with $i \in \{1, 2, \ldots, n\}$ satisfies $Q = (P_i', P_{-i}) \in \mathscr{L}_C$. Since n is even and $P, Q \in \mathscr{L}_C$, $f_C(P) = f_C(Q)$. If $i \neq 1$, then $Q_1 = P_1$ thus it follows that $g(P) = g(Q)$. But if $i = 1$ we have two possibilities. First, if $g(P) = r_2(P_1)$ then $f_C(P) = r_1(P_1) = f_C(Q)$. If $g(Q) \neq f_C(Q)$ then $g(Q) = g(P) = r_2(P_1)$ or $g(P)P_1g(Q)$. In either case, no manipulation occurs. Second, if $g(P) = r_1(P_1)$ then either $g(Q) = g(P) = r_1(P_1)$ or $g(P)P_1g(Q)$. Thus again, no manipulation occurs. Additionally g is clearly not anonymous nor does it satisfy unanimity.

Definition 2.8 allows us to arrive at the following characterization.

Theorem 2.5 *For any $m \geq 4$ and for any even integer $n \geq 4$, $f : \mathscr{L}_C \to A$ is strategy-proof, neutral, and violates unanimity if and only if f is a majority avoiding dictatorship.*

Proof:

Let $f : \mathscr{L}_C \to A$ be strategy-proof, neutral, and violate unanimity. Suppose $P = (P_1, \ldots, P_n) \in \mathscr{L}_C$. By Lemma 2.2, $f(P) \neq f_C(P) = x$.

Define $f_x : L(A \setminus \{x\})^N \to A \setminus \{x\}$ by

$$f_x(R_1, R_2, \ldots, R_n) = f(R^x)$$

Clearly f_x satisfies strategy-proofness as f is assumed to be strategy-proof. Also, f_x is surjective as f satisfies neutrality. Since $|A \setminus \{x\}| \geq 3$, it follows from the Gibbard-Satterthwaite Theorem that f_x is a dictatorship. Say $j \in N$ is the dictator. Thus $f_x(R_1, R_2, \ldots, R_n) = r_1(R_j)$.

Let P^x be the profile obtained from P by moving x up to the top of every ordering. Then $f(P^x) = f_x(P'_1, P'_2, \ldots, P'_n) = r_1(P'_j) \neq x$ where $P'_i = P_i|_{A\setminus\{x\}}$. By strategy-proofness, we can move x down to its original position in P_i for each i and stay in the domain, $\mathscr{L}_C$, and not change the output of f. Thus we have $f(P) = r_1(P_j|_{A-f_C(P)})$. Since f is a neutral function, the choice of $f_C(P) = x$ is arbitrary as we can apply a permutation $\phi : A \to A$ to conclude this holds for any $y \in A$ when $y = f_C(P)$.

Assume that f is a majority avoiding dictatorship with $j \in N$ the dictator. First f clearly violates unanimity since for all $P \in \mathscr{L}_C$ such that $r_1(P_i) = x$ for $i = 1, \ldots, n$, $f_C(P) = x$. Then by definition, $f(P) = r_1(P_j|_{A-f_C(P)}) \neq x$.

Now let $P \in \mathscr{L}_C$ and let $\phi : A \to A$ be a permutation. Since majority rule is neutral we have that $f_C(\phi(P)) = \phi(f_C(P))$. By our observation in Example 2.2 and the fact that ϕ is a one-to-one function we have

$$r_1(\phi(P_j)|_{A-f_C(\phi(P))}) = \phi(r_1(P_j)|_{A-f_C(\phi(P))})$$

Then

$$\begin{aligned} f(\phi(P)) &= r_1(\phi(P_j)|_{A-f_C(\phi(P))}) \\ &= \phi(r_1(P_j)|_{A-f_C(\phi(P))}) \\ &= \phi(f(P)) \end{aligned}$$

Thus f is neutral.

Finally we show that f is strategy-proof. Suppose $P \in \mathscr{L}_C$ and $P'_i \in L(A)$ with $i \in \{1, 2, \ldots, n\}$ satisfies, $Q = (P'_i, P_{-i}) \in \mathscr{L}_C$. We have that $f_C(P) = f_C(Q)$ since $P, Q \in \mathscr{L}_C$ and n is even. If $i \neq j$, then $Q_j = P_j$ and it follows that $f(Q) = f(P)$. If $i = j$ we have two possibilities. If $f(P) = r_2(P_j)$, then $f_C(P) = r_1(P_j)$ by definition so $f_C(Q) = f_C(P) = r_1(P_j)$. We know $f(Q) \neq f_C(Q)$ so

$f(Q) = f(P)$ or $f(Q)$ is some lower ranked alternative for voter j, i.e. $f(P)\ P_j\ f(Q)$. Otherwise, $f(P) = r_1(P_j)$ and again $f(Q) = f(P)$ or $f(P)\ P_j\ f(Q)$. In any case, no manipulation occurs therefore f is strategy-proof.

$\square$

The above characterization holds for four or more alternatives. To present the case for three alternatives we introduce the idea of voting by committees on a set of two alternatives. The definition below is due to Larsson and Svensson [13].[2]

Definition 2.9 *Let $W \subseteq 2^N$ be a nonempty class of nonempty subsets of N. W is a committee if $S \in W$ and $S \subseteq T \subseteq N$ imply that $T \in W$. Given a domain, D, and $A = \{a_1, a_2\}$, a social-choice function $f : D \to A$ is called voting by committees if for all $P \in D$*

$$f(P) = a_1 \text{ if and only if } \{i \in N : \ a_1\ P_i\ a_2\} \in W$$

With this definition in mind, Larsson and Svensson characterized voting by committees as the only strategy-proof and surjective social-choice function when there are two alternatives. We define our own concept of a committee based rule below for a larger alternative set.

Definition 2.10 *If $m = 3$ then a voting rule $f : \mathscr{L}_C \to A$ will be called a* ***majority avoiding committee rule*** *if there exists a subset W of 2^N satisfying*

i) for any $I \subseteq N$, $|W \cap \{I, N \setminus I\}| = 1$ $(\diamond)$

ii) for any $I \subseteq J \subseteq N$, $I \in W$ implies $J \in W$ $(\star)$

such that for any profile $P \in \mathscr{L}_C$, $f(P) = x$ if $f_C(P) \neq x$ and $N_{xy}(P) \in W$ where $y \in A \setminus \{f_C(P), x\}$. To emphasize the dependence of W, we denote f by f_W. Also we will use the abbreviation (MAC) for majority avoiding committee rule. We will refer to these two properties again as property $\diamond$ and property $\star$.

[2]This is also called a simple game with sets in W called winning coalitions.

Below is an example of a majority avoiding committee rule.

Example 2.5 *For this example,* $A = \{x, y, z\}$*,* $N = \{1, 2, 3, 4\}$*, and*

$$W = \{I \subseteq N : |I| \geq 3 \text{ or } |I| = 2 \text{ and } I \subseteq \{1, 2, 3\}\}$$

It is easy to check that W *satisfies property* $\diamond$ *and* $\star$*. If*

$$P = \begin{pmatrix} x & x & y & z \\ y & z & x & x \\ z & y & z & y \end{pmatrix}$$

then $f_C(P) = x$*. Since* $N_{yz}(P) = \{1, 3\} \in W$ *it follows that* $f_W(P) = y$*. It turns out that* f_W *is strategy-proof. In fact, no single individual can always force their best or second best choice to be the social outcome.*

Theorem 2.6 *If* $m = 3$ *and* $n \geq 4$ *is even, then* $f : \mathscr{L}_C \to A$ *is strategy-proof, neutral, and violates unanimity if and only if* f *is a majority avoiding committee rule.*

Proof:

Let $f : \mathscr{L}_C \to A$ be strategy-proof, neutral, and violates unanimity. Since $m = 3$ we may assume that $A = \{x, a, b\}$. By Lemma 2.2, $f(P) \neq f_C(P)$ for every $P \in \mathscr{L}_C$.

Define $f_x : L(\{a, b\})^N \to \{a, b\}$ by

$$f_x(R_1, R_2, \ldots, R_n) = f(R^x)$$

Clearly f_x is strategy-proof and neutral as f is assumed to be. In particular, f_x must be surjective. Then applying Theorem 2 in Larsson and Svensson's work [13], f_x must be voting by committees. This means that if

$$W = \{N_{ab}(R) : R \in L(\{a, b\})^N \text{ and } f_x(R) = a\}$$

then

$$f_x(R) = \begin{cases} a & \text{if } N_{ab}(R) \in W \\ b & \text{if } N_{ab}(R) \notin W \end{cases}$$

Thus W satisfies property $\star$.

Let $I \subseteq N$ and consider the following two cases. First, if $I \in W$ then there exists $P \in L(\{a,b\})^N$ such that $N_{ab}(P) = I$ and $f_x(P) = a$. Define a transposition $\phi : \{a,b\} \to \{a,b\}$ by $\phi = (ab)$. Then neutrality implies that $f_x(\phi(P)) = \phi(f_x(P)) = b$ so $N_{\phi(a)\phi(b)}(\phi(P)) = N_{ba}(\phi(P)) = I$ and $N_{\phi(b)\phi(a)}(\phi(P)) = N_{ab}(\phi(P)) = N \setminus I$. Then since $f_x(\phi(P)) = b$, by definition of W, $N \setminus I \notin W$. Second, if $I \notin W$ then there exists $P \in \mathscr{L}_C$ such that $N_{ab}(P) = I$ and $f_x(P) = b$. For the same ϕ, neutrality implies $f_x(\phi(P)) = \phi(f_x(P)) = a$ and $N_{\phi(a)\phi(b)}(\phi(P)) = N_{ba}(\phi(P)) = I$ and $N_{\phi(b)\phi(a)}(\phi(P)) = N_{ab}(\phi(P)) = N \setminus I$. Thus by definition of W, $N \setminus I \in W$. Therefore W satisfies property $\diamond$.

Suppose $Q \in \mathscr{L}_C$ is such that $f_C(Q) = x$. By Lemma 2.2, $f(Q) \neq f_c(Q)$. By strategy-proofness, $f(Q) \neq x$ implies $f(Q^x) \neq x$. Furthermore, $f(Q) = f(Q^x)$. Beginning with the profile Q^x, one by one move x down in each Q_i^x to where it is in Q_i while maintaining the relative rankings of a and b. Note that at each step the output is never $f_C(Q) = x$. By strategy-proofness, the output must remain $f(Q^x)$. Suppose at some step the output changes to some $y \in A \setminus \{f(Q^x)\}$. If $yQ'_j f(Q^x)$ then a manipulation has occurred by voter j. If on the other hand $f(Q^x)R'_j y$, by changing Q'_j back to Q_j^x the output remains $f(Q^x)$ and again a manipulation occurred by voter j. Thus $f(Q) = f(Q^x)$.

Consequently, for $A = \{x, a, b\}$, if $\mathscr{D}_x = \{P \in \mathscr{L}_C : f_C(P) = x\}$. then for any $Q \in \mathscr{D}_x$

$$f(Q) = f(Q^x) = \begin{cases} a & \text{if } N_{ab}(Q) \in W \\ b & \text{if } N_{ab}(Q) \notin W \end{cases}$$

Now suppose $U \in \mathscr{L}_C$ satisfies $f_C(U) \neq x$. Let ϕ be a permutation on A such that $\phi(f_C(U)) = x$. Since majority rule is neutral, $f_C(\phi(U)) = x$. So $\phi(U) \in \mathscr{D}_x$. Therefore

$$f(\phi(U)) = \begin{cases} a & \text{if } N_{ab}(\phi(U)) \in W \\ b & \text{if } N_{ab}(\phi(U)) \notin W \end{cases}$$

Using the fact that f is neutral we have

$$f(U) = \phi^{-1} f(\phi(U)) = \begin{cases} \phi^{-1}(a) & \text{if } N_{ab}(\phi(U)) \in W \\ \phi^{-1}(b) & \text{if } N_{ab}(\phi(U)) \notin W \end{cases}$$

$$= \begin{cases} \phi^{-1}(a) & \text{if } N_{\phi^{-1}(a)\phi^{-1}(b)}(U) \in W \\ \phi^{-1}(b) & \text{if } N_{\phi^{-1}(a)\phi^{-1}(b)}(U) \notin W \end{cases}$$

Thus we have for any profile $P \in \mathscr{L}_C$, $f(P) = w$ if $f_C(P) \neq w$ and $N_{wy} \in W$ where $y \in A \setminus \{w, f_C(P)\}$.

Conversely suppose that W satisfies property $\diamond$ and property $\star$ and that $f(P) = f_W(P|_{A-f_C(P)})$ for $P \in \mathscr{L}_C$. We will first show that f violates unanimity. Let $P = (P_1, \ldots, P_n) \in \mathscr{L}_C$ such that $r_1(P_i) = x$ for all $i = 1, \ldots, n$. Then clearly $f_C(P) = x$ and by assumption $f(P) = f_W(P|_{A-f_C(P)}) \neq x$. Thus f clearly violates unanimity.

Now let us show that f_W is neutral. By the definition of f_W, $f_W(P) \neq f_C(P)$ for all $P \in \mathscr{L}_C$. Define a transposition $\phi : A \to A$. Suppose $\phi = (xy)$. Then $f_C(\phi(P)) = \phi(f_C(P)) = f_C(P)$ since ϕ fixes $f_C(P)$. Notice that $N_{yx}(\phi(P)) = N_{xy}(P)$ as $\phi(x) = y$ thus $N_{yx}(\phi(P)) \in W$ and $f_W(\phi(P)) = \phi(f_W(P)) = y$. Now suppose $\phi = (xz)$ for $z = f_C(P)$. Then $f_C(\phi(P)) = \phi(f_C(P)) = x$ so $f_W(\phi(P)) \neq x$. Notice again that $N_{zy}(\phi(P)) = N_{xy}(P)$ since $\phi(x) = z$. Thus $N_{zy}(\phi(P)) \in W$ and $f_W(\phi(P)) = \phi(f_W(P)) = z$. Finally suppose that $\phi = (zy)$. Since $f_C(\phi(P)) = y$, $f_W(\phi(P)) \neq y$. Again we see that $N_{xz}(\phi(P)) = N_{xy}(P) \in W$ thus $f_W(\phi(P)) = \phi(f_W(P)) = x$. Therefore we conclude that f_W is neutral when ϕ is a transposition. But since every permutation can be written as a composition of transpositions, we can see by looking at the transpositions one at a time that at each step the function remains neutral. Therefore f_W satisfies neutrality.

Using property $\star$ we can conclude that f_W must be strategy proof. Suppose that $P \in \mathscr{L}_C$ and $P_i' \in L(A)$ with $i \in \{1, \ldots, n\}$ satisfies $Q = (P_i', P_{-i}) \in \mathscr{L}_C$. We have that $f_C(P) = f_C(Q)$ since $P, Q \in \mathscr{L}_C$ and n is even, thus $f_W(Q) \in \{f_W(P), y\}$ for $y \in A \setminus \{f_W(P), f_C(P)\}$. Suppose $f_W(P)\ P_i\ y$ and $y\ P_i'\ f_W(P)$. Then either $f_W(Q) = f_W(P)$ or $f_W(Q) = y$, but no manipulation occurs. Suppose $y\ P_i\ f_W(P)$ and $f_W(P)\ P_i'\ y$. Let $I = \{i \in N : f_W(P)\ P_i\ y\}$ and $J = \{i \in N : f_w(P)\ P_i'\ y\}$. Hence $I \subseteq J \subseteq N$ Then since W satisfies property $\star$, $f_W(Q) = f_W(P)$. Therefore f_W is strategy-proof.

$\square$

One way to summarize the results given in this section is by rephrasing them into the following significant theorem. This gives a complete characterization of strategy-proof and neutral social choice functions on the Condorcet domain.

Theorem 2.7 *Assume that $m \geq 3$ and $n \geq 3$. A social choice function $f : \mathscr{L}_C \to A$ is strategy-proof and neutral if and only if one of the following holds:*

i) f satisfies unanimity and f is either a dictatorship or majority rule;

ii) f does not satisfy unanimity, $m \geq 4$, n is even, and f is a MAD;

iii) f does not satisfy unanimity, $m = 3$, n is even, and f is a MAC rule.

2.4 Two Alternatives

Thus far we have focused on social choice functions that are defined only when the size of the alternative set is three or greater. We now turn our attention to the case where $m = 2$.

Proposition 2.1 *A social choice function $f : \mathscr{L}_c \to A$, where $A = \{a_1, a_2\}$, is strategy-proof, neutral, and anonymous if and only if*

(1) n is odd and f is majority rule

(2) n is even and there exists an integer $k^* \in [\frac{n}{2}+1, n+1]$ such that

$$f_{k^*}(P) = \begin{cases} a_1 & \text{if } |N_{a_1a_2}(P)| \in (n-k^*, \frac{n}{2}-1] \cup [k^*, n+1] \\ a_2 & \text{otherwise} \end{cases}$$

In Proposition 2.1, if $k^* = \frac{n}{2}+1$, then $f = f_C$. If $k^* = n+1$, then f is the rule that selects the Condorcet loser (recall this is the alternative that is not the Condorcet alternative).

Before presenting the long proof for Proposition 2.1, we will illustrate the type of rule described in item (2) with the following example.

Example 2.6 *Let $n = 10$ so $k^* \in [6, 11]$. Choose $k^* = 8$. If $|N_{a_1a_2}(P)| \in (2, 4] \cup [8, 11]$ then $f(P) = a_1$. Let*

$$P = \begin{pmatrix} a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 & a_2 \\ a_2 & a_2 & a_2 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 & a_1 \end{pmatrix}$$

Notice how no voter has any motivation to lie. If, for example, P changed to $P' = (P_3', P_{-3})$ where $r_1(P_3') = a_1$, $f(P') \not{P}_3 f(P)$ hence no manipulation occurs. Furthermore, any voter with $r_1(P_i) = a_1$ already has their most preferred social output.

Example 2.6 is a social choice function that satisfies the three conditions of strategy-proofness, neutrality, and anonymity but is distinct from majority rule. This illustrates why the characterization in Proposition 2.1 is dependent on the parity of n.

Proof:

For (1) we will first show that when $m = 2$ the strategy-proof condition is equivalent to monotonicity [3] .

Assume f is strategy-proof and let $P \in \mathscr{L}_C$ such that $f(P) = a_1$. Let $Q \in \mathscr{L}_C$ such that $N_{a_1a_2}(P) \subseteq N_{a_1a_2}(Q)$. Beginning with the profile P, one by one, move a_1 up to the top in any P_j such that $r_1(P_j) = a_2$ but $r_1(Q_j) = a_1$. Since n is odd, at each step we know the resulting profile will remain in our domain. Then by strategy-proofness of f, at each step the output must remain a_1. Hence, upon the final change we can conclude $f(Q) = a_1$. The labeling of the alternatives in this step of the proof was arbitrary, hence an analogous argument can be made where the roles of a_1 and a_2 are interchanged. Thus f satisfies monotonicity.

Assume f satisfies monotonicity. Suppose by way of contradiction that f is not strategy-proof. Then let $P, P' \in \mathscr{L}_C$ be such that $P' = (P'_j, P_{-j})$ for some $j \in N$ and $f(P')\ P_j\ f(P)$. Since $f(P')\ P_j\ f(P)$, $r_1(P_j) \neq f(P)$ but since $m = 2$, $r_1(P'_j) = f(P)$. Thus $f(P)$ is only ranked higher in P' contradicting that f satisfies monotonicity. Thus f must be strategy-proof.

Now we wish to show that a social choice function $f : \mathscr{L}_C \to A$ is neutral, anonymous, and satisfies monotonicity if and only if n is odd and f is majority rule, i.e. $f = f_C$.

Assume that there exits a function $f : \mathscr{L}_C \to A$ that is neutral, anonymous, and satisfies monotonicity yet $f \neq f_C$. Then there exists a profile R such that $f(R) \neq f_c(R)$. Since f and f_C satisfy neutrality, we can assume $f(R) = a_1$ and $f_C(R) = a_2$. Now $f_C(R) = a_2$ implies that $|N_{a_1a_2}(R)| < |N_{a_2a_1}(R)|$. Choose a profile $R' \in \mathscr{L}_C$ such that $|N_{a_1a_2}(R')| = |N_{a_2a_1}(R)|$ and $N_{a_1a_2}(R) \subseteq N_{a_1a_2}(R')$.

[3] A social choice function $f : \mathscr{L}_C \to A$ satisfies monotonicity if, for all $P \in \mathscr{L}_C$ such that $f(P) = x$, if $P' \in \mathscr{L}_C$ is another profile such that xP'_iy whenever xP_iy for each $i \in \{1, \ldots, n\}$ and every $y \in A \setminus x$, then $f(P') = x$ as well.

Since $|N_{a_1a_2}(R)| < |N_{a_1a_2}(R')|$, monotonicity implies $f(R) = f(R') = a_1$.

Let $\sigma : N \to N$ map $N_{a_1a_2}(R')$ onto $N_{a_2a_1}(R)$. Observe that the profile R'_σ can also be formed by taking the profile R and applying the transposition $\phi : A \to A$ given by $\phi = (a_1a_2)$. Then using anonymity and neutrality we get

$$a_1 = f(R) = f(R') = f(R'_\sigma) = f(\phi(R)) = \phi(f(R)) = a_2$$

Which is impossible, so $f = f_C$.

For (2), assume $g : \mathscr{L}_c \to A$ is a strategy-proof, neutral, and anonymous voting rule. We wish to show that $g = f_{k^*}$ for some integer $k^* \in [\frac{n}{2}+1, n+1]$.

Let $k^* = \min\{k \in [\frac{n}{2}+1, n] : g(P') = a_1, P \in \mathscr{L}_c, \text{ and } |N_{a_1a_2}(P')| \geq \frac{n}{2}+1\}$. First, assume there does not exist a profile which realizes this minimal k^*, i.e. for all $P^* \in \mathscr{L}_c$ such that $g(P^*) = a_1$, $N_{a_1a_2}(P^*) \leq \frac{n}{2} - 1$. Let $k^* = n+1$ so we can write $f_{k^*} = f_{n+1}$. Now for any $Q \in \mathscr{L}_c$ with $\frac{n}{2}+1 \leq |N_{a_1a_2}(Q)| < n+1$ it is clear by assumption that $g(Q) = a_2$. Therefore $g(Q) = f_{n+1}(Q)$ in this case. Now let $Z \in \mathscr{L}_c$ such that $0 \leq |N_{a_1a_2}(Z)| \leq \frac{n}{2} - 1$. We see that $g(Z) = a_1$ by neutrality (since, for $\phi = (a_1a_2)$, $\phi(Z)$ would be a profile like Q above for which $g(Q) = a_2$). Z is such that $n - k^* < |N_{a_1a_2}(Z)| \leq \frac{n}{2} - 1$ for $k^* = n+1$ thus by definition $g(Z) = f_{k^*}(Z)$.

Now let $P \in \mathscr{L}_c$ be a profile that satisfies this minimal k^*. Hence $g(P) = a_1$. Let $R \in \mathscr{L}_c$ be a profile such that $N_{a_1a_2}(P) \subset N_{a_1a_2}(R)$. Since g is anonymous, we can assume the $i \in N_{a_1a_2}(P)$ satisfy $i = 1, \ldots, |Na_1a_2(P)|$. We can do a similar rearrangement of the profile R for $i = 1, \ldots, |N_{a_1a_2}(R)|$. Then one at a time, move a_1 up in each P_i such that $|N_{a_1a_2}(P)| < i \leq |N_{a_1a_2}(R)|$. Since $k^* \geq \frac{n}{2}+1$ we stay in the domain $\mathscr{L}_C$ at each step. Also since g is strategy-proof, at each step the output must remain a_1. Upon the final change we can conclude that $g(R) = a_1$. Hence $g(R) = f_{k^*}(R)$ for all profiles such that $|N_{a_1a_2}(R)| \geq k^*$.

Suppose $R' \in \mathscr{L}_C$ satisfies $|N_{a_1a_2}(R')| \leq n - k^*$. Let $\phi = (a_1a_2)$ and notice that $\phi(R') \in \mathscr{L}_C$. Since $|N_{a_1a_2}(\phi(R'))| \geq k^*$, by the previous arguement, $g(\phi(R'))| = a_1$. Since g satisfies neutrality, $g(R') = f_{k^*}(R') = a_2$.

Next consider a profile $P' \in \mathscr{L}_C$ with $g(P') = a_1$ and $|N_{a_1a_2}(P')| \ngeq k^*$. We know from above that $n - k^* < |N_{a_1a_2}(P')|$. Then $|N_{a_1a_2}(P)| \ngeq k^*$, the minimality of k^*, and $P' \in \mathscr{L}_C$ together imply that $|N_{a_1a_2}(P)| \leq \frac{n}{2} - 1$. Finally we can conclude that for $P'' \in \mathscr{L}_C$ with $g(P'') = a_2$ and $|N_{a_1a_2}(P'')| \nleq n - k^*$ that $\frac{n}{2} + 1 \leq |N_{a_1a_2}(P'')| < k^*$ since k^* is minimal and $P'' \in \mathscr{L}_C$.

Now assume f_{k^*} is a function as defined in the proposition. We wish to show that f_{k^*} is anonymous, neutral, and strategy-proof. Anonymity is clear as $f_{k^*}(P)$ is determined by $|N_{a_1a_2}(P)|$ and for any permutation $\sigma : N \to N$, $|N_{a_1a_2}(P)| = |N_{a_1a_2}(P_\sigma)|$. Therefore $f(P) = f_{k^*}(P_\sigma)$.

To see f_{k^*} is neutral, suppose $P \in \mathscr{L}_C$ is such that $f_{k^*}(P) = a_1$ and let $\phi = (a_1a_2)$. We have two possible cases for $|N_{a_1a_2}(P)|$. First assume $|N_{a_1a_2}(P)| \geq k^*$. Note that $|N_{a_2a_1}(P)| = n - |N_{a_1a_2}(P)|$ and $|N_{a_2a_1}(P)| = |N_{a_1a_2}(\phi(P))|$. Then

$$\begin{aligned} |N_{a_1a_2}(\phi(P))| &= n - |N_{a_1a_2}(P)| \\ &\leq n - k^* \end{aligned}$$

So by the definition of f_{k^*}, $f_{k^*}(\phi(P)) = a_2$.

For the second case assume $n - k^* < |N_{a_1a_2}(P)| \leq \frac{n}{2} - 1$. Then multiplying through this inequality by -1, we have

$$-n + k^* > -|N_{a_1a_2}(P)| \geq -\frac{n}{2} + 1$$

Then adding n to the above we have $k^* > n - |N_{a_1a_2}(P)| \geq \frac{n}{2} + 1$. As noted above, $n - |N_{a_1a_2}(P)| = |N_{a_1a_2}(\phi(P))|$ giving $k^* > |N_{a_1a_2}(\phi(P))| \geq \frac{n}{2} + 1$. This implies $f_{k^*}(\phi(P)) = a_2$.

Now suppose $P \in \mathscr{L}_C$ is such that $f_{k^*}(P) = a_2$. Again we have two cases two consider. First assume $\frac{n}{2} + 1 \leq |N_{a_1a_2}(P)| < k^*$. Then multiplying this inequality through by -1 and adding n to all parts gives

$$\frac{n}{2} - 1 \geq n - |N_{a_1a_2}(P)| > n - k^*$$

Thus $f_{k^*}(\phi(P)) = a_1$ since $n - |N_{a_1a_2}(P)| = |N_{a_1a_2}(\phi(P))|$.

For the second case assume $|N_{a_1a_2}(P)| \leq n - k^*$. Then $|N_{a_2a_1}(P)| = n - |N_{a_1a_2}(P)|$ and $|N_{a_2a_1}(P)| = |N_{a_1a_2}(\phi(P))|$ gives that $|N_{a_1a_2}(\phi(P))| \geq k^*$ which implies $f_{k^*}(\phi(P)) = a_1$.

To see that f_{k^*} is strategy-proof, suppose that $P = (P_1, \ldots, P_n)$ and $i \in N$ such that $f_{k^*}(P) = \phi(r_1(P_i))$ and $Q = (\phi(P_i), P_{-i}) \in \mathscr{L}_C$. Consider the following cases.

First suppose $f_{k^*}(P) = a_1$ and $r_1(P_i) = a_2$. If $|N_{a_1a_2}(P)| \geq k^*$ then $|N_{a_1a_2}(Q)| = |N_{a_1a_2}(P)|+1 \geq k^*$ so clearly $f(Q) = a_1$. If $n-k^* < |N_{a_1a_2}(P)| \leq \frac{n}{2}-1$. Adding 1 to the inequality gives $n - k^* + 1 < |N_{a_1a_2}(P)| + 1 \leq \frac{n}{2}$, which implies $|N_{a_1a_2}(Q)| \leq \frac{n}{2}$. But by definition of our domain, we can conclude $|N_{a_1a_2}(Q)| \leq \frac{n}{2}-1$ which gives $f_{k^*}(Q) = a_1$.

Second, suppose $f_{k^*}(P) = a_2$ and $r_1(P_i) = a_1$. Using neutrality we have that $f_{k^*}(\phi(P)) = a_1$ and $r_1(\phi(P_i)) = a_2$. Let $R = \phi(P)$ and $R_i = \phi(P_i)$. Then by case 1, for $Q' = (\phi(R_i), R_{-i})$, $f_{k^*}(Q') = a_1$. Then a second application of neutrality gives $f_{k^*}(\phi(Q')) = a_2$ and $\phi(Q) = \phi((\phi(R_i), R_{-i})) = (\phi(P_i), P_{-i})$. Therefore f is strategy-proof.

$\square$

Let us now consider again what our functions look like if we drop the condition of anonymity but still want to satisfy strategy-proofness and neutrality. The results below are given on the domain, $\mathscr{L}_C$, explicitly defined below for the case of $m = 2$.

$$\mathscr{L}_C = \left\{P \in L(\{a_1, a_2\})^N : |N_{a_1a_2}(P)| \neq \frac{n}{2}\right\}$$

These results are an adaptation of the work of Larsson and Svensson on voting by committees for two alternatives but we replace the condition of surjectivity with the more strict condition of neutrality [13].

Lemma 2.3 *Let $A = \{a_1, a_2\}$. Given a voting rule $f : \mathscr{L}_C \to A$, let*

$$W_f = \{N_{a_1a_2}(P) : P \in \mathscr{L}_C \text{ and } f(P) = a_1\}.$$

Then f is neutral if and only if W_f satisfies for all $I \subseteq N$, $|W_f \cap \{I, N \setminus I\}| = 1$.

Proof:

Suppose f is neutral and let ϕ be the transposition given by $\phi = (a_1a_2)$. Let $I \subseteq N$ and consider the following two cases. First, if $I \in W_f$ then there exists $P \in \mathscr{L}_C$ such that $N_{a_1a_2}(P) = I$ and $f(P) = a_1$. Then neutrality implies that $f(\phi(P)) = a_2$ so

$$N_{\phi(a_1)\phi(a_2)}(\phi(P)) = N_{a_2a_1}(\phi(P)) = I$$

and

$$N_{\phi(a_2)\phi(a_1)}(\phi(P)) = N_{a_1a_2}(\phi(P)) = N \setminus I.$$

Then since $f(\phi(P)) = a_2$, by the definition of W_f, $N \setminus I \notin W_f$. Second, if $I \notin W_f$ then there exists $P \in \mathscr{L}_C$ such that $N_{a_1a_2}(P) = I$ and $f(P) = a_2$. Then neutrality implies $f(\phi(P)) = a_1$ and

$$N_{\phi(a_1)\phi(a_2)}(\phi(P)) = N_{a_2a_1}(\phi(P)) = I$$

and

$$N_{\phi(a_2)\phi(a_1)}(\phi(P)) = N_{a_1a_2}(\phi(P)) = N \setminus I.$$

Thus by definition of W_f, $N \setminus I \in W_f$. Therefore W_f satisfies for all $I \subseteq N$, $|W \cap \{I, N \setminus I\}| = 1$.

Conversely suppose that W_f satisfies for all $I \subseteq N$, $|W \cap \{I, N \setminus I\}| = 1$ and consider the following two cases. First, suppose $P \in \mathscr{L}_C$ is such that $f(P) = a_1$.

Let $I = N_{a_1a_2}(P)$ then $I \in W_f$ by definition. Consider $\phi(P)$. Then

$$N_{\phi(a_2)\phi(a_1)}(\phi(P)) = N_{a_1a_2}(\phi(P)) = N \setminus I \notin W_f$$

thus by definition $f(\phi(P)) = a_2$. Second, suppose $P \in \mathscr{L}_C$ is such that $f(P) = a_2$. Then $N_{a_1a_2}(P) \notin W_f$. Let $I = N_{a_1a_2}(P)$. Consider $\phi(P)$. Since $I \notin W_f$,

$$N_{\phi(a_2)\phi(a_1)}(\phi(P)) = N_{a_1a_2}(\phi(P)) = N \setminus I \in W_f.$$

Thus $f(\phi(P)) = a_1$. Therefore f is neutral.

□

Theorem 2.8 *Let $A = \{a_1, a_2\}$. A voting rule $f : \mathscr{L}_C \to A$ is strategy-proof and neutral if and only if there exists a subset W of 2^N such that*

$$f(P) = a_1 \quad \textit{if and only if } N_{a_1a_2}(P) \in W$$

where W satisfies the following two conditions:

i) for any $I \subseteq N$, $|W \cap \{I, N \setminus I\}| = 1$ $\qquad(\triangle)$

ii) for any $I \subseteq J \subseteq N$, if $I \in W$ and $\frac{n}{2} \notin [|I|, |J|] \cap \mathbb{Z}$, then $J \in W$. $\qquad(\boxplus)$

Proof:

First let $f : \mathscr{L}_C \to A$ be a strategy-proof and neutral voting rule. Define W as follows,

$$W = \{N_{a_1a_2}(P) : P \in \mathscr{L}_C \text{ and } f(P) = a_1\}$$

We wish to show that W satisfies conditions $\triangle$ and $\boxplus$. Since f is a neutral voting rule we know immediately by Lemma 2.3 that W must satisfy condition $\triangle$. Now suppose $P \in \mathscr{L}_C$ is such that $f(P) = a_1$, then $N_{a_1a_2}(P) = I \in W$. Let $I \subseteq J \subseteq N$ where $\frac{n}{2} \notin [|I|, |J|] \cap \mathbb{Z}$. Now suppose $P' \in \mathscr{L}_C$ is such that $N_{a_1a_2}(P') = J$. Since $I \subseteq J$ and $\frac{n}{2} \notin [|I|, |J|] \cap \mathbb{Z}$, we can one at a time move a_1 up to be top ranked in each $P_j \in J \setminus I$ and note that at each step the output must be a_1 since $f(P) = a_1$ and f is strategy-proof. Throughout this one at a time process we know that each profile $(P'_j, P_{-j}) \in \mathscr{L}_C$ since by the definition of the domain $\frac{n}{2} \notin \{|I|, |J|\} \cap \mathbb{Z}$. Upon the final change of $P_j \in J \setminus I$, we have the profile P' and by strategy-proofness, $f(P') = a_1$. Therefore $J \in W$ and W satisfies condition $\boxplus$.

Conversely suppose W satisfies conditions $\triangle$ and $\boxplus$. From Lemma 2.3, f must be neutral since W satisfies condition $\triangle$. To see that f is strategy-proof, suppose $P = (P_1, \ldots, P_n) \in \mathscr{L}_C$ satisfies $f(P) = a_1$ and so $I = N_{a_1 a_2}(P) \in W$. If $(P'_j, P_{-j}) \in \mathscr{L}_C$ for some $j \in N \setminus I$ and $P'_j \neq P_j$, then $r_1(P_j) = a_2$ and $r_1(P'_j) = a_1$. By the definition of $\mathscr{L}_C$, $\frac{n}{2} \notin \{|I|, |I|+1\}$. By $\boxplus$, $I \in W$ along with $I \subseteq J \subseteq N$ we have that $J = \{I\} \cup \{j\} \in W$. Therefore $f((P_{-j}, P'_j)) = a_1$. By neutrality, the argument above holds for $f(P) = a_2$ as well. Therefore f is strategy-proof.

$\square$

2.5 One Final Remark

It is interesting to note that throughout this entire chapter our results have always assumed that the number of voters is three or more. If we wish to consider the case where $n = 2$, $P = (P_1, P_2) \in \mathscr{L}_C$, it is necessary that $r_1(P_1) = r_1(P_2)$ otherwise $P \notin \mathscr{L}_C$. Thus for $n = 2$, we can consider a strategy-proof social-choice function on $\mathscr{L}_C$ which can satisfy anonymity but it must be dictatorial with both voters as the dictator in this case.

CHAPTER 3
THE SINGLE-PEAKED DOMAIN

The previous chapter was focused on characterizing majority rule on the largest domain on which the majority element is always defined, that is, the Condorcet domain. We now turn our attention to another important domain, namely the single-peaked domain. The notion of this domain has its roots in the work of Duncan Black [3] and was later formalized by the work of Kenneth Arrow [1]. Many researchers have expanded upon this initial work to investigate strategy-proof social choice functions on single-peaked domains - such as Moulin's work on the set of real numbers $\mathbb{R}$ [18] and Weymark's generalization of this [24]. We begin this chapter with preliminary definitions and notation as well as a general discussion of what it means for a preference to be single-peaked. We then turn our attention to results on the single-peaked domain.

3.1 Preliminaries

Within this work we consider single-peakedness as it relates to an underlying linear ordering of the alternatives, yet there are other ways to define a preference as single-peaked (see Ballester and Haeringer for a necessary and sufficient condition to define single-peakedness and sources therein for other definitions [2]). Below is a formal definition of this notion of single-peaked. The peak of a linear order, P_i, is that preference's most preferred alternative and will be denoted $\tau(P_i)$.

Definition 3.1 *Let $>$ be a linear order on the set A of alternatives. A preference* *order* $P_i \in L(A)$ *is* **single-peaked** *with respect to* $>$ *if for all* $a, b \in A$,

$$[\tau(P_i) > a > b \text{ or } b > a > \tau(P_i)] \Rightarrow aP_ib.$$

Let $SP_>(A)$*, or more simply* $SP(A)$*, denote the set of all single-peaked preferences with respect to* $>$.

Put more simply, a preference is single-peaked if it is the single preference that is strictly increasing or the single preference that is strictly decreasing, otherwise the preference is first strictly increasing then strictly decreasing. This behavior can be seen in the preferences that are represented graphically in Figure 3.1.[1]

First, we give some motivating examples for single-peaked preferences - a realistic scenario in which single-peakedness can be applied. Then we show what single-peaked preferences look like graphically. The interpretations in Examples 3.1 and 3.2 come from [25]. Further interpretations as well as a readable summary of what it means for a preference to be single-peaked can be found in the work of Black [3], a seminal piece within the work of the single-peaked domain.

Example 3.1 *Consider political ideologies on a spectrum. Liberal verses conservative is often interpreted as left verses right. Then a voter would have a single-peaked preference if they have an ideal balance between these two opposing viewpoints and prefer viewpoints that are closer to this ideal than those that are farther away from this ideal. The ideal can be one of the endpoints of the spectrum, e.g. left-wing or right-wing views, or could be somewhere in the middle, e.g. moderate views.*

Example 3.2 *We can think of the set of alternatives,* $\{a_1, a_2, \ldots, a_m\}$ *as locations on a street where each* a_i *is the address of an individual. If only one bus stop is to be built on the street, it is reasonable to assume that each individual would prefer to*

[1] There is also a notion of preferences that are not strictly increasing (or strictly decreasing) which is referred to as "single-plateaued" [2] [24].

walk the least amount to that bus stop. Then these preferences will be single-peaked, where each individual i has a peak of a_i. As the other locations, a_j for $j \neq i$, move farther away from this location in either direction, the individual dislikes the location the farther it is from a_i.

Example 3.3 *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let the underlying ordering, $>$, on A be as follows:*

$$a_5 > a_4 > a_3 > a_2 > a_1$$

The graphs in Figure 3.1 show six of the possible 16 total single-peaked preferences

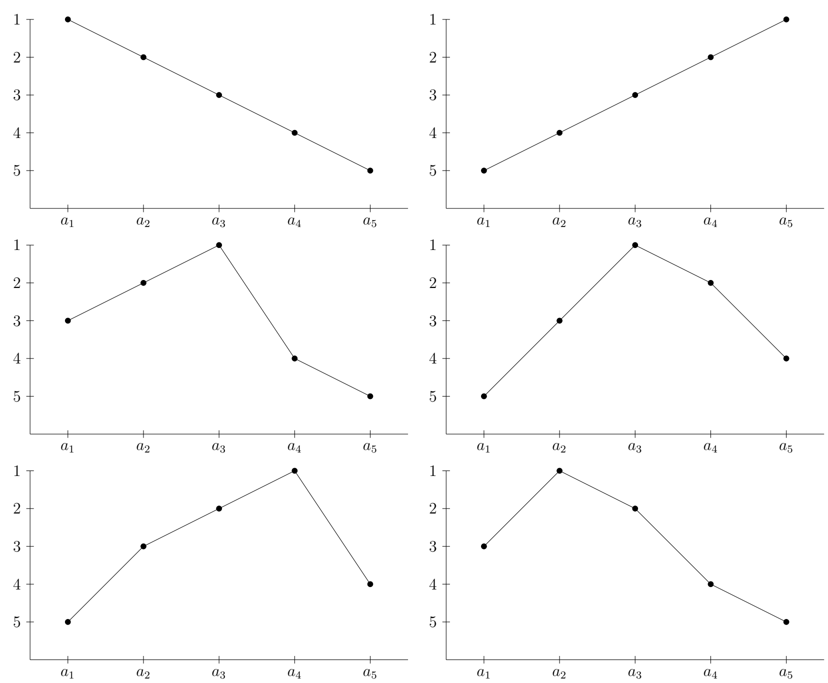

Figure 3.1: Some single-peaked preferences when $m = 5$

on A with respect to $>$. The vertical axis shows the number corresponding to the ranking of each alternative, with one being the most preferred, i.e. the peak. Notice how in each preference as we move away from the peak we move further down in the ranking of each alternative.[2]

[2]Note, the lines on these graphs are only added for visual purposes, but they suggest ways that

For the next definition, there will be a fixed linear order $>$ on the set of alternatives A.

Definition 3.2 *The set of all single-peaked orderings on A with respect to $>$ will be denoted $SP(A)$. The set of all profiles of linear orders where the preferences are from $SP(A)$ is called the* ***single-peaked domain*** *and is denoted $SP(A)^N$.*

Below is a concrete example of the permissible linear orderings within $SP(A)^N$ for a specific set of alternatives.

Example 3.4 *Let $A = \{a_1, a_2, a_3, a_4\}$ and let the underlying ordering on A be as follows:*

$$a_4 > a_3 > a_2 > a_1$$

Then for this specific case we have

$$SP(A) = \left\{ \begin{matrix} a_1 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & a_4 \\ a_2 & a_1 & a_3 & a_3 & a_4 & a_2 & a_2 & a_3 \\ a_3 & a_3 & a_1 & a_4 & a_2 & a_4 & a_1 & a_2 \\ a_4 & a_4 & a_4 & a_1 & a_1 & a_1 & a_4 & a_1 \end{matrix} \right\}$$

For small m it is easy to list all linear orders of A and then check which ones satisfy the underlying ordering and are thus in $SP(A)^N$. To understand the size of the single-peaked domain, and to verify Examples 3.3 and 3.4, below is an argument to show that $|SP(A)| = 2^{m-1}$.

First note the following two observations:

(1) If $A = \{a_1, a_2, \ldots, a_m\}$ and the ordering on A is defined by

$$a_m > a_{m-1} > \ldots > a_2 > a_1$$

then, for any $P_i \in SP(A)$, $r_m(P_i) \in \{a_1, a_m\}$.

---

the size of the alternative set can grow even to an extension into an infinite case.

(2) If $m \geq 3$, then for any $P_i \in SP(A)$ and for any $x \in A$,

$$P_i|_{A-\{x\}} \in SP(A - \{x\}).$$

We will include some explanation for why (2) holds. First, suppose x is any alternative other than the peak, $\tau(P_i)$. By Definition 3.1,

$$[\tau(P_i) > a > b \text{ or } b > a > \tau(P_i)] \;\Rightarrow\; aP_ib.$$

holds for all $a, b \in A$, thus it still holds for all $a, b \in A - \{x\}$. If $x = \tau(P_i)$, then $\tau(P_i|_{A-\{x\}}) = r_2(P_i)$ and we have

$$[\tau(P_i|_{A-\{x\}}) > a > b \text{ or } b > a > \tau(P_i|_{A-\{x\}})] \;\Rightarrow\; a\ P_i|_{A-\{x\}}\ b.$$

for all $a, b \in A - \{x\}$.

Proposition 3.1 *Let $A = \{a_1, a_2, \ldots, a_m\}$, then $|SP(A)| = 2^{m-1}$.*

We will use observations (1) and (2) and induction for the proof below. See Escoffier et al. [8] for a combinatorial proof of Proposition 3.1.

Proof:

First, consider the base case where $m = 2$. There are two linear orders on two elements and both are single-peaked, trivially.

Suppose by way of induction that for $m \geq 2$, $|SP(A)| = 2^{m-1}$. We wish to count the number of single-peaked linear orders on $A' = A \cup \{a_{m+1}\}$ where the underlying ordering on A is extended to A' so that $a_{m+1} > a_m > \ldots > a_1$. By (1), we know that for all $P_i \in SP(A')$, $r_{m+1}(P) \in \{a_1, a_{m+1}\}$. Hence we can take all $P_j \in SP(A)$ and append the alternative a_{m+1} to the bottom of the linear order and get 2^{m-1} single-peaked linear orders on $m+1$ alternatives. Now take the set A' and form all single-peaked linear ordering with respect to $>$. By (2), we can take each $P_i \in SP(A')$ and remove the alternative a_1, maintaining the relative orders of the other m elements and know that each $P_i|_{A'-\{a_1\}} \in SP(A' - \{a_1\})$. But now $|A' - \{a_1\}| = m$ so by the induction hypotheses, we know there are 2^{m-1} *distinct* single-peaked orderings on $A' - \{a_1\}$. By (1) we know that $r_m(P_i|_{A'-\{a_1\}}) \in \{a_2, a_{m+1}\}$. Thus we can append a_1 to the bottom of each distinct $P_i|_{A'-\{a_1\}}$ and still have a single-peaked ordering. Thus we have

$$|SP(A')| = 2(2^{m-1}) = 2^m = 2^{(m+1)-1}.$$

$\square$

Most of the definitions of properties of social choice functions given in Chapter 2 can be easily translated to this chapter simply by changing the domain of the function from $\mathscr{L}_C$ to $SP(A)^N$. This is not the case for neutrality, so we supply a new definition of what it means to be neutral in the context of the single-peaked domain.

Definition 3.3 *A social choice function $f : SP(A)^N \to A$ is neutral if for any profile $P \in SP(A)^N$ and for any permutation ϕ on A,*

$$\phi(P) \in SP(A)^N \Rightarrow f(\phi(P)) = \phi(f(P))$$

Notice that the implication in Definition 3.3 depends on $\phi(P) \in SP(A)^N$ which makes it distinct from Definition 2.5 which held for any permutation. In fact, there exist single-peaked profiles belonging to $SP(A)^N$ and permutations on A such that the permuted profiles do not belong to $SP(A)^N$.

Example 3.5 *Let $A = \{a_1, a_2, a_3, a_4\}$. For $a_4 > a_3 > a_2 > a_1$, the full domain, $SP(A)$, can be seen in Example 3.4 above. Consider the profile $P \in SP(A)^3$ below.*

$$P = \begin{pmatrix} a_1 & a_2 & a_3 \\ a_2 & a_3 & a_2 \\ a_3 & a_1 & a_4 \\ a_4 & a_4 & a_1 \end{pmatrix}$$

Let $\phi = (a_2 a_3)$ *be a permutation on* A*. Then if we apply* ϕ *to* P *we get*

$$\phi(P) = \begin{pmatrix} a_1 & a_3 & a_2 \\ a_3 & a_2 & a_3 \\ a_2 & a_1 & a_4 \\ a_4 & a_4 & a_1 \end{pmatrix}$$

Notice that $\phi(P_1) \notin SP(A)$ *hence* $\phi(P) \notin SP(A)^3$*.*

We now introduce an idea that is pivotal to establishing many of the results in the next section. Let $A = \{a_1, a_2, \ldots, a_m\}$ be the set of alternatives $m \geq 2$ with the underlying order on A as follows: $a_i > a_j$ if and only if $i > j$. From here on, we will assume this is the underlying ordering. We want to work with the permutation $\phi_k : A \to A$ for any integer $k \in [2, m]$ defined by

$$\phi_k(a_i) = a_{k+1-i} \text{ for } i = 1, \ldots, k$$

and

$$\phi_k(a_i) = a_i \text{ for } i = (k+1), \ldots, m.$$

We will refer to the permutation ϕ_k as the *k-inversion permutation*. In particular, ϕ_2 is the transposition given by $(a_1 a_2)$.

The k-inversion permutation is constructed so that preferences of a particular structure can be inverted and still be single-peaked. Note that if $P_i \in SP(A)$, $\phi_k(P_i)$ is not always necessarily a single-peaked preference as well (consider applying ϕ_2 in Example 3.5). But since Definition 3.3 relies on the fact that $\phi(P) \in SP(A)$, we will show that if $P_i \in SP(A)$, then $\phi_m(P_i) \in SP(A)$. That is, the m-inversion permutation will always maintain the single-peaked structure of any preference.

Notice that for any a, b in A,

$$a < b \iff \phi_m(a) > \phi_m(b).$$

For any $P_i \in L(A)$,

$$a \; P_i \; b \iff \phi_m(a) \; \phi_m(P_i) \; \phi_m(b).$$

In particular, a is ranked first in P_i if and only if $\phi_m(a)$ is ranked first in $\phi_m(P_i)$.

Proposition 3.2 *Let $A = \{a_1, a_2, \ldots, a_m\}$ where $m \geq 2$. If $P_i \in SP(A)$, then $\phi_m(P_i) \in SP(A)$.*

Proof:

Let $P_i \in SP(A)$. We know that the peak of P_i is $\tau(P_i)$ and the peak of $\phi_m(P_i)$ is $\phi_m(\tau(P_i))$. Thus,

$$\tau(\phi_m(P_i)) = \phi_m(\tau(P_i)).$$

Consider the situation

$$\tau(\phi_m(P_i)) > a > b.$$

Now $a = \phi_m(x)$ and $b = \phi_m(y)$ for some $x, y \in A$. So the previous relation becomes

$$\phi_m(\tau(P_i)) > \phi_m(x) > \phi_m(y).$$

Since ϕ_m inverts the underlying order $>$ on A it follows that

$$y > x > \tau(P_i).$$

Since P_i is single-peaked, it follows that xP_iy. Finally,

$$x \; P_i \; y \implies \phi_m(x) \; \phi_m(P_i) \; \phi_m(y).$$

Thus, $a \; \phi_m(P_i) \; b$ and $\phi_m(P_i)$ is single-peaked.

$\square$

3.2 Majority Rule on the Single-Peaked Domain

The focus of this section is on presenting results that help to characterize majority rule on the single-peaked domain. This question was brought up by a reviewer for Powers and Wells' recent paper and thus many of the proofs can be found in that publication [20]. Here we highlight key ideas that are needed to arrive at these previously published results.

We start off with a lemma that only requires a social-choice function, defined on the single-peaked domain, to satisfy strategy-proofness and neutrality.

Lemma 3.1 (Powers and Wells, 2023) *If $f : SP(A)^N \to A$ is strategy-proof and neutral, then f satisfies unanimity.*

The key to establishing Lemma 3.1 above is the k-inversion permutation defined above. Beginning with a single-peaked profile that has the same top-ranked alternative for all preferences, we can get the same profile using the k-inversion permutation and with our assumption of neutrality, this leads to a contradiction. To do this, we must be particularly careful about the structure of the preferences so that $P_i \in SP(A)$ and $\phi_k(P_i) \in SP(A)$. Specifically, the preference below is utilized. Notice that $\phi_k(P_i)$ will be single-peaked for all $k \in [2, m]$.

$$P_i = \begin{pmatrix} a_k \\ a_{k-1} \\ a_{k-2} \\ \vdots \\ a_2 \\ a_1 \\ a_{k+1} \\ a_{k+2} \\ \vdots \\ a_m \end{pmatrix} \qquad \phi_k(P_i) = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ \vdots \\ a_{k-1} \\ a_k \\ a_{k+1} \\ a_{k+2} \\ \vdots \\ a_m \end{pmatrix}$$

Lemma 3.1 presents an interesting contrast to results presented in the previous chapter, particularly those in Section 2.3. Recall that we had results that characterized strategy-proof and neutral social choice functions depending on whether or not the condition of unanimity was satisfied. Here we see that we cannot get a characterization for a strategy-proof and neutral social choice function that violates unanimity.

It is not true that the converse of Lemma 3.1 holds. Below is an example of a rule that satisfies strategy-proofness and unanimity but is not neutral.

Example 3.6 *Let $A = \{a_1, a_2, a_3\}$ and let the underlying ordering be $a_3 > a_2 > a_1$. Define $f : SP(A)^3 \to A$ be as follows: for any $P = (P_1, P_2, P_3) \in SP(A)^3$,*

$$f(P) = \min\{r_1(P_1), r_1(P_2), r_1(P_3)\}.$$

In other words, f outputs the element of A which is top ranked by some voter and has minimum index. It is clear that f satisfies unanimity.

It is not hard to see that f satisfies strategy-proofness since the social output is based on the minimum index. The only way a voter, say j, can change the output of f is by changing their top ranked alternative so that $r_1(P'_j) < \min\{r_1(P_1), r_1(P_2), r_1(P_3)\}$. But that means $r_1(P_j)\ P_j\ r_1(P'_j)$ so no manipulation would occur.

To see that f is not neutral, consider the following profile.

$$P = \begin{pmatrix} a_1 & a_2 & a_2 \\ a_2 & a_1 & a_1 \\ a_3 & a_3 & a_3 \end{pmatrix}$$

Then $f(P) = a_1$. Let $\phi = (a_1 a_3)$. Then applying ϕ to P we have $\phi(P) \in SP(A)^3$ shown below.

$$\phi(P) = \begin{pmatrix} a_3 & a_2 & a_2 \\ a_2 & a_3 & a_3 \\ a_1 & a_1 & a_1 \end{pmatrix}$$

Then $f(\phi(P)) = a_2$. Thus, $f(\phi(P)) \neq \phi(f(P))$ and f is not neutral.

Before presenting the next result, we need some notation. Let $P \in SP(A)^N$. The profile P_{a_m} is obtained from P by moving a_m to the bottom for each P_i. Notice that $P_{a_m} \in SP(A)^N$, as displayed in Figure 3.2. The peak of the preferences, $\tau(P_i)$, remains the same for all preferences where $\tau(P_i) \neq a_m$. Otherwise, the peak changes to a_{m-1}.

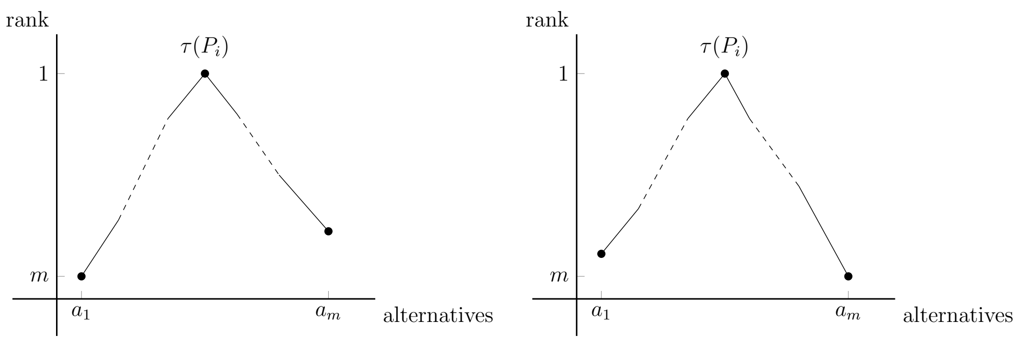

Figure 3.2: Forming P_{a_m} by moving a_m to the bottom of each P_i

The k-inversion permutation played an important role in establishing the next proposition which is Proposition 8 in Powers and Wells [20].

Proposition 3.3 (Powers and Wells, 2023) *Assume $f : SP(A)^N \to A$ is neutral, strategy-proof, and $m \geq 3$. If $g : SP(A)^N \to A$ is either a dictatorship or majority rule and*

$$f(P_{a_m}) = g(P_{a_m})$$

for all $P \in SP(A)^N$, then $f = g$.

What we know from Proposition 3.3 is that if f agrees with a dictatorship or majority rule on all P_{a_m}, then f is either a dictatorship or majority rule on all $P \in SP(A)^N$. This idea, along with Lemma 3.1, is used by Powers and Wells to establish the following result which presents a characterization of majority rule [20].

Theorem 3.1 (Powers and Wells, 2023) *Let $m \geq 2$. A social choice function $f : SP(A)^N \to A$ is strategy-proof, anonymous, and neutral if and only if n is odd and f is majority rule.*

Theorem 3.1 is basically Theorem 2.4 where the Condorcet domain, $\mathscr{L}_C$ is restricted to its subdomain, $SP(A)^N$ which necessitates the fact that n must be odd. We cannot have a function that outputs a single alternative and satisfies these three properties when n is even. A proof of this claim will follow in the next section.

If we wish to replace the condition of anonymity with the weaker condition that f must be non-dictatorial, we have a new result about majority rule that only holds for three voters.

Theorem 3.2 *Let $|A| = m \geq 2$. If $f : SP(A)^3 \to A$ is strategy-proof, neutral, and non-dictatorial, then f is majority rule.*

Proof:

Consider the base case where $m = 2$. Suppose $f : SP(A)^3 \to A$ is strategy-proof, neutral, and non-dictatorial. Then, by Lemma 3.1, f satisfies unanimity. So for any $P = (P_1, P_2, P_3) \in SP(A)^3$ where $r_1(P_i)$ agree for all i, $f(P) = r_1(P_i)$, which is clearly the majority element. So we must look at profiles that are not unanimous. Consider the profile

$$P = \begin{pmatrix} a_1 & a_2 & a_2 \\ a_2 & a_1 & a_1 \end{pmatrix}$$

We wish to show that $f(P) = a_2$, the majority element. Suppose by way of contradiction that $f(P) = a_1$. Then by strategy-proofness, a_1 can be moved up in P_2 or P_3 and maintain that the output is a_1. Hence $f(P) = a_1$ implies that $f(Q) = a_1$ for any $Q \in SP(A)^3$ where $r_1(Q_1) = a_1$. Furthermore, neutrality implies that for $\phi = (a_1 a_2)$, $f(\phi(P)) = \phi(a_1) = a_2$ and again by strategy-proofness we get that $f(Q) = a_2$ for any $Q \in SP(A)^3$ where $r_1(Q_1) = a_2$. Thus we get a contradiction with the assumption that f is non-dictatorial. A similar argument can be made in the cases where $r_1(P_2) \neq r_1(P_1), r_1(P_3)$ and $r_1(P_3) \neq r_1(P_1), r_1(P_2)$ resulting in a dictator for the disagreeing preference, which is always a contradiction. Hence f is majority rule.

Suppose by way of induction that for $m - 1 \geq 2$, if $f : SP(A)^3 \to A$ is strategy-proof, neutral, and non-dictatorial, then f is majority rule. We wish to show this holds true for m. Suppose $|A| = m$, $f : SP(A)^3 \to A$ is strategy-proof, neutral, and non-dictatorial, and $P \in SP(A)^3$ is an arbitrary profile. We wish to show that $f(P_{a_m}) = f_c(P_{a_m})$. Let $B = \{a_1, \ldots, a_{m-1}\}$ and define $h : SP(B)^3 \to B$ as follows: for any $Q \in SP(B)^3$,

$$h(Q) = f \begin{pmatrix} Q \\ a_m \end{pmatrix}$$

where $\begin{pmatrix} Q \\ a_m \end{pmatrix} \in SP(A)^3$ is formed by adjoining to $Q \in SP(B)^3$ $r_m(Q_i) = a_m$ for all $i \in \{1, 2, 3\}$. Since h is defined from f, h inherits the properties of neutrality and strategy-proof and we have that h is either a dictatorship or it is not.

Note that $f(P_{a_m}) = h(Q)$ where $P_{a_m} \in SP(A)^3, Q \in SP(B)^3$, and $Q = P|_B$. Suppose that h is a dictatorship. Then f is a dictatorship on all P_{a_m} and by Proposition 3.3 f is dictatorship on all $P \in SP(A)^3$, contradicting that f is non-dictatorial. Thus h must be non-dictatorial and applying Proposition 3.3 again, h must be majority rule. Hence $f(P_{a_m}) = h(Q) = f_C(Q)$ for any P_{a_m} and by Proposition 3.3 f is majority rule on all $P \in SP(A)^3$.

□

3.3 Additional Results on the Single-Peaked Domain

We begin this section with a simple example to show why we must restrict the previous result, Theorem 3.2, to $n = 3$.

Example 3.7 *Let $|A| = m \geq 2$ and $n \geq 4$. Define $f : SP(A)^N \to A$ to be the majority element on the first three preferences. This rule is strategy-proof, neutral, and non-dictatorial yet is not majority rule. This rule is well-defined as there will always be a majority element on the first three preferences since a profile of odd single-peaked preferences always gives a Condorcet winner.*

To see that this rule is distinct from majority rule consider the following profile

$$P = \begin{pmatrix} a_1 & a_1 & a_2 & \dots & a_2 \\ a_2 & a_2 & a_1 & \dots & a_1 \\ a_3 & a_3 & a_3 & \dots & a_3 \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_m & a_m & a_m & \dots & a_m \end{pmatrix}$$

The profile

$$P = \begin{pmatrix} a_1 & a_1 & a_2 & a_2 \\ a_2 & a_2 & a_1 & a_1 \\ a_3 & a_3 & a_3 & a_3 \\ \vdots & \vdots & \vdots & \vdots \\ a_m & a_m & a_m & a_m \end{pmatrix}$$

belongs to $SP(A)^N$ and the majority element of P is not defined. If $n \geq 5$, clearly the majority element is a_2 as it is ranked first for all P_i with $i = 3, \dots, n$ but the majority element on the first three preferences is a_1. In either case, $f(P) = a_1 \neq f_C(P)$.

Since f is defined as the majority element on the first three preferences it is not hard to see that f is neutral and non-dictatorial as majority rule is neutral and non-dictatorial. To verify that f is indeed strategy-proof, suppose to the contrary. That is suppose there exists $i \in \{1, 2, \dots, n\}$ and $P_i' \in SP(A)$ such that $f(P_i', P_{-i})\ P_i\ f(P_i, P_{-i})$. Since the only voters who have an affect on the social out-

come are the first three, we can assume that $i \in \{1,2,3\}$. Since $f(P_i', P_{-i})\ P_i\ f(P_i, P_{-i})$, the new output, $f(P_i', P_{-i})$, must have been $r_1(P_i) \neq f(P_i, P_{-i})$. Additionally, the original output, $f(P_i, P_{-i})$, must be the top ranked alternative for P_j and P_k where $j, k \in \{1,2,3\} \setminus \{i\}$. So the only change that can be made from P_i to P_i' is to change $r_1(P_i')$ to agree with $r_1(P_j) = r_1(P_k)$. Thus $f(P_i', P_{-i}) = f(P_i, P_{-i})$ contradicting the assumption that $f(P_i', P_{-i})\ P_i\ f(P_i, P_{-i})$.

In contrast to the example above which shows there is a strategy-proof, neutral, and non-dictatorial function that is distinct from majority rule, the proposition below shows this is not the case when we switch the condition of non-dictatorial for that of anonymity.

Proposition 3.4 *Let $m \geq 2$ and $n = 2k$ for some positive integer k. There is no social choice function $f : SP(A)^N \to A$ such that f is strategy-proof, neutral, and anonymous.*

Proposition 3.4 is a consequence of Theorem 3.1. Recall that Theorem 3.1 required n to be odd to get a characterization of majority rule. For cohesiveness, a proof of Proposition 3.4 is given below. The idea behind the proof is to construct a profile that has some symmetry and then use anonymity and neutrality to leverage this "symmetry" and get a contradiction.

Proof:

Let $m \geq 2$. Assume to the contrary that there does exist a function f that is strategy-proof, neutral, and anonymous that outputs a single alternative when n is even. First we will show that $|A| = m$ must be odd.

Let $P = (P_1, \ldots, P_n)$ be the single-peaked profile where the first $\frac{n}{2}$ entries are $a_1 \ldots a_m$ and the second $\frac{n}{2}$ entries are $a_m \ldots a_1$. So P looks like the profile shown below:

$$P = \begin{pmatrix} a_1 & \dots & a_1 & a_m & \dots & a_m \\ a_2 & \dots & a_2 & a_{m-1} & \dots & a_{m-1} \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots \\ a_m & \dots & a_m & a_1 & \dots & a_1 \end{pmatrix}$$

Apply the m-inversion permutation ϕ_m to the profile P and notice that $a_1 \dots a_m$ and $a_m \dots a_1$ each occur $\frac{n}{2}$-times. Hence, $\phi_m(P)$ is a permutation of the original profile P. Then anonymity implies $f(\phi_m(P)) = f(P)$. By neutrality, $f(\phi_m(P)) = \phi_m(f(P))$. Thus, $\phi_m(f(P)) = f(P)$. The only way the m-inversion permutation fixes an element of A is if m is odd and the element is $a_{\frac{m+1}{2}}$. Thus,

$$f(P) = a_{\frac{m+1}{2}}.$$

Let ℓ be the following single-peaked linear order on A:

$$\ell = a_{\frac{m+1}{2}} a_{\frac{m+1}{2}+1} \cdots a_m a_{\frac{m+1}{2}-1} \cdots a_1$$

So $r_1(\ell) = a_{\frac{m+1}{2}}$ and $r_2(\ell) = a_{\frac{m+1}{2}+1}$. Next, let ϕ be the transposition $(a_{\frac{m+1}{2}} \ a_{\frac{m+1}{2}+1})$ and note that $\phi(\ell) \in SP(A)$, $r_1(\phi(\ell)) = r_2(\ell)$ and $r_2(\phi(\ell)) = r_1(\ell)$. We now introduce the profile $Q = (Q_1, \dots, Q_n)$ defined by $Q_i = \ell$ for each $i = 1, 2, \dots, \frac{n}{2}$ and $Q_i = \phi(\ell)$ for all $i = \frac{n}{2} + 1, \dots, n$. By strategy-proofness,

$$f(P) = a_{\frac{m+1}{2}} \text{ implies } f(Q) = a_{\frac{m+1}{2}}.$$

Consider the profile $\phi(Q)$. Neutrality implies that $f(\phi(Q)) = \phi(f(Q)) = a_{\frac{m+1}{2}+1}$. But since $\phi(Q)$ is permutation of Q, anonymity implies $f(\phi(Q)) = f(Q) = a_{\frac{m+1}{2}}$, hence a contradiction.

$\square$

Proposition 3.4 is a result on the single-peaked domain but there could be a class of functions that are strategy-proof, neutral, and anonymous for $m \geq 3$ if we consider a slight restriction on the domain. In general,

$$SP(A)^N \subseteq \mathscr{L}_C$$

if and only if n is odd. If n is odd we know from the work of Black [3] that the Condorcet alternative exists. If n is even, then we could consider the domain

$$SP(A)_C = SP(A)^N \cap \mathscr{L}_C.$$

In other words, we can look at single-peaked profiles with an even number of voters in which there is a Condorcet winner. The class of functions that satisfy strategy-proofness, anonymity, and neutrality for $m = 2$ on $SP(A)_C$ were characterized in the previous chapter. The case for $m \geq 3$ on the domain $SP(A)_C$ still needs to explored.

All of the results thus far on the single-peaked domain have assumed our social choice function satisfies neutrality. If we no longer require that the functions satisfy neutrality, we can get a characterization that follows directly from the work of Weymark [24].

To understand the language of the following definition and result, the idea of a median must be discussed. The median of an ordered list of an odd number of elements is simply the middle element. That is, if $\{a_1, \ldots, a_t\}$ is an ordered list is of t alternatives, then the *median* is the element a_j such that

$$\left|\{i : a_j \geq a_i\}\right| \geq \frac{t}{2} \text{ and } \left|\{i : a_j \leq a_i\}\right| \geq \frac{t}{2}.$$

We will use the notation below to indicate that a_j is the median of the list $a_1, \ldots, a_t$.

$$med(a_1, \ldots, a_t) = a_j$$

When it comes to the median of a set of alternatives, the underlying ordering $>$ is used to establish the order that the alternatives are listed. The example below illustrates how we choose the median of a set of alternatives.

Example 3.8 *Let $A = \{a_1, a_2, a_3, a_4, a_5\}$ and let $f : SP(A)^N \to A$ be the function that outputs the median of the top ranked alternative for all voters. Suppose P is* *the profile shown below:*

$$P = \begin{pmatrix} a_1 & a_5 & a_5 & a_3 & a_3 \\ a_2 & a_4 & a_4 & a_2 & a_4 \\ a_3 & a_3 & a_3 & a_1 & a_5 \\ a_4 & a_2 & a_2 & a_4 & a_2 \\ a_5 & a_1 & a_1 & a_5 & a_1 \end{pmatrix}$$

The list of top ranked alternatives for this profile is as follows: a_1, a_5, a_5, a_3, a_3. *If we arrange this list with respect to the underlying ordering* $a_1 > a_2 > a_3 > a_4 > a_5$ *we have:* a_1, a_3, a_3, a_5, a_5. *Hence the median of this list is* a_3. *Therefore,* $f(P) = med(a_1, a_3, a_3, a_5, a_5) = a_3$.

Below is Weymark's definition of an $(n-1)$-parameter generalized median social choice function adapted to our notation.

Definition 3.4 *A social-choice function* $f : SP(A)^N \to A$ *is an* $(n-1)$*-parameter generalized median social choice function if there exist* $b_i \in A$ *for* $i = 1, \ldots, n-1$ *(the parameters) such that, for any* $P = (P_1, \ldots, P_n) \in SP(A)^N$, $f(P)$ *is the median element with respect to the ordering* $>$ *on* A *of the list*

$$r_1(P_1), \ldots, r_1(P_n), b_1, \ldots, b_{n-1}.$$

We denote this median element by

$$med(r_1(P_1), \ldots, r_1(P_n), b_1, \ldots, b_{n-1}).$$

We can use this definition to characterize all functions on the single-peaked domain that satisfy strategy-proofness, anonymity, and unanimity. This result is due to Weymark [24] but rephrased into the language and notation of our results.

Theorem 3.3 (Weymark, 2011) *A social choice function* $f : SP(A)^N \to A$ *is strategy-proof, anonymous, and satisfies unanimity if and only if* f *is an* $(n-1)$*-parameter generalized median social choice function.*

If n is odd, then the Condorcet rule f_C is an $(n-1)$-parameter generalized median social choice function where half of the parameters equal a_1 and the other half equal a_m. Choosing these as the parameters in this specific case is necessary for the function to satisfy neutrality, which we know must always hold by Theorem 3.1. An example where n is odd is illustrated below.

Example 3.9 *Let $n = 5$, $m = 4$, and $f : SP(A)^N \to A$ be an $(n-1)$-parameter generalized median social choice function. Consider the profile $P = (P_1, \dots, P_n) \in SP(A)^N$ below.*

$$P = \begin{pmatrix} a_1 & a_2 & a_3 & a_2 & a_3 \\ a_2 & a_3 & a_4 & a_1 & a_2 \\ a_3 & a_4 & a_2 & a_3 & a_4 \\ a_4 & a_1 & a_1 & a_4 & a_1 \end{pmatrix}$$

Then for $b_1, b_2 = a_1$ and $b_3, b_4 = a_4$ we get

$$f(P) = med(a_1, a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_4) = a_2 = f_C(P)$$

To illustrate that neutrality holds in this case consider the k-inversion permutation with $k = 4$. Then applying ϕ_4 to the profile P we have the following profile

$$\phi_4(P) = \begin{pmatrix} a_4 & a_3 & a_2 & a_3 & a_2 \\ a_3 & a_2 & a_1 & a_4 & a_3 \\ a_2 & a_1 & a_3 & a_2 & a_1 \\ a_1 & a_4 & a_4 & a_1 & a_4 \end{pmatrix}$$

Then for the same parameters as above we see that

$$f(\phi_4(P)) = med(a_1, a_1, a_2, a_2, a_3, a_3, a_4, a_4, a_4) = a_3 = f_C(\phi_4(P)).$$

Additionally, $f(\phi_4(P)) = \phi_4(f(P)) = \phi_4(a_2) = a_3$.

3.4 Conclusion

Both this chapter and the previous chapter had much discussion of domains for which we could get our characterization of majority rule as the only strategy-proof, anonymous, and neutral function. But we know that this characterization does not always hold when we alter the domain of interest. As pointed out earlier in the chapter, we wonder what the class of functions that satisfy these three properties will look like on $SP(A)_C$. Recall $SP(A)_C$ is defined for even n. When n is odd, we can construct a restriction of $SP(A)^N$ that produces a rule that is strategy-proof, anonymous, and neutral, but is not f_C. We conclude this chapter with an interesting example to illustrate this idea.

Example 3.10 *Let $A = \{a_1, a_2, \ldots, a_m\}$ with our standard underlying ordering. Let Σ be the subset of $SP(A)$ defined by*

$$\Sigma = \{a_1 a_2 \ldots a_{m-1} a_m,\ a_m a_{m-1} \ldots a_2 a_1\}.$$

In other words, the profiles will only consist of the above two single-peaked preferences - the one that is strictly decreasing and the one that is strictly increasing.

Assume n is odd. For $m \geq 3$ and odd, define $f : \Sigma^N \to A$ by $f(P) = a_{\frac{m+1}{2}}$ for all $P \in \Sigma^N$. Since we are only considering profiles where each preference is one of two linear orders, it is not difficult to prove that f is strategy-proof, anonymous, and neutral.

CHAPTER 4
EXPANDING THE DOMAIN TO ALLOW FOR INDIFFERENCE

Thus far, all results have been on profiles that contain preferences that are strict linear orders. In this chapter, we introduce the idea of voter indifference. A voter is indifferent between two alternatives, say $x, y \in A$, if x is as at least as good as y and y is at least as good as x. Allowing for voter indifference expands the domain that we are considering since we now want to allow for weak orders of the alternatives.

4.1 Preliminaries

As usual, $A = \{a_1, a_2, \ldots, a_m\}$ is the set of alternatives. A binary relation, R, on A is a subset of $A \times A$. We will write $a_i R a_j$ if the ordered pair (a_i, a_j) belongs to R. In addition, we will write $a_i \not R\, a_j$ if (a_i, a_j) does not belong to R. We begin by giving a formal definition of a weak order.

Definition 4.1 *A* ***weak order*** *on A is a reflexive, complete, and transitive binary relation, R, on A.*

By reflexive, we mean that aRa for all $a \in A$. By complete, we mean that for all $a_i, a_j \in A$, $a_i R a_j$ and/or $a_j R a_i$. Finally, transitive means for all $a_i, a_j, a_k \in A$, if $a_i R a_j$ and $a_j R a_k$, then $a_i R a_k$.

Below we give a formal statement of what we mean by indifference.

Definition 4.2 *We say that alternative a_i and alternative a_j are* ***indifferent*** *based* *on the weak order R on A if $a_i R a_j$ and $a_j R a_i$. Moreover, the* ***indifference class*** *containing a_i is the set of all elements from A that are indifferent to a_i.*

When working with strict linear orderings we used the notation $P = (P_1, \ldots, P_n)$ for a profile of linear preferences and xP_iy to say voter i prefers alternative x to alternative y. In the context of weak orders, we change this notation to $P = (R_1, \ldots, R_n)$ for a profile and xR_iy to say that x is at least as good as y.

A linear ordering is a weak ordering, but not all weak orderings are linear orderings. Thus the domain of all weak orderings on A has larger cardinality than the domain of linear orderings on A. In this chapter, we are generally interested in exploring domains that are between the domain of linear orders and of weak orders. Let $W(A)$ denote the set of weak orders on A. Let D be a pre-domain in the following sense:

$$L(A) \subseteq D \subseteq W(A).$$

To go from the pre-domain, D, to the collection of profiles we will be working with in this chapter we need some notation. First, we define rank in the context of weak orders. Let k be a positive integer. The notation $r_k(R_i)$ represents the kth ranked indifference class of individual i. Since there can be indifference in preferences, $r_k(R_i)$ can be one single alternative or a set of alternatives. The first, or top, ranked indifference class is denoted $r_1(R_i)$ and is defined as follows

$$r_1(R_i) = \{x \in A : xR_iy \text{ for all } y \in A\}.$$

Next,

$$r_2(R_i) = \{x \in A : xR_iy \text{ for all } y \in A \setminus r_1(R_i)\}.$$

This generalizes as follows

$$r_k(R_i) = \{x \in A : xR_iy \text{ for all } y \in A \setminus r_1(R_i) \cup r_2(R_i) \cup \ldots \cup r_{k-1}(R_i)\}.$$

Below we give an example that further illustrates some of the definitions and notation that we have presented.

Example 4.1 *Let* $A = \{a_1, a_2, a_3, a_4, a_5\}$*. Consider the following profile* $P = (R_1, R_2, R_3)$*.*

$$P = \begin{pmatrix} a_1 & a_1a_2 & a_3a_5 \\ a_2a_4 & a_3a_4 & a_1 \\ a_3 & a_5 & a_4 \\ a_5 & & a_2 \end{pmatrix}$$

Then $r_2(R_2) = \{a_3, a_4\}$ *whereas* $r_2(R_3) = a_1$*. Additionally,* R_2 *shows that* a_3 *is indifferent to* a_4 *so we can call* $\{a_3, a_4\}$ *the indifference containing* a_4 *for* R_2*. Notice that* R_1 *has a different indifference class containing* a_4*, namely* $r_2(R_1) = \{a_2, a_4\}$*.*

We need some more notation. For any profile, $P = (R_1, R_2, \ldots, R_n)$, belonging to D^N and for any subset $\{x, y\}$ of A,

$$N_{xy}(P) = \{i \in N : xR_iy \text{ and } y \not R_i\, x\}$$

is the set of individuals that rank x in an indifference class above y in the profile P. Additionally,

$$N_{[xy]}(P) = \{i \in N : x \text{ is indifferent to } y\}$$

is the set of individuals that rank x and y in the same indifference class in the profile P.

We can now define the domain of interest. The notation $\mathscr{D}_C$ is for the set of all profiles of weak orders $P = (R_1, R_2, \ldots, R_n) \in D^N$ such that there exists $x \in A$ satisfying

$$|N_{xy}(P)| > |N_{yx}(P)|$$

for all $y \in A \setminus \{x\}$. In particular, $\mathscr{W}_C$ denotes the set of all profiles of weak orders $P = (R_1, R_2, \ldots, R_n) \in W(A)^N$ such that there exists $x \in A$ satisfying

$$|N_{xy}(P)| > |N_{yx}(P)|$$

for all $y \in A \setminus \{x\}$. Observe that

$$L(A) \subseteq D \subseteq W(A) \implies \mathscr{L}_C \subseteq \mathscr{D}_C \subseteq \mathscr{W}_C.$$

We now extend the Condorcet rule from $\mathscr{L}_C$ to $\mathscr{D}_C$ in the obvious way.

Definition 4.3 *The function $f_C : \mathscr{D}_C \to A$ is defined as follows: for any profile $P \in \mathscr{D}_C$, $f_C(P) = x$ where x is the Condorcet alternative with respect to P. Then f_C is the **Condorcet rule** on $\mathscr{D}_C$.*

The domain $\mathscr{D}_C$ can be equal to $\mathscr{L}_C$ or equal to $\mathscr{W}_C$. Later we give examples of a $\mathscr{D}_C$ that is strictly between $\mathscr{L}_C$ and $\mathscr{W}_C$. Below we give an example of a different domain that is between $\mathscr{L}_C$ and $\mathscr{W}_C$ but not a $\mathscr{D}_C$.

Example 4.2 *Let I denote the complete indifference relation on a set A. An individual chooses I if they have no strict preferences among the alternatives in A. Given $n \geq 3$ and a profile $P = (R_1, R_2, \ldots, R_n) \in (L(A) \cup \{I\})^N$, let top$(P)$ be the set of alternatives that are ranked first by at least one individual in the profile P. Notice that top$(P) = \emptyset$ if and only if $R_i = I$ for all i. Let*

$$Z = \{P \in (L(A) \cup \{I\})^n : |\{i : R_i = I\}| \geq (n-2) \text{ and } |top(P)| = 1\}.$$

So Z contains profiles where at least $n-2$ individuals choose complete indifference and either one or two individuals choose a linear order on A with the same top ranked alternative. The Condorcet rule is well defined on Z by [20] thus we have the domain $\mathscr{L}_C \cup Z$ and can see that

$$\mathscr{L}_C \subseteq \mathscr{L}_C \cup Z \subseteq \mathscr{W}_C.$$

The addition of weak orders to the domain, once again requires us to define neutrality for this domain where $\phi : A \to A$ is a permutation. For this definition we use the notation, $\phi(P)$. By this we mean, for any profile $P = (R_1, R_2, \ldots, R_n)$ and for any $\phi : A \to A$, $\phi(P) = (\phi(R_1), \phi(R_2), \ldots, \phi(R_n))$.

Definition 4.4 *A social choice function $f : \mathscr{D}_C \to A$ is neutral if for any profile $P \in \mathscr{D}_C$ and any permutation $\phi : A \to A$, if $\phi(P) \in \mathscr{D}_C$ then $f(\phi(P)) = \phi(f(P))$.*

Similar to Definition 3.3 for neutrality in the context of single-peakedness, Definition 4.4 depends on the permuted profile, $\phi(P)$, remaining in the domain. Indeed, we can construct domains where $\phi(P)$ is not always an element of the domain for some permutations ϕ. This is illustrated below.

Example 4.3 *Let $A = \{a_1, a_2, a_3, a_4\}$. Let our pre-domain be $D = L(A) \cup \{\hat{R}\}$ where*

$$\hat{R} = \begin{pmatrix} a_1 \\ a_2 \\ a_3a_4 \end{pmatrix}$$

Let $n = 3$ and suppose we have the following profile:

$$P = \begin{pmatrix} a_1 & a_2 & a_1 \\ a_2 & a_1 & a_2 \\ a_4 & a_4 & a_3a_4 \\ a_3 & a_3 & \end{pmatrix}$$

Then $P \in \mathscr{D}_C$ with $f_C(P) = a_1$. If $\phi = (a_3a_4)$, then $\phi(P) \in \mathscr{D}_C$. However, if $\phi' = (a_1a_4)$, then $\phi'(P) \notin \mathscr{D}_C$ since a_1 and a_3 are not indifferent in the pre-domain.

$$\phi(P) = \begin{pmatrix} a_1 & a_2 & a_1 \\ a_2 & a_1 & a_2 \\ a_3 & a_3 & a_3a_4 \\ a_4 & a_4 & \end{pmatrix} \qquad \phi'(P) = \begin{pmatrix} a_4 & a_2 & a_4 \\ a_2 & a_4 & a_2 \\ a_1 & a_1 & a_3a_1 \\ a_3 & a_3 & \end{pmatrix}$$

4.2 Majority Rule with Weak Orders

In this section we present a characterization of majority rule on the domain $\mathscr{D}_C$. This question was brought up by a reviewer for Powers and Wells recent paper [20] and this is an expansion of that work which gives a more robust result.

Theorem 4.1 *Let D be a pre-domain such that*

$$L(A) \subseteq D \subseteq W(A)$$

for $m \geq 3$ and $n \geq 2$. A social choice function $f : \mathscr{D}_C \to A$ is strategy-proof, neutral, and anonymous if and only if $f = f_C$.

Since majority rule always satisfies strategy-proofness, neutrality, and anonymity, our proof focuses on the forward direction of Theorem 4.1.

Proof:

We will give the following proof in two cases.

First, consider the case for $n = 2$. Assume $P = (R_1, R_2) \in \mathscr{D}_C$, with $f_C(P) = x$, $f(P) = y$, and $x \neq y$. Let ϕ be the permutation on A equaling the transposition (xy) and let ℓ be a linear order on A such that x is ranked first and y is ranked second. Since $f_C(P) = x$, we know that x must be ranked on top in both preferences.

If $r_1(R_1) = r_1(R_2) = \{x\}$, then, one at a time, replace R_1 and R_2 with the linear order ℓ. Strategy-proofness implies $f_C(\ell, \ell) = x$ and $f(\ell, \ell) = y$. Now consider the profile $P' = (\phi(\ell), \phi(\ell))$. Strategy-proofness implies that $f(\phi(\ell), \phi(\ell)) = y$ but neutrality implies $f(\phi(\ell), \phi(\ell)) = \phi(f(\ell, \ell)) = x$, a contradiction.

If $r_1(R_1) = \{x\}$ and $\{x\} \subset r_1(R_2)$, then we can replace R_1 with ℓ and maintain that $f(\ell, R_2) = y$ by strategy-proofness. Apply ϕ and see that neutrality implies $f(\phi(\ell), R_2) = \phi(y) = x$ when $2 \in N_{[xy]}(P)$ and $f(\phi(\ell), \phi(R_2)) = \phi(y) = x$ when $2 \notin N_{[xy]}(P)$. On the other hand, strategy-proofness implies that $f(\phi(\ell), R_2) = f(\phi(\ell), \phi(R_2)) = y$, a contradiction.

Finally, consider $\{x\} \subset r_1(R_1)$ and $\{x\} \subset r_1(R_2)$. Since $f_C(P) = x$ we know that x can be the only alternative that is contained in both top ranked indifference classes. Without loss of generality, we will assume $y \notin r_1(R_1)$. We can replace R_1 with ℓ and maintain that $f(\ell, R_2) = y$. But this has already been established by the previous two cases so this completes the proof for $n = 2$.

For our second case, assume $n \geq 3$. Let $P = (R_1, \ldots, R_n)$ be a profile belonging to $\mathscr{D}_C$. Recall that if $P \in \mathscr{L}_C$ then $f(P) = f_C(P)$ by Theorem 2.4 and we are done. We are interested in profiles in which at least one $R_i \in W(A) \setminus L(A)$.

Assume there exists $P \in \mathscr{D}_C$ such that $f_C(P) = x, f(P) = y$, and $x \neq y$. Let $|N_{xy}(P)| = k,\ |N_{yx}(P)| = h$ with $k > h$. Since f is anonymous we can arrange the preferences in P so that $\{1, \ldots, k\} = N_{xy}(P)$, $\{k+1, \ldots, k+h\} = N_{yx}(P)$, and $\{(k+h), \ldots, n\} = N_{[xy]}(P)$.

From $P = (R_1, \ldots, R_n)$ obtain $P' = (R'_1, \ldots, R'_n)$ by one at a time replacing each R_i in P in the following way:

$$R'_i = \begin{cases} \ell & \text{if } i \in N_{xy}(P) \\ \phi(\ell) & \text{if } i \in N_{yx}(P) \\ R_i & \text{if } i \in N_{[xy]}(P) \end{cases}$$

Notice that $N_{xy}(P) = N_{xy}(P')$ and $N_{yx}(P) = N_{yx}(P')$. Using the fact that f_C and f are strategy-proof it follows that

$$f_C(P') = x \text{ and } f(P') = y.$$

If $N_{[xy]}(P) = \emptyset$, then $N_{[xy]}(P') = \emptyset$ as well. Upon the final change from R_i to R'_i we get $P' \in \mathscr{L}_C$. This results in a contradiction with Theorem 2.4 as $f(P') = y \neq f_C(P')$.

If $N_{[xy]}(P) \neq \emptyset$, proceed as follows. Apply ϕ to P' to get the profile $\phi(P') \in$

$\mathscr{D}_C$ shown below. Recall that R_i is such that $i \in N_{[xy]}(P)$ so $R_i = \phi(R_i)$.

$$\phi(P') = \begin{pmatrix} y & \dots & y & x & \dots & x & R_i & \dots \\ x & \dots & x & y & \dots & y & & \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots \end{pmatrix}$$
$$\underbrace{\qquad\quad}_{k} \quad \underbrace{\qquad\quad}_{h} \quad \underbrace{\qquad}_{n-(k+h)}$$

By neutrality, $f_C(\phi(P')) = \phi(f_C(P')) = y$ and $f(\phi(P')) = \phi(f(P')) = x$. Notice that $|N_{yx}(\phi(P'))| = k, |N_{xy}(\phi(P'))| = h$, and $|N_{[xy]}(\phi(P'))| = n - (k + h)$.

Let $r = k - h > 0$. First assume r is odd. We can obtain a rearrangement of P' from $\phi(P')$ by one at a time moving x up in the first r preferences. Since r is odd, at each step either x or y will be a Condorcet winner thus we always stay in the domain. Call the resulting profile R which is shown below.

$$R = \begin{pmatrix} x & \dots & x & y & \dots & y & x & \dots & x & R_i & \dots \\ y & \dots & y & x & \dots & x & y & \dots & y & & \\ \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots & \vdots & \vdots & \dots \end{pmatrix}$$
$$\underbrace{\qquad\quad}_{r} \quad \underbrace{\qquad\quad}_{k-r} \quad \underbrace{\qquad\quad}_{h} \quad \underbrace{\qquad}_{n-(k+h)}$$

At each step, strategy-proofness implies $f(R) = x$. Notice that R is a rearrangement of the profile P' since $|N_{xy}(R)| = r + h = k$ and $|N_{yx}(R)| = k - r = h$. By anonymity, $f(R) = f(P')$ which contradicts our earlier assumption that $f(P') = y$.

Now assume $r = k-h$ is even. So $r \geq 2$. Choose one R_i from the profile $\phi(P')$ such that $i \in N_{[xy]}(P)$ and replace R_i with the linear order ℓ. Call the resulting profile W. Now $|N_{xy}(W)| = h+1$ and $|N_{yx}(W)| = k$. Let $r' = k-(h+1) = r-1$. Since r is even, $|N_{yx}(W)| > |N_{xy}(W)|$, hence $f_C(W) = f_C(\phi(P')) = y$. By strategy-proofness, $f(W) = x$. If $|N_{[xy]}(P)| = 1$, W is a profile such that $N_{[xy]}(W) = \emptyset$. Then, as above, we get a contradiction with Theorem 2.4. On the other hand, if $|N_{[xy]}(P)| \geq 2$, notice that by breaking the tie to form the profile W we have that

$r' = r - 1$ must be an odd value. Hence we get a contradiction by the argument above. Therefore we can conclude $f(P) = f_C(P)$ for all $P \in \mathscr{D}_C$.

$\square$

Similar to the work of Chapter 2, this characterization of majority rule requires that the size of the alternative set be three or more. For the $m = 2$ case, we know how strategy-proof, neutral, and anonymous functions would behave if $D = L(A)$ (see Proposition 2.1). But for any larger pre-domain, D, the question remains open. The recent work of Lahiri and Pramanik [12] does give some insight into voting over two alternatives with indifference. However, their domain does not require that a strict Condorcet alternative exist. Additionally, they look at the surjective condition rather than neutrality.

Consider the following example to help illustrate Theorem 4.1 and the domain $\mathscr{D}_C$.

Example 4.4 *Let $A = \{a_1, a_2, a_3\}$, $n = 3$, and let $D = L(A) \cup \{I\}$ where I is the complete indifference relation on A. Then*

$$\mathscr{D}_C = \{P = (R_1, R_2, R_3) \in D^3 : |\{i : R_i = I\}| \leq 2 \text{ and } f_C(P) \text{ exists}\}$$

Suppose $f : \mathscr{D}_C \to A$ is strategy proof, neutral, and anonymous. Consider the profile below

$$P = \begin{pmatrix} a_1 & a_1 & \\ a_2 & a_3 & I \\ a_3 & a_2 & \end{pmatrix}$$

Suppose $f(P) \neq f_C(P) = a_1$. Let $\phi = (a_2 a_3)$. Then we have the profile

$$\phi(P) = \begin{pmatrix} a_1 & a_1 & \\ a_3 & a_2 & I \\ a_2 & a_3 & \end{pmatrix}$$

Notice that $\phi(P)$ is just a rearrangement of the profile P and since f is anonymous, $f(P) = f(\phi(P))$. Since f is neutral, $f(P)$ must be fixed by ϕ. Hence $f(P) = f_C(P) = a_1$.

Notice that the set $\mathscr{D}_C$ given in the previous example is similar to $\mathscr{L}_C \cup Z$, the domain defined in Example 4.2. Even though that is not a $\mathscr{D}_C$ domain, we can still get a characterization of majority rule. This result was discussed in the remarks of Powers and Wells' paper [20].

Proposition 4.1 (Powers and Wells 2023) *For any $m \geq 3$ and any $n \geq 3$, a voting rule $f : \mathscr{L}_C \cup Z \to A$ is strategy-proof, neutral, and anonymous, if and only if $f = f_C$.*

Below we present a simple example to show that Theorem 4.1 does not hold if we replace anonymity with the non-dictatorial assumption.

Example 4.5 *Let $A = \{a_1, a_2, \ldots, a_m\}$, $m \geq 2$ and let $D = L(A) \cup \{I\}$ where I is the complete indifference relation on A. For $n = 4$, define $f : \mathscr{D}_C \to A$ as follows: for any profile $P = (R_1, R_2, R_3, R_4)$,*

$$f(P) = \begin{cases} r_1(R_1) & \text{if } \{i : R_i = I\} = \{4\}; \\ f_c(P) & \text{otherwise.} \end{cases}$$

Clearly the above example is not majority rule. Let us verify that it is indeed not anonymous but is non-dictatorial, neutral, and strategy proof.

We can see that f is not anonymous through the following example. Let $P \in \mathscr{D}_C$ be the following profile.

$$P = \begin{pmatrix} a_1 & a_2 & a_2 & I \\ a_2 & a_1 & a_1 & \end{pmatrix}$$

By definition, $f(P) = a_1$. Let $\sigma : N \to N$ be given by $\sigma = (34)$. Then $P_\sigma \in \mathscr{D}_C$ and $f(P_\sigma) = f_C(P_\sigma) = a_2 \neq f(P)$, hence f is not anonymous.

Referring to the profile, P_σ, above we can see that f is non-dictatorial. Since $f(P) = r_1(R_1)$ whenever $\{i : R_i = I\} = \{4\}$, voter 1 is the only possible dictator. But since there exists a profile in the domain where the output is not the first ranked alternative of voter 1, clearly f is non-dictatorial.

It is easy to see that neutrality holds. If $f(P) = r_1(R_1)$, then applying any permutation, $\phi : A \to A$, does not change that $R_4 = I$. Furthermore, since a dictatorship is neutral, $f(\phi(P)) = \phi(f(P)) = \phi(r_1(R_1))$. If $f(P) = f_C(P)$, then the neutrality of majority rule guarantees the result.

Finally we show that f is strategy-proof. First suppose $P \in \mathscr{D}_C$ is such that $\{i : R_i = I\} = \{4\}$. Then $f(P) = r_1(R_1)$. If $P' = (R'_j, R_{-j}) \in \mathscr{D}_C$ is such that $\{i : R'_i = I\} = \{4\}$, so that the only change from P to P' occurs in R_1, R_2, or R_3, then f is strategy-proof in this case as a dictatorship is strategy-proof. Similarly, suppose $P \in \mathscr{D}_C$ is such that $\{i : R_i = I\} \neq \{4\}$. Then $f(P) = f_C(P)$. If $P' \in \mathscr{D}_C$ is such that $\{i : R'_i = I\} \neq \{4\}$ then f is strategy-proof in this case as majority rule is strategy-proof. Thus the only case we need to be concerned of a possible manipulation is when P changes to P' in such a way that $\{i : R_i = I\} = \{4\}$ changes to $\{i : R'_i = I\} \neq \{4\}$ and vice versa.

Suppose $P \in \mathscr{D}_C$ is such that $\{i : R_i = I\} = \{4\}$ and $P' = (R'_j, R_{-j}) \in \mathscr{D}_C$ is such that $\{i : R'_i = I\} \neq \{4\}$. If $R'_j = R'_4$, then $f(P')$ is no better for voter 4 since $R_4 = I$ originally. Since $P \in \mathscr{D}_C$ with say $f_C(P) = x$, for some $x \in A$, then $|N_{xy}(P)| > |N_{yx}(P)|$ for all $y \in A \setminus \{x\}$. If R_j for $j = 1, 2$, or 3 changes their preference to $R'_j = I$ then since $P' \in \mathscr{D}_C$, $|N_{xy}(P')| > |N_{yx}(P')|$ for all $y \in A \setminus \{x\}$. Finally, suppose $P \in \mathscr{D}_C$ is such that $\{i : R_i = I\} \neq \{4\}$ and $P' = (R'_j, R_{-j})$ is such that $\{i : R'_i = I\} = \{4\}$. If $r_1(R'_1) = f_c(P)$ then $f(P) = f(P')$. If $r_1(R'_1) \neq f_C(P)$, with say $r_1(R'_1) = y$ and $f_C(P) = x$, then $|N_{xy}(P)| = 2$ implying $x\ R_4\ y$. Thus no manipulation occurs in this case. Therefore f is strategy-proof.

4.3 Strategy-proof and Neutral Social Choice Functions

We begin by defining a restriction of the domain $\mathscr{D}_C$. We are interested in the subset of profiles where all preferences have a singleton as the top ranked alternative *and* a singleton as the second ranked alternative but can have indifference elsewhere. We will use the following notation for these preferences:

$$W_2^1(A) = \{R_i \in W(A) : \text{a singleton ranked on top and second}\}$$

Then our domain is as follows:

$$\mathscr{D}_C^* = \{P \in \mathscr{D}_C : R_i \in W_2^1(A) \text{ for all } i \in \{1, \ldots, n\}\}$$

Since $L(A) \subseteq W_2^1(A)$, our chain of domains now look like this

$$\mathscr{L}_C \subseteq \mathscr{D}_C^* \subseteq \mathscr{W}_C$$

Below is an example of the type of domain we are interested in.

Example 4.6 *Let $A = \{a_1, a_2, a_3, a_4\}$. Then $\mathscr{D}_C^*$ consists of profiles that only have preferences that are strict linear orders or come from the set below.*

$$\left\{\begin{array}{cccccccccccc} a_1 & a_1 & a_1 & a_2 & a_2 & a_2 & a_3 & a_3 & a_3 & a_4 & a_4 & a_4 \\ a_2 & a_3 & a_4 & a_1 & a_3 & a_4 & a_1 & a_2 & a_4 & a_1 & a_2 & a_3 \\ a_3a_4 & a_2a_4 & a_2a_3 & a_3a_4 & a_1a_4 & a_1a_3 & a_2a_4 & a_1a_4 & a_1a_2 & a_2a_3 & a_1a_3 & a_1a_2 \end{array}\right\}$$

By defining such a domain, we hope to be able to extend the results of Section 2.3 to include some indifference in the preferences. Consider the function shown below.

Example 4.7 *Let $m \geq 3$ and $n = 2k \geq 4$. Define $g : \mathscr{D}_C^* \to A$ by*

$$g(P) = r_1\big(R_1|_{A - f_C(P)}\big) = \begin{cases} r_2(R_1) & \textit{if } f_C(P) = r_1(R_1) \\ r_1(R_1) & \textit{if } f_C(P) \neq r_1(R_1) \end{cases}$$

This rule is neutral and strategy-proof. Below we prove that this is indeed true.

Let $P \in \mathscr{D}_C^*$ and let ϕ be a permutation on A such that $\phi(P) \in \mathscr{D}_C^*$. Since majority rule is neutral, $f_C(\phi(P)) = \phi(f_C(P))$. Then by the definition of g we have

$$g(\phi(P)) = \begin{cases} r_2(\phi(R_1)) & \text{if } f_C(\phi(P)) = r_1(\phi(R_1)) \\ r_1(\phi(R_1)) & \text{if } f_C(\phi(P)) \neq r_1(\phi(R_1)) \end{cases}$$

Then we see $r_2(\phi(R_1)) = \phi(r_2(R_1))$ and $r_1(\phi(R_1)) = \phi(r_1(R_1))$. Also

$$f_C(\phi(P)) = r_1(\phi(R_1)) \iff \phi(f_C(P)) = \phi(r_1(R_1)) \iff f_C(P) = r_1(R_1)$$

since ϕ is a one-to-one function. Thus we can re-write $g(\phi(P))$ as

$$g(\phi(P)) = \begin{cases} \phi(r_2(R_1)) & \text{if } f_C(P) = r_1(R_1) \\ \phi(r_1(R_1)) & \text{if } f_C(P) \neq r_1(R_1) \end{cases} = \phi(g(P))$$

Now to see that g is strategy-proof, suppose $P \in \mathscr{D}_C^*$ and $R_i' \in W_2^1(A)$ with $i \in \{1, 2, \ldots, n\}$ satisfies $Q = (R_i', R_{-i}) \in \mathscr{D}_C^*$. Since n is even and $P, Q \in \mathscr{D}_C^*$, $f_C(P) = f_C(Q)$. If $i \neq 1$, then $Q_1 = R_1$ thus it follows that $g(P) = g(Q)$. But if $i = 1$ we have two possibilities. First, if $g(P) = r_2(R_1)$ then $f_C(P) = r_1(R_1) = f_C(Q)$. If $g(Q) \neq f_C(Q)$ then $g(Q) = g(P) = r_2(R_1)$ or $g(P)R_1g(Q)$. In either case, no manipulation occurs. Second, if $g(P) = r_1(R_1)$ then either $g(Q) = g(P) = r_1(R_1)$ or $g(P)R_1g(Q)$. Thus again, no manipulation occurs. Additionally g is clearly not anonymous nor does it satisfy unanimity.

Example 4.7 rephrases Example 2.4 within the context of the domain $\mathscr{D}_C^*$. This example gives rise to the idea of a majority avoiding dictatorship. Below we extend the definition of majority avoiding dictatorship from $\mathscr{L}_C$ to $\mathscr{D}_C^*$.

Definition 4.5 *We will say that $g : \mathscr{D}_C^* \to A$ is a* ***majority avoiding dictatorship*** *(MAD) if there exists $j \in N$ such that $g(P) = r_1(R_j|_{A - f_C(P)})$.*

Recall that in Section 2.3 the characterization of strategy-proof and neutral rules was dependent on the parity of n. This was due to the fact that a strategy-proof and surjective function defined on the Condorcet domain, $\mathscr{L}_C$, must satisfy unanimity when n is odd (as shown in the proof of Theorem 1 in Campbell and Kelly [4]). We will show this also holds in our current domain. Since profiles in $\mathscr{D}_C^*$ all have single elements on top, our definition of unanimity from the introduction still holds.

Lemma 4.1 *For any $m \geq 3$ and for any odd integer $n \geq 3$, if $f : \mathscr{D}_C^* \to A$ is strategy-proof and surjective then f satisfies unanimity.*

Proof:

Suppose $f : \mathscr{D}_C^* \to A$ is strategy-proof and surjective but does not satisfy unanimity. Then there exists a profile $P = (R_1, \ldots, R_n) \in \mathscr{D}_C^*$ such that $r_1(R_i) = x$ for $i = 1, \ldots, n$ but $f(P) \neq x$. Say $f(P) = y$ for some $y \in A \setminus \{x\}$. Note that $f_C(P) = x$. By strategy-proofness we can move y up to be the second ranked element in each preference and maintain $f(P) = y$ and $f_C(P) = x$. Let $Q = (Q_1, \ldots, Q_n)$ be a profile where each $Q_i \in L(A)$ with $r_1(Q_i) = x$ and $r_2(Q_i) = y$. Then $Q \in \mathscr{L}_C$ with $f_C(Q) = x$. Let $\{P^t : t = 0, \ldots, n\}$ be the standard sequence of profiles from P to Q. Note that $f_C(P^t) = x$ for $t = 0, \ldots, n$ and so all these profiles belong to the domain $\mathscr{D}_C^*$. Moreover, by strategy-proofness, $f(P^t) = y$ implies $f(P^{t+1}) = y$ for $t = 0, \ldots, n-1$. But $P^n = Q \in \mathscr{L}_C$ thus we get a contradiction since $f(Q) = x$ by Campbell and Kelly [4].

$\square$

With Lemma 4.1 established, we would like to show that when n is even and our function is strategy-proof, neutral, and does not satisfy unanimity then the output is never the Condorcet alternative; the function in Example 4.7 is one such function.

Lemma 4.2 *For any $m \geq 3$ and for n an even integer, if $f : \mathscr{D}_C^* \to A$ is strategy-proof, neutral, and violates unanimity, then $f(P) \neq f_C(P)$ for all $P \in \mathscr{D}_C^*$.*

Proof:

Assume that there exists a profile $P = (R_1, \ldots, R_n)$ belonging to $\mathscr{D}_C^*$ such that $f(P) = f_C(P) = x$. From $P \in \mathscr{D}_C^*$, we wish to derive a new profile that maintains that the Condorcet alternative is x. Let ℓ be a linear order on A such that $r_1(\ell) = x$. For each R_i from the profile P such that $r_2(R_i) = x$, construct a linear order, R_i', where $r_1(R_i') = r_1(R_i)$ and $r_2(R_i') = r_2(R_i) = x$. Consider a new profile $P'' \in \mathscr{D}_C^*$ formed in the following way:

$$R_i'' = \begin{cases} R_i' & \text{if } r_2(R_i) = x \\ \ell & \text{otherwise} \end{cases}$$

Let $\{P^t : t = 0, \ldots, n\}$ be the standard sequence of profiles from P to P''. Note that $f_C(P^t) = x$ for $t = 0, \ldots, n$ and so all the profiles belong to the domain $\mathscr{D}_C^*$. Then by strategy-proof $f(P^t) = x$ implies $f(P^{t+1}) = x$ for $t = \{0, \ldots, n-1\}$. But $P^n = P'' \in \mathscr{L}_C$ thus by Lemma 2.2, $f(P) \neq f_C(P)$ for all $P \in \mathscr{L}_C$. This contradiction completes the proof.

□

Utilizing Lemma 4.2 and Lemma 4.1, we believe that a generalization of the characterization of strategy-proof and neutral social choice functions can be extended from $\mathscr{L}_C$ to $\mathscr{D}_C^*$. It seems likely that strategy-proof and neutral functions would behave similarly in this domain. One reason, is we know we can construct a function that will be dictatorial (as we did in the proof of Theorem 2.4). This is due to the fact that the Gibbard-Satterthwaite Theorem extends nicely to voting with weak orders. Below is the definition of a weak dictatorship followed by the theorem. Both of these are taken from the work of Taylor [23] and translated into our current notation.

Definition 4.6 *We say that a social choice $f : W(A)^N \to A$ is a **weak dictatorship** if there exists a voter j such that for all $P \in W(A)^N$ $f(P) \in r_1(R_j)$. Then we call voter j a **weak dictator**.*

Theorem 4.2 (The Gibbard-Satterthwaite Theorem with Weak Orders) *For n a positive integer and $m \geq 3$, then any social choice function $f : W(A)^N \to A$ that is surjective and strategy-proof is a weak dictatorship.*

Theorem 4.2 requires that the number of alternatives be at least three, just as the original phrasing of the Gibbard-Sattherwaite. Therefore to utilize Theorem 4.2 to get a characterization of strategy-proof and neutral rules, we look at the number of alternatives being at least four so that we can define a function on a restriction of the domain $\mathscr{D}_C^*$. Below we present some notation for how we think such a restriction will need to be constructed.

Let $W(A)_2^1|_{A\setminus\{x\}}$ be the preferences from $W(A)_2^1$ with the alternative x deleted from the ordering with all other relative rankings maintained. To illustrate how these preferences will look consider the following example. Since we know how linear orders look when we delete a single alternative, we focus on preferences that have indifference.

Example 4.8 *Let $A = \{w, x, y, z\}$. Then the following preferences are three of the types of preferences with indifference from $W(A)_2^1$*

$$\begin{pmatrix} x & y & z \\ y & x & y \\ wz & wz & wx \end{pmatrix}$$

Now if we remove x from the ordering but maintain all other relative rankings we have the following preferences in $W(A)_2^1|_{A\setminus\{x\}}$

$$\begin{pmatrix} y & y & z \\ wz & wz & y \\ & & w \end{pmatrix}$$

Now we want to define a restriction of our domain. We have

$$\mathscr{D}_{C\setminus x}^* = \{P \in \mathscr{D}_C^* : R_i \in W_2^1(A)|_{A\setminus\{x\}} \text{ for all } i \in \{1, \ldots, n\}\}$$

With this notation in mind we present the following conjecture.

Conjecture 4.1 *For any* $m \geq 4$ *and for any even integer* $n \geq 4$, $f : \mathscr{D}_C^* \to A$ *is strategy-proof, neutral, and violates unanimity if and only if* f *is a majority avoiding dictatorship.*

Notice this conjecture is for $m \geq 4$. We do not know yet what strategy-proof and neutral functions will look like for $m = 3$ and an even number of voters. We hope to explore this question in the future.

CHAPTER 5
CONCLUSIONS

In this dissertation we explored social choice functions on Condorcet domains, paying particular attention to functions that satisfy the property of strategy-proofness. Throughout our work we considered what happens when we add or remove other desirable properties of social choice functions such as unanimity, neutrality, and whether a function satisfies anonymity or must be non-dictatorial.

Much of this work focused on presenting a characterization of majority rule as the only strategy-proof, neutral, and anonymous social choice function on the relevant domain ($\mathscr{L}_C, SP(A)^N$, and $\mathscr{D}_C$). Yet even though majority rule always satisfies these properties, there are certain cases where a function, distinct from majority rule, is characterized by these properties (see Proposition 2.1 and Example 3.10). Below we present one more example of a function defined on a subdomain of $\mathscr{L}_C$ that is strategy-proof, neutral, and anonymous yet is distinct from majority rule.

Example 5.1 *Let $A = \{a_1, a_2, a_3, a_4\}$ and*

$$\Sigma = \left\{ \begin{matrix} a_1 & a_2 & a_1 & a_2 \\ a_2 & a_1 & a_2 & a_1 \\ a_3 & a_3 & a_4 & a_4 \\ a_4 & a_4 & a_3 & a_3 \end{matrix} \right\}$$

Assume n is an odd integer then $\Sigma^N \subseteq \mathscr{L}_C$. Define $f : \Sigma^N \to A$ to be majority rule on $\{a_3, a_4\}$. Then f is strategy-proof, neutral, and anonymous. Clearly $f(P) \neq$

$f_C(P)$ for any profile $P \in \Sigma^N$.

Reducing the desired conditions for the social choice functions to just strategy-proofness and neutrality allowed us to arrive at a significant theorem that completely characterizes the class of functions on $\mathscr{L}_C$ that satisfy these two properties (see Section 2.3). This characterization raised the question of what this class of functions might look like when we allow for indifference (see Section 4.3), though we have not yet arrived at a complete characterization in this context.

We now conclude this dissertation with ideas for future work.

5.1 Future Work

5.1.1 Extending the MAC Rule

We believe that there is an extension to our definition of a majority avoiding committee rule that does not satisfy the condition of neutrality. Thus only our second condition in the definition would hold. We believe that this extension would require a committee that is formed for each alternative rather than a general committee structure which is what allows for neutrality to hold. Below is an example of what this type of rule may look like for $m = 3$. We believe this could extend relatively naturally to any size set of alternatives.

Example 5.2 *Let $A = \{a_1, a_2, a_3\}$ and define a social choice function $f : \mathscr{L}_C \to A$ as follows:*

If $f_C(P) = a_1$, then

$$f(P) = \begin{cases} a_2 & \text{if } N_{a_2 a_3}(P) = N \\ a_3 & \text{otherwise} \end{cases}$$

If $f_C(P) = a_2$, *then*

$$f(P) = \begin{cases} a_3 & \text{if } N_{a_3 a_1}(P) = N \\ a_1 & \text{otherwise} \end{cases}$$

If $f_C(P) = a_3$, *then*

$$f(P) = \begin{cases} a_1 & \text{if } N_{a_1 a_2}(P) = N \\ a_2 & \text{otherwise} \end{cases}$$

To see that this function is not neutral, consider the profile P *given below*

$$P = \begin{pmatrix} a_1 & a_1 & \dots & a_1 \\ a_2 & a_3 & \dots & a_3 \\ a_3 & a_2 & \dots & a_2 \end{pmatrix}$$

By definition of f, $f(P) = a_3$ *Let* $\phi = (a_2 a_3)$. *Then*

$$\phi(P) = \begin{pmatrix} a_1 & a_1 & \dots & a_1 \\ a_3 & a_2 & \dots & a_2 \\ a_2 & a_3 & \dots & a_3 \end{pmatrix}$$

Neutrality would imply that $f(\phi(P)) = a_2$ *but since* $N_{a_2 a_3} \neq N$, $f(\phi(P)) = a_3$.

5.1.2 Classes of Functions That Are Strategy-Proof, Anonymous, and Neutral

As mentioned in Chapter 3, we wonder what the class of functions that satisfy strategy-proofness, anonymity, and neutrality look like when we look at the domain $SP(A)_C$, the restriction of the single-peaked domain for and even number of voters where the Condorcet alternative always exists.

In Chapter 4, we noted that for two alternatives, we have yet to fully characterize the class of functions that satisfies strategy-proofness, anonymity, and neutrality when $m = 2$ and indifference is allowed. We have some work on this question that can be expanded upon to see if we can get a complete characterization.

5.1.3 Strategy-proof, Neutral and Non-Dictatorial

Our work often presented characterizations where we replaced the condition of anonymity with the weaker condition of a non-dictatorship. In the context of the single-peaked domain, we were able to present a characterization of majority rule as the only rule that satisfied these properties but that result was only for three voters (see Theorem 3.2). This leads to the following open problem:

- For $n \geq 3$, characterize the rules $f : SP(A)^N \to A$ which satisfy strategy-proofness, neutrality, and non-dictatorial.

Theorem 3.2 and Example 3.7 are first steps in solving this problem.

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